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On the Wiener integral with respect to a sub-fractional Brownian motion on an interval

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ABSTRACT

The domain $\Lambda_{k,T}^{sf}$ of the Wiener integral with respect to a sub-fractional Brownian motion $(S_t^k)_{t\in[0,T]}$, $k\in(-\frac{1}{2},\frac{1}{2})$, $k\neq 0$, is characterized. The set $\Lambda_{k,T}^{sf}$ is a Hilbert space which contains the class of elementary functions as a dense subset. If $k\in(-\frac{1}{2},0)$, any element of $\Lambda_{k,T}^{sf}$ is a function and if $k\in(0,\frac{1}{2})$, the domain $\Lambda_{k,T}^{sf}$ is a space of distributions.

1. Introduction

The *fractional Brownian motion* (fBm for short) is the best known and most used process with long-dependence property for models in telecommunication, turbulence, finance, etc. This process was first introduced by Kolmogorov [9] and later studied by Mandelbrot and his coworkers [10]. The fBm is a continuous centered Gaussian process $(B_t^k)_{t\in R}$, starting from zero, with covariance

$$C_{B^k}(s,t)=\frac{1}{2}\left(|s|^{2k+1}+|t|^{2k+1}-|t-s|^{2k+1}\right),\quad s,t\in R,\tag{1.1}$$

where $k\in(-\frac{1}{2},\frac{1}{2})$ ($H=k+\frac{1}{2}$ is called Hurst parameter). The case $k=0$ corresponds to the Brownian motion.

The self-similarity and stationarity of the increments are two main properties for which fBm enjoyed success as a modeling tool. The fBm is the only continuous Gaussian process which is self-similar and has stationary increments.

An extension of Bm which preserves many properties of the fBm, but not the stationarity of the increments, has been proposed in [1,4].

In [1] the so-called *sub-fractional Brownian motion* (*sfBm* for short) is introduced as a continuous Gaussian process $(S_t^k)_{t\geqslant 0}$, starting from zero, with covariance

$$C_{S^k}(s,t)=s^{2k+1}+t^{2k+1}-\frac{1}{2}\left[(s+t)^{2k+1}+|t-s|^{2k+1}\right],\quad s,t\geqslant 0.\tag{1.2}$$

For $k>0$ the sfBm arises from occupation time fluctuations of branching particle systems (see [1–3]).

Some basic properties of sub-fBm are given in [1,4,19].

In particular in [4] the even and parts of fBm (the even part is the sfBm up to a multiplicative constant) are considered and series expansions of the covariance functions and also of each of the above mentioned processes are obtained by means of Fourier–Bessel theory.

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The sfBm has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths, the variation and the renormalized variation and it is neither a Markov processes nor a semimartingale).

In comparison with the fBm the sfBm has non-stationary increments and the increments over non-overlaping intervals are more weakly correlated and their covariance decays polynomially at a higher rate in comparison with fBm (for this reason in [1] is called the sfBm):

(i_1) *Covariance*: For all $s, t \geqslant 0$,

$$C_{S^k}(s,t) > C_{B^k}(s,t) \quad \text{if } k \in \left(-\frac{1}{2}, 0\right),$$
$$C_{S^k}(s,t) < C_{B^k}(s,t) \quad \text{if } k \in \left(0, \frac{1}{2}\right).$$

(i_2) *Non-stationarity of increments*: For all $s \leqslant t$,

$$E\big[\big|S_t^k - S_s^k\big|^2\big] = -2^{2k}\big(t^{2k+1} + s^{2k+1}\big) + (t+s)^{2k+1} + (t-s)^{2k+1},$$
$$E\big(\big|S_t^k\big|^2\big) = \big(2 - 2^{2k}\big)t^{2k+1}.$$

(i_3) *Correlation of increments*: For $0 \leqslant u < v \leqslant s < t$, define

$$R_{u,v,s,t}^k = E\big[\big(B_v^k - B_u^k\big)\big(B_t^k - B_s^k\big)\big],$$
$$C_{u,v,s,t}^k = E\big[\big(S_v^k - S_u^k\big)\big(S_t^k - S_s^k\big)\big].$$

Then

$$C_{u,v,s,t}^k = \frac{1}{2}\big[(t+u)^{2k+1} + (t-u)^{2k+1} + (s+v)^{2k+1} + (s-v)^{2k+1} - (t+v)^{2k+1} - (t-v)^{2k+1} - (s+u)^{2k+1} - (s-u)^{2k+1}\big],$$
$$R_{u,v,s,t}^k < C_{u,v,s,t}^k < 0 \quad \text{if } k \in \left(-\frac{1}{2}, 0\right),$$
$$0 < C_{u,v,s,t}^k < R_{u,v,s,t}^k \quad \text{if } k \in \left(0, \frac{1}{2}\right),$$

and

$$R_{u,v,s+\tau,t+\tau}^k \sim k(2k+1)(t-s)(v-u)\tau^{2k-1} \quad \text{as } \tau \to \infty,$$
$$C_{u,v,s+\tau,t+\tau}^k \sim k(2k+1)(1-2k)\big(v^2 - u^2\big)\tau^{2(k-1)} \quad \text{as } \tau \to \infty.$$

The above mentioned properties make sfBm a possible candidate for models which involve long-dependence, self-similarity and non-stationarity.

It is well known that in the Brownian case $L^2(R)$ is the space of integrands for the Wiener integral. In the case of fBm in [14,15] the same problem is addressed. For $k \in (-\frac{1}{2}, 0)$ in [15] it is found the entire domain and for $k \in (0, \frac{1}{2})$ an enough large classes of functions as integrands is determined. The corresponding families of integrands are defined in terms of fractional integrals and derivatives. So far, it was not clear in the case $k \in (0, \frac{1}{2})$ which is the whole domain of the Wiener integral for fBm. Recently, in [7] the problem of the domain of the Wiener integral for fBm is completely solved for every k. In the case $k \in (0, \frac{1}{2})$ the domain is a space of tempered distributions in some Sobolev space.

For sfBm with the infinite time interval R_+, the domain of the corresponding Wiener integral is described in [20].

In this paper we characterize the domain of the Wiener integral with respect to a sfBm motion $(S_t^k)_{t\in[0,T]}$, $k \in (-\frac{1}{2}, \frac{1}{2})$, $k \neq 0$ (Theorem 3.2). The domain $\Lambda_{k,T}^{sf}$ is a Hilbert space which contains the class of elementary functions as a dense subset. If $k \in (-\frac{1}{2}, 0)$, any element of $\Lambda_{k,T}^{sf}$ is a function and if $k \in (0, \frac{1}{2})$, the domain $\Lambda_{k,T}^{sf}$ is a space of distributions. In the later case we provide relevant examples of subspaces of functions included in the domain. One area of applications of the Wiener integral, involves prediction problems.

