Effects of soil–water characteristic curve and relative permeability equations on estimation of unsaturated permeability function

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Abstract

Unsaturated permeability function can be estimated by theoretical models from soil–water characteristic curve (SWCC). To date, there are numerous estimation models that can be used to obtain the unsaturated permeability function from SWCC. However, each model results in a different estimation curve. The reason for this difference is not well understood.

In this study, the available SWCC equations and the available relative permeability ($k_r$) equations were combined to form a matrix of unsaturated permeability estimation models. The matrix of unsaturated permeability estimation models was used to study the effect of SWCC equations and relative permeability ($k_r$) equations as controlling factors in the estimation of unsaturated permeability function. The study was conducted using twenty sets of published experimentally measured SWCC and unsaturated permeability data ($k_w$). The effects of the SWCC equations and $k_r$ equations on a variation of estimated unsaturated permeability functions are presented in this paper. It was found that the SWCC equation has a more significant effect on the estimation of unsaturated permeability function than that of the $k_r$ equation.

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1. Introduction

Flow through the unsaturated zone of soil contributes to a variety of geotechnical and geo-environmental problems. The unsaturated permeability function is the most important hydraulic property governing the flow process. Therefore, knowledge of the unsaturated permeability function is crucial in the analysis of the flow process in the unsaturated zone. The unsaturated permeability function can be directly measured in soil laboratories; however, a reliable measurement of the permeability function for an unsaturated soil is challenging due to the time-consuming nature and high cost of taking direct measurements (van Genuchten, 1980; Agus et al., 2003; Chaminda et al., 2013). To overcome the high cost and the other challenges associated with taking direct measurements, the unsaturated permeability function can be estimated by theoretical models derived from the soil–water characteristic curve (SWCC). There are numerous estimation models that can be used to obtain the unsaturated permeability function from the SWCC. However, each model results in a different estimation curve for the unsaturated permeability function, and no unified model which can be used for all soil types has been put forward to date (Mualem, 1986). The reason for this difference is not well understood.

The objective of this study is to provide an understanding of the underlying reasons behind this variation. In this study, the
available SWCC equations and the available relative permeability \( k_r \) equations were combined to form a matrix of unsaturated permeability estimation models. The matrix of unsaturated permeability estimation models was used to study the effect of the SWCC equations and relative permeability \( k_r \) equations as factors controlling the estimation of the unsaturated permeability function. The study was conducted on twenty sets of published data which had experimentally measured SWCC and unsaturated permeability \( k_w \) data.

2. Background

2.1. Combination of soil–water characteristic curve and relative permeability equations

Unsaturated permeability estimation models usually combine knowledge of the soil–water characteristic curve (SWCC) with a flow equation and derive an equation to estimate the unsaturated permeability function. This procedure is in fact an integration of the SWCC with a relative permeability \( k_r \) equation. The outcome of this integration (or combination) is an unsaturated permeability estimation model. Researchers have proposed different unsaturated permeability estimation models by combining a specific SWCC equation with a specific \( k_r \) equation (Brooks and Corey, 1964; Brutsaert, 1966; van Genuchten, 1980; Fredlund et al., 1994). However, there is significant variation between the estimated unsaturated permeability functions using these models. It appears that the SWCC and the \( k_r \) equations are among the factors controlling this variation and a standard procedure is required to investigate their effects.

2.2. Selection of relative permeability \( k_r \) equations

Unsaturated permeability estimation models can be divided into uniform pore-size and parallel models, known as macroscopic models, and series-parallel models, known as statistical models. The uniform pore-size models (Kozeny, 1927; Carman, 1937; Averjanov, 1950; Yuster, 1951; Irmay, 1954; Carman, 1956) have been shown to have limited applicability (Wyllie and Spangler, 1952). Since the parallel models (Purcell, 1949; Burdine et al., 1950; Gates and Tempelaar Lietz, 1950; Fatt and Dykstra, 1951; Wyllie and Spangler, 1952; Burdine, 1953; Wyllie and Gardner, 1958; Rowe, 1960) overestimated the unsaturated permeability at high suction values, the concept of tortuosity was introduced to compensate for the poor fit between the measured and the predicted values (Burdine, 1953; Fatt and Dykstra, 1951; Gates and Tempelaar Lietz, 1950; Wyllie and Spangler, 1952). The series-parallel models (Childs and Collis-George, 1950; Marshall, 1958; Millington and Quirk, 1959; Kunze et al., 1968; Mualem, 1976; Assouline, 2001) accounted for the random distribution of pore-sizes in the direction of flow by introducing a “cutting and rejoicing” concept. While these models may underestimate the relative permeability at low moisture contents (Brutsaert, 1966), they do appear more theoretical, and they are well-suited for practical use due to the fewer empirical factors required.

