Differential Geometry and its Applications 3 (1993) 191–201
North-Holland

η-invariants on manifolds with cylindrical end*

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Communicated by P. B. Gilkey
Received 7 May 1992
Revised 8 August 1992

Klimek S. and K. P. Wojciechowski, η-invariants on manifolds with cylindrical end, Diff. Geom. Appl. 3 (1993) 191–201.

Abstract: We study the η-function of an operator A of Dirac type on a non-compact Riemannian manifold X_∞, which is obtained from a compact manifold X with boundary Y by attaching the infinite cylinder $X_\infty = (-\infty, 0] \times Y \cup_Y X$. We assume that the metric structure is a product on the cylinder and that the operator B, the tangential part of the operator A on the cylinder, is non-singular. We show that the η-function $\eta_A(s)$ shares all the properties of the η-function of an operator of Dirac type defined on a closed manifold. In particular, $\eta_A(s)$ is a holomorphic function for $\mathrm{Re}(s) > -2$.

Keywords: Operator of Dirac type, eta-function, heat equation, Duhamel's principle.

MS classification: 58G25, 58G10, 53G99, 35J99.

Introduction

In this paper we continue our study of the η-invariant on a manifold with boundary, initiated in [3] (see also [5,8]). We investigate the η-invariant of an operator of Dirac type on a manifold with cylindrical end. Our result implies the stronger version of the main results of paper [5, Theorem 0.1 and Theorem 0.2]. This version will be discussed elsewhere. The results of this paper are used in the new proof of the Holonomy Theorem of Witten (see [10]).

Let X denote an odd-dimensional compact Riemannian manifold with boundary Y and let S denote a bundle of Clifford modules over X. Let $A\colon C^\infty(X;S) \to C^\infty(X;S)$ denote a compatible Dirac operator type acting on the sections of S (see [2,4]; for detailed definitions). We assume that there exists a collar neighbourhood $N \cong [0,1] \times Y$ of the boundary Y in X such that the Riemannian structure on X and the Hermitian structure on S are product in N (i.e. they do not depend on the normal coordinate u,

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* Work supported in part by NSF grants no. DMS–9206936 and no. DMS–9105057.

when restricted to $Y_u = \{u\} \times Y$). The operator A has the following form on N:

$$A = G(\partial_u + B), \tag{0.1}$$

where $G\colon S\,|_Y \to S\,|_Y$ is a bundle isomorphism (Clifford multiplication by the inward normal vector) and $B\colon C^\infty(Y; S\,|_Y) \to C^\infty(Y; S\,|_Y)$ is the corresponding Dirac operator on Y. *In this paper we consider only the case of* $\ker B = \{0\}$ *i.e. B is an invertible elliptic operator.* G and B do not depend on the normal coordinate u in N and they satisfy the following identities

$$\begin{aligned} G^2 &= -\mathrm{Id}, \quad G \cdot B = -B \cdot G \\ G^* &= -G, \quad B^* = B. \end{aligned} \tag{0.2}$$

G is a skew-adjoint involution and S, the bundle of spinors, decomposes in N into $\pm i$-eigenspaces of G, $S = S^+ \oplus S^-$. It follows that (0.1) leads to the following representation of the operator A in N

$$A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \cdot \left(\partial_u + \begin{bmatrix} 0 & B_- = B_+^* \\ B_+ & 0 \end{bmatrix} \right), \tag{0.3}$$

where $B_+\colon C^\infty(Y; S^+) \to C^\infty(Y; S^-)$ maps spinors of positive "chirality" into the spinors of negative "chirality".