Like in the case of fBm, the finite time interval set-up considered here is quite different from that of infinite interval. For instance, the infinite time relies mainly upon Riemann–Liouville fractional derivatives and integrals and for finite interval, the Erdély–Kober-type fractional derivatives and integrals and Hankel transform represent a basic tool.

Multiple subfractional integrals can be studied along the same lines.

2. Preliminaries

For $k \in (-\frac{1}{2}, \frac{1}{2})$, $k \neq 0$, we consider a sfBm $(S_t^k)_{t\in[0,T]}$.

Let $f : [0,T] \to R$ be a measurable application and $\alpha \in R$, $\sigma, \eta \in R$. We define the *Erdély–Kober-type fractional integral*

$$\left(I^{\alpha}_{T-,\sigma,\eta} f\right)(s) = \frac{\sigma s^{\sigma\eta}}{\Gamma(\alpha)} \int_s^T \frac{t^{\sigma(1-\alpha-\eta)-1} f(t)}{(t^{\sigma} - s^{\sigma})^{1-\alpha}}\, dt, \quad s \in [0,T],\ \alpha > 0, \tag{2.1}$$

$$\left(I^{\alpha}_{T-,\sigma,\eta} f\right)(s) = s^{\sigma\eta}\left(-\frac{d}{\sigma s^{\sigma-1}\, ds}\right)^n s^{\sigma(n-\eta)}\left(I^{\alpha+n}_{T-,\sigma,\eta-n} f\right)(s), \quad s \in [0,T],\ \alpha > -n, \tag{2.2}$$

$$\left(I^{\alpha}_{0+,\sigma,\eta} f\right)(s) = \frac{\sigma s^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^s \frac{t^{\sigma(1+\eta)-1} f(t)}{(t^{\sigma} - s^{\sigma})^{1-\alpha}}\, dt, \quad s \in [0,T],\ \alpha > 0, \tag{2.3}$$

$$\left(I^{\alpha}_{0+,\sigma,\eta} f\right)(s) = s^{-\sigma(\alpha+\eta)}\left(\frac{d}{\sigma s^{\sigma-1}\, ds}\right)^n s^{\sigma(n+\alpha+\eta)}\left(I^{\alpha+n}_{0+,\sigma,\eta} f\right)(s), \quad s \in [0,T],\ \alpha > -n. \tag{2.4}$$

The basic properties of fractional integrals of deterministic functions can be found in [16].

We recall the following properties:

(i$_1$) (*Continuity*) The operators (2.1), (2.3) are continuous in $L^p([0,T])$ for every $1 \leqslant p < \infty$.

(i$_2$) (*Inverse operators*)

$$\left(I^{\alpha}_{T-,\sigma,\eta}\right)^{-1} = I^{-\alpha}_{T-,\sigma,\eta+\alpha}, \qquad \left(I^{\alpha}_{0+,\sigma,\eta}\right)^{-1} = I^{-\alpha}_{0+,\sigma,\eta+\alpha}.$$

(i$_3$) (*Integration by parts formula*)

$$\int_0^T x^{\sigma-1} f(t) I^{\alpha}_{0+,\sigma,\eta} g(t)\, dt = \int_0^T x^{\sigma-1} g(t) I^{\alpha}_{b-,\sigma,\eta} g(t)\, dt.$$

We introduce the following kernel

$$n(t,s) = \frac{\sqrt{\pi}}{2^k} I^k_{T-,2,\frac{1-k}{2}}\left(u^k 1_{[0,t)}\right)(s), \tag{2.5}$$

and we fix a Brownian motion $(W_t)_{t\in[0,T]}$.

By using the Hankel transform in [4] the following result is established (see [13] for the case of fBm).

Theorem 2.1. *We have the Wiener integral representation*

$$S_t^k \overset{d}{=} c_k \int_0^1 n(t,s)\, dW_s, \quad t \in [0,T], \tag{2.6}$$

$$c_k^2 = \frac{\Gamma(2k+2)\sin\pi(k+\frac{1}{2})}{\pi} \tag{2.7}$$

($\overset{d}{=}$ *means the equality in law*).

Next we assume that S^k is a sfBm given pathwise by the right-hand side of (2.6).

Denote $\mathcal{E}_T$ the family of elementary functions $f : [0,T] \to R$,

$$f = \sum_{j=1}^{N-1} a_j 1_{[t_j,t_{j+1})}, \quad 0 = t_0 < t_1 < \cdots < t_N = T,\ a_j \in R. \tag{2.8}$$

For f as above we define the Wiener integral $\int_0^T f(t)\, dS_t^k$ in the natural way by

$$\int_0^T f(t)\, dS_t^k = \sum_{j=1}^{N-1} a_j\left(S^k_{t_{j+1}} - S^k_{t_j}\right). \tag{2.9}$$

We endow $\mathcal{E}_T$ with the inner product

$$\langle f, g\rangle_{\Lambda^{sf}_{k,T}} = E\left[\int_0^T f(t)\,dS^k_t \int_0^T g(t)\,dS^k_t\right] \tag{2.10}$$

(we identify the elements f and g when $\langle f-g, f-g\rangle_{\Lambda^{sf}_{k,T}} = 0$).

For $f \in \mathcal{E}_T$ as in (2.8) we introduce the following signed measure

$$\mu_f = \sum_{j=1}^{N-1} (a_j - a_{j-1})\varepsilon_{t_j} + f(0+)\varepsilon_0 + f(T-)\varepsilon_T, \tag{2.11}$$

ε_a being the Dirac measure concentrated at a, that is $\varepsilon_a(A) = 1_A(a)$.

Remark 2.2. It is easily seen that

$$\int_0^T f(t)\,dS^k_t = -\int_0^T S^k_t\,d\mu_f(t), \quad f \in \mathcal{E}_T, \tag{2.12}$$

and therefore

$$\langle f, g\rangle_{\Lambda^{sf}_{k,T}} = \int_0^T\int_0^T C_k(s,t)\,d\mu_f(s)\,d\mu_g(t). \tag{2.13}$$

Next, if $f:[0,T]\to R$ we denote by $f^o: R\to R$ its odd extension ($f^o(x) = -f(-x)$ if $x\in[-T,0]$, $f^o(x)=0$ if $x \in R\backslash[-T,T]$).