Mualem and Dagan (1978) generalized Childs and Collis-George, Burdine and Mualem models as statistical models as follows:

### Generalized Childs and Collis-George (C)

\[
k_r(\theta) = S_w^0 \left[ \int_0^\theta \frac{d\theta}{\psi^{1+b}} / \int_0^\theta \frac{d\theta}{\psi^{1+b}} \right]^{2}
\]

### Generalized Burdine (B)

\[
k_r(\theta) = S_w^0 \left[ \int_0^\theta \frac{d\theta}{\psi^{1+b}} / \int_0^\theta \frac{d\theta}{\psi^{1+b}} \right]
\]

### Generalized Mualem (M)

\[
k_r(\theta) = S_w^0 \left[ \int_0^\theta \frac{d\theta}{\psi^{1+b}} / \int_0^\theta \frac{d\theta}{\psi^{1+b}} \right]
\]

where \( k_r \) is the relative permeability, \( \psi \) is suction, kPa, \( \theta \) is the effective volumetric water content defined as \( \theta = \theta - \theta_r \), \( \theta_r \) is the residual volumetric water content, \( \theta_e \) is the saturated volumetric water content, \( \theta_z \) is the slope of the SWCC, \( \phi_e \) is effective saturation and \( b \) and \( n \) are parameters accounting for tortuosity. It should be noted that the Burdine model was considered as a statistical model (Mualem and Dagan, 1978) even though it is a parallel or macroscopic model which incorporates the random distribution of pore-sizes by means of a tortuosity factor. These equations (i.e., (1), (2) and (3)) were selected as relative permeability equations \( k_r \) in this study.

2.3. Selection of soil–water characteristic curve equations

The soil–water characteristic curve (SWCC), which is defined as the relationship between the amount of water in the soil and soil suction (Fredlund, 2002), is a key factor in the estimation of unsaturated permeability function. The SWCC can be obtained in the laboratory through measurements of the soil water content at different suction values. A mathematical equation can be used to best-fit the measured SWCC data. To date, there have been numerous best-fitting equations proposed for the soil–water characteristic curve of unsaturated soil (Brooks and Corey, 1964; Brutsaert, 1966; Fredlund and Xing, 1994; Gardner, 1958; McKee and Bumb, 1984; McKee and Bumb, 1987; van Genuchten, 1980). Among all the equations, those proposed by van Genuchten (1980) and Fredlund and Xing (1994) give more flexibility to the equation to best-fit the measured data (Leong and Rahardjo, 1997). Fredlund and Xing (1994) equation is expressed as:

\[
\theta_w = C(\psi) \left( \frac{\theta_s}{\ln(\exp(1) + (\frac{\psi}{c})^n)^m} \right)
\]

where \( C(\psi) = 1 - \frac{\ln(1 + \frac{\psi}{c})}{\ln(c)} \) is a correction factor, \( \psi_r \) is the suction corresponding to the residual water content, \( a \) is the fitting parameter related to the air-entry value of the soil (kPa), \( n \) is the fitting parameter related to the slope of the SWCC, \( m \) is the fitting parameter related to the residual water content of the soil, \( e \) is the Euler number, 2.71828, \( \psi \) is soil suction or total suction, (kPa). Leong and Rahardjo (1997) concluded that
$C(\psi)$ in Fredlund and Xing equation can be assumed to be equal to unity without affecting the initial portion of the SWCC. Furthermore, this serves to reduce the number of parameters in the equation. Therefore, Eq. (4) takes the following form:

$$\theta_w = \frac{\theta_S}{[\ln(\exp(1) + \left(\frac{\theta_r}{\theta_w}\right)^n)]^m}$$  \hspace{1cm} (5)

van Genuchten (1980) proposed Eq. (6) to best-fit SWCC data.

$$\theta_w = \theta_r + \frac{(\theta_s - \theta_r)}{[1 + (\alpha \psi)^{n/3}]^{1/2}}$$  \hspace{1cm} (6)

where $\alpha$, $n$ and $m$ are fitting parameters. van Genuchten constrained $m = 1 - 1/n$ in his SWCC equation and derived a closed form solution for the estimation of the unsaturated permeability function. It should be noted that the flexibility of Eq. (6) will be less when the number of fitting parameters is reduced (Leong and Rahardjo, 1997). The limited form of van Genuchten equation (1980) is as follows:

$$\theta_w = \theta_r + \frac{(\theta_s - \theta_r)}{[1 + (\alpha \psi)^{n}]^{1/2}}$$  \hspace{1cm} (7)

Eqs. (4)–(7) are selected as the best-fit SWCC equations to be investigated in this study.