In [8] we studied the η-function of an operator A on X, where A was subject to certain boundary conditions. Here we attach the infinite cylinder $(-\infty, 0] \times Y$ to the manifold X and as a result we obtain a complete Riemannian manifold

$$X_\infty = (-\infty, 0] \times Y \cup_Y X. \tag{0.4}$$

We extend the bundle S to X_∞ in a natural way. The Riemannian structure on X and the Hermitian structure on S are product in N, hence we can extend them to X_∞ in a natural way and, at the end, we can extend the operator A to X_∞ by using formula (0.1). It is well-known that A has a unique self-adjoint extension in $L^2(X_\infty; S)$ (see [7]), which we also denote by A. We want to study the η-function of the operator A. We follow the standard definition and define $\eta_A(s)$ by the formula

$$\eta_A(s) = \Gamma\left(\frac{s+1}{2}\right) \cdot \int_0^\infty t^{(s-1)/2} \cdot \mathrm{Tr}(Ae^{-tA^2})dt. \tag{0.5}$$

This formula needs certain justification. First of all, the operator e^{-tA^2} has a smooth kernel, but now it is not a trace class operator. Fortunately, we have to deal with the trace of the operator Ae^{-tA^2} and here the situation is different. Let us consider the kernel $k(t; x, y)$ of the operator Ae^{-tA^2}. We obtain $k(t; x, y)$ by patching together $k_1(t; x, y)$, the kernel of the operator A extended to the closed double $\tilde{X}$ of X and $k_2(t; x, y)$, the kernel of the operator A on the infinite cylinder. It follows from (0.3) (see also [8, Lemma 2.2]) that $\mathrm{tr}\, k_2(t; x, x) = 0$. Therefore

$$\mathrm{Tr}\, Ae^{-tA^2} = \int_{X_\infty} \mathrm{tr}\, k(t; x, x)dx$$

consists of the interior contribution and the "error" term. We use Duhamel's principle to show that the "error" term does not provide singularities to the η-function *when B is a non-singular operator*. Therefore in this case the η-function shares all basic properties of the η-function of Dirac operators on closed manifolds. Let us also remark that in the even-dimensional case η-function of the compatible Dirac operator is identically 0.

Acknowledgments

The authors would like to thank Peter Gilkey for numerous discussions on the subject of this paper, which helped to improve the final formulation of the main results. Thanks are also due to Bernie Morrel. Discussions with him led us to the formulation and proof of Theorem 1.1.

1. Elementary estimates on $k(t;x,y)$

We have to establish an elementary estimate on the kernel $k(t;x,y)$ to be able to apply Duhamel's principle. Here is the result we need:

Theorem 1.1. *Let us assume that $d(x,y) > \delta > 0$, where $d(x,y)$ denotes the Riemannian distance between x and y. There exist positive constants c_1, c_2 such that for any $t > 0$*

$$\|k(t;x,y)\| \leqslant c_1 \cdot e^{-c_2 t}. \tag{1.1}$$

Theorem 1.1 is a consequence of two results. The first result describes the behavior of the kernel outside the diagonal, and for small time.

Proposition 1.2. *Let us assume that $d(x,y) > \delta > 0$ and that $0 < t < 1$. Then there exist positive constants c_1 and c_3 such that*

$$\|k(t;x,y)\| \leqslant c_1 \cdot e^{-c_3/t}. \tag{1.2}$$

The proof of this result is a straightforward application of Duhamel's Principle. To make this exposition complete we present this proof at the beginning of Section 2, once we have established the necessary notation.

The second result depends on the assumption that B is a non-singular operator. Let λ_0^2 denote the lowest eigenvalue of the operator B^2. It is well-known (see, for instance [5, Lemma 6.2]) that in this case the operator A has only discrete spectrum in the interval $(-\lambda_0, \lambda_0)$. Let μ_0^2 denote the smallest non-zero eigenvalue of the operator A^2 First let us observe the following elementary estimate:

Proposition 1.3. *For each $k = 0, 1, 2, \dots$, $A^k e^{-tA^2}$ is a bounded operator on $L^2(X_\infty, S)$ and there exists positive constant c such that following estimate holds*

$$\|A^k e^{-tA^2}\| \leqslant c \cdot t^{-k/2} \cdot e^{-t\mu_0^2}. \tag{1.3}$$

Proof. From the functional calculus we obtain

$$\|A^k e^{-tA^2}\| \leqslant \sup_{\lambda \geqslant \mu_0} (\lambda^k e^{-t\lambda^2}). \tag{1.4}$$