Remark 2.3. If $(B^k_t)_{t\in R}$ is a fBm, then it is known that (see [14,17])

$$E\left(B^k_t B^k_s\right) = \frac{c_k^2}{2}\int_R \widehat{1_{[0,t)}}(x)\overline{\widehat{1_{[0,s)}}(x)}|x|^{-2k}\,dx,$$

and by using the equality $S^k_t \overset{d}{=} \frac{B^k_t + B^k_{-t}}{\sqrt{2}}$, we arrive to the relation,

$$E\left(S^k_t S^k_s\right) = \langle 1_{[0,t)}, 1_{[0,s)}\rangle_{\Lambda^{sf}_{k,T}} = \frac{c_k^2}{2}\int_R \widehat{1^o_{[0,t)}}(x)\overline{\widehat{1^o_{[0,s)}}}(x)|x|^{-2k}\,dx, \quad s,t \geqslant 0,$$

which shows that

$$\langle f, g\rangle_{\Lambda^{sf}_{k,T}} = \frac{c_k^2}{2}\int_R \widehat{f^o}(x)\overline{\widehat{g^o}}(x)|x|^{-2k}\,dx, \quad f,g \in \mathcal{E}_T. \tag{2.14}$$

Definition 2.4. The completion of $(\mathcal{E}_T, \langle\cdot,\cdot\rangle_{\Lambda^{sf}_{k,T}})$ is called *the domain of the Wiener integral* and we denote it by $(\Lambda^{sf}_{k,T}, \langle\cdot,\cdot\rangle_{\Lambda^{sf}_{k,T}})$. For $f \in \Lambda^{sf}_{k,T}$, we define *the Wiener integral of f with respect to S^k* by

$$\int_0^\infty f(t)\,dS^k_t = L^2(\Omega, \mathcal{F}, P) - \lim_{n\to\infty}\int_0^\infty f_n(t)\,dS^k_t, \tag{2.15}$$

where $(f_n)_n \subset \mathcal{E}_T$ is such that $\langle f_n - f, f_n - f\rangle_{\Lambda^{sf}_{k,T}} \to 0$.

3. Main result

Let $\mathcal{D}((0,T))$ be the set of C^∞-functions defined on $[0,T]$ with compact support included in $(0,T)$ and we denote by $\mathcal{D}((0,T))'$ the corresponding space of distributions.

The symbol $\mathcal{S}$ denotes the Schwartz space of all rapidly decreasing real valued functions defined on R and let $\mathcal{S}'$ be the corresponding space of tempered distributions.

Given $f^* \in \mathcal{S}'$ we introduce its restriction to $\mathcal{D}((0,T))$ as the distribution $f^*|_{[0,T]} \in \mathcal{D}((0,T))'$ defined by

$$(f^*|_{[0,T]})(\varphi) = f^*(\varphi 1_{[0,T]}).$$

We shall denote $\mathcal{S}_T = \{f|_{[0,T]}\colon\ f \in \mathcal{S}\}$ and $(Qh)(x) = h(-x)$.

We say that $f^* \in \mathcal{S}'$ is *odd* if $f^*(-x) = -f^*(x)$, or equivalently, if $f^*(\varphi) = -f^*(Q\varphi)$.

The next lemma is necessary in order to have that the objects appearing in the statement of Theorem 3.2 are well defined.

Lemma 3.1. *Let f^*, $g^* \in S'$ be odd distributions such that*

$$f^*|_{[0,T]} = g^*|_{[0,T]}, \qquad \operatorname{supp}(f^*) \subset [-T,T], \qquad \operatorname{supp}(g^*) \subset [-T,T],$$
$$\int_R \left|\widehat{f^*}(x)\right|^2 |x|^{-2k}\,dx < \infty, \qquad \int_R \left|\widehat{g^*}(x)\right|^2 |x|^{-2k}\,dx < \infty.$$

Then $f^* = g^*$.

Proof. Define $h^* = f^* - g^*$. Then h^* is odd and $h^* = 0$ on $R\backslash[-T,T]$. If $\operatorname{supp}(\varphi) \subset [0,T]$, then by hypothesis $h^*(\varphi) = 0$ and if $\operatorname{supp}(\varphi) \subset [-T,0]$, then $h^*(\varphi) = -h^*(Q\varphi) = 0$.

Therefore $h^* = 0$ on $R\backslash\{-T,0,T\}$ and then $\operatorname{supp}(h^*) = \{-T,0,T\}$. It is known that such a distribution is a combination of Dirac distributions or derivatives of Dirac distributions and such a combination does not have the integrability property $\int_R |\widehat{h^*}(x)|^2|x|^{-2k}\,dx < \infty$. Then all the coefficients of a such combination are zero and consequently $h^* = 0$. $\square$

The main result of this section is the following

Theorem 3.2.

(i) *If $-\frac{1}{2} < k < 0$, then the space $(\Lambda^{sf}_{k,T}, \langle\cdot,\cdot\rangle_{\Lambda^{sf}_{k,T}})$, where*

$$\Lambda^{sf}_{k,T} = \left\{ f : [0,T] \to R\colon\ \exists \varphi_f \in L^2([0,T]),\ I^{-k}_{T-,2,\frac{k+1}{2}}\left(\frac{2^k}{\sqrt{\pi}}\varphi_f\right)(t) = t^k f(t) \right\}, \tag{3.1}$$

$$\langle f,g\rangle_{\Lambda^{sf}_{k,T}} = c_k^2 \int_0^T \varphi_f(t)\varphi_g(t)\,dt, \tag{3.2}$$

is the domain of the Wiener integral and

$$\int_0^T f(t)\,dS^k_t \stackrel{d}{=} c_k \int_0^T \varphi_f(t)\,dW_t. \tag{3.3}$$

In particular the Gaussian space of S^k is

$$H(S^k) = \left\{ \int_0^T f(t)\,dS^k_t \right\}_{f \in \Lambda^{sf}_{k,T}}.$$

(ii) *If $0 < k < \frac{1}{2}$, then the space $(\Lambda^{sf}_{k,T}, \langle\cdot,\cdot\rangle_{\Lambda^{sf}_{k,T}})$, where*

$$\Lambda^{sf}_{k,T} = \left\{ f \in \mathcal{D}(0,T)'\colon\ \exists f^* \in \mathcal{S}',\ f^* \text{ odd},\ \operatorname{supp}(f^*) \subset [-T,T],\ f^*|_{[0,T]} = f,\ \int_R \left|\widehat{f^*}(x)\right|^2 |x|^{-2k}\,dx < \infty \right\}, \tag{3.4}$$

$$\langle f,g\rangle_{\Lambda^{sf}_{k,T}} = \frac{c_k^2}{2} \int_R \widehat{f^*}(x)\overline{\widehat{g^*}(x)}|x|^{-2k}\,dx, \tag{3.5}$$

is the domain of the Wiener integral.
If we define

$$|\Lambda|^{sf}_{k,T} = \left\{ f : [0,T] \to R\colon\ I^k_{T-,2,\frac{1-k}{2}}\left(u^k|f|\right) \in L^2([0,T]) \right\}, \tag{3.6}$$

then we have the strict inclusion $|\Lambda|^{sf}_{k,T} \subset \Lambda^{sf}_{k,T}$ and

$$\int_0^T f(t)\,dS_t^k \stackrel{d}{=} c_k \int_0^T I^k_{T-,2,\frac{1-k}{2}}\left(\frac{\sqrt{\pi}}{2^k}u^k f\right)(t)\,dW_t, \quad f \in L^2\big([0,T]\big). \tag{3.7}$$

In particular the inclusion $\{\int_0^T f(t)\,dS_t^k\}_{f\in\Lambda^{sf}_{k,T}} \subset H(S^k)$ *is strict.*

Remark 3.3.