3. Methodology

The selected best-fit SWCC equations and the $k_r$ equations form a matrix as shown in Table 1. A combination of each pair (i.e., one SWCC equation and one $k_r$ equation) will result in one unsaturated permeability estimation model. The matrix shown in Table 1 was created as the standard procedure to study the effect of SWCC and $k_r$ equations. The designated names of each model as shown in Table 1 are based on the SWCC and $k_r$ equations used in the model. In addition, two existing permeability estimation models, a combination of limited form of van Genuchten SWCC equation with Mualem $k_r$ equation as proposed by van Genuchten (1980) (shown as VG-1980 in this study) and a combination of the Fredlund and Xing SWCC equation and the Childs and Collis-George $k_r$ equation as proposed by Fredlund et al. (1994) (shown as F&X-1994 in this study), are shown under the appropriate column in the matrix in Table 1.

3.1. Procedure of combining SWCC and $k_r$ equations

The SWCC can be represented by any function which describes the relationship between the volumetric water content, $\theta_v$ and suction, $\psi$, of a soil. The function $\theta_v = f(\psi)$ expresses the volumetric water content as a function of suction, while the inverse of $f(\psi)$ expresses the suction as a function of volumetric water content, $\psi = f^{-1}(\theta_v) = g(\theta_v)$ (see Fig. 1). A generalized equation which defines the relative permeability, $k_r$, equation of a soil as a function of volumetric water content can be expressed as follows:

$$k_r(\theta_v) = \left[ \int_{\theta_v}^{\theta_{s,\ell}} f(\psi)d\theta_v / \int_{\theta_v}^{\theta_{s,\ell}} f(\psi)d\psi \right]^d$$  \hspace{1cm} (8)

where, $\theta_{s,\ell}$ is the lower limit of integration for volumetric water content and $d$ is a parameter which varies according to the model. Eq. (8) can be transformed to a form that describes

![Fig. 1. Integration limits for $d\theta_v$ and $d\psi$.](image)

Table 1

Matrix of unsaturated permeability estimation models

<table>
<thead>
<tr>
<th>SWCC</th>
<th>$k_r$</th>
<th>Childs and Collis-George</th>
<th>Burdine</th>
<th>Mualem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fredlund and Xing (1994)</td>
<td>$C$</td>
<td>(Fredlund et al., 1994), FCM&lt;sup&gt;a&lt;/sup&gt;</td>
<td>FBM</td>
<td>FMM</td>
</tr>
<tr>
<td>Fredlund and Xing (1994) with $C(\psi) = 1$</td>
<td>$C$</td>
<td>FCM ($C(\psi) = 1$)</td>
<td>FBM ($C(\psi) = 1$)</td>
<td>FMM ($C(\psi) = 1$)</td>
</tr>
<tr>
<td>Van Genuchten (1980)</td>
<td>$V$</td>
<td>VCM</td>
<td>VBM</td>
<td>VMM</td>
</tr>
<tr>
<td>Van Genuchten (1980) with $m = 1 - 1/n$</td>
<td>$V$, $m = 1 - 1/n$</td>
<td>VCM ($m = 1 - 1/n$)</td>
<td>VBM ($m = 1 - 1/n$)</td>
<td>(van Genuchten, 1980), VMM ($m = 1 - 1/n$)</td>
</tr>
</tbody>
</table>

<sup>a</sup>FCM means that the model was resulted from the combination of Fredlund and Xing’s best-fit SWCC equation and Childs and Collis-George relative permeability equation. The capital M at the end of each name stands for model. Therefore, FCM refers to the Fredlund and Xing-Childs and Collis-George model. The rest of the names can be interpreted according to this description.
the relative permeability equation as a function of suction:

\[
k_r(\psi) = \left[ \int_{\theta(\psi_i)}^{\theta(\psi_f)} f(\psi) f'(\psi) d\psi / \int_{\theta(\psi_i)}^{\theta(\psi_f)} f(\psi) f'(\psi) d\psi \right]^{d}
\]  \tag{9}

Fig. 1 shows the integration limits for both the \( k_r(\theta_w) \) and \( k_r(\psi) \) equations.