If $t \leqslant k/2\mu_0^2$, then

$$\sup_{\lambda \geqslant \mu_0} (\lambda^k e^{-t\lambda^2}) = \left(\frac{k}{2t}\right)^{k/2} \cdot e^{-k/2}$$

while if $t \geqslant k/2\mu_0^2$, then

$$\sup_{\lambda \geqslant \mu_0} (\lambda^k e^{-t\lambda^2}) = \mu_0^k \cdot e^{-t\mu_0^2}$$

and the claim follows. □

A special feature of a manifold with cylindrical end is that we may carry the theory of Sobolev spaces exactly as in the case of $\mathbb{R}^n$. The point is that we can choose the covering of X_∞ by a finite number of coordinate charts. We can also choose a finite trivialization of the bundle S. Let $\{U_m, \Phi_m\}_{m=1}^K$ denote such a trivialization, where $\Phi_m \colon S\ |_{U_m} \to U_m \times \mathbb{C}^N$ is a bundle isomorphism and U_m denotes an open (possibly non-compact) subset of $\mathbb{R}^n$. Let $\{\varphi_m\}$ denote the corresponding partition of unity. We assume that for any m the derivatives of the function φ_m are bounded. We say that section s of the bundle S belongs to the kth Sobolev space if and only if $\varphi_m \cdot s$ belongs to $H^k(\mathbb{R}^n; \mathbb{C}^N)$ for any m. We define the kth Sobolev norm

$$\|s\|_k = \sum_{m=1}^{K} \|(\mathrm{Id} + \Delta_m)^{k/2}(\varphi_m \cdot s)\|_{L^2}, \tag{1.5}$$

where Δ_m denotes the Laplacian on the trivial bundle $U_m \times \mathbb{C}^N \subset \mathbb{R}^n \times \mathbb{C}^N$.

Lemma 1.4. *The operator*

$$\mathrm{Id} + A^2 \colon H^2(X_\infty; S) \to L^2(X_\infty; S)$$

is an isomorphism of Hilbert spaces.

Proof. Let s denote an element of $H^2(X_\infty; S)$. We have

$$\begin{aligned} \|A^2 s\|_{L^2} &= \|\sum_{m=1}^{K} A^2(\varphi_m s)\| \leqslant \sum_{m=1}^{K} \|A^2(\varphi_m s)\| \\ &\leqslant c \cdot \sum_{m=1}^{K} \|(\mathrm{Id} + \Delta_m)(\varphi_m s)\| \leqslant c \cdot \|s\|_{H^2}. \end{aligned} \tag{1.6}$$

Now let us assume that $s \in \mathrm{Dom}(A^2)$. We have

$$\begin{aligned}
\|A^2(\varphi_m s)\| &= \|\varphi_m(A^2 s) + 2\cdot[A;\varphi_m](As) + [A;[A;\varphi_m]]s\| \\
&\leqslant \|\varphi_m(A^2 s)\| + 2\cdot\|[A;\varphi_m](As)\| + \|[A;[A;\varphi_m]]s\| \\
&\leqslant \|\varphi_m(A^2 s)\| + 2\cdot\|[A;\varphi_m]\cdot(A(\mathrm{Id}+A^2)^{-1})\cdot(\mathrm{Id}+A^2)s)\| \\
&\quad + \|[A;[A;\varphi_m]]s\| \\
&\leqslant c_1\|A^2 s\| + c_2\|(\mathrm{Id}+A^2)s\| + c_3\|s\| \leqslant c_4\cdot\{\|A^2 s\| + \|s\|\},
\end{aligned} \tag{1.7}$$

for certain positive constants c_1, c_2, c_3, c_4. Hence $\varphi_m s \in \mathrm{Dom}(A^2)$, but, on U_m, we have the obvious inequality

$$\|(\mathrm{Id}+A^2)(\varphi_m s)\| \geqslant c_5\cdot\|(\mathrm{Id}+\Delta_m)(\varphi_m s)\|. \tag{1.8}$$