(a) According to (i_2) we have

$$\varphi_f(t) = I^k_{T-,2,\frac{1-k}{2}}\left(\frac{\sqrt{\pi}}{2^k}u^k f\right)(t). \tag{3.8}$$

(b) The Hilbert space $\Lambda^{sf}_{k,T}$ contains distributions that are not functions.
To justify this fact, choose an arbitrary distribution g with $\mathrm{supp}(g) \subset [0,T]$ which is not a function, in the Sobolev space

$$W^{-k,2} = \left\{ f \in \mathcal{S}'\colon \int_R \big|\hat{f}(x)\big|^2\big(1+|x|^2\big)^{-k}\,dx < \infty \right\}.$$

Consider its odd part $g^o(x) = \frac{g(x)-g(-x)}{2}$. Then $g^o|_{[0,T]} \in \Lambda^{sf}_{k,T}$ and $g^o|_{[0,T]}$ is not a function.

(c) From (ii) it follows also the strict inclusion $L^2([0,T]) \subset \Lambda^{sf}_{k,T}$.

(d) In the case $-\frac{1}{2} < k < 0$ the set (3.1) can be defined in terms of distributions as above for the case $0 < k < \frac{1}{2}$.

(e) In the case of fBm a similar result as in Theorem 3.2 holds.

The case $-\frac{1}{2} < k < 0$ is solved in [15]. The domain of the fractional Wiener integral is

$$\Lambda^f_{k,T} = \big\{ f:[0,T]\to R\colon \exists \varphi_f \in L^2\big([0,T]\big),\ I^{-k}_{T-}\big(s^k\varphi_f\big)(t) = t^k f(t)\big\}, \tag{3.9}$$

$$\langle f,g\rangle_{\Lambda^f_{k,T}} = c_k^2 \int_0^T \varphi_f(t)\varphi_g(t)\,dt. \tag{3.10}$$

The case $0 < k < \frac{1}{2}$ is solved in [8]. The domain of the fractional Wiener integral is

$$\Lambda^f_{k,T} = \left\{ f \in \mathcal{D}(0,T)'\colon \exists f^* \in \mathcal{S}',\ \mathrm{supp}(f^*) \subset [0,T],\ f^*|_{[0,T]} = f,\ \int_R \big|\widehat{f^*}(x)\big|^2 |x|^{-2k}\,dx < \infty \right\}, \tag{3.11}$$

$$\langle f,g\rangle_{\Lambda^f_{k,T}} = \frac{c_k^2}{2}\int_R \widehat{f^*}(x)\overline{\widehat{g^*}(x)}|x|^{-2k}\,dx \tag{3.12}$$

(the above description of the domain of the Wiener integral is also true for $-\frac{1}{2} < k < 0$: see [8]).

The proof of the theorem requires a few lemmas which we give below.

Lemma 3.4. *If* $f, g \in \mathcal{S}_T$, *then*

$$\int_0^T\int_0^T C_k(s,t) f'(s) g'(t)\,ds\,dt = \frac{c_k^2}{2}\int_R \widehat{f^o}(x)\overline{\widehat{g^o}}(x)|x|^{-2k}\,dx = \langle f,g\rangle_{\Lambda^{sf}_{k,T}}. \tag{3.13}$$

Moreover, if $k \in (0, \frac{1}{2})$,

$$\begin{aligned}\langle f,f\rangle_{\Lambda^{sf}_{k,T}} &= c_k^2 \left\| I^k_{T-,2,\frac{1-k}{2}}\left(\frac{\sqrt{\pi}}{2^k}u^k f\right)\right\|^2_{L^2([0,T])} \\ &= k(2k+1)\int_0^T\int_0^T f(u)f(v)\big[|u-v|^{2k-1} - (u+v)^{2k-1}\big]\,du\,dv.\end{aligned} \tag{3.14}$$

Proof. From [7, Lemma 3.1] we have the relation

$$\int_R \widehat{f}(x)\overline{\widehat{g}}(x)|x|^{-2k}\,dx = -c_k^{-2}\int_{R^2}|s-t|^{2k+1}f'(s)g'(t)\,ds\,dt.$$

If we apply the above equality to Qg we obtain

$$\int_R \widehat{f}(x)\overline{\widehat{Qg}}(x)|x|^{-2k}\,dx = c_k^{-2}\int_{R^2}|s+t|^{2k+1}f'(s)g'(t)\,ds\,dt,$$

where from

$$\begin{aligned}\int_R \widehat{f}(x)\big(\overline{\widehat{(g-Qg)}(x)}\big)|x|^{-2k}\,dx &= -c_k^{-2}\int_{R^2}\big(|s-t|^{2k+1}+|s+t|^{2k+1}\big)f'(s)g'(t)\,ds\,dt\\ &= 2c_k^{-2}\int_{R^2}C_k(s,t)f'(s)g'(t)\,ds\,dt,\end{aligned}$$

and this equality implies (3.13).

Assume now that $k\in(0,\frac{1}{2})$. From

$$\frac{\partial^2}{\partial s\partial t}C_k(s,t)=k(2k+1)\big[|s-t|^{2k-1}-(s+t)^{2k-1}\big],$$

and Theorem 2.1 we obtain the equality

$$k(2k+1)\big[|s-t|^{2k-1}-(s+t)^{2k-1}\big]=\left[\frac{\sqrt{\pi}}{2^{k+\frac{1}{2}}\Gamma(k)}c_k\right]^2\int_0^{s\wedge t}u^{2(1-k)}\big(s^2-u^2\big)^{k-1}\big(t^2-u^2\big)^{k-1}\,du.$$

The previous equality and Fubini's theorem imply the second equality in (3.14). From (3.13) and using integration by parts we obtain the first relation in (3.14). □

Lemma 3.5. *The inclusion $\mathcal{S}_T\subset\Lambda_{k,T}^{sf}$ holds and*

$$\int_0^T f(t)\,dS_t^k = f(T)S_T^k-\int_0^T S_t^k f'(t)\,dt,\quad f\in\mathcal{S}_T. \tag{3.15}$$

Moreover for each $f\in\mathcal{S}_T$ there exists $(f_n)_n\subset\mathcal{E}_T$ such that $f_n\xrightarrow{\Lambda_{k,T}^{sf}}f$.