In general, SWCC can be divided into a finite number of intervals along the volumetric water content or suction axis. The relationship between two subsequent points on the curve can be described by a polygon or linear equation as follows:

\[
\theta_w = f(\psi) = \theta_{wi} + \frac{(\theta_{wi} + 1 - \theta_{wi})}{(\psi_{i+1} - \psi_i)} (\psi - \psi_i)
\]  \tag{10}

\[
\psi = g(\theta_w) = \psi_i + \frac{(\psi_{i+1} - \psi_i)}{((\theta_{wi} + 1 - \theta_{wi}) (\theta_w - \theta_{wi}))} (\theta_w - \theta_{wi})
\]  \tag{11}

Fig. 2 shows the division of SWCC into a finite number of intervals along the volumetric water content or suction axis and their respective equations. If Eq. (11) is substituted into Eq. (8) or Eq. (10) is substituted into Eq. (9) and the integration is performed numerically, a series can be obtained as an unsaturated permeability estimation model which can be expressed as a function of the volumetric water content or suction, respectively.

In order to obtain the \( \theta_w = f(\psi) \) relationship, a SWCC equation (i.e., Eqs. (4)–(7)) was used to best-fit the experimentally measured SWCC data with the least square method. The best-fit curve was divided into a finite number of intervals along the suction axis. Eqs. (10) or (11) was then used to represent the relationship between two subsequent points on the curve and substituted into a \( k_r \) equation (i.e., Eqs. (12)–(14)). By performing numerical integration along the respective axis, three general permeability estimation models are obtained. Table 2 shows the models based on their \( k_r \) equations.

3.2. Assumptions made in the combined models

1) All the models considered in this study were developed based on actual volumetric water content (\( \theta_w \)) and not on effective water content \( \theta = \theta_w - \theta_r \). The residual volumetric water content, \( \theta_r \), is required in order to compute the effective water content, \( \theta = \theta_w - \theta_r \). The residual volumetric water content is defined qualitatively as the water content below which a large increase in suction is required to remove additional water (Fredlund et al., 1994). However, there is no theoretical definition for this parameter. The common practice for determining the residual water content is by the graphical method (Fredlund and Xing, 1994; Vanapalli et al., 1998) and there is no independent procedure for determining the residual water content (van Genuchten, 1980). If an effective volumetric water content (normalized) is used in estimating the unsaturated permeability of soil, the value of relative permeability, \( k_r \), at residual volumetric water content, \( \theta_r \), is zero (Fredlund et al., 1994). However, in soil physics, the unsaturated permeability at \( \theta_r \) cannot be zero (Brutsaert, 1966). Therefore, the models presented in this study were based on actual volumetric water content and not on the normalized volumetric water content.

2) Parameters \( b \) and \( n_1 \) were introduced into Eqs. (1)–(3) to provide relative permeability models with more flexibility (Mualem and Dagan, 1978). However, these two parameters need to be determined empirically from measured data. Therefore, the value of these two parameters depends

<table>
<thead>
<tr>
<th>( k_r )</th>
<th>Series</th>
<th>Models name</th>
</tr>
</thead>
</table>
| **Childs and Collis-George (C)** | \[
\begin{align*}
k_r(\psi) &= \sum_{i=1}^{n} \left[ \frac{\theta_{wi} + 1 - \theta_{wi}}{\psi_{i+1} - \psi_i} \right] \left[ \frac{\theta_{wi} + 1 - \theta_{wi}}{\theta_{wi} - \theta_{wi}} \right]^{b} \left[ \frac{\theta_{wi} + 1 - \theta_{wi}}{\theta_{wi} - \theta_{wi}} \right]^{n_1} \left[ \frac{\theta_{wi} + 1 - \theta_{wi}}{\theta_{wi} - \theta_{wi}} \right]^{2} \\
\end{align*}
\] | FCM |
| | | FCM \( C(\psi) = 1 \) |
| | | VCM |
| | | VCM \( m = 1 - 1/n \) |
| **Burdine (B)** | \[
\begin{align*}
k_r(\psi) &= \sum_{i=1}^{n} \left[ \frac{\theta_{wi} + 1 - \theta_{wi}}{\psi_{i+1} - \psi_i} \right] \left[ \frac{\theta_{wi} + 1 - \theta_{wi}}{\theta_{wi} - \theta_{wi}} \right]^{b} \left[ \frac{\theta_{wi} + 1 - \theta_{wi}}{\theta_{wi} - \theta_{wi}} \right]^{n_1} \left[ \frac{\theta_{wi} + 1 - \theta_{wi}}{\theta_{wi} - \theta_{wi}} \right]^{2} \\
\end{align*}
\] | FBM |
| | | FBM \( C(\psi) = 1 \) |
| | | VBM |
| | | VBM \( m = 1 - 1/n \) |
| **Mualem (M)** | \[
\begin{align*}
k_r(\psi) &= \left[ \sum_{i=1}^{n} \left[ \frac{\theta_{wi} + 1 - \theta_{wi}}{\psi_{i+1} - \psi_i} \right] b \left[ \frac{\theta_{wi} + 1 - \theta_{wi}}{\theta_{wi} - \theta_{wi}} \right] \left[ \frac{\theta_{wi} + 1 - \theta_{wi}}{\theta_{wi} - \theta_{wi}} \right]^{n_1} \right]^{2} \\
\end{align*}
\] | FMM |
| | | FMM \( C(\psi) = 1 \) |
| | | VMM |
| | | VMM \( m = 1 - 1/n \) |
on the soil database under consideration. For instance, values of 0, 2 and 0.5 were proposed for parameter \( n_1 \) by Childs Collis-George (1950), Burdine (1953) and Mualem (1976), respectively, based on their soil database. As the objective of this paper was to investigate the variation between the different models fairly independent from soil database, correction factor was not considered. Therefore, the value of zero was considered for parameter \( b \) and \( n_1 \) in Eqs. (1)–(3).