Therefore $s = \sum_{m=1}^{K}\varphi_m s$ is an element of $H^2(X_\infty; S)$. □

It is obvious that for any k, $A^k\colon H^{m+k}(X_\infty;S) \to H^m(X_\infty;S)$ is a bounded operator and that $(\mathrm{Id}+A^2)^k\colon H^{m+2k}(X_\infty;S) \to H^m(X_\infty;S)$ is an isomorphism of Hilbert spaces. After this brief discussion of Sobolev spaces we prove the following theorem.

Theorem 1.5. *There exists N, such that the following estimate holds for any $x, y \in X_\infty$ and any $t > 0$:*

$$\|k(t;x,y)\| \leqslant c_1\cdot t^{-N/2}\cdot e^{-t\mu_0^2}. \tag{1.9}$$

Proof. Let $\dim X_\infty = 2k-1$. For any $x \in X_\infty$, the Dirac delta function δ_x is an element of the space $H^{-k}(X_\infty;S)$. Let $\delta_{\epsilon,x}$ denote an approximation of δ_x by smooth functions with compact supports. There exists a constant $c > 0$ such that

$$\|(\mathrm{Id}+A^2)^{-k/2}\delta_{\epsilon,x}\|_{L^2} \leqslant c, \tag{1.10}$$

for any x and any $0 < \epsilon < 1$, and we have

$$\begin{aligned}
\|k(t;x,y)\| &= \|\langle\delta_x; Ae^{-tA^2}(\delta_y)\rangle\| \\
&= \|\langle(\mathrm{Id}+A^2)^{-k/2}(\delta_x); (\mathrm{Id}+A^2)^k Ae^{-tA^2}((\mathrm{Id}+A^2)^{-k/2}(\delta_y))\rangle\| \\
&= \lim_{\epsilon\to 0}\|((\mathrm{Id}+A^2)^{-k/2}(\delta_{\epsilon,x}); (\mathrm{Id}+A^2)^k Ae^{-tA^2}\cdot((\mathrm{Id}+A^2)^{-k/2}(\delta_{\epsilon,y})))\| \\
&\leqslant \lim_{\epsilon\to 0}((\mathrm{Id}+A^2)^{-k/2}(\delta_{\epsilon,x})\|\cdot\|(\mathrm{Id}+A^2)^k Ae^{-tA^2}\|\cdot\|(\mathrm{Id}+A^2)^{-k/2}(\delta_{\epsilon,y})\| \\
&\leqslant c^2\cdot\|(\mathrm{Id}+A^2)^k Ae^{-tA^2}\| \\
&\leqslant c_1\cdot t^{-(2k+1)/2}\cdot e^{-t\mu_0^2}. \quad □
\end{aligned} \tag{1.11}$$

Theorem 1.1 follows directly from Proposition 1.2 and Theorem 1.5.

2. Duhamel's principle. The existence of the η-invariant

Now we are ready to use Duhamel's principle in order to study the behavior of the η-function. We follow the strategy of [5] (see Section 2 of [5]). Let $E_1(t;x,y)$ denote the kernel of the operator $e^{-t\tilde{A}^2}$, where $\tilde{A}$ denotes the double of the operator A (see [5, Section 2]; see also [4,9]). The operator $\tilde{A}$ lives on the closed double $\tilde{X}$ of the manifold X. We also introduce $E_2(t;x,y)$, the kernel of the operator $e^{-tA^2_\infty}$, where A_∞ denotes the operator $G(\partial_u + B)$ on the infinite cylinder $(-\infty,+\infty)\times Y$. We define a "parametrix" $Q(t;x,y)$ for the kernel $\mathcal{E}(t;x,y)$ of the operator e^{-tA^2} on X_∞ as