Proof. Let Δ: $0=t_0<t_1<\cdots<t_N=T$ with $\|\Delta\|\to 0$. Denote by λ the Lebesgue measure and define

$$\mu_f=f'.\lambda,\qquad f_\Delta=\sum_{j=0}^{N-1}f(t_j)1_{[t_j,t_{j+1})}\in\mathcal{E}_T.$$

It is clear that μ_{f_Δ} converges weakly to μ_f ($\mu_{f_\Delta}\Longrightarrow\mu_f$ for short) as $\|\Delta\|\to 0$ and hence

$$\mu_{f_\Delta}\otimes\mu_{f_\Delta}\Longrightarrow\mu_f\otimes\mu_f,\qquad \mu_{f_\Delta}\otimes\mu_f\Longrightarrow\mu_f\otimes\mu_f,$$

which imply the convergences

$$\int_0^T f_\Delta(t)\,dS_t^k=-\int_0^T S_t^k\,d\mu_{f_\Delta}(t)\to-\int_0^T S_t^k\,d\mu_f(t)=f(T)S_T^k-\int_0^T S_t^k f'(t)\,dt,$$

$$\int_0^T\int_0^T C_k(s,t)\,d\mu_{f_\Delta}(s)\,d\mu_{f_\Delta}(t)\to\int_0^T\int_0^T C_k(s,t)\,d\mu_f(s)\,d\mu_f(t),$$

$$\int_0^T\int_0^T C_k(s,t)\,d\mu_{f_\Delta}(s)\,d\mu_f(t)\to\int_0^T\int_0^T C_k(s,t)\,d\mu_f(s)\,d\mu_f(t).$$

Now, by Lemma 3.4, we deduce that $\mathcal{E}_T$ is dense in $\mathcal{S}_T$, $f\in\Lambda_{k,T}^{sf}$ and (3.15) is satisfied. □

Lemma 3.6.

(a) *If $f^* \in \mathcal{S}'$ is odd,*

$$\operatorname{supp}(f^*) \subset [-T, T], \qquad \int_R \left|\widehat{f^*}(x)\right|^2 |x|^{-2k}\, dx < \infty,$$

then there exist a family $(f^*_\varepsilon)_{\varepsilon>0} \subset \mathcal{S}'$ such that f^*_ε is odd,

$$\operatorname{supp}\big(f^*_\varepsilon\big) \subset [-T+\varepsilon, T-\varepsilon], \qquad \int_R \widehat{f^*_\varepsilon}(x)|x|^{-2k}\, dx < \infty,$$

$$\lim_{\varepsilon\to 0} \int_R \left|\widehat{f^*_\varepsilon}(x) - \widehat{f^*}(x)\right|^2 |x|^{-2k}\, dx = 0.$$

(b) *The space $\mathcal{S}_T$ is dense in $\Lambda^{sf}_{k,T}$.*

Proof. (a) Define

$$\rho_\varepsilon : [-T, T] \to [-T+\varepsilon, T-\varepsilon], \qquad \rho_\varepsilon(x) = \frac{T-\varepsilon}{T} x,$$

and $f^*_\varepsilon \in \mathcal{S}'$, $f^*_\varepsilon(\varphi) = f^*(\varphi \circ \rho_\varepsilon)$. By using the fact that $\widehat{f^*_\varepsilon}(x) = \widehat{f^*}(\rho_\varepsilon(x))$, it is not difficult to see that $(f^*_\varepsilon)_{\varepsilon>0}$ satisfies the conclusion.

(b) By (a) it is enough to choose $f^* \in \mathcal{S}'$, f^* odd, $\operatorname{supp}(f^*) \subset [-a, a] \subset (-T, T)$ and $\int_R |\widehat{f^*}(x)|^2 |x|^{-2k}\, dx < \infty$.

Then the regularization $f^*_\varepsilon = f^* * h_\varepsilon$, where $h_\varepsilon(x) = \varepsilon^{-1} h(x\varepsilon^{-1})$, $h \in C^\infty$, h even, $\operatorname{supp}(h) \subset [-1, 1]$, $\int_R h(x)\, dx = 1$, satisfies the requirements. □

Recall, that if $f : [0, T] \to R$, $\alpha > 0$, the *Riemann–Liouville fractional integral* $I^\alpha_{T-} f$ is defined by

$$\big(I^\alpha_{T-} f\big)(s) = \frac{1}{\Gamma(\alpha)} \int_s^T \frac{f(t)}{(t-s)^{1-\alpha}}\, dt, \quad s \in [0, T].$$

Lemma 3.7. *Assume $-\frac{1}{2} < k < 0$, $a > 1$. Then there exists a sequence $(l_n)_n \subset \mathcal{E}_a$ such that*

$$\int_0^a u^{\frac{1}{2}-k} \left|1_{[0,1)}(u) - I^k_{a-}\big(x^{-1} l_n\big)(u)\right|^2 du \to 0.$$

Proof. We proceed as in [14, Lemma 8.1]. Since

$$1_{[0,1)}(s) = I^k_{a-}\big(x^{-1} f\big)(s), \qquad f(s) = \frac{1}{\Gamma(1-k)} s(1-s)^{-k}_+,$$

all we have to prove is that

$$h_n = \int_0^a u^{\frac{1}{2}-k} \left|I^k_{a-}\big(x^{-1} f\big)(u) - I^k_{a-}\big(x^{-1} l_n\big)(u)\right|^2 du \to 0, \tag{3.16}$$

for some $(l_n)_n \subset \mathcal{E}_a$. In fact we prove (3.16) with

$$l_n(u) = \frac{1}{\Gamma(1-k)} \sum_{j=2}^{n-2} \frac{j}{n} \left(1 - \frac{j}{n}\right)^{-k} 1_{[\frac{j}{n}, \frac{j+1}{n})}(u). \tag{3.17}$$

For $0 < b < c < a$, an easy computation yields

$$\begin{aligned} &\Gamma(1+k) I^k_{a-}\big(u^{-1} 1_{[b,c)}\big)(s) \\ &\quad = -\frac{d}{ds} \int_b^c u^{-1}(u-s)^k_+\, du = \int_s^1 u^{-2}(u-s)^k 1_{[b,c)}(u) + c^{-1}(c-s)^k_+ - b^{-1}(b-s)^k_+. \end{aligned} \tag{3.18}$$