Therefore, Eqs. (1)–(3) take the following forms: Childs and Collis-George (C)

\[
k_r(\theta_w) = \int_0^\theta_w \frac{d\theta}{\psi^2} / \int_0^\theta \frac{d\theta}{\psi^2}
\]

(12)

Burdine (B)

\[
k_r(\theta_w) = \int_0^\theta_w \frac{d\theta}{\psi^2} / \int_0^\theta \frac{d\theta}{\psi^2}
\]

(13)

Mualem (M)

\[
k_r(\theta_w) = \left[ \int_0^\theta \frac{d\theta}{\psi} / \int_0^\theta \frac{d\theta}{\psi} \right]^2
\]

(14)

3) Vapor permeability was assumed as a lower limit for the estimated relative permeabilities in the study. The vapor permeability was computed based on modified form of Fick’s law and it was normalized with respect to the saturated permeability of the soil (i.e., relative vapor permeability). The details on the lower limit of permeability can be found in Ebrahimi et al. (2004) and Peters and Durner (2008).

4. Results and discussions

4.1. General assessment

Twenty sets of published data which had experimentally measured SWCC and permeability data as shown in Table 3 were selected from the literature. The measured SWCC data of each soil was best-fit by Eqs. (4)–(7) by the least square method. Table 4 shows the values of SWCC fitting parameters for Silt loam (S8) and Guelph loam (S1) soils. The unsaturated permeability functions were then estimated by the matrix of unsaturated permeability estimation models shown in Table 1. Therefore, fourteen unsaturated permeability functions (2 existing models and 12 models combined in this study) were estimated for each of the soils selected in the study. Fig. 3 shows the results of best-fit SWCCs and the estimated unsaturated permeability functions (i.e., shown as relative permeability) of Silt loam. The root mean square error (RMSE) was used as a statistical measure to evaluate the fit of the estimation models to the measured data for the soil database as presented in Table 5. The RMSE is defined according to Eq. (15).

\[
RMSE = \sqrt{\frac{\sum_{i=1}^{X} (\log(k_{ri}) - \log(\hat{k}_{ri}))^2}{X}}
\]

(15)

Where, \( k_r \) is the measured unsaturated permeability value, \( \hat{k}_r \) is the estimated unsaturated permeability value, \( X \) is the number of measured data points and \( i \) is a counter.

The RMSE values show the deviation between the measured unsaturated permeability and estimated values. An overall comparison of the RMSE values for all 14 estimation models (matrix of unsaturated permeability estimation models) suggests that the VMM \( m = 1 - 1/n \), VCM \( m = 1 - 1/n \) and F&X-1994 models resulted in the lowest average and standard