$$Q(t;x,y) = \varphi_1(x)E_1(t;x,y)\psi_1(y) + \varphi_2(x)E_2(t;x,y)\psi_2(y), \tag{2.1}$$

where $\{\psi_1,\psi_2\}$ is a smooth partition of unity on X_∞, such that $\psi_1 = 1$ on $X\backslash N$ and $\psi_2 = 1$ on $(-\infty,\frac{1}{2}]\times Y$, and $\varphi_i\colon X_\infty \to [0,1]$, such that $\varphi_i = 1$ on $\operatorname{supp}(\psi_i)$, $\psi_i(x) = 0$ for $d(x,\operatorname{supp}(\psi_i)) \geqslant 1$. Furthermore we assume that there exists $\epsilon > 0$, such that $d(\operatorname{supp}(\partial\varphi_i/\partial u);\operatorname{supp}(\psi_i)) > \epsilon$, where $d(x,y)$ denotes the Riemannian distance.

Lemma 2.1. *There exist positive constants c_1, c_2, c_3 such that we have the following estimate on the kernels $E_i(t;x,y)$*

$$||E_i(t;x,y)|| \leqslant c_1 t^{-n} e^{-c_2 t} e^{-c_3 d^2(x,y)/t}. \tag{2.2}$$

Remark 2.2. We refer to [5, Section 1] for the proof of (2.2) in the case of a closed manifold and invertible operator. The invertibility of the operator $\tilde{A}$ on the manifold $\tilde{X}$ follows from the construction of $\tilde{A}$ and Green's formula (see [9]; see [4] for a detailed exposition of the related topics). This covers the case of $E_1(t;x,y)$ and implies the result for $E_2(t;x,y)$.

Now we have

$$\begin{aligned} A\mathcal{E}(t;x,x) \\ &= AQ(t;x,x) + \int_0^t ds \int_{\operatorname{supp}_z C(t;z,x)} dz A\mathcal{E}(s;x,z)\cdot C(t-s;z,x) \qquad (2.3) \\ &= AQ(t;x,x) + (A\mathcal{E}\#C))(t;x,x). \end{aligned}$$

Formula (2.3) needs explanation. The operator A acts here on the first space variable of the kernel, hence we really have

$$AQ(t;x,x) = A_x\mathcal{E}(t;x,y)_{|x=y}.$$

$C(t;x,y)$ denotes the "error" term in the formula

$$\mathcal{E}(t;x,y) = Q(t;x,y) + \int_o^t ds \int_{\operatorname{supp}_z C(t;z,x)} dz\mathcal{E}(s;x,z)\cdot C(t-s;z,y). \tag{2.4}$$

Direct computation gives the following expression for $C(t;x,y)$

$$
\begin{aligned}
C(t;x,y) = -\sum_{k=1}^{2}\{&(\partial^2\varphi_i/\partial u)(x)E_i(t;x,y)\psi_i(y) \\
&+ 2\cdot(\partial\varphi_i/\partial u)(x)(\partial E_i/\partial u)(t;x,y)\psi_i(y)\},
\end{aligned}
\tag{2.5}
$$

where u denotes the normal variable of the coordinate x.

The next lemma follows immediately from Lemma 2.1, the choice of functions φ_i and ψ_i and (2.5).

Lemma 2.3. *$C(t;x,y) = 0$ for $d(x,y) > \epsilon$. The support of C with respect to the variable x is contained in the collar N and there exist positive constants c_1, c_2, c_3 (we can assume that those are the constants from the estimate (2.2)) such that:*

$$\|AC(t;x,y)\| \leqslant c_1 e^{-c_2 t} e^{-c_3 d^2(x,y)/t}. \tag{2.6}$$

Proof of Proposition 1.2. We have

$$k(t;x,y) = AQ(t;x,y) + \sum_{k=1}^{\infty}(AQ\#C_k)(t;x,y), \tag{2.7}$$

where

$$C_1(t;x,y) = C(t;x,y) \quad \text{and} \quad C_k = C_{k-1}\#C,$$

and we define

$$\alpha\#\beta(t;x,y) = \int_0^t ds \int_{X_\infty} dz\, \alpha(s;x,z)\beta(t-s;z,y).$$