By using (3.18) we can write

$$\Gamma(1-k)\Gamma(1+k)I^k_{a-}\big(x^{-1}l_n\big)(s)$$
$$= \Gamma(1-k)\Gamma(1+k)I^k_{1-}\big(x^{-1}l_n\big)(s) = \Gamma(1+k)\sum_{j=2}^{n-2}\frac{j}{n}\left(1-\frac{j}{n}\right)^{-k} I^k_{1-}\big(u^{-1}1_{[\frac{j}{n},\frac{j+1}{n})}\big)(s) = g_n^{(1)}(s) + g_n(s), \tag{3.19}$$

$$g_n^{(1)}(s) = \sum_{j=2}^{n-2}\frac{j}{n}\left(1-\frac{j}{n}\right)^{-k}\int_s^1 u^{-2}(u-s)^k 1_{[\frac{j}{n},\frac{j+1}{n})}(u)\,du,$$

$$g_n(s) = \sum_{j=2}^{n-2}\frac{j}{n}\left(1-\frac{j}{n}\right)^{-k}\left[\left(\frac{j+1}{n}\right)^{-1}\left(\frac{j+1}{n}-s\right)_+^k - \left(\frac{j}{n}\right)^{-1}\left(\frac{j}{n}-s\right)_+^k\right].$$

The equality

$$\sum_{j=2}^{n-2} a_j(b_{j+1}-b_j) = \sum_{j=2}^{n-2} b_j(a_j - a_{j-1}) + a_{n-2}b_{n-1} - a_1 b_2$$

implies

$$g_n(s) = \sum_{j=2}^{5} g_n^{(j)}(s),$$

$$g_n^{(2)}(s) = -\sum_{j=2}^{n-2}\frac{1}{n}\left(\frac{j}{n}\right)^{-1}\left(\frac{j}{n}-s\right)_+^k\left(1-\frac{j}{n}\right)^{-k},$$

$$g_n^{(3)}(s) = -\sum_{j=2}^{n-2}\left(\frac{j}{n}\right)^{-1}\left(\frac{j}{n}-s\right)_+^k \frac{j-1}{n}\left[\left(1-\frac{j}{n}\right)^{-k} - \left(1-\frac{j-1}{n}\right)^{-k}\right],$$

$$g_n^{(4)}(s) = \frac{n-2}{n-1}\left(1-\frac{n-2}{n}\right)^{-k}\left(\frac{n-1}{n}-s\right)_+^k, \qquad g_n^{(5)}(s) = \frac{n-1}{2n}\left(\frac{2}{n}-s\right)_+^k.$$

The conclusion of the lemma follows from the following easily seen relations:

$$\int_0^1 s^{\frac{1}{2}-k}\big|g_n^{(5)}(s)\big|^2\,ds \to 0, \tag{3.20}$$

$$\Gamma(1-k)\Gamma(1+k)I^k_{a-}\big(u^{-1}l_n\big)(s) - g_n^{(5)}(s) = \sum_{j=1}^{4} g_n^{(j)}(s) \to \Gamma(1-k)\Gamma(1+k)1_{[0,1)}(s), \tag{3.21}$$

$$\sup_n \big|g_n^{(j)}(s)\big| \leqslant g^{(j)}(s), \quad 1 \leqslant j \leqslant 4, \tag{3.22}$$

$$\int_0^1 s^{\frac{1}{2}-k}\big|g_n^{(j)}(s)\big|^2\,ds < \infty, \quad 1 \leqslant j \leqslant 4 \tag{3.23}$$

(whose proof is left to the reader). □

Proof of Theorem 3.2. (i) The fact that (3.2) is a scalar product follows easily. Let $(f_n)_n$ be a Cauchy sequence in $\Lambda^{sf}_{k,T}$.

Since $f_n(t) = t^{-k}I^{-k}_{-T,2,\frac{1+k}{2}}(\frac{2^k}{\sqrt{\pi}}\varphi_n)(t)$, $\varphi_n \in L^2([0,T])$, we have that $(\varphi_n)_n$ is a Cauchy sequence in $L^2([0,T])$, and thus $\varphi_n \to \varphi$ in $L^2([0,T])$.

If we define $f(t) = t^{-k}I^{-k}_{-T,2,\frac{1+k}{2}}(\frac{2^k}{\sqrt{\pi}}\varphi)(t)$, then we deduce that $f_n \xrightarrow{\Lambda^{sf}_{k,T}} f$.

Next we prove that $\mathcal{E}_T$ is dense in $\Lambda^{sf}_{k,T}$. Without loss of generality we assume that $T > 1$. Let $f \in \Lambda^{sf}_{k,T}$, $f = \frac{2^k}{\sqrt{\pi}}I^{-k}_{-T,2,\frac{1+k}{2}}\varphi$, $\varphi \in L^2([0,T])$. The general relation

$$I^\alpha_{c-,\sigma,\eta}f(x) = x^{\sigma\eta}I^\alpha_{c^\sigma}s^{-\alpha-\eta}f\big(s^{\frac{1}{\sigma}}\big)\big(x^\sigma\big)$$

becomes in our case

$$\varphi(t) = \frac{\sqrt{\pi}}{2^k} t^{1-k} I^k_{T^2-}\big(x^{-1} f(\sqrt{x})\big)\big(t^2\big).$$

So, we have to prove that there exists $(l_n)_n \subset \mathcal{E}_{T^2}$ such that

$$\int_0^T t^{2(1-k)} \big| I^k_{T^2-}\big(x^{-1} l_n(\sqrt{x})\big)\big(t^2\big) - I^k_{T^2-}\big(x^{-1} f(\sqrt{x})\big)\big(t^2\big)\big|^2 \, dt \to 0,$$

or equivalently,

$$\int_0^{T^2} t^{\frac{1}{2}-k} \big| I^k_{T^2-}\big(x^{-1} l_n(\sqrt{x})\big)(t) - I^k_{T^2-}\big(x^{-1} f(\sqrt{x})\big)(t)\big|^2 \, dt \to 0. \tag{3.24}$$

Since $t^{\frac{1-2k}{4}} I^k_{T^2-}(x^{-1} f(\sqrt{x}))(t) \in L^2([0,T^2])$ can be approximated in $L^2([0,T^2])$ by functions of the form $g(t) = t^{\frac{1-2k}{4}} \sum_j a_j 1_{[b_j,c_j)}(t)$, the convergence in (3.24) reduces to showing that there is a sequence $(l_n)_n \subset \mathcal{E}_{T^2}$ such that

$$\int_0^{T^2} t^{\frac{1}{2}-k} \big| 1_{[b,c)}(t) - I^k_{T^2-}\big(x^{-1} l_n\big)(t)\big|^2 \, dt \to 0.$$

Now, it remains to apply Lemma 3.7.