<table>
<thead>
<tr>
<th>Soil number</th>
<th>Reference</th>
<th>Soil name</th>
<th>( k_r ) (m/s)</th>
<th>( \theta_s )</th>
<th>Last measured suction value of SWCC (kPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>Elrick and Bowman (1964)</td>
<td>Guelph Loam</td>
<td>( 3.917 \times 10^{-6} )</td>
<td>0.519</td>
<td>95.56</td>
</tr>
<tr>
<td>S2</td>
<td>van Genuchten (1980)</td>
<td>Beit Netofa Clay</td>
<td>( 9.491 \times 10^{-6} )</td>
<td>0.447</td>
<td>1385.53</td>
</tr>
<tr>
<td>S3</td>
<td>Moore (1939)</td>
<td>Yolo light clay</td>
<td>( 1.230 \times 10^{-7} )</td>
<td>0.373</td>
<td>63.38</td>
</tr>
<tr>
<td>S4</td>
<td>Brooks and Corey (1964)</td>
<td>Toucheit silt loam</td>
<td>( 3.507 \times 10^{-5} )</td>
<td>0.469</td>
<td>37.24</td>
</tr>
<tr>
<td>S5</td>
<td>Brooks and Corey (1964)</td>
<td>Columbia Sandy loam</td>
<td>N.A.</td>
<td>0.455</td>
<td>8.28</td>
</tr>
<tr>
<td>S6</td>
<td>Brooks and Corey (1964)</td>
<td>Hygiene Sandstone</td>
<td>( 1.250 \times 10^{-5} )</td>
<td>0.250</td>
<td>18.91</td>
</tr>
<tr>
<td>S7</td>
<td>Richards (1952)</td>
<td>Superstition Sand</td>
<td>( 1.830 \times 10^{-5} )</td>
<td>0.47</td>
<td>20.00</td>
</tr>
<tr>
<td>S8</td>
<td>van Genuchten (1980)</td>
<td>Silt Loam</td>
<td>( 5.741 \times 10^{-7} )</td>
<td>0.396</td>
<td>96.32</td>
</tr>
<tr>
<td>S9</td>
<td>Meerdink and Benson (1996)</td>
<td>Wenatchee Silty Clay</td>
<td>( 2.200 \times 10^{-5} )</td>
<td>0.360</td>
<td>500.00</td>
</tr>
<tr>
<td>S10</td>
<td>Meerdink and Benson (1996)</td>
<td>Live Oak Red Clay</td>
<td>( 3.200 \times 10^{-8} )</td>
<td>0.520</td>
<td>249.27</td>
</tr>
<tr>
<td>S11</td>
<td>Ng and Leung (2011)</td>
<td>CDT</td>
<td>( 2.52 \times 10^{-7} )</td>
<td>0.419</td>
<td>66.78</td>
</tr>
<tr>
<td>S12</td>
<td>Rassam and Williams (1999)</td>
<td>Mine Tailings</td>
<td>( 9.19 \times 10^{-7} )</td>
<td>0.391</td>
<td>500.00</td>
</tr>
<tr>
<td>S13</td>
<td>Samingan et al. (2003)</td>
<td>UP-1 (Residual soil)</td>
<td>( 1.21 \times 10^{-8} )</td>
<td>0.443</td>
<td>9275.14</td>
</tr>
<tr>
<td>S14</td>
<td>Samingan et al. (2003)</td>
<td>UP-2 (Residual soil)</td>
<td>( 6.65 \times 10^{-8} )</td>
<td>0.494</td>
<td>8955.86</td>
</tr>
<tr>
<td>S15</td>
<td>Samingan et al. (2003)</td>
<td>UP-3 (Residual soil)</td>
<td>( 6.25 \times 10^{-10} )</td>
<td>0.310</td>
<td>9361.88</td>
</tr>
<tr>
<td>S16</td>
<td>Samingan et al. (2003)</td>
<td>UP-4 (Residual soil)</td>
<td>( 9.48 \times 10^{-7} )</td>
<td>0.594</td>
<td>1000.00</td>
</tr>
<tr>
<td>S17</td>
<td>Valiantzas (2011)</td>
<td>Weld Silty Clay</td>
<td>( 5.671 \times 10^{-6} )</td>
<td>0.461</td>
<td>19.09</td>
</tr>
<tr>
<td>S18</td>
<td>Valiantzas (2011)</td>
<td>Fine Sand</td>
<td>( 1.250 \times 10^{-5} )</td>
<td>0.360</td>
<td>11.93</td>
</tr>
<tr>
<td>S19</td>
<td>Valiantzas (2011)</td>
<td>Volcanic Sand</td>
<td>( 8.102 \times 10^{-5} )</td>
<td>0.343</td>
<td>18.54</td>
</tr>
<tr>
<td>S20</td>
<td>Van Genuchten and Nielsen (1985)</td>
<td>G.E. No.2 Sand</td>
<td>N.A.</td>
<td>0.376</td>
<td>4.65</td>
</tr>
</tbody>
</table>
deviation of RMSE values (i.e., Ave = 0.665 and SD = 0.8, Ave = 0.694 and SD = 0.807 and Ave = 0.732 and SD = 0.669, respectively) for the soil database used in this study. On the other hand, the Burdine based estimation models (i.e., FBM, FBM $C(\psi) = 1$, VBM and VBM $m = 1 − 1/n$) resulted in the highest RMSE values for the 16 soils. It should be noted that the RMSE values show how good the estimated unsaturated permeabilities fit the experimental data within a limited suction range where experimental data were available. However, the performance of the estimation models could not be evaluated for the entire suction range due to the limited laboratory measurement data available in the literature.

