Malliavin Calculus and Euclidean Quantum Mechanics.
I. Functional Calculus

ANA BELO CRUZEIRO

Centro de Matemática e Aplicações Fundamentais (INIC),
Av. Prof. Gama Pinto, 2, 1699 Lisboa Codex, Portugal

AND

JEAN-CLAUDE ZAMBRINI

The Royal Institute of Technology, Department of Mathematics,
S-10044 Stockholm, Sweden

Received May 1989, revised November 1989

We give a rigorous version of the functional calculus developed by R. Feynman in relation to his path integral formulation of Quantum Mechanics. Our approach is Euclidean but distinct from the one founded on the Feynman-Kac formula. It uses two basic ingredients: a new probabilistic interpretation of the classical heat equation, introduced recently in the framework of Euclidean Quantum Mechanics, and an infinite dimensional differential calculus adapted to functionals of the diffusion processes relevant for this interpretation.

1. PATH INTEGRATION AND QUANTUM DYNAMICS

Let $H$ be a lower bounded self-adjoint quantum mechanical energy operator (Hamiltonian) on $L^2(\mathbb{R}^3)$. For the simplest situation of a unit mass particle in a scalar potential $V$ it is of the form $H = -(\hbar^2/2) \Delta + V$, where $\Delta$ denotes the Laplacian in $\mathbb{R}^3$ and $\hbar$ is a positive number called the Planck constant. The dynamics is described by the initial value problem for the Schrödinger equation on $L^2(\mathbb{R}^3)$,

$$i\hbar \frac{\partial \psi}{\partial \tau} = H\psi$$

$$\psi(x, 0) = \chi(x),$$

(1.1)
where \( \chi \) belongs to the domain \( \mathcal{D}(H) \) of \( H \) in \( L^2(\mathbb{R}^3) \). The Euclidean version of (1.1) is defined, in mathematical physics, by

\[
-H \frac{\partial \eta^*}{\partial t} = H\eta^*,
\]

\[
\eta^*(x, 0) = \chi(x),
\]

i.e., as the initial value problem for the heat equation on \( L^2(\mathbb{R}^3) \). In quantum mechanics, a fundamental interpretation, due to Born, is associated with (1.1), namely,

\[
\int_{\mathbb{R}^3} \overline{\psi} \psi(x, \tau) \, dx
\]

represents the probability of finding the physical system in the Borelian \( B \) at time \( \tau \). However, the traditional (Von Neumann's) axiomatic of the theory does not specify mathematically the meaning of (1.3). Actually, no probability space is introduced in this formulation [21].

R. Feynman introduced another approach in which the relation with probabilistic concepts is, heuristically, rather simple [9]. The solution of (1.1) is represented by a functional integration (a "Path integral") and we have

\[
(e^{-i\tau H/\hbar}) \chi(y) = \int_{\mathcal{D}(H)} \chi(\omega(0)) \, e^{i\int_{\mathcal{L}} L(\dot{\omega}(s), \omega(s)) \, ds} \, d\omega,
\]

where \( e^{-i\tau H/\hbar} \) is the dynamical one-parameter group of unitary operators on \( L^2(\mathbb{R}^3) \), \( \mathcal{D}(H) \) is the space of paths conditioned in the future, \( \mathcal{D}(H) = \{ \omega \in C([0, \tau]; \mathbb{R}^3) \mid \omega(\tau) = y \} \), \( \mathcal{D}(H) = \prod_{0 \leq s \leq \tau} d\omega(s) \) is used as a measure on \( \mathcal{D}(H) \), and \( L \) is the Lagrangian of the corresponding classical system, here \( L(\dot{\omega}, \omega) = \frac{1}{2} \|\dot{\omega}\|^2 - V(\omega) \).

Actually, the fundamental tool of Feynman's analysis is the concept of (complex) transition amplitude between two states \( \chi \) and \( \psi \) in \( L^2(\mathbb{R}^3) \),

\[
\langle \psi | e^{-i\tau H/\hbar} \chi \rangle = \int_{\mathcal{D}(H)} \chi(\omega(0)) \, e^{i\int_{\mathcal{L}} L(\dot{\omega}(s), \omega(s)) \, ds} \, d\omega(\omega),
\]

where \( \langle \cdot | \cdot \rangle_2 \) denotes the scalar product in \( L^2(\mathbb{R}^3) \) and \( \mathcal{D}(H) \) is now the space of all the paths, \( \mathcal{D}(H) = \{ \omega \in C([0, \tau]; \mathbb{R}^3) \} \). It has been possible to give a mathematical meaning to such path integrals [1.2, 6, 17.1] but not to the underlying stochastic process itself (neither \( d\omega \) nor \( e^{i\tau H/\hbar} \) are well...
defined measures). M. Kac has shown [14] that, after analytical continuation in the time parameter, \(\tau \to it\), a Euclidean analogue of (1.4) is

\[
(e^{-iH/h}\chi)(y) = \int_{Q'} \chi(\omega(0)) e^{-1/2\lambda V(\omega(x))} \, ds \, d\mu_\omega(\omega), \quad (1.6)
\]

where \(\mu_\omega\) is the Wiener measure, which includes formally the kinetic energy term \(\frac{1}{2}||\dot{\omega}||^2\). So the Euclidean Lagrangian is now proportional to \(L(\dot{\omega}, \omega) - \frac{1}{2}||\dot{\omega}||^2 + V(\omega)\). The stochastic process underlying (1.6) is, of course, well defined, but its properties are completely different from the ones expected in quantum dynamics. The time symmetry of the theory is lost, there is no analogue of (1.3), and the potential \(V\) plays now a role (of “killing term”) meaningless from a physical point of view.

An alternative probabilistic interpretation of the heat equation (1.2) has been proposed recently, in which these difficulties disappear [23]. We give, in Section 2, an expository summary of this new approach (“Euclidean Quantum Mechanics”).

A common puzzling aspect of (1.4) and (1.6) is to be associated with formal Euler–Lagrange (respectively Euclidean Euler–Lagrange) equations. Consider for example the question: what is the equation of motion of a free \((V = 0)\), one dimensional, process? According to (1.6) the relevant process is the Brownian motion \(w\). For any deterministic \(\phi: [0, T/2] \to \mathbb{R}\) such that \(\phi \in L^2([0, T/2], dt)\) and \(\phi(0) = 0\), the expectation \(E_0[\int_0^{T/2} \phi(t) \, dw(t)\] = 0.

For differentiable trajectories \(t \to w(t)\) this would mean

\[
E_0 \left[ \phi \left( \frac{T}{2} \right) w \left( \frac{T}{2} \right) - \int_0^{T/2} \phi(t) \, \dot{w}(t) \, dt \right] = 0. \quad (1.7)
\]

But, actually, almost all sample paths of the Brownian motion \(w\) are nowhere differentiable and (1.7) can only be looked upon as a formal free Newton equation, with Neumann boundary conditions, in a weak sense.

It is clearly not obvious how to give a realistic dynamical meaning to formulas like (1.7). First, one should define relevant concepts of velocity and acceleration for diffusion processes. Moreover, according to (1.6), to each given potential \(V\) is associated a single process, in striking contrast with the richer dynamical structure of classical mechanics, the formal limit \(h \to 0\) of quantum mechanics. Also notice that Euler-Lagrange equations like (1.7) do not follow, actually, from a variational principle as in classical mechanics, but from the principle of stationary phase in the limit \(h \to 0\), for (1.4) or from the Laplace method, for (1.6) (cf. [9]).

One of the aims of this work is to show that Euclidean Euler–Lagrange laws of motion still make sense in the quantum context, in a way consistent with the properties of Brownian trajectories, the dynamical content of classical mechanics, and the functional calculus developed by R. Feynman.
We shall use the calculus of variations in the sense of Malliavin [16], an infinite dimensional calculus adapted to Wiener functionals natural in this context. Actually, for the cases treated in this work, the generality of Malliavin Calculus is not strictly necessary; but it enables us to obtain easily other results and is very suggestive for the extension to Euclidean field theory.

2. BERNSTEIN DIFFUSION AND QUANTUM MECHANICS

After Jamison [13], let us define a Bernstein transition probability as a function of six variables \( H(s, x, t, B, u, z) \), \(-T/2 \leq s \leq t \leq u \leq T/2\), \( x, z \in \mathbb{R}^3 \), and \( B \) in \( \mathcal{B}^3 \), the Borel sigma algebra of \( \mathbb{R}^3 \), such that

\[
\begin{align*}
(B1) \quad & \forall x, z \text{ in } \mathbb{R}^3, -T/2 \leq s < t < u \leq T/2, \\
& B \rightarrow H(s, x, t, B, u, z) \text{ is a probability measure on } \mathcal{B}^3; \\
(B2) \quad & \forall B \text{ fixed in } \mathcal{B}^3, -T/2 \leq s < t < u \leq T/2, \\
& (x, z) \rightarrow H(s, x, t, B, u, z) \text{ is } \mathcal{B}^3 \times \mathcal{B}^3 \text{ measurable}; \\
(B3) \quad & \forall B_1, B_2 \text{ in } \mathcal{B}^3, -T/2 \leq s < t < u < v \leq T/2, \\
& \int_{B_2} H(s, x, t, B_1, u, \xi) H(t, y, u, B_2, v, z) d\xi = \int_{B_1} H(s, x, t, B_1, u, \xi) H(t, y, u, B_2, v, z). 
\end{align*}
\]

Let \( z_t \equiv z(t) \) be an \( \mathbb{R}^3 \)-valued stochastic process indexed by \( I = [-T/2, T/2] \), defined on \( (\Omega, \sigma_I, P) \), where \( \sigma_I \) is the Borel \( \sigma \) algebra of \( \Omega \). Let \( \mathcal{F}_t \), respectively \( \mathcal{F}_t^- \), for \( t \) in \( I \), be an increasing (decreasing) filtration for \( z_t \). Jamison has called "reciprocal" transition probability a function \( H \) verifying (B1)-(B3) and proved the following

2.1. THEOREM. Let \( H(s, x, t, B, u, z) \) be a Bernstein transition probability and \( m \) a probability measure on \( \mathcal{B}^3 \times \mathcal{B}^3 \). Then there is a unique probability measure \( P_m \) such that, with respect to \( (\Omega, \sigma_I, P_m) \), \( z_t \), \( t \) in \( I \), satisfies

\[
\begin{align*}
(1) \quad & E[f(z_t)|\mathcal{F}_u \cup \mathcal{F}_s] = E[f(z_s)|z_s, z_u], \ s < t < u \text{ in } I \text{ and } f \text{ bounded measurable (here } E[\cdots|\mathcal{A}] \text{ denotes a conditional expectation given } \mathcal{A}). \\
(2) \quad & P_m(z_t \in B_S, z_{-T/2} \in B_E) = m(B_S \times B_E) \text{ for } B_S, B_E \text{ in } \mathcal{B}^3. \\
(3) \quad & P_m(z_t \in B|z_s, z_u) = H(s, z_s, t, B, u, z_u), \forall -T/2 < s \leq t \leq u < T/2, B \text{ in } \mathcal{B}^3.
\end{align*}
\]
A process $z_{t}, t \in I,$ satisfying (1) is called a Bernstein process (or a local Markov process). A Bernstein process $z_{t}, t \in I,$ with values in $\mathbb{R}^{3}$ and almost certainly continuous sample paths is called a Bernstein Bridge if its transition probability satisfies the following conditions, for every $x, z$ in $\mathbb{R}^{3}$:

(a) \[ \lim_{t \downarrow s} \frac{1}{u - t} \int_{\mathbb{R}^{3}} H(s, x, t, dy, u, z) = 0 \] and

(a*) \[ \lim_{t \uparrow a} \frac{1}{u - t} \int_{\mathbb{R}^{3}} H(s, x, t, dy, u, z) = 0, \]

(b) There is an $\mathbb{R}^{3}$-valued function $B^{y}(x, s)$ such that

(b*) $B^{y}(z, u)$ such that

$$\lim_{t \downarrow s} \frac{1}{u - t} \int_{\mathbb{R}^{3}} (y - x) H(s, x, t, dy, u, z) = B^{y}(x, s)$$

(c) There is a $d \times d$ matrix-valued function $C(x, s)$ such that

(c*) $C^{y}(z, u)$ such that

$$\lim_{t \downarrow s} \frac{1}{u - t} \int_{\mathbb{R}^{3}} (y - x)(y - x)^{t} H(s, x, t, dy, u, z) = C^{y}(x, s),$$

where $t$ denotes the transpose

(c*) $C^{y}(z, u)$ such that

$$\lim_{t \uparrow a} \frac{1}{u - t} \int_{\mathbb{R}^{3}} (z - y)(z - y)^{t} H(s, x, t, dy, u, z) = C^{y}(z, u),$$

(d) There is a $\delta > 0$ such that

(d*) $\delta > 0$ such that

$$\lim_{t \downarrow s} \frac{1}{u - t} \int_{\mathbb{R}^{3}} \|y - x\|^{2 + \delta} H(s, x, t, dy, u, z) = 0$$

(d*) $\delta > 0$ such that

$$\lim_{t \uparrow a} \frac{1}{u - t} \int_{\mathbb{R}^{3}} \|z - y\|^{2 + \delta} H(s, x, t, dy, u, z) = 0.$$
The function $B'$ (respectively $B^*_t$) is called the forward (backward) drift of the Bernstein Bridge and $C'$ (resp. $C^*_t$) its forward (respectively backward) diffusion matrix.

Let the potential $V$ be such that the integral kernel of $e^{-(t-s)H/h}$ on $L^2(\mathbb{R}^3)$, denoted by $h(s, x, t, y) = h(x, t-s, y)$, is jointly continuous in $x, y \in \mathbb{R}^3$ and $t-s > 0$, and strictly positive. This is a large class of potentials investigated, in particular, by Kato (cf. [2]). For $s \leq t \leq u, x, y, z$ in $\mathbb{R}^3$, one verifies easily [23] that

$$h(s, x, t, y, u, z) = \frac{h(s, x, t, y) h(t, y, u, z)}{h(s, x, u, z)} \tag{2.1}$$

is the density of a Bernstein transition probability, and that it satisfies (a)–(d) and (a*)–(d*). Notice that, since the kernel $h$ is not, in general, the density of any Markovian probability transition, (2.1) is not just an $h$-transform in the sense of Doob. According to Theorem 2.1, the additional data of a joint probability $m$ produces a Bernstein diffusion $z, t$ in $I$. One shows that a particular choice of $m$ produces a Markovian diffusion, namely

$$m(B_s \times B_E) = \int_{B_s \times B_E} \Theta^*_{T/2}(x) h(x, T, y) \Theta_{T/2}(y) \, dx \, dy \tag{2.2}$$

for $\Theta^*_{T/2}$ and $\Theta_{T/2}: \mathbb{R}^3 \to \mathbb{R}$ two arbitrary bounded measurable positive functions. Equation (2.2) will be the Euclidean (real) version of Feynman’s transition amplitude (1.5) on $I = [-T/2, T/2]$. $\Theta^*_{T/2}$ and $\Theta_{T/2}$ are used to regularize the Bernstein Bridges as follows.

After substitution of (2.2) in Theorem 2.1(4), the finite dimensional distributions of an arbitrary Bernstein diffusion $z, t \in I$, reduce to

$$P_m(dy_1, t_1, ... , dy_n, t_n) = \int \Theta^*_{T/2}(x) h(x, t_1 + \frac{T}{2}, y_1)$$

$$\cdots h \left( y_n, \frac{T}{2} - t_n, z \right) \Theta_{T/2}(z) \, dx \, dy_1 \cdots dy_n \, dz \tag{2.3}$$

for $-T/2 \leq t_1 < t_2 < \cdots < t_n \leq T/2$.

The data of a joint probability $m$ is not very natural but if we are given a pair of boundary probability densities $p_{-T/2}$ and $p_{T/2}$ instead, the marginals of (2.2) constitute a system of equations for $\Theta^*_{T/2}$ and $\Theta_{T/2}$.

$$\Theta^*_{T/2}(x) \int_{\mathbb{R}} h(x, T, z) \Theta_{T/2}(z) \, dz = p_{-T/2}(x) \tag{2.4}$$

$$\Theta_{T/2}(z) \int_{\mathbb{R}} \Theta^*_{T/2}(x) h(x, T, z) \, dx = p_{T/2}(z),$$
whose existence and uniqueness of positive (not necessarily integrable) solutions, when $p_{-T/2}$ and $p_{T/2}$ are strictly positive, has been shown [4].

In this way, to each potential $V$, that is, to each Hamiltonian $H$, is associated an infinite family of Bernstein diffusions, indexed by their boundary probabilities (or equivalently by the initial condition of (1.2), in contrast with Kac's point of view). It is easy to verify (cf. [23]) that the resulting Bernstein diffusion $z_t$ is without killing, time reversible, generally not stationary, with forward (respectively backward) drift, and diffusion matrix given, for $y$ in $\mathbb{R}^3$, $t$ in $I$, by

$$B(y, t) = \frac{\nabla \eta(y, t)}{\eta(y, t)}, \quad B^*(y, t) = -\frac{\nabla \eta^*(y, t)}{\eta^*(y, t)} \quad (2.5)$$

$$C(y, t) = C^*(y, t) = h I, \quad (2.6)$$

where $I$ is the $3 \times 3$ identity matrix and $\eta, \eta^*$ are defined formally by

$$\eta(y, t) = (e^{i(T/2)H/h} \Theta_{T/2})(y) \quad (2.7)$$

$$\eta^*(y, t) = (e^{-(i(T/2)H/h} \Theta^*_{T/2})(y) \quad (2.8)$$

for $t$ in the compact interval $I$.

The proofs of the existence of $\eta$ and $\eta^*$ can be found in [2]. We recall that $V$ is taken to be in the Schrödinger class (essentially this is a slight improvement of the above-mentioned Kato class) and also that $\Theta_{T/2}$ and $\Theta^*_{T/2}$ are chosen to be, resp., of the form $e^{i(T/2)H/\phi}$ and $e^{-i(T/2)H/\phi^*}$, with $\phi, \phi^*$ strictly positive functions in $L^2(\mathbb{R}^3)$.

Moreover, the probability for $z_t$ to be in the Borelian $B$ at time $t$ is given by

$$\int_B \eta^*(y, t) \, dy, \quad t \in I \quad (2.9)$$

This is the Euclidean version of Born interpretation (1.3), missing in Kac's approach. Notice that $\eta^*$ is a solution of the heat equation (1.2) and that $\eta$ is a solution of the backward heat equation. The theory is time-symmetric since $\eta \leftrightarrow \eta^*$ under time reversal.

The concept of Bernstein diffusion has been introduced in [23.1] on the basis of partial results of Bernstein, Fortet, Beurling, and Jamison. It enables us to realize an old idea of E. Schrödinger [18], used recently as a probabilistic framework for a new Euclidean approach to Quantum Mechanics [23.2]. A summary of the probabilistic construction can be found in [23.3] and the analytical part of the theory is considered in [2].
3. MALLIAVIN CALCULUS AND AN INTEGRATION BY PARTS FORMULA

Let us consider the space $\Omega_0$ of continuous paths starting from zero (the classical Wiener space), namely $\Omega_0 = \{ \omega \in C([0, T/2]; \mathbb{R}^3) : \omega(0) = 0 \}$. We denote by $\mathcal{H}$ the associated Cameron–Martin space,

$$\mathcal{H} = \left\{ \phi \in \Omega_0 : \phi \text{ exists and } \int_0^{T/2} \|\phi(s)\|^2 \, ds < +\infty \right\},$$

which is a Hilbert Space with respect to the scalar product $(\cdot | \cdot)_1$, defined by $(\phi_1 | \phi_2)_1 = \int_0^{T/2} \phi_1(s) \cdot \phi_2(s) \, ds$. The space $\Omega_0$ is endowed with the Wiener measure of parameter $\lambda > 0$, $\mu_\lambda^\omega$, which is the Gaussian measure supported by $\Omega_0$ and with characteristic functional

$$I_{\omega} = \exp\left( -\frac{\lambda}{2} \|l\|_1^2 \right), \quad \forall l \in \Omega_0^* \subset \mathcal{H}, \quad (3.1)$$

where $\Omega_0^*$ is the dual of $\Omega_0$.

Since we shall be working with functionals of the Brownian motion, we shall need to use an infinite dimensional calculus providing good differentiation techniques and, in particular, an integration by parts formula. Although for sufficiently smooth functionals (and therefore in the cases treated in this work) those kinds of formulas were known for a certain time, we shall use here a more general version in the framework of Malliavin Calculus. This is an infinite dimensional calculus which is adapted to non-"regular" functionals like those coming from Itô calculus (cf. [16]). It is indeed known that they are in general not even continuous with respect to the natural topologies of $\Omega_0$ (either the uniform-norm topology or the Hölderian norm one). In this paragraph we recall some of the basic notions needed afterwards.

In Malliavin Calculus, the space $\mathcal{H}$ plays a central rôle. For a given $\phi \in \mathcal{H}$, the Cameron–Martin theorem assures us that the probability measure induced by the translation $\omega \rightarrow \omega + \phi$ is absolutely continuous with respect to $\mu_\lambda^\omega$. Therefore, for any Wiener functional $F : \Omega_0 \rightarrow \mathcal{E}$, where $\mathcal{E}$ is a real Hilbert space, one can consider the following (a.s.) limit, for $\phi \in \mathcal{H}$:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{ F(\omega + \varepsilon \phi) - F(\omega) \}. \quad (3.2)$$

This is, by definition (when the limit exists), the derivative of $F$ along the direction of $\phi$ and will be denoted by $D_\phi F[\omega]$. Accordingly, the gradient of $F$ is the linear operator on $\mathcal{H}$, $\nabla F[\omega] \in L(\mathcal{H}, \mathcal{E})$, defined by:

$$\nabla F[\omega](\phi) = D_\phi F[\omega].$$
When $\nabla F[\omega]$ is a Hilbert–Schmidt operator, namely when
\[
\|\nabla F[\omega]\|_{H.S.}^2 = \sum_{k=1}^{\infty} \|\nabla F[\omega](e_k)\|_{\mathbb{C}}^2 < +\infty,
\]
where $\{e_k\}$ is an orthonormal basis of $\mathcal{H}$, the functional $\nabla F$ takes also values in a Hilbert space (the space of Hilbert–Schmidt operators on $\mathcal{H}$).