The assumption $d(x,y) > \delta$ together with (2.2) implies the estimate (2.6) for the kernel $AQ(t;x,y)$ and we have

$$
\begin{aligned}
&\|(AQ\#C)(t;x,y)\| \\
&\quad \leqslant \int_0^t ds \int_{\mathrm{supp}_z C(t-s;z,y)} dz\, c_1 e^{c_2 s} \cdot e^{-c_3 d^2(x,z)/s} \\
&\qquad \cdot c_1 e^{-c_2(t-s)} \cdot e^{-c_3 d^2(z,y)/(t-s)} \\
&\quad \leqslant c_1^2 \cdot e^{-c_2 t} \cdot e^{-c_3 d^2(x,y)/t} \cdot \mathrm{vol}(\mathrm{supp}_z C(t-s;z,y)) \cdot \int_0^t ds \\
&\quad \leqslant c_1(c_1 \cdot t \cdot \mathrm{vol}(Y)) \cdot e^{-c_2 t} \cdot e^{-c_3 d^2(x,y)/t}.
\end{aligned}
\tag{2.8}
$$

In the same way we obtain the estimate

$$\|AQ\#C_k)(t;x,y)\| \leqslant c_1 \frac{(c_1 \cdot t \cdot \mathrm{vol}(Y))^k}{k!} \cdot e^{-c_2 t} \cdot e^{-c_3 d^2(x,y)/t}. \tag{2.9}$$

We apply (2.7) and (2.9) in order to obtain the estimate

$$\|k(t;x,y)\| \leqslant c_1 \cdot e^{c_4 t} \cdot e^{-c_3 d^2(x,y)/t} \leqslant c_1 \cdot e^{c_4 t} \cdot e^{-c_3 \delta^2/t}, \tag{2.10}$$

and the Proposition 1.2 follows. □

Now, we are ready to prove the crucial technical result. We want to use formula (2.3) in order to show that the leading contribution to $\eta_A(s)$ comes from the term $\operatorname{Tr} AQ(t;x,y)$, hence from the corresponding trace of the heat kernel on the closed manifold.

Theorem 2.4. *We have the following estimate of the "error" term in the formula* (2.3)

$$\|(A\mathcal{E}\#C)(t;x,x)\| \leqslant c_4 \cdot e^{-c_5 t} \cdot e^{-c_3 \cdot \max(d(x;\operatorname{supp}_z C);\epsilon)/t}. \tag{2.11}$$

Proof. Let us first consider the case $x \in X$. We use here the fact $C(t;z,x)$ is 0 for $d(z,x) < \epsilon$.

$$\begin{aligned} &\|(A\mathcal{E}\#C)(t;x,x)\| \\ &\leqslant \int_0^t ds \int_{\substack{\operatorname{supp}_z C(t-s;z,x) \\ d(x,z)>\epsilon}} dz\, c_1 e^{-c_2 s} \cdot c_1 e^{-c_2(t-s)} \cdot e^{-c_3 d^2(z,x)/(t-s)} \\ &\leqslant c_1^2 \cdot e^{-c_2 t} \cdot \operatorname{vol}(Y) \cdot \int_0^t e^{-c_3 \epsilon^2/(t-s)} \\ &\leqslant c_1^2 \cdot e^{-c_2 t} \cdot \operatorname{vol}(Y) \cdot \int_0^t e^{-c_3 \epsilon^2/s} ds \\ &\leqslant c_1^2 \cdot e^{-c_2 t} \cdot \operatorname{vol}(Y) \cdot t \cdot e^{-c_3 \epsilon^2/t} \leqslant c_4 \cdot e^{-c_5 t} \cdot e^{-c_5 \cdot \epsilon^2/t}. \end{aligned} \tag{2.12}$$