4.2. Effects of SWCC and $k_r$ equations

It seems that, based solely on RMSE values, the conclusion of what model gives the best or worst estimation is soil database dependent and may vary with different databases. Therefore, it is important to evaluate the variation between the estimation models independent from the soil database and to evaluate the effect of the SWCC equation and $k_r$ equation.

All the permeability estimation models were categorized into a SWCC equation category (i.e., same SWCC equation and different $k_r$ equations to study the effect of $k_r$ equations) and $k_r$ equation category (i.e., same $k_r$ equation and different SWCC equations to study the effect of SWCC equations). The SWCC equation category contained four groups, namely Fredlund and Xing (1994), Fredlund and Xing (1994) with $C(\psi) = 1$, van Genuchten (1980) and van Genuchten (1980) with $m = 1 − 1/n$ based models. The $k_r$ equation category contained three groups, namely the Childs Collis-George, Burdine and Mualem based models. This is later shown when presenting the results in Tables 5–7.

The permeability values estimated by the estimation models in each group of the respective category were then compared. For instance, the permeability values estimated by the FCM, FMM and FBM models in the Fredlund and Xing (1994) group in the SWCC category were compared. The comparison was done by computing the logarithmic difference between the maximum and minimum permeability values estimated by the models at the last measured suction value of the SWCC data point. For example, the last measured SWCC data point for Guelph loam soil (i.e., S1) was at a suction value of 95.56 kPa, so all of the difference indices were computed and compared at this suction. The results for all of the soils are presented in Table 6. A smaller value for the difference index meant that the models of the respective group estimated more or less the same relative permeability values, while a larger value meant the models of the respective group estimated different relative permeability values. As presented in Table 6, the average of the difference indices for the SWCC category (i.e., effect of $k_r$,

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**Table 4**

<p>| SWCC fitting parameters of Silt loam and Guelph loam soils |
|---------------------------------|----------|--------|--------|----------|--------|--------|</p>
<table>
<thead>
<tr>
<th><strong>Best-Fit SWCC equation</strong></th>
<th><strong>Fitting parameters</strong></th>
<th><strong>Soil</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Fredlund and Xing (1994)</td>
<td>$\theta_s$ 0.396, $a$ 18.580, $m$ 0.531, $n$ 2.253, $C(\psi)$ 92.007, $\alpha$ N.A., $\theta_f$ N.A.</td>
<td>Silt Loam</td>
</tr>
<tr>
<td>Fredlund and Xing (1994) $C(\psi) = 1$</td>
<td>$\theta_s$ 0.396, $a$ 19.065, $m$ 0.637, $n$ 2.659, $C(\psi)$ 1.000, $\alpha$ N.A., $\theta_f$ N.A.</td>
<td></td>
</tr>
<tr>
<td>van Genuchten (1980) $m = 1 − 1/n$</td>
<td>$\theta_s$ 0.396, $a$ N.A., $m$ 3.504, $n$ 1.721, $C(\psi)$ N.A., $\alpha$ 0.014, $\theta_f$ 0.179</td>
<td></td>
</tr>
<tr>
<td>van Genuchten (1980) $C(\psi) = 1$</td>
<td>$\theta_s$ 0.396, $a$ N.A., $m$ 0.525, $n$ 2.107, $C(\psi)$ N.A., $\alpha$ 0.043, $\theta_f$ 0.128</td>
<td></td>
</tr>
<tr>
<td>Fredlund and Xing (1994) $C(\psi) = 1$</td>
<td>$\theta_s$ 0.519, $a$ 5.938, $m$ 0.354, $n$ 2.449, $C(\psi)$ 99.959, $\alpha$ N.A., $\theta_f$ N.A.</td>
<td>Guelph Loam</td>
</tr>
<tr>
<td>van Genuchten (1980) $m = 1 − 1/n$</td>
<td>$\theta_s$ 0.519, $a$ 6.230, $m$ 0.412, $n$ 2.249, $C(\psi)$ 1.000, $\alpha$ N.A., $\theta_f$ N.A.</td>
<td></td>
</tr>
<tr>
<td>van Genuchten (1980) $C(\psi) = 1$</td>
<td>$\theta_s$ 0.519, $a$ N.A., $m$ 0.207, $n$ 2.642, $C(\psi)$ N.A., $\alpha$ 0.180, $\theta_f$ 0.170</td>
<td></td>
</tr>
<tr>
<td>van Genuchten (1980) $m = 1 − 1/n$</td>
<td>$\theta_s$ 0.519, $a$ N.A., $m$ 0.498, $n$ 1.991, $C(\psi)$ N.A., $\alpha$ 0.116, $\theta_f$ 0.220</td>
<td></td>
</tr>
</tbody>
</table>

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![Typical results of best-fit SWCCs and estimated unsaturated permeability functions](image)
SWCC category, which were much smaller than the 1.420 and became quite clear as shown in Table 7. As Table 7 shows, the

When the Burdine based models were excluded from the study. The high RMSE values of the Burdine based models are due to the underestimation of the unsaturated permeability values at a relatively lower suction range and overestimation at a higher suction range. If a correction factor is considered for Burdine based models, the computed RMSE values will increase and it becomes apparent that considering a tortuosity factor is a modification in the wrong direction (Rahimi, 2015).