In this case it is possible to define $\nabla^2 F$ in an analogous way. By iteration, the following Sobolev spaces can be considered, for $1 \leq r, p < +\infty$,
\[
W^r_p(\Omega_0, d\mu^\mu_0; \mathbb{E}) = \{F \in L^p(\Omega_0, d\mu^\mu_0; \mathbb{E}): E_{\mu^\mu_0} \|\nabla^r F\|_{H.S.}^p < +\infty \ \forall 1 \leq i \leq r\},
\]
where $\|\nabla^r F[\omega]\|_{H.S.}^p = \sum_{k_1, \ldots, k_r = 1}^{+\infty} \|\nabla^r F[\omega](e_{k_1}, \ldots, e_{k_r})\|_{\mathbb{C}}^p$.

For a functional $\Phi \in L^2(\Omega_0, d\mu^\mu_0; \mathcal{H})$, the divergence of $\Phi$, $\mathcal{D}_{\mu^\mu_0}\Phi$, is defined as the adjoint of the gradient operator in $L^2$ with respect to the Wiener measure. In other words, the divergence of $\Phi$ is the functional $\mathcal{D}_{\mu^\mu_0}\Phi \in L^2(\Omega_0, d\mu^\mu_0; \mathbb{R})$, when it exists, verifying the relation
\[
E_{\mu^\mu_0}[\nabla F[\Phi]] = E_{\mu^\mu_0}[F \mathcal{D}_{\mu^\mu_0}\Phi]
\]
for every $F \in W^r_p(\Omega_0, d\mu^\mu_0; \mathbb{R})$.

Although it is possible to consider more general functionals $\Phi$ (cf. [11]), for $\Phi[\omega](\cdot)$ an adapted process (we shall restrict ourselves to this case), the divergence coincides with an Itô's integral and we have
\[
\mathcal{D}_{\mu^\mu_0}\Phi[\omega] = \int_0^{T/2} \frac{d}{dt} \Phi[\omega](t) \lambda^{-1} \, d\omega(t).
\]

Let us consider a $\mu^\mu_0$-integrable functional $\rho > 0$. With respect to the measure $\rho \mu^\mu_0$, we may as well define the divergence $\mathcal{D}_{\rho \mu^\mu_0}$ as the adjoint of the gradient and write the analogous relation of (3.3). We have the following integration by parts formula:

3.1. **Proposition.** Let $\rho \in W^{1+\epsilon}_1(\Omega_0, d\mu^\mu_0; \mathbb{R})$ for some $\epsilon > 0$, $\rho > 0$. Let $\Phi \in W^{2p_0}_1(\Omega_0, d\mu^\mu_0; \mathcal{H})$, where $p_0$ is the conjugate exponent of $1 + \epsilon$. Then
\[
E_{\rho \mu^\mu_0}[\nabla F[\Phi]] = E_{\rho \mu^\mu_0}[F(\mathcal{D}_{\rho \mu^\mu_0}\Phi - (\nabla \log \rho \mid \Phi))] \quad (3.5)
\]
for every functional $F \in W^{2p_0}_1(\Omega_0, d\mu^\mu_0; \mathbb{R})$.

**Proof.** The integration by parts formula (3.5) means that $\mathcal{D}_{\rho \mu^\mu_0}\Phi$ and $\mathcal{D}_{\rho \mu^\mu_0}$ are related by
\[
\mathcal{D}_{\rho \mu^\mu_0}\Phi = \mathcal{D}_{\rho \mu^\mu_0}\Phi - (\nabla \log \rho \mid \Phi),
\]
We recall that any functional belonging to a Sobolev space can be
approximated by bounded functionals of bounded derivatives. Let $F$ be such a functional and $G_k$ a sequence of bounded functionals with first bounded derivative approximating $F\rho$ in $W^1_{1+\epsilon}$. By definition of the divergence, we have

$$E_{\mu_\rho}[\langle\nabla G_k|\Phi\rangle_1] = E_{\mu_\rho}[G_k\nabla_\rho\Phi].$$

Since $\Phi \in W^{2\rho}_{1}(\Omega_0, d\mu_\rho; \mathcal{H})$, we have $\nabla_\rho\Phi \in L^2(\Omega_0, d\mu_\rho; \mathbb{R})$: therefore, using Hölder inequality, we get, when $G_k \to F\rho$,

$$E_{\mu_\rho}[\langle\nabla(F\rho)|\Phi\rangle_1] = E_{\mu_\rho}[F\rho\nabla_\rho\Phi]$$

and the result follows from the equality $\nabla(F\rho) = F \nabla \rho + \rho \nabla F$, for $F$ bounded. The general result is obtained by approximation. 

4. Functional Calculus

For the sake of generality, it will be convenient to consider a Hamiltonian more general than the one considered before, namely the Euclidean version of the Hamiltonian for a unit mass and charged particle in an electromagnetic field,

$$H_A = -\frac{\hbar^2}{2}\left(\nabla - \frac{A}{\hbar}\right)^2 + V$$

$$= -\frac{\hbar^2}{2} \mathcal{A} + \hbar \mathcal{A} \cdot \nabla + \frac{\hbar}{2} \nabla \cdot \mathcal{A} - \frac{1}{2} \|\mathcal{A}\|^2 + V, \quad (4.1)$$

where $\mathcal{A}: \mathbb{R}^3 \to \mathbb{R}^3$ is called the vector potential. We also define the adjoint

$$H_A^* = -\frac{\hbar^2}{2} \left(\nabla - \frac{A}{\hbar}\right)^2 + V \quad (4.2)$$

and we shall impose on $\mathcal{A}$ and $V$ the following conditions:

(1) $\exists \epsilon_0 > 0$ s.t. $H_A - \epsilon_0$ generates a contraction semigroup.

(2) $\mathcal{A}, \|\mathcal{A}\|^2$, and $V$ belong to the Schrödinger class, namely the class of potentials $\mathcal{V}$ such that the kernel $e^{-(t-s)(\hbar/2, \mathcal{A} + \mathcal{V})}$, $t \geq s$, is well defined, jointly continuous in $x, y$, and $(t-s)$, strictly positive, and, moreover, verifies

$$\sup_x E_x \left(\exp x \int_0^{T/2} |\mathcal{V}|(\hbar^{1/2} \mathcal{W}(t)) \, dt\right) < +\infty$$
and

\[ \sup_x E_x \left( \exp x \int_{-\frac{r}{2}}^0 |\hat{V}|(h^{1/2}w_*(t)) \, dt \right) < +\infty, \quad \text{for some } x > 1, \]

where \( w(t) \) is a \( \mathcal{F}_t \)-Brownian motion and \( w_*(t) \) is an \( \mathcal{F}_t \)-Brownian motion (cf. [3] for such estimations).

(III) \( A, \nabla \cdot A, \|A\|, \) and \( V \) belong to the class, \( \mathcal{K}_p \cap \mathcal{K}_p^* \) defined by

\[ \mathcal{K}_p = \left\{ g : \sup_x E_x \int_{0}^{T/2} \|\nabla g\| \rho(h^{1/2}w(t)) \, dt < +\infty \right\} \]

\[ \mathcal{K}_p^* = \left\{ g : \sup_x E_x \int_{0}^{T/2} \|\nabla g\| \rho(h^{1/2}w_*(t)) \, dt < +\infty \right\} \]

for some \( p > 1 \).

(IV) \( A \) and \( V \) are such that the function \( \eta \) defined in (4.6) belongs to \( \mathcal{K}_p \) and \( \eta^* \) (cf. 4.8) belongs to \( \mathcal{K}_p^* \), for some \( p > 1 \).

These hypotheses allow us to construct positive solutions of the heat (resp. backward heat) equation \( \eta^* \) and \( \eta \), with \( t \in I \), for appropriate initial conditions (cf. [2] for an analytic vector argument in the self-adjoint case). They also allow us to have a representation of the semigroup \( e^{-(t-s)H^*_*} \) by using Girsanov and Feynman–Kac formulae. The method of construction of the Bernstein processes is then analogous to the case where \( A = 0 \) (cf. [23.3]). It follows from this construction that \( z(t) \), for smooth drifts, solves simultaneously the (Itô’s) \( \mathcal{P}_t \) and \( \mathcal{F}_t \) stochastic differential equations

\[ dz(t) = h^{1/2} \, dw(t) + \left( \frac{\nabla \eta}{\eta} - A \right) \, dt \]  

\[ d_* z(t) = h^{1/2} \, d_* w_*(t) - \left( \frac{\nabla \eta^*}{\eta^*} + A \right) \, dt, \]

where \( d_* \) denotes the backward differential, namely \( d_* F(t) = F(t) \, F(t - dt) \), \( \eta \) and \( \eta^* \) are formally given by (2.7) and (2.8), and the boundary conditions satisfy (2.4). For simplicity, we shall restrict ourselves to Bernstein processes with smooth drift (smooth diffusions), although the method is valid in more general situations where \( z(t) \) is not a strong solution of (4.3) and (4.4).

4.1. Proposition. Let \( z(t), \ t \in I, \) be a smooth Bernstein process associated with (4.1). The law of \( z \) on the space \( C([0; T/2]; \mathbb{R}^3) \) is absolutely continuous with respect to the Wiener measure of parameter \( h \) and initial
distribution given by the law of \( z(0) \). The corresponding Radon–Nikodym density is given by

\[
\rho[z] = \frac{\eta(z(T/2), T/2)}{\eta(z(0), 0)} \exp - \frac{1}{\hbar} \int_0^{T/2} \left[ A(z(t)) \circ dz(t) + V(z(t)) dt \right] \quad (4.5)
\]

or, equivalently,

\[
\rho[z] = \frac{\eta(z(T/2), T/2)}{\eta(z(0), 0)} \exp - \frac{1}{\hbar} \int_0^{T/2} \left[ A(z(t)) \cdot dz(t) + \frac{\hbar}{2} \nabla \cdot A(z(t)) dt + V(z(t)) dt \right], \quad (4.5)^1
\]

where \( o \) is the Fisk–Stratonovich integral \([24]\), \( \eta = \eta_x \) denotes the solution of the backward heat equation,

\[
\hbar \frac{\partial \eta}{\partial t} = H_A \eta \quad t \in [0, T/2]
\]

\[
\eta(x, o) = \chi(x)
\]

and \( \chi \) is a positive bounded vector in \( e^{T/2H_A}(\mathcal{D}(e^{T/2H_A})) \).