(ii) From Lemmas 3.5, 3.6 we obtain that $\mathcal{E}_T$ is dense in $\Lambda^{sf}_{k,T}$. The completeness of $\Lambda^{sf}_{k,T}$ follows along the same arguments as in the case of fBm (see [7, Theorem 3.3]) and we omit the details.

To complete the proof we have to show that

$$c_k \left\| I^k_{T-,2,\frac{1-k}{2}} \left(\frac{\sqrt{\pi}}{2^k} t^k f \right) \right\|_{L^2([0,T])} = \|f\|_{\Lambda^{sf}_{k,T}}, \quad f \in |\Lambda|^{sf}_{k,T}. \tag{3.25}$$

First, observe that the second equality in (3.14) still holds for $f \in |\Lambda|^{sf}_{k,T}$ (with the same arguments). In particular it follows that $f \in L^1([0,T])$ and hence $f^o \in L^1(R)$. Then, a density argument shows that for (3.25) it is enough to take $f \in \mathcal{S}_T$.

In order to prove this fact we use Hankel transforms. Recall that the Hankel transform of order $-1 < \nu < -\frac{1}{2}$ of a function $f : R_+ \to R$ is defined by

$$(H_\nu f)(\lambda) = \int_0^\infty \sqrt{\lambda t}\, J_\nu(\lambda t) f(t)\, dt,$$

where for $\nu \neq -1, -2, \ldots,$ J_ν is the Bessel function of order ν defined by

$$J_\nu(t) = \sum_{n=0}^\infty \frac{(-1)^n (\frac{t}{2})^{\nu+2n}}{\Gamma(n+1)\Gamma(\nu+n+1)}, \quad t > 0.$$

Details on the Hankel transform can be found in [11,18,22].

First, we prove that

$$H_{\frac{1}{2}-k}\big(I^k_{T-,2,\frac{1-k}{2}}\big(t^k f\big) 1_{[0,T]}\big)(\lambda) = \frac{2^{k+\frac{1}{2}}}{\sqrt{\pi}} \lambda^{-k} \int_0^T f(t) \sin(\lambda t)\, dt. \tag{3.26}$$

By using the integration by parts (i_3) for Erdély–Kober-type fractional derivatives and integrals we have

$$\begin{aligned} H_{\frac{1}{2}-k}\big(I^k_{T-,2,\frac{1-k}{2}}\big(t^k f\big) 1_{[0,T]}\big)(\lambda) &= \int_0^T \sqrt{\lambda t}\, J_\nu(\lambda t) I^k_{T-,2,\frac{1-k}{2}}\big(t^k f\big)(t)\, dt \\ &= \int_0^T t^{k+1} f(t) I^k_{0+,2,1-k}\left(\sqrt{\frac{\lambda}{t}} J_\nu(\lambda t)\right)(t)\, dt. \end{aligned}$$

From [4, page 46] we have the relation

$$I^k_{0+,2,\frac{1-k}{2}}\left(\sqrt{\frac{\lambda}{t}}J_\nu(\lambda t)\right)=\frac{2^{k+\frac{1}{2}}}{\sqrt{\pi}}\lambda^{-k}t^{-k-1}\sin\lambda t,$$

where from (3.26) follows easily.

From Parseval's equality for Hankel transforms we obtain

$$\begin{aligned}\left\|I^k_{T-,2,\frac{1-k}{2}}\left(\frac{\sqrt{\pi}}{2^k}t^k f\right)\right\|^2_{L^2([0,T])}&=\left\|H_{\frac{1}{2}-k}\left(I^k_{T-,2,\frac{1-k}{2}}\left(t^k f\right)1_{[0,T]}\right)(\lambda)\right\|^2_{L^2(R_+)}\\&=2\int_0^\infty\lambda^{-2k}\left[\int_0^T f(s)\sin(\lambda s)\,ds\right]\left[\int_0^T f(t)\sin(\lambda t)\,dt\right]d\lambda\\&=2\int_0^T\int_0^T f(s)f(t)\left[\int_0^\infty\lambda^{-2k}\sin(\lambda s)\sin(\lambda t)\,d\lambda\right]ds\,dr\\&=\int_0^T\int_0^T f(s)f(t)\left[\int_0^\infty\lambda^{-2k}\left(\cos\left(\lambda(s-t)\right)-\cos\left(\lambda(s+t)\right)\right)d\lambda\right]ds\,dr.\end{aligned}$$

If we use now above the equality (see [5, formula 3.823])

$$\int_0^\infty\frac{1-\cos ux}{x^{2k}}\,dx=d_k|u|^{2k-1},\quad u>0,$$

$$d_k=-\Gamma(1-2k)\cos\pi\left(\frac{1}{2}-k\right),$$

we deduce, by integration by parts and Lemma 3.4,

$$\begin{aligned}\left\|I^k_{T-,2,\frac{1-k}{2}}\left(\frac{\sqrt{\pi}}{2^k}t^k f\right)\right\|^2_{L^2([0,T])}&=d_k\int_0^T\int_0^T f(s)f(t)\left[|s-t|^{2k-1}-(s+t)^{2k-1}\right]ds\,dt\\&=d_k k(2k+1)\int_0^T\int_0^T f(s)f(t)\frac{\partial^2 C_k(s,t)}{\partial s\partial t}\,ds\,dt\\&=d_k k(2k+1)\int_0^T\int_0^T C_k(s,t)f'(s)f'(t)\,ds\,dt=c_k^{-2}\|f\|^2_{\Lambda^{sf}_{k,T}}.\qquad\square\end{aligned}$$

The relation between the domain of the Wiener integral associated with fBm (see [8,14]) and sfBm is given in the following result.

Proposition 3.8. *The inclusion $\Lambda^f_{k,T}\subset\Lambda^{sf}_{k,T}$ holds.*

Proof. For $f\in\Lambda^f_{k,T}$ choose a sequence $(f_n)_n\subset\mathcal{E}_T$ such that $f_n\xrightarrow{\Lambda^f_{k,T}}f$.