In (2.12) we have used the elementary inequality

$$\int_0^t e^{-c/s} ds \leqslant t \cdot e^{-c/t} \quad \text{for} \quad c > 0. \tag{2.13}$$

Now we have to consider the case of $x = (u,y) \in (-\infty,0] \times Y$. In this case we have the inequality

$$d(x, \operatorname{supp}_z C(t;z,x)) \geqslant \epsilon - u. \tag{2.14}$$

We apply (2.14) to obtain

$$\begin{aligned} &\|(A\mathcal{E}\#C)(t;x,x)\| \\ &\leqslant \int_0^t ds \int_{\operatorname{supp}_z C(t-s;z,x)} dz\, c_1 e^{-c_2 s} \cdot c_1 e^{-c_2(t-s)} \cdot e^{-c_3 d^2(z,x)/(t-s)} \\ &\leqslant c_1^2 \cdot e^{-c_2 t} \cdot \operatorname{vol}(Y) \int_0^t e^{-c_3(\epsilon-u)/(t-s)} ds \\ &\leqslant c_1^2 \cdot e^{-c_2 t} \cdot t \cdot e^{-c_3(\epsilon-u)/t} \leqslant c_4 \cdot e^{-c_5 t} \cdot e^{-c_3(\epsilon-u)/t}. \end{aligned} \tag{2.15}$$

(2.12) and (2.15) imply (2.11). □

Corollary 2.5. *The function $f_A(s)$ given by formula*

$$f_A(s) = \int_0^\infty t^s(\operatorname{Tr} Ae^{-tA^2} - \operatorname{Tr} AQ(t))dt \tag{2.16}$$

is a holomorphic function of s on the whole complex plane.

Proof. We have the following estimate

$$\begin{aligned} |f_A(s)| &\doteq \left|\int_0^\infty t^s(\operatorname{Tr} Ae^{-tA^2} - \operatorname{Tr} AQ(t))dt\right| \\ &\leqslant \left|\int_0^\infty t^s(\operatorname{Tr}(A\mathcal{E}\#C)(t)dt\right| \\ &\leqslant \int_0^\infty t^{\mathrm{Re}(s)}dt \int_{X_\infty} \|(A\mathcal{E}\#C)(t;x,x)\|dx \\ &\leqslant \int_0^\infty t^{\mathrm{Re}(s)} \cdot \left\{\int_X \|(A\mathcal{E}\#C)(t;x,x)\|dx \right. \\ &\quad \left. + \int_{(-\infty,0]\times Y} \|(A\mathcal{E}\#C)(t;x,x)\|dx\right\}dt. \end{aligned}$$

It follows from (2.11) that the integral $\int_X \|(A\mathcal{E}\#C)(t;x,x)\|dx$ is bounded by $c_4e^{-c_5t}\cdot e^{-c_3\delta^2/t}\cdot \mathrm{vol}(X)$ and now we use (2.15) to estimate the second integral

$$\begin{aligned} &\int_{(\infty,0]\times Y} \|(A\mathcal{E}\#C)(t;x,x)\|dx\}dt \\ &\leqslant c_4 \cdot^{-c_5t}\cdot \mathrm{vol}(Y)\cdot \int_0^\infty e^{-c_3(\epsilon+u)^2/t}du \\ &\leqslant c_4\cdot e^{-c_5t}\cdot \mathrm{vol}(Y)\cdot \int_\epsilon^\infty e^{-c_3u^2/t}du \\ &\leqslant c_4e^{-c_5t}\cdot (t/c_3)^{1/2}\cdot \int_{(c_3\epsilon^2/t)^{1/2}}^\infty e^{-v^2}dv \\ &\leqslant (c_4/2c_3\epsilon)\cdot t\cdot e^{-c_5t}\cdot \int_{(c_3\epsilon^2/t)^{1/2}}^\infty (e^{-v^2})'dv \\ &\leqslant c_5\cdot e^{-c_6t}\cdot e^{-c_3\epsilon^2/t}. \end{aligned} \tag{2.17}$$

It follows that for any s we have the estimate

$$\|f_A(s)\| \leqslant c_7\cdot \int_0^\infty t^{\mathrm{Re}(s)}\cdot e^{-c_6t}\cdot e^{-c_3\epsilon^2/t}dt. \quad \square \tag{2.18}$$

Now we can establish the desired properties of the η-function.