**Remark.** Hypothesis (I) and the choice of initial conditions assure the solvability of the system (4.6).

**Proof.** It follows from the construction of the Bernstein processes that their (forward) transition probability densities w.r.t. \( dy \) are given by

\[
p(o, x, t, y) = \frac{\eta(y, t)}{\eta(x, o)} h(o, x, t, y),
\]

where \( h \) is the integral kernel associated to the semigroup \( e^{tH_A} \). By hypothesis (II) we have an integral representation of this kernel using Feynman–Kac and Girsanov formulae. This gives precisely (4.5) (a formula somewhat reminiscent of some results of \([8]\)).

We remark that one could also look at the s.d.e. solved by \( z(t) \). Girsanov's theorem would give directly the expression for the density

\[
\rho[\tilde{w}] = \exp \left[ \int_0^{T/2} \left( \frac{\nabla \eta}{\eta} + \frac{A}{\hbar} \right)(\tilde{w}(s)) \cdot d\tilde{w}(s) - \frac{1}{2\hbar} \int_0^{T/2} \left\| \frac{\nabla \eta}{\eta} + A \right\|^2 (\tilde{w}(s)) \, ds \right].
\]
where \( \tilde{w} = h^{1/2}w \), once the equality \( E\rho(\tilde{w}) = 1 \) was established. According to Itô’s calculus,

\[
\int_0^{T/2} \frac{\nabla \eta}{\eta} \cdot d\tilde{w}(s) + \int_0^{T/2} \frac{\dot{\eta}}{\eta} \, ds = \int_0^{T/2} \frac{\nabla \eta}{\eta} \cdot d\tilde{w}(s) + \frac{h}{2} \int_0^{T/2} \left[ \frac{\nabla^2 \eta}{\eta^2} - \frac{2}{h} \frac{\ddot{\eta}}{\eta} \right] \, ds.
\]

By using (4.6) this formula would give (4.5), after exponentiation.

4.2. PROPOSITION. The law of \( z \) is absolutely continuous with respect to the \((\mathcal{F})\) Wiener measure (corresponding to the Brownian motion \( h^{1/2}w_*(t) \)) with density given by

\[
\rho_*(z) = \frac{\eta^*(z(-T/2), -T/2)}{\eta^*(z(0), 0)} \exp \left\{ \frac{1}{h} \int_{-T/2}^0 [A(z(t))o d_*(z(t)) + V(z(t)) \, dt] \right\},
\]

where \( \eta^* = \eta^* \chi' \) is the solution of the forward heat equation, for \( \chi' \) a positive bounded vector in \( \mathcal{D}(e^{T/2H}) \):

\[
-h \frac{\partial \eta^*}{\partial t} = \mathcal{H}_* \eta^* \quad t \in \left[ -\frac{T}{2}, 0 \right].
\]

\[
\eta^*(x, o) = \chi'(x).
\]

The proof is analogous. Notice that the Hamiltonian \( \mathcal{H}_* \) is not the same as in (4.6); the vector potential \( A \) changes its sign. This is due to the fact that, since the underlying Schrödinger equation in terms of the physical vector potential \( a \) is

\[
\frac{i\hbar}{2} \frac{\partial \psi}{\partial \tau} = \frac{\hbar^2}{2} \left[ \nabla - \frac{ia}{\hbar} \right]^2 \psi + V\psi
\]

\[
\equiv -\frac{\hbar^2}{2} \Delta \psi + \frac{i\hbar}{2} \nabla \cdot a\psi + i\hbar a \nabla \psi + \frac{1}{2} a^2 \psi + V\psi,
\]

its Euclidean version (involving \( a \rightarrow iA \)) is \( \mathcal{H}_* \eta^* \) given in (4.8). On the other hand,

\[
-\frac{i\hbar}{2} \frac{\partial \tilde{\psi}}{\partial \tau} = \frac{\hbar^2}{2} \Delta \tilde{\psi} - \frac{i\hbar}{2} \nabla \cdot a\tilde{\psi} - i\hbar a \nabla \tilde{\psi} + \frac{1}{2} a^2 \tilde{\psi} + V\tilde{\psi}
\]

whose Euclidean version is \( \mathcal{H}_\mathcal{A} \eta \), as defined in (4.1).
It is an interesting open problem to find in which sense Propositions 4.1 and 4.2 still hold when $x$ or $x'$ have zeroes since, even in this case (the quantum mechanical "stationary states"), the Bernstein diffusions are well defined (cf. [1.1, 2, 7] and ulterior references in [5]).

For simplicity, we shall restrict ourselves to processes with smooth drifts. Also we shall consider the conditioned Bernstein diffusion such that $z(0) = 0$. In fact, this does not change the kind of results we are going to obtain but this particular choice allows as to use directly the Wiener measure $\mu^h_{\omega}$. Considering the Sobolev spaces defined in Section 3, we have:

4.3. Lemma. Let $\varepsilon > 0$. Then, for some $\alpha$ and $p$ in hypotheses (II)–(IV), the functionals $\rho$ and $\rho^*$ given by the last two propositions belong, resp., to the spaces $W_1^{1+\varepsilon}(\Omega_0, d\mu^h_{\omega}; \mathbb{R})$ and $W_1^{1+\varepsilon}(\Omega_0^*, d\mu^h_{\omega^*}; \mathbb{R})$, where $\Omega_0^* = \{\omega \in C([-T/2, 0]; \mathbb{R}^3): \omega(0) = 0\}$.

Proof. First notice that

$$E_{\mu^h_{\omega}} \left( \exp \left[ \int_0^T A(z) \cdot dz \right] \right)$$

and that the first factor in the r.h.s. is equal to one. Then it follows from hypothesis (II), the expression of $\rho$, and the representation of $\eta(\eta^*)$ in terms of Girsanov and Feynman–Kac formulae, that $\rho(\rho^*)$ belongs to $L^{1+\varepsilon}$, and that $\nabla \rho$ will also belong to $L^{1+\varepsilon}$ (by using some Hölder inequality estimates).

With respect to the gradient of $\rho$, we have

$$\nabla \rho [z] = \left[ \frac{\nabla \eta(z(T/2), T/2, \chi(o))}{\chi(o)} \nabla z(T/2) \right]$$

$$+ \frac{\eta(z(T/2), T/2, \chi(o))}{\chi(o)} \left[ -\frac{1}{\hbar} \int_0^{T/2} [A(z(t)) \circ dz(t) + V(z(t)) \, dt] \right]$$

$$\times \left[ \exp \left[ \int_0^{T/2} [A(z(t)) \circ dz(t) + V(z(t)) \, dt] \right] \right]$$

By choosing $\alpha$ and $p$ big enough, $\nabla \rho$ will also belong to $L^{1+\varepsilon}$ (cf. for example, [20] for the Sobolev estimates of stochastic integrals).
Let us recall the following definition of forward and backward derivatives, for \( f \) regular enough,

\[
D_f(z(t), t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E_t[f(z(t + \Delta t), t + \Delta t) - f(z(t), t)] \quad (4.9)
\]

\[
D_*f(z(t), t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E_t^*[f(z(t), t) - f(z(t - \Delta t), t - \Delta t)], \quad (4.10)
\]

where \( E_t = E[\mathcal{F}_t] \) and \( E_t^* = E[\mathcal{F}_t^*] \). We denote by \( E \) the (complete) expectation with respect to the process \( z(t) \).

4.4. LEMMA. Let \( \delta z(t) \) be a \( \mathcal{F}_t \)-adapted process of bounded variation such that \( \delta z(s) = 0 \) and let \( f(z(t), t) \) be a \( \mathcal{F}_t \)-adapted process such that \( Df(z(t), t) \) exists and \( t \to Df(z(t), t) \) is continuous. Then, for \( s < u \),

\[
E_s \int_s^u \frac{d}{dt} \delta z(t) f(z(t), t) \, dt = -E_s \int_s^u \delta z(t) Df(z(t), t) \, dt + E_s[\delta z(u) f(z(u), u)].
\]

Proof. By Itô's formula, and since \( \delta z(t) \) is of bounded variation,

\[
d(\delta z(t) f(z(t), t)) = d \delta z(t) f(z(t), t) + \delta z(t) df(z(t), t).
\]

Taking the conditional expectation \( E_s \), and after time integration, the conclusion follows from definition (4.9). \( \square \)

We are now able to show that:

4.5. THEOREM. Let \( A \) and \( V \) satisfy (I)–(IV) for suitable \( \alpha \) and \( p \) and let \( 0 \leq s \leq T/2 \). For every \( \mathcal{F}_t \)-adapted functional \( \delta z(\cdot) \in \bigcap_{p \geq 1} W^p_1(\Omega_0, d\mu^h_0; \mathcal{H}) \) such that \( \delta(z)(t) = 0 \) for \( t \) in \( [0, s] \) and for every \( F \in \bigcap_{p \geq 1} W^p_1(\Omega_0, d\mu^h_0; \mathbb{R}) \) we have

\[
E((\nabla F[z] | \delta z)_s) = E \left[ F[z] \left( \int_s^{T/2} \frac{d}{dt} \delta z(t) \frac{1}{h} \, dz(t) \right. \right.
\]

\[
\left. + \frac{1}{h} \int_s^{T/2} \delta z(t) \nabla A(z(t)) \cdot dz(t) \right. \right.
\]

\[
\left. + \int_s^{T/2} \delta z(t) \left[ \frac{1}{2} \nabla(\nabla \cdot A) + \frac{\nabla V}{h} \right](z(t)) \, dt \right. \right.
\]

\[
\left. + \int_s^{T/2} \frac{A}{h}(z(t)) \frac{d}{dt} \delta z(t) \, dt - \delta z(T/2) \frac{\nabla \eta}{\eta} (z(T/2), T/2) \right].
\]

(4.11)
Similarly, for every \( \mathcal{F}_t \)-adapted \( \mathcal{H} \)-valued functional \( \delta_\star z \in \bigcap_{p \geq 1} W_p^p(\Omega_0^\star, d\mu_w^h) \) such that \( \delta_\star z(t) = 0 \) for \( t \in [s, 0], -T/2 \leq s \leq 0 \), and every \( F_\star \in \bigcap_{p \geq 1} W_p^p(\Omega_0^\star, d\mu_w^h) \),

\[
E((\nabla F_\star[z] | \delta_\star z)_1) = E\left[ F_\star[z] \left( \int_{-T/2}^s \frac{d}{dt} \delta_\star z(t) \frac{1}{\hbar} d_\star z(t) \right) \right.
\]

\[
+ \frac{1}{\hbar} \int_{-T/2}^s \delta_\star z(t) \nabla A(z(t)) \cdot d_\star z(t) \nabla_\star V(z(t)) dt
\]

\[
+ \int_{-T/2}^s \frac{A}{\hbar} \frac{d}{dt} \delta_\star z(t) dt
\]

\[
- \left. \frac{\nabla \eta^\star}{\eta^\star} (z(-T/2), -T/2) \right| \delta_\star z(t) - \delta_\star z(0) \right] \right]. \quad (4.12)
\]

Remarks. (1) \( \delta z(\cdot) \) is called the "variation of \( z \)," by analogy with the terminology of classical calculus of variation.