Case $-\frac{1}{2}<k<0$. By [8, Theorem 3.3] (see also Remark 3.3) we have that

$$\int_R\left|\widehat{f_n 1_{[0,T]}}(x)-\widehat{f1_{[0,T]}}(x)\right|^2|x|^{-2k}\,dx\to 0.$$

In particular

$$\int_R\left|\widehat{f_n^o}(x)-\widehat{f^o}(x)\right|^2|x|^{-2k}\,dx\to 0,\tag{3.27}$$

and then by (2.14)

$$\|f_m-f_n\|^2_{\Lambda^f_{k,T}}=\frac{c_k^2}{2}\int_R\left|\widehat{f_m^o}(x)-\widehat{f_n^o}(x)\right|^2|x|^{-2k}\,dx\to 0,\quad m,n\to\infty.$$

Since $\Lambda^f_{k,T}$ is complete there exists $g\in\Lambda^f_{k,T}$ such that $f_n\xrightarrow{\Lambda^{sf}_{k,T}}g$.

The convergencies $f_n \xrightarrow{\Lambda^f_{k,T}} f$ and $f_n \xrightarrow{\Lambda^{sf}_{k,T}} g$ imply that $f_n \xrightarrow{L^2([0,T])} f, g$ and consequently $f = g \in \Lambda^{sf}_{k,T}$.

Case $0 < k < \frac{1}{2}$. Applying [8, Theorem 3.3] again, we find $f_n^*, f^* \in \mathcal{S}'$ with

$$\operatorname{supp}(f_n^*) \subset [0,T], \qquad \operatorname{supp}(f^*) \subset [0,T],$$

$$f_n^*|_{[0,T]} = f_n, \qquad f^*|_{[0,T]} = f, \qquad \int\limits_R \big|\widehat{f_n^*}(x) - \widehat{f^*}(x)\big|^2 |x|^{-2k}\,dx \to 0.$$

The odd parts $(f_n^*)^o$, $(f^*)^o$ of f_n^*, f^* satisfy

$$\operatorname{supp}(f_n^*)^o \subset [-T,T], \qquad \operatorname{supp}(f^*)^o \subset [-T,T],$$

$$(f_n^*)^o|_{[0,T]} = f_n, \qquad (f^*)^o|_{[0,T]} = f, \qquad \int\limits_R \big|\widehat{(f_n^*)^o}(x) - \widehat{(f^*)^o}(x)\big|^2 |x|^{-2k}\,dx \to 0,$$

and consequently $f \in \Lambda^{sf}_{k,T}$. □

Remark 3.9.

(a) Gripenberg and Norros [6] established the following prediction formula for fBm (see also [12]) in terms of fractional Wiener integral:
For any $k \in (-\frac{1}{2}, \frac{1}{2})$ *and* $0 < a < t$,

$$E\big[B_t^k \mid B_s^k, 0 \leqslant s \leqslant a\big] = B_a^k + \int\limits_0^a \Phi_{a,t}(u)\,dB_u^k, \tag{3.28}$$

$$\Phi_{a,t}(u) = \frac{\sin \pi k}{\pi} u^{-k}(a-u)^{-k} \int\limits_a^t \frac{(z-a)^k}{z-u} z^k\,dz. \tag{3.29}$$

By using properties of the Erdély–Kober-type fractional integrals, the following *prediction formula* holds for sfBm in terms of sub-fractional Wiener integrals (see [21]):
For any $k \in (-\frac{1}{2}, \frac{1}{2})$ *and* $0 < a < t$,

$$E\big[S_t^k \mid S_s^k, 0 \leqslant s \leqslant a\big] = S_a^k + \int\limits_0^a \Psi_{a,t}(u)\,dS_u^k, \tag{3.30}$$

$$\Psi_{a,t}(u) = \frac{2\sin \pi k}{\pi} u\big(a^2-u^2\big)^{-k} \int\limits_a^t \frac{(z^2-a^2)^k}{z^2-u^2} z^k\,dz. \tag{3.31}$$

(b) The sub-fractional Wiener integral arises also in the linear filtering with a sfBm in the signal and the noise in the observation process being a linear combination of independent Bm and sfBm (see [21]).

(c) In [4] series expansions of the fBm, even part (that is sfBm up to a multiplicative constant) and odd part are obtained by means of Fourier–Bessel theory. The mean square convergence of the Riemann–Stieltjes integral for some classes of deterministic integrands with respect to the partial sums of such expansions to the corresponding Wiener integrals may be obtained.

(d) As mentioned in the introduction sfBm arises from occupation time fluctuations of some particle systems for $k \in (0, \frac{1}{2})$ as weak limit in the space of continuous functions with values tempered distributions [1–3]. From such results the weak convergence of weighted occupation times to the sub-fractional Wiener integral could be deduced.

(e) Taking the multidimensional forms of the operators appearing in the sub-fractional Wiener integral as a Wiener integral (Theorem 3.2) one obtains the multiple sub-fractional Wiener integrals. Based on such multiple integrals, the orthogonal decomposition of square integrable functionals, which are measurable with respect to sfBm, in series of multiple sub-fractional Wiener integrals allows the development of the stochastic calculus with respect to sfBm.

Remark 3.10. In the case of fBm in [12] the so-called the fundamental martingale is introduced as a fractional Wiener integral and a version of Girsanov theorem is obtained.

A corresponding sub-fractional fundamental martingale is also naturally attached in the sfBm case. This is the process $(M_t^k)_{t\in[0,T]}$ defined by the relation

$$M_t^k = d_k \int_0^t s^{-k}\, dW_s,$$

$$d_k = \frac{2^k}{c_k \Gamma(1-k)\sqrt{\pi}}. \tag{3.32}$$

This martingale generates the same filtration as S^k.

For a function $f:[0,T] \to R$ with $\int_0^T f^2(s)s^{-2k}\,ds < \infty$ we define the probability Q_f by

$$\begin{aligned}\frac{dQ_f}{dP}\bigg|_{\mathcal{F}_t^{S^k}} &= \exp\left(\int_0^t f(s)\,dM_s^k - \frac{1}{2}\int_0^t f^2(s)\langle M^k\rangle_s\right)\\ &= \exp\left(\int_0^t f(s)\,dM_s^k - \frac{d_k^2}{2}\int_0^t f^2(s)s^{-2k}\,ds\right),\end{aligned} \tag{3.33}$$

and the function

$$(\Psi_k f)(s) = \frac{1}{\Gamma(1-k)} I^k_{0+,2,-k} f(s). \tag{3.34}$$

Then the following *sub-fractional Girsanov theorem* is satisfied: *the process*

$$S_t^k - \int_0^t (\Psi_k f)(s)\,ds, \quad t \in [0,T],$$

is a Q_f-sfBm.

In particular if $f \equiv a \in R$ it follows that the process $(S_t^k - at)_{t\in[0,T]}$ is Q_a-sfBm.

Remark 3.11. Since the main representation formula of Theorem 2.1 is the basic result which it is used in our approach and a corresponding formula holds for the odd part of the fBM (see [4]), the results can be extended in an analogous manner to the case of odd part of the fBm.

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