Theorem 2.6. *$\eta_A(s)$, the η-function of the operator A on the manifold X_∞, is a well-defined holomorphic function for* $\mathrm{Re}(s) > -2$. *It has a meromorphic extension to the* *whole complex plane with simple poles at the points* $-2k = -2, -4, \ldots$. *The residue of* $\eta_A(s)$ *at the point* $s = -2k$ *is determined in the following way. Let* $r_{2k}(\tilde{A})(x)$ *denote the residue of* $\eta_{\tilde{A}}(s;x)$, *the "local"* η*-function of the operator* $\tilde{A}$ *on the manifold* $\tilde{X}$. *Then* $r_{2k}(\tilde{A})(x)$ *are given by formulas in terms of the total symbol of the operator* $\tilde{A}$ *at the point* x *(see* [2,6]*). In particular* $r_{2k}(x) = 0$ *in the collar neighbourhood* N. *We have*

$$r_{2k}(A) = \mathrm{Res}_{s=-2k}\, \eta_A(s) = \int_X r_{2k}(\tilde{A})(x)dx. \tag{2.19}$$

Proof. We have

$$\begin{aligned}
\eta_A(s) &= \Gamma\Big(\frac{s+1}{2}\Big) \cdot \int_0^\infty t^{(s-1)/2} \cdot \mathrm{Tr}\, Ae^{-tA^2} dt \\
&= \Gamma\Big(\frac{s+1}{2}\Big) \cdot \int_0^\infty t^{(s-1)/2} \cdot \mathrm{Tr}(AQ(t) + (A\mathcal{E}\#C)(t))dt \\
&= \Gamma\Big(\frac{s+1}{2}\Big) \cdot \int_0^\infty t^{(s-1)/2} \cdot \mathrm{Tr}(A\mathcal{E}\#C)(t)))dt \\
&\quad + \Gamma\Big(\frac{s+1}{2}\Big) \cdot \int_0^\infty dt \cdot t^{(s-1)/2} \\
&\quad \cdot \int_{X_\infty} \mathrm{Tr}\, A_x(\varphi_1(x)E_1(t;x,y)\psi_1(y))\,|_{x=y}\, dx \qquad (2.20) \\
&= \Gamma\Big(\frac{s+1}{2}\Big) \cdot \int_0^\infty t^{(s-1)/2} \cdot \mathrm{Tr}(A\mathcal{E}\#C)(t))dt \\
&\quad + \Gamma\Big(\frac{s+1}{2}\Big) \cdot \int_o^\infty dt \cdot t^{(s-1)/2} \cdot \int_{X_\infty} \varphi_1(x) \cdot \mathrm{tr}(A_x E_1(t;x,y)\psi_1(y))\,|_{x=y}\, dx \\
&\quad + \Gamma\Big(\frac{s+1}{2}\Big) \cdot \int_o^\infty dt \cdot t^{(s-1)/2} \\
&\quad \cdot \int_{X_\infty} (\delta\varphi_1/\delta u)(x)) \cdot \psi_1(x) \cdot \mathrm{tr}\, G(x)E_1(t;x,x)dx
\end{aligned}$$

The last term is 0 because $d(\mathrm{supp}(\delta\varphi_1/\delta u), \mathrm{supp}(\psi_1)) > \epsilon$ and we are left with the η-density of the Dirac operator on a closed manifold, which is holomorphic for $\mathrm{Re}(s) > -2$ (see [2,6]). Now the result follow from the analysis on a closed manifold (see [2, Theorem 0.2, Theorem 3.4 and Theorem 3.11]). □

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