(2) Although, to be consistent, we should denote by \( \mathcal{H}^\star \) the Cameron–Martin space of \( \Omega_0^\star \), we do not distinguish these spaces.

Proof: We shall only prove the first equality, since the proof of (4.12) is similar.

Let us consider the Wiener space \( \Omega_0 \) endowed with the measure \( \rho \mu_w^h \). By Lemma 4.3, \( \rho \in W_1^{1+\epsilon}(\Omega_0, d\mu_w^h; \mathbb{R}) \) for suitable \( \epsilon \) and \( p \). In order to apply formula (3.5) for the variation \( \Phi = \delta z \) (we use this notation in order to keep close to those of the classical calculus of variations), and with respect to the measure \( \rho \mu_w^h \), we now compute the term \( (\nabla \log \rho[z] | \delta z)_1 \). We have, according to the expression of \( \rho \) given in Proposition 4.1, (4.5),

\[
(\nabla \log \rho[z] | \delta z)_1 = \frac{\nabla \eta}{\eta} (z(T/2), T/2) \delta z(T/2) - \frac{\nabla \eta}{\eta} (z(0), 0) \delta z(0)
\]

\[
- \left( \nabla \int_0^{T/2} \frac{A}{\hbar} (z(t)) \frac{ez(t)}{\hbar} (z(t)) dt \right)_1
\]

\[
- \left( \nabla \int_0^{T/2} \left[ \frac{1}{2} \nabla \cdot A + \frac{V}{\hbar} \right] (z(t)) dt \delta z \right)_1,
\]

therefore:
\[
\left(\nabla \log \rho[z] \right)_{\delta z} = \frac{\nabla \eta}{\eta} (z(T/2), T/2) \delta z(T/2)
\]

\[
- \int_{s}^{T/2} \left[ \frac{1}{2} \nabla (\nabla \cdot A) + \frac{\nabla V}{h} \right] (z(t)) \delta z(t) \, dt
\]

\[
- \int_{s}^{T/2} \frac{A}{h} (z(t)) \, d\delta z(t)
\]

\[
- \frac{d}{ds} \int_{s}^{T/2} \frac{A}{h} (z(t) + \varepsilon \delta z(t)) \, dz(t). \tag{4.13}
\]

Since we have \(\mathcal{Q}_{t+h} \delta z = \int_{s}^{T/2} (d/dt) \delta z(t)(1/h) \, dz(t)\), the integration by parts formula (3.5) reduces in this case to (4.11) (cf. [20] for related computations). \(\square\)

4.6. Theorem. In the hypothesis of Theorem 4.5 and assuming, furthermore, that \(F\) is a \(\mathcal{P}_{c}\)-adapted functional (resp. \(F_*\) is \(\mathcal{F}_{c}\)-adapted), we have

\[
E \left[ F[z] \left( \int_{s}^{T/2} \delta z(t) \left[ -\frac{1}{h} DDz + \frac{1}{h} Dz \wedge \text{rot} \, A 
\right. \right. \right.
\]

\[
\left. \left. + \frac{1}{2} \text{rot} \, \text{rot} \, A + \frac{\nabla V}{h} \right] (z(t), t) \, dt \right.
\]

\[
\left. + \delta z(T/2) \left[ \frac{1}{h} Dz(T/2) - \frac{\nabla \eta}{\eta} (z(T/2), T/2) + A(z(T/2)) \right] \right) = 0.
\tag{4.14}
\]

where \(\wedge\) stands for the exterior product in \(\mathbb{R}^3\) and,

\[
E \left[ F[z] \left( \int_{-T/2}^{s} \delta_* z(t) \left[ -\frac{1}{h} D_* D_* z + \frac{1}{h} D_* z \wedge \text{rot} \, A \right. \right. \right.
\]

\[
\left. \left. - \frac{1}{2} \text{rot} \, \text{rot} \, A + \frac{\nabla V}{h} \right] (z(t)) \, dt \right.
\]

\[
\left. + \delta_* z(-T/2) \left[ -\frac{1}{h} D_* z(-T/2) - \frac{\nabla \eta^*}{\eta^*} (z(-T/2), -T/2) \right. \right.
\]

\[
\left. \left. - A(z(-T/2)) \right) \right) = 0. \tag{4.15}
\]

Proof. Because \(F\) is \(\mathcal{P}_{c}\)-adapted.
\[
E \left[ F[z] \int_s^{T/2} \frac{d}{dt} \delta z(t) \frac{1}{\hbar} dz(t) \right] \\
= E \left[ F[z] E_s \int_s^{T/2} \frac{d}{dt} \delta z(t) \frac{1}{\hbar} dz(t) \right] \\
= E \left[ F[z] \int_s^{T/2} \frac{d}{dt} \delta z(t) \frac{1}{\hbar} Dz(t) dt \right] \\
= E \left[ \frac{1}{\hbar} F[z] [\delta z(T/2) Dz(T/2) - \delta z(o) Dz(o)] \right] \\
- E \left[ F[z] \int_s^{T/2} \delta z(t) \frac{1}{\hbar} DDz(t) dt \right]
\]

by Lemma 4.4. On the other hand, the last two terms in the expression for \((\nabla \log \rho[z] | \delta z)\), given by (4.13) lead, in the integration by parts formula, to

\[
- E \left[ F[z] \int_s^{T/2} A(\delta z(t)) d(\delta z)(t) \right] \\
- E \left[ \frac{F[z]}{\hbar} \int_s^{T/2} \nabla A \cdot Dz(z(t)) \delta z(t) dt \right] \\
= - E \left[ \frac{F[z]}{\hbar} (z(T/2)) \delta z(T/2) \right] \\
+ E \left[ \frac{F[z]}{\hbar} \int_s^{T/2} A(z(t)) \delta z(t) dt \right] \\
- E \left[ \frac{F[z]}{\hbar} \int_s^{T/2} \nabla A \cdot Dz(z(t)) \delta z(t) dt \right],
\]

where \( (\nabla A \cdot Dz)_j \triangleq \sum_i \delta_{ij} A^i Dz^i \).

Finally, we get

\[
E[\delta F[z](\delta z)] = E \left[ F[z] \int_s^{T/2} \delta z(t) \left[ - \frac{1}{\hbar} DDz + \frac{1}{2} \nabla(\nabla \cdot A) \right. \right. \\
\left. + \frac{\nabla V}{\hbar} - \frac{DA}{\hbar} + \frac{1}{\hbar} \nabla A \cdot Dz \right] (z(t)) dt \right] \\
+ E \left[ F[z] \delta z(T/2) \left[ \frac{1}{\hbar} Dz(T/2) \right. \right. \\
\left. - \frac{\nabla \eta}{\eta} (z(T/2), T/2) + \frac{A}{\hbar} (z(T/2)) \right].
\]
Since \( DA = (\hbar/2) AA + (Dz \cdot \nabla)A \), by Itô calculus, we have
\[
- \frac{1}{\hbar} DDz + \frac{1}{2} \nabla(\nabla \cdot A) + \frac{\nabla V}{\hbar} - \frac{DA}{\hbar} + \frac{1}{\hbar} \nabla A \cdot Dz
\]
\[= - \frac{1}{\hbar} DDz + \frac{1}{\hbar} Dz \wedge \text{rot} A + \frac{\nabla V}{\hbar} + \frac{1}{2} \text{rot rot} A.
\]

On the other hand, by the hypothesis on \( F \) and \( \delta z \), \( \delta F[z](\delta z) = 0 \) and therefore we have (4.14). The proof of (4.15) is analogous.

Remarks. (1) The choice of zero at time zero as initial condition was just made in order to simplify the computation. If we choose more general conditioning, the integration with respect to the corresponding Wiener measures is easily reduced to the case already treated.

(2) In (4.12) and (4.15) \( E \) denotes, somewhat ambiguously, the expectation associated with the decreasing filtration, namely for a process in \((\Omega, \mathcal{F}, \mathcal{F}_t \mu^h_{\omega})\), where \( \omega \) is an \( \mathcal{F}_t \)-Wiener process and \( \mu^h_{\omega} \) is the Radon-Nikodym density analogue to (4.7) for this filtration.

(3) After Feynman [9, (7.30)], integration by parts formulae related with (3.5) have been used in several independent mathematical contexts, probabilistic or not (cf. for example, [12, 15, 19]).

5. Applications

According to Feynman [9, Eq. (7.30)], the "real time version" of (4.11) could be regarded as a starting point to define the laws of quantum mechanics. We are going to show that the same is true in our Euclidean framework.

5.1. Equations of Motion and Least Action Principle

Let us consider first the case where \( F=1 \) in formula (4.14). Since the variation \( \delta z(\cdot) \) is an arbitrary \( \mathcal{F}_t \)-adapted \( \mathcal{H} \)-valued functional in \( \bigcap_{p>1} W^p_t(\Omega, d\mu^h_{\omega}) \), we obtain, in the case where \( z(0) = 0 \), the following equation of motion:

5.1. Proposition. The forward Newton equation with boundary condition holds, for \( t \in [0, T/2] \):
\[ DDz(t) = Dz(t) \wedge \text{rot } A + \nabla V(z(t)) + \frac{\hbar}{2} \text{rot rot } A \]
\[ z(0) = 0 \quad \text{a.s.} \tag{5.1} \]
\[ Dz(T/2) = \left[ h \left( \frac{\nabla \eta}{\eta^*} \right) - A \right] (z(T/2), T/2) \]

In an analogous way, the (backward) Newton equation also holds for \( t \in [-T/2, 0] \):
\[ D_\ast D_\ast z(t) = D_\ast z(t) \wedge \text{rot } A + \nabla V(z(t)) - \frac{\hbar}{2} \text{rot rot } A \]
\[ z(0) = 0 \quad \text{a.s.} \tag{5.2} \]
\[ D_\ast z(-T/2) = \left[ -h \left( \frac{\nabla \eta^*}{\eta} \right) + A \right] (z(-T/2), -T/2) \]

**Proof.** The equations follow from (4.14) (respectively (4.15)) by taking \( F[z] = 1 \) (resp. \( F_\ast = 1 \)). Let us prove (5.1). For every \( \mathcal{F}_t \)-adapted functional \( \delta z(\cdot) \) we have
\[
E \left[ \int_0^{T/2} \delta z(t) \left[ \frac{1}{\hbar} Dz(t) + \frac{1}{\hbar} Dz(t) \wedge \text{rot } A + \frac{1}{2} \text{rot rot } A + \frac{\nabla V}{\hbar} \right] \right] = E \left( \delta z(T/2) \left[ \frac{1}{\hbar} Dz(T/2) - \frac{\nabla \eta}{\eta} (z(T/2), T/2) + A(z(T/2)) \right] \right) .
\]

By the construction of the process \( z(t) \) (or, equivalently, of the measure \( \rho_{\mu_\ast}^{\hbar} \)), the right-hand side of this equality is zero and, since \( \delta z \) is arbitrary, we have
\[
E \left( \frac{1}{\hbar} Dz(t) - \frac{1}{\hbar} Dz(t) \wedge \text{rot } A - \frac{1}{2} \text{rot rot } A - \frac{\nabla V}{\hbar} \right) (z(t), t) = 0 \quad \text{a.s.}
\]
By the definition of \( D \), the expression under the conditional expectation is \( \mathcal{F}_t \)-adapted, and (5.1) follows.

Notice that these results are stronger than the result of Feynman and Kac, since they hold without expectations.

Although Feynman’s approach has been strongly suggested by the least action principle of classical mechanics, his description of quantum dynamics does not follow from a variational principle. As suggested by the proposition, this is not so in our Euclidean context.
5.2. Definition. If \( (\nabla F[z] \delta z)_t = 0 \) a.s. for every \( \mathcal{F}_t \)-adapted \( \mathcal{H} \)-valued variation \( \delta z \), we say that the process \( z \) is an extremal for the functional \( F \) on the set of diffusions of type \( d\tilde{z}(t) = h^{1/2} dw(t) + b(\tilde{z}(t)) \, dt \) having an absolutely continuous law w.r.t. \( \mu^b \) (one can define in complete analogy a concept of extremality associated to an \( \mathcal{F}_t \)-adapted functional \( F^*_b \)).

Let us now consider the functional of the Bernstein diffusion

\[
J[z] = E_0 \int_0^{T/2} L(z(s), Dz(s), s) \, ds + E_0 I(z(T/2), T/2),
\tag{5.3}
\]

where \( I(z, s) = -h \log \eta(z, s) \). The function \( L: \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R} \) is called the Lagrangian and \( J \) is called the forward action functional with final boundary condition. All our results will be, of course, independent of the choices \( s = 0 \) as initial condition.

In the situation considered before, it is natural to choose

\[
L(z, Dz, s) = \frac{1}{2} \|Dz\|^2 + V + A \cdot Dz + \frac{h}{2} \nabla \cdot A \tag{5.4}
\]

and therefore, by Itô calculus,

\[
J[z] = E_0 \int_0^{T/2} \left[ \frac{1}{2} \|Dz(s)\|^2 + V(z(s)) \right] \, ds + E_0 I(z(T/2), T/2).
\tag{5.5}
\]

Then we have the following result:

5.3. Theorem. A necessary and sufficient condition for a smooth Bernstein diffusion \( z \) to be extremal of the action functional (5.5) on the set of \( \mathbb{R}^3 \)-valued Markovian diffusions \( \tilde{z}(s), 0 \leq s \leq T/2 \), with diffusion matrix \( hI \), having an absolutely continuous law w.r.t. \( \mu^b \), and such that \( \tilde{z}(0) = 0 \), \( D\tilde{z}(T/2) = (h(\nabla \eta/\eta) - A)(\tilde{z}(T/2), T/2) \), is that the following forward Newton equation holds, for \( t \in [0, T/2] \):

\[
DDz(t) = Dz(t) \wedge \text{rot} A + \nabla V(z(t)) + \frac{h}{2} \text{rot rot} A \quad \text{a.s.} \tag{5.6}
\]

Proof. We have
\[
\frac{d}{dz} \mid_{z=0} \left( E_0 \int_0^{T/2} \left[ \frac{1}{2} \|D(z + \varepsilon \delta z)\|^2 + V(z + \varepsilon \delta z) + A(z + \varepsilon \delta z) \cdot D(z + \varepsilon \delta z) + \frac{\hbar}{2} \nabla \cdot A(z + \varepsilon \delta z) \right] dt \right)
\]
\[+ E_0 \nabla I(z(T/2)) \delta z(T/2) \]
\[= E_0 \int_0^{T/2} \left[ Dz \cdot D \delta z + \nabla V(z) \cdot \delta z + A(z) \cdot D \delta z + \nabla A \cdot Dz \delta z + \frac{\hbar}{2} \nabla(\nabla \cdot A)(z) \cdot \delta z \right] dt + E_0 \nabla I(z(T/2)) \delta z(T/2) \]
\[= -E_0 \int_0^{T/2} \delta z(t) \left[ DDz - \nabla V - Dz \wedge \text{rot} A - \frac{\hbar}{2} \text{rot} \text{rot} A \right] (z(t), t) dt
\]
\[+ E_0 \delta z(T/2) \left[ Dz(T/2) + A(z(T/2)) + \nabla I(z(T/2)) \right]. \]

Now, taking into account that \( DA = (\hbar/2) \Delta A + (Dz \cdot \nabla) A \) we have that \( z \) is an extremal of the action iff for every \( \mathcal{P}_t \)-adapted variation \( \delta z \),

\[ E_0 \int_0^{T/2} \delta z(t) \left[ DDz - \nabla V - Dz \wedge \text{rot} A - \frac{\hbar}{2} \text{rot} \text{rot} A \right] (z(t), t) dt = 0 \]

or equivalently (by Lemma 4.4),

\[ E_0 \int_0^{T/2} \frac{d}{dt} \delta z(t) \left[ Dz(t) - \int_0^t \left( \nabla V + Dz \wedge \text{rot} A + \frac{\hbar}{2} \text{rot} \text{rot} A \right) (z(s), s) ds \right] dt - E_0 \delta z(T/2)
\]
\[\times \left[ Dz(T/2) - \int_0^{T/2} \left( \nabla V + Dz \wedge \text{rot} A + \frac{\hbar}{2} \text{rot} \text{rot} A \right) ds \right] = 0. \]

The result follows from Choosing \( \delta z(t) = \tilde{\delta} z(t) - \delta z(T/2) \), where

\[ \tilde{\delta} z(t) = \int_0^t g(s) \left[ Dz(s) - \int_0^s \left( \nabla V + Dz \wedge \text{rot} A + \frac{\hbar}{2} \text{rot} \text{rot} A \right) (z(\tau), \tau) d\tau \right] ds \]

with \( g \) an arbitrary smooth function.
Starting with the time-reversed functional of (5.3), namely the action functional with initial boundary condition

\[ J_*(z) = E_0 \int_{-T/2}^{0} L(z(s), D_* z(s), s) \, ds + E_0 I_*(z(-T/2), -T/2), \tag{5.7} \]

where \( I_*(z, s) = -h \log \eta^*(z, s) \), and working on the space \((\Omega_0^*, \rho_* \mu_{\omega_*}^h)\), we have the corresponding result:

5.4. THEOREM. A necessary and sufficient condition for a Bernstein diffusion \( \eta \) to be extremal of \( J_*(\eta) \) on the set of Markovian diffusions \( \tilde{\eta}(s), -T/2 \leq s \leq 0 \), with diffusion matrix \( hI \) and such that \( \tilde{\eta}(0) = 0 \), \( D_* \tilde{\eta}(-T/2) = (-\tilde{h}(\nabla \eta^*/\eta^*) + A)(\tilde{\eta}(-T/2), -T/2) \) is that the following backward Newton equation holds, for \( t \in [ -T/2, 0 ] \):

\[ D_* D_* \tilde{\eta}(t) = D_* \tilde{\eta}(t) \wedge \text{rot} A + \nabla V(z(t)) - \frac{h}{2} \text{rot} \text{rot} A \quad \text{a.s.} \tag{5.8} \]

Remark. The case where \( I_*(y, t) = -h \log h(-T/2, x, t, y) \) (i.e., the case of a Bernstein Bridge) introduces a singularity in the action functional \( J_*(\eta) \). The backward Newton equation still holds but the initial velocity is divergent.

5.5. COROLLARY. Any Bernstein diffusion which is an extremal of the action functional (5.3) is also a solution of the Newton equation, for \( t \in [-T/2, T/2] \),

\[ \frac{1}{2}(DDz(t) + D_* D_* z(t)) = \frac{1}{2}(Dz(t) + D_* z(t)) \wedge \text{rot} A + \nabla V(z(t)). \tag{5.9} \]

Proof. Clearly the action functional \( J \) of (5.5) involves the increasing filtration \( \mathcal{F}_t \) associated with the Bernstein diffusion \( z_* \), \( s \in [0, T/2] \). The decreasing filtration \( \mathcal{F}_t^- \) is such that \( \mathcal{F}_t^- \) is another increasing filtration associated with the time reversed process \( \tilde{z}(s) \equiv z(-s) \). On the other hand, the l.h.s. of (5.5) can also be regarded as a function of \( z(o) \), namely \( I(z(o), o) \), where \( I(y, s) = -h \log \eta(y, s) \). Denoting by \( \wedge \) the time reversed function, one verifies that

\[ \hat{I}(\tilde{z}(s), s) = -I_*(z(-s), -s), \quad \forall s \in \left[ 0, \frac{T}{2} \right], \]

where \( I_*(y, s) = -h \log \eta^*(y, s) \), as defined in (5.7).

This implies in particular the relation between drifts

\[ D \tilde{z}(s) + A(\tilde{z}(s)) = -D_* z(-s) - A(z(-s)). \]
According to Theorem 5.3, the forward Newton equation (5.6) holds for \( z(s), s \in [0, T/2] \). By Theorem 5.4, involving the time reversed action, the time reversal of (5.6) holds, for \( s \in [-T/2, 0] \). Since the Bernstein diffusion is time symmetric by construction, we can also consider only \( z(s) \), for any \( s \in [-T/2, T/2] \). Then the sum of Eqs. (5.6) and (5.8), i.e., the time symmetric Newton equation (5.9) holds.

The existence of a critical point is easily verified. Given any positive regular solution \( \eta^* = \eta^*(x, t) \) of the initial value problem (4.8), the change of variable \( I_\eta(x, t) = -\hbar \log \eta^*(x, t) \) produces the (backward) stochastic Hamilton-Jacobi equation

\[
\frac{\partial I_\eta}{\partial t} - \frac{\hbar}{2} A I_\eta + (\nabla I_\eta - A) \nabla I_\eta = \frac{1}{2} |\nabla I_\eta - A|^2 + V + (\nabla I_\eta - A) \cdot A + \frac{\hbar}{2} \nabla \cdot A.
\]

Taking the gradient of this relation, and using the forward drift

\[
B_\eta(x, t) = (\nabla I_\eta - A)(x, t)
\]

we obtain the backward Newton equation (5.8). Notice that the P.D.E. for \( I_\eta \) is a variant of an equation familiar in Optimal Stochastic Control theory [10, p. 461]. Moreover, in this context, one shows that the critical point of \( J_\eta \), for example, is actually a minimum of the action (cf. [23.3]).

The Newton equation (5.9) is interpreted as a generalization of the (Euclidean) classical equation of motion of a particle in an electromagnetic field. Indeed, if \( B \) and \( E \) denote, respectively, the magnetic and electric vector fields, we have

\[
B = \text{rot} A
\]

\[
E = \nabla V
\]

so that the right-hand side of (5.9) reduces to the generalized Lorentz force,

\[
E + \frac{1}{2}(Dz(t) + D_\eta z(t)) \wedge B.
\]

Notice the symmetric form taken by the generalized velocity and acceleration in (5.9). The change of sign in front of the scalar potential term is a familiar feature of the Euclidean point of view. So it is natural to see (5.9) as the Euler–Lagrange equation for the considered Lagrangian. In particular, one observes that nothing but the classical (Euclidean) Lagrangian is used in order to obtain the equation of motion (5.9).

When \( \hbar = 0 \) the two stochastic differential equations (4.3), (4.4) specifying the underlying Bernstein process \( z \) reduce to a pair of ordinary differential equations and the two derivatives (4.9), (4.10) coincide with the
(strong) ordinary derivative of smooth trajectories. Then the actions $J$ and $J_\star$ reduce to classical functionals, the variational derivatives reduce to their classical counterparts, and Theorem 5.3 (or 5.4) is nothing but a familiar result of classical calculus of variations. The advantages of our extension of the classical variational approach should be obvious by now. In contrast with the formal law of motion (1.7), for example, the one-dimensional Wiener process $w$ starting from the point $x$ at time $-T/2$ is now characterized as the solution of the free (regularized) Newton laws (i.e., for $V = A = 0$ in (5.4))

$$DDw(t) = 0, \quad Dw\left(\frac{T}{2}\right) = 0. \quad (5.10)$$

On the other hand, it is also true that $D_\star D_\star w(t) = 0$.

Notice that the backward velocity is singular at $t = -T/2$, since the drift is $D_\star w(t) = B_\star(w(t), t) = (w(t) - x)/(t + T/2)$, $\forall t \in [-T/2, T/2]$. In physical terms, this is just an expression of the fact that, since the position of the free particle is exactly known to be $x$ at $t = -T/2$, its momentum has to be undefined (Uncertainty principle).

Remarks. (1) It is an interesting open problem to find the most general class of classical Lagrangians $L$ for which Euler-Lagrange equations similar to those of Theorem 5.3 make sense. Clearly, this class is much more restricted than in classical mechanics but its seems to contain many of the physically relevant situations for quantum physics.

(2) The idea of a stochastic calculus of variations for diffusion processes associated with quantum mechanics is due to Yasue [22]. It has suggested several interesting works (see [17,21] and [5] for some references). The Euclidean version developed here can be regarded as a Euclidean elaboration of this idea. Notice that no analogue of Feynman functional calculus is known in this context, namely in the context of Nelson's Stochastic Mechanics, whose dynamical structure is different from our Euclidean framework (cf. [2] and conclusion of the present work).

5.2. Commutation Relations

In this section we shall only consider the case $A = 0$.

5.6. Theorem. A Bernstein diffusion $z(t), t \in I$, verifies the relation

$$E[z'(t) D_\star z'(t) - z'(t) Dz'(t)] = h\delta_{ij} \quad (5.11)$$

where $i, j = 1, 2, 3$ denote the components.
Proof. Let us write the integration by parts formula (4.11) for $F[z] = \z(t)$ with $0 < t < T/2$ and for

$$\delta z(\tau) = \begin{cases} 0, & \tau < t - \Delta t \\ 1 + \frac{\tau - t}{\Delta t}, & t - \Delta t \leq \tau \leq t \\ 1, & \tau > t. \end{cases}$$

Then, we get (in one dimension, for simplicity)

$$h = E \left[ z(t) \left\{ \int_{t - \Delta t}^{T/2} \frac{1}{\Delta t} dz(t) + \int_{t - \Delta t}^{T/2} \delta z(t) \nabla V(z(t)) \, dt - Dz(T/2) \right\} \right]$$

$$= E \left[ z(t) E^* \left[ \frac{z(t) - z(t - \Delta t)}{\Delta t} \right] \right]$$

$$+ E \left[ z(t) \left\{ \int_{t - \Delta t}^{T/2} \delta z(\tau) \nabla V(z(\tau)) \, d\tau - Dz(T/2) \right\} \right].$$

For $\Delta t \to 0$ we obtain

$$h = E \left[ z(t) \left\{ D z(t) - D z(T/2) + \int_{t}^{T/2} \nabla V(z(\tau)) \, d\tau \right\} \right]. \quad (5.12)$$

Now, if we write (4.11) for the same functional $F$ and for

$$\delta z(\tau) = \begin{cases} 0, & \tau < t \\ 1 - \frac{\tau - t}{\Delta t}, & t \leq \tau \leq t + \Delta t \\ 1, & \tau > t + \Delta t, \end{cases}$$

analogous reasoning yields

$$0 = E \left[ z(t) \left\{ D z(t) - D z(T/2) + \int_{t}^{T/2} \nabla V(z(\tau)) \, d\tau \right\} \right]. \quad (5.13)$$

The subtraction of (5.12) and (5.13) gives the conclusion. \qed

Remark. From this theorem we also deduce the familiar relation

$$\lim_{\Delta t \to 0} E \left( \frac{(z_{s,\Delta}(t + \Delta t) - z_{s,\Delta}(t))^2}{\Delta t} \right) = h \mathbf{1}, \quad (5.14)$$
where \( 1 \) denotes the \( 3 \times 3 \) identity matrix and the numerator means 
\( (z_{x,v}(t + \Delta t) - z_{x,v}(t)) \otimes (z_{x,v}(t + \Delta t) - z_{x,v}(t)) \).

It has been shown in [2] that there is a family of Hilbert spaces which are natural in Euclidean Quantum Mechanics, and denoted by \( \mathcal{F}^{*} \). These spaces are the completion of \( \mathcal{F}^{*} = \{ \eta^{*}(t) = e^{-iH} \chi: \chi \in \mathcal{D}(e^{T/2H}) \} \), with respect to the scalar product

\[
\langle \eta^{*}(t) | \eta^{*}(t) \rangle, - \langle U^{-1} \eta^{*}(t) | U^{-1} \eta^{*}(t) \rangle_{L^{2}} = \langle \chi_{1} | \chi_{2} \rangle_{L^{2}}.
\]

where \( U^{-1}: \mathcal{F}^{*} \to \mathcal{D}(e^{T/2H}) \), \( U^{-1} \eta^{*}(t) = \chi \) (actually, \( \mathcal{F}^{*} \) is unitarily equivalent to \( L^{2} \) via an unitary extension of \( U^{-1} \)).

In this context, the (Euclidean) position and momentum observables in \( L^{2}(\mathbb{R}^{3}) \), \( Q \) and \( P \), were defined (in one dimension, for simplicity) by

\[
Q: \mathcal{D}(Q) = \{ \chi \in L^{2}: x\chi \in L^{2} \} \to L^{2}, \quad Q\chi = x\chi \in L^{2}.
\]

and

\[
P: \mathcal{D}(P) = \left\{ \chi \in L^{2}: \int \|\nabla \chi\|^{2} \, dx < +\infty \right\} \to L^{2}, \quad P\chi = -\nabla \chi.
\]

It is easy to check, after an integration by parts, that the left-hand side of (5.11) reduces to

\[
\int \eta(y, t) (QP - PQ) \eta^{*}(y, t) \, dy.
\]

If we write \( [Q, P] = \mathcal{Q}(P - PQ), \) which is the commutator of \( Q \) and \( P \) (defined on a dense domain in \( L^{2}(\mathbb{R}^{3}) \)), we have then proved, after reintroduction of the 3-dimensional notation, that

\[
(\eta^{*}(t)) [Q_{i}, P_{j}] \eta^{*}(t)) = \hbar \delta_{ij},
\]

i.e., the (Euclidean) commutation relation between position and momentum observables.

5.3. Correlation Functions

As in 5.2 we restrict ourselves to the case \( A = 0 \). According to the integration by parts formula (4.11) for \( F[z] = z(\tau) \) and

\[
\delta z(s) = \begin{cases} s - t, & s > t \\ 0, & s \leq t, \end{cases}
\]

where \( t \) is fixed between 0 and \( T/2 \), we have

\[
\delta z(\tau) \hbar = E \left\{ z(\tau) \left\{ \int_{s}^{T/2} dz(s) + \int_{s}^{T/2} (s - t) \nabla V(z(s)) \, ds - Dz(T/2)(T/2 - t) \right\} \right\}.
\]
Therefore, when $\tau \leq t$,

$$0 = E \left[ z(\tau) \left\{ z \left( \frac{T}{2} \right) - z(t) + \int_{\tau}^{T/2} (s-t) \nabla V(z(s)) \, ds - Dz \left( \frac{T}{2} \right) (T/2 - t) \right\} \right]$$

(5.15)

and when $\tau \geq t$,

$$(\tau - t) \hbar = E \left[ z(\tau) \left\{ z \left( \frac{T}{2} \right) - z(t) + \int_{\tau}^{T/2} (s-t) \nabla V(z(s)) \, ds - Dz \left( \frac{T}{2} \right) (T/2 - t) \right\} \right].$$

(5.15')

The time derivative yields, when $\tau < t$,

$$\frac{d}{dt} E[z(\tau)z(t)] = -E \left[ z(\tau) \left\{ \int_{\tau}^{T/2} \nabla V(z(s)) \, ds + Dz \left( \frac{T}{2} \right) \right\} \right]$$

and when $\tau > t$,

$$\frac{d}{dt} E[z(\tau)z(t)] = -E \left[ z(\tau) \left\{ \int_{\tau}^{T/2} \nabla V(z(s)) \, ds + Dz \left( \frac{T}{2} \right) \right\} \right] + \hbar.$$

This means that the first derivative of the correlation is discontinuous at $\tau - t$. Moreover, for any $\tau \neq t$, we have

$$\frac{d^2}{dt^2} E[z(t)z(\tau)] = E[\nabla V(z(t)) \cdot z(\tau)].$$

(5.16)

One could summarize symbolically our information about the correlations by

$$\frac{d^2}{dt^2} E[z(t)z(\tau)] = \hbar \delta(t - \tau) + E[\nabla V(z(t)) \cdot z(\tau)].$$

(5.16')

and regard this result as the Euclidean version of Feynman's formula (7.59) [9].

Consider, for example, the free case $V = 0$, for $z$ the Brownian Bridge $z(\cdot) = z_{0,\hbar}(\cdot)$. We can easily solve (5.16') if we introduce the boundary conditions

$$E[z_{0,\hbar}(0) z_{0,\hbar}(\tau)] = x E[z_{0,\hbar}(\tau)] = x z(\tau)$$

$$E[z_{0,\hbar}(t) z_{0,\hbar}(v)] = z E[z_{0,\hbar}(t)] = z z(t),$$

(5.17)
where $\tilde{z}(s)$ denotes the solution of the classical free Newton equation with the same boundary conditions as the Brownian Bridge. Considering first the case $t < \tau$, then the case $\tau < t$, the result is

$$f(t, \tau) \equiv E[z_{\tilde{z}, \tau}(t) z_{\tilde{z}, \tau}(\tau)]$$

$$= \begin{cases} \tilde{z}(t) \tilde{z}(\tau) + \frac{h}{v} t(v - \tau), & t < \tau \\ \tilde{z}(t) \tilde{z}(\tau) + \frac{h}{v} \tau(v - t), & \tau < t. \end{cases} \quad (5.18)$$

As observed by Feynman, the $\hbar$ dependent term, of purely quantum mechanical nature, represents the contribution of all the possible (non-classical) paths between the two fixed endpoints. Correlations like (5.18) can, as well, follow from the introduction of the (backward) characteristic functional

$$E_i \left[ \exp i \int_0^t f(\tau) z(\tau) d\tau \right] = v_i[f],$$

where $f$ is any deterministic function such that the left-hand side makes sense. Then, one verifies that

$$E[z(\tau)] = (-i) \frac{\delta}{\delta f(\tau)} v_i[f] |_{f=0},$$

and

$$E[z(t) z(\tau)] = (-i)^2 \frac{\delta^2}{\delta f(t) \delta f(\tau)} v_i[f] |_{f=0}.$$

More general situations can be treated in the same way. To conclude, let us stress that (4.11) enables us to treat also functionals which are much more irregular than the one considered by Feynman. For example, a functional like

$$F[z] = \int_{t/2}^{T/2} f(z(s), s) \, dz(s) \quad (5.19)$$

for any $f$ s.t. $E_i[f^2(z(s), s)] < \infty, \ t \leq s \leq T/2$, makes sense in the present variational context.

5.4. Examples

The easiest method to produce explicit examples of Bernstein diffusions is by analytical continuation of solutions of the Schrodinger equation (1.1) [2]. As shown in the present section, nothing is lost, along this way, in the dynamical structure of Feynman's approach, i.e, in the dynamical structure of quantum mechanics; nevertheless the resulting dynamical theory,
Euclidean Quantum Mechanics, is only local in time. Let us consider a few examples. We put $\hbar \equiv 1$ for simplicity.

Suppose $A = 0$, $V$ continuous, bounded from below, and $H = -\frac{1}{2}\hbar A + V$ essentially self-adjoint. Then $h(s, x, t, y) = \text{kernel}\{e^{-(t-i)H}\}$ is strictly positive and fulfills the conditions of Section 2. If $H$ has a ground state (or vacuum state) $\chi_0$ in $L^2(\mathbb{R}^3)$ (eigenvector corresponding to the lowest eigenvalue $E_0$), $\chi_0$ is unique up to a phase, and can be chosen strictly positive. The two relevant solutions of the forward and backward heat equations are, respectively,

$$\eta_{\chi_0}^*(y, t) = \chi_0(y) e^{-E_0t}$$

and

$$\eta_{\chi_0}(y, t) = \chi_0(y) e^{E_0t},$$

where, in contrast with (2.7)-(2.8), we start from a common initial condition $\chi_0$ at time 0. The resulting Bernstein diffusion $z_1$ is stationary, with an invariant probability density (the integrand of (2.9)) given by

$$\rho(y, t) dy = \|\chi_0(y)\|^2 dy$$

and forward and backward drifts

$$B(y) = -B_*(y) = \frac{\nabla \chi_0}{\chi_0}(y).$$

In particular, $z_1$ is an extremal of the action functional (5.5), for $L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 + V$ the classical Euclidean Lagrangian with $A = 0$, and therefore it is a particular solution of the Newton equation (5.6) and its time reversal (5.8). When $V$ is the harmonic potential $V(y) = \frac{1}{2}|y|^2$, $z_1$ is the one dimensional free Euclidean field (zero space dimensions), basically the only diffusion process associated with this potential according to the usual Euclidean approach of constructive field theory [12], founded on the point of view that the vacuum is sufficient to understand the dynamics.

Now consider the case $A = (-iB y_2, iB y_1, 0)$, for $B$ a constant, and $V = 0$. The Hamiltonian (4.1), or equivalently

$$H_A^* = -\frac{1}{2}(\nabla + A)^2, $$

describes a free particle in a homogeneous magnetic field $\text{rot} A = (0, 0, B)$. Then the associated integral kernel satisfies the conditions of Section 4. It is explicitly given by

$$h(s, x, t, y) = \frac{B}{4\pi \sinh(1/2)B(t-s)} \left( \frac{1}{2\pi(t-s)} \right)^{1/2} e^{-(y_3 - x_3)^2/2(t-s)} \cdot e^{-B/4 \left((y_3 - x_3)^2 + (y_2 - x_2)^2\right)} \cotgh (B/2)(t-s) + (B/2)(x_1 y_2 - x_2 y_1).$$
We can consider, for example, the Bernstein Bridge $z(t) \equiv z_{\infty}(t)$ associated with this situation. It is a Bernstein diffusion, defined on $t \in [s, u]$, with probability density given by

$$
\rho(y, t) dy = h(s, x, t, y) dy
$$

according to (2.9), where $\mathcal{N}(x, z) = h(s, x, u, z)$. Its forward and backward drifts are

$$
B(y, t) = \left( \frac{B}{2} (z_1 - y_1) \coth \frac{B}{2} (u - t) + \frac{B}{2} z_2, \right.
$$

$$
\left. \frac{B}{2} (z_2 - y_2) \coth \frac{B}{2} (u - t) - \frac{B}{2} z_1, \frac{z_3 - y_3}{u - t} \right)
$$

and

$$
B_*(y, t) = \left( \frac{B}{2} (y_1 - x_1) \coth \frac{B}{2} (t - s) + \frac{B}{2} x_2, \right.
$$

$$
\left. \frac{B}{2} (y_2 - x_2) \coth \frac{B}{2} (t - s) - \frac{B}{2} x_1, \frac{y_3 - x_3}{t - s} \right),
$$

respectively. $z_1, t \in [s, u[, is an extremal of the action (5.5) for $L(q, \dot{q}) = \frac{1}{2} \| \dot{q} \|^2 + A \cdot \dot{q}$, and then a particular solution of the Newton equations (5.6) and (5.8). The initial and final velocities are divergent since the associated positions are exactly known.

Finally let us consider a free case $A = V = 0$, built up from the explicit solution of the Schrödinger equation (1.1) on $L^2([\mathbb{R}])$, with initial condition

$$
\chi(y) = (\pi a)^{-1/4} \exp \left\{ -\frac{y^2}{2a} + iv_0 y \right\},
$$

where $a$ and $v_0$ are two constants. The analytical continuation of $\chi$ and its complex conjugate $\bar{\chi}$ gives us two (distinct!) positive vectors

$$
\chi_c(y) = (\pi a)^{-1/4} \exp \left\{ -\frac{y^2}{2a} - V_0 y \right\}
$$

$$
\bar{\chi}_c(y) = (\pi a)^{-1/4} \exp \left\{ -\frac{y^2}{2a} + V_0 y \right\}
$$

(notice that $V_0 = -iv_0$), initial conditions of the forward and backward free heat equations used, in EQM, to produce the associated Bernstein diffusion $z_t$. The underlying integral kernel $h(s, t, y)$ is the (one dimensional) free
kernel, compatible with the conditions of Section 2. The diffusion $z_t$ has the Gaussian probability density

$$
\rho(y, t) \, dy = \left(2\pi \frac{s^2 - t^2}{2a}\right)^{-1/2} \exp\left(-\frac{(y - V_0 t)^2}{(a^2 - t^2)/a}\right) \, dy
$$

and its forward and backward drifts are

$$
B(y, t) = V_0 - \frac{y - V_0 t}{a - t}
$$

and

$$
B_*(y, t) = V_0 - \frac{y - V_0 t}{a + t}.
$$

$z_t$ is extremal of the free action (5.5) with $L(q, \dot{q}) = \frac{1}{2} |\dot{q}|^2$ and it is a particular solution of the relevant Newton equations (5.6) and (5.8). However, since $\tilde{\chi}_e$ is an analytic vector of the free Hamiltonian $H_0 = -\frac{1}{2}(d^2/dy^2)$ with a convergence radius $a$, $z_t$ is not defined anymore for $|t| > a$.

The locality in time of Euclidean Quantum Mechanics seems to be the price to pay in order to preserve the dynamical structure of Quantum Physics along the line discovered by Feynman.

6. CONCLUSION

A few elementary applications of Feynman's functional calculus, in our Euclidean approach, have been given in Section 5. A wealth of information is still hidden in the integration by parts formula (4.11).

The simplest method to associate a Bernstein diffusion $z$ to each regular solution $\psi$ of the Schrödinger equation (1.1) has been described in [2] (see also [23]). In particular, if the given initial condition of the Schrödinger equation (1.1) is a smooth vector $\tilde{\chi}$ in $\mathcal{D}(H)$ of the form

$$
\tilde{\chi}(x) = (e^{R + iS})(x)
$$

the pair of relevant initial conditions $\chi$ and $\chi'$ for the Euclidean equations (4.6) and (4.8) is chosen to be

$$
\chi(x) = (e^{R - S})(x) \quad \text{and} \quad \chi'(x) = (e^{R + S})(x),
$$

where $\bar{R}(x) = R(x)$ and $\bar{S}(x) = -iS(x)$. 

It follows from this method that the analytical continuation in the time parameter of the Schwinger function

\[ E[z(t_1) \cdots z(t_n)], \quad -T/2 \leq t_1 \leq \cdots \leq t_n \leq T/2, \]

(6.3)

where \( I = \left[-T/2, T/2\right] \) is contained in the time interval of existence of the process \( z \), and that it produces the quantum mechanical analogue of the moments,

\[ \langle Q(t_1) \cdots Q(t_n) \rangle_{L^2}, \]

(6.4)

where \( Q(t) \) denotes the evolution of the position observable \( Q \) under the Heisenberg automorphism of linear operators on \( L^2(\mathbb{R}^3) \) and \( \langle \cdot \rangle \) is the expectation in this Hilbert space. This means that the analytical continuation of Euclidean Quantum Mechanics yields, indeed, Quantum Mechanics. In particular, this is not Nelson's Stochastic Mechanics, a real time approach to quantum phenomena, in which the function (6.4) has no probabilistic interpretation.

The aim of the new Euclidean functional calculus initiated here lies in its extension to infinite dimensional processes, namely to Quantum field theory. This will be examined in future publications.

Acknowledgments

This work was done during an enjoyable stay of the first author (A.B.C.) at the Royal Institute of Technology. She wishes to thank Prof. L. Carleson and the STUF for their scientific and material support. The second author (J.C.Z) was supported by the Göran Gustafsson Foundation. Both authors thank Profs. Malliavin, Kolsrud, and Andersson for many useful comments regarding this research program.

References