These models are reported to be in less agreement with measured data (Mualem, 1976; van Genuchten, 1980).

In order to illustrate the effect of SWCC equations on the estimation of unsaturated permeability function, the results
### Table 6
Computed difference index at suction value of last measured SWCC data point for entire soil database

| Soils | S1   | S2   | S3   | S4   | S5   | S6   | S7   | S8   | S9   | S10  | S11  | S12  | S13  | S14  | S15  | S16  | S17  | S18  | S19  | S20  | Average | Standard deviation |
|-------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|---------|-------------------|
| **SWCC Category** | Fredlund and Xing (1994) | 0.115 | 1.643 | 0.130 | 0.163 | 0.119 | 0.142 | 0.444 | 0.661 | 0.832 | 0.413 | 0.717 | 0.362 | 0.865 | 0.223 | 0.774 | 0.295 | 0.769 | 0.582 | 1.330 | 0.254 | 0.542 | 0.418 |
| | Fredlund and Xing (1994), $C(\psi) \equiv 1$ | 0.287 | 1.546 | 0.356 | 0.676 | 0.328 | 0.561 | 1.099 | 0.207 | 0.727 | 0.581 | 0.807 | 0.665 | 0.804 | 0.260 | 0.713 | 0.277 | 0.784 | 1.154 | 1.416 | 0.656 | 0.695 | 0.377 |
| | van Genuchten (1980) | 0.395 | 1.601 | 0.449 | 0.861 | 0.825 | 0.944 | 2.387 | 0.676 | 0.739 | 0.556 | 0.904 | 0.960 | 0.909 | 0.769 | 0.293 | 0.632 | 1.247 | 1.972 | 0.654 | 0.927 | 0.521 |
| | van Genuchten (1980), $m = 1 - 1/n$ | 0.527 | 1.534 | 0.547 | 1.544 | 0.650 | 1.704 | 2.832 | 0.219 | 0.614 | 0.249 | 0.611 | 0.699 | 1.042 | 3.220 | 3.377 | 0.195 | 1.209 | 1.727 | 3.954 | 0.664 | 1.356 | 1.139 |
| **k, Category** | Mualem | 0.938 | 0.619 | 1.000 | 1.678 | 0.931 | 2.236 | 3.529 | 1.656 | 0.221 | 0.390 | 0.132 | 0.725 | 1.557 | 3.519 | 1.058 | 0.067 | 1.201 | 1.337 | 5.232 | 0.372 | 1.420 | 1.323 |
| | Childs and Collis-George | 0.732 | 0.678 | 0.853 | 1.617 | 0.874 | 2.177 | 3.484 | 1.524 | 0.306 | 0.359 | 0.258 | 0.653 | 1.692 | 0.617 | 2.021 | 0.111 | 1.102 | 1.277 | 5.044 | 0.299 | 1.284 | 1.209 |
| | Burdine | 0.509 | 0.690 | 0.775 | 0.796 | 0.435 | 1.038 | 1.677 | 0.921 | 0.420 | 0.050 | 0.336 | 0.916 | 1.514 | 1.428 | 1.990 | 0.193 | 0.624 | 0.470 | 2.514 | 0.296 | 0.880 | 0.645 |

### Table 7
Computed difference index at suction value of last measured SWCC data for entire soil database-No Burdine.

| Soils | S1   | S2   | S3   | S4   | S5   | S6   | S7   | S8   | S9   | S10  | S11  | S12  | S13  | S14  | S15  | S16  | S17  | S18  | S19  | S20  | Average | Standard deviation |
|-------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|---------|-------------------|
| **SWCC Category** | Fredlund and Xing (1994) | 0.115 | 0.312 | 0.082 | 0.034 | 0.022 | 0.002 | 0.059 | 0.223 | 0.183 | 0.150 | 0.250 | 0.182 | 0.231 | 0.188 | 0.402 | 0.118 | 0.070 | 0.019 | 0.094 | 0.175 | 0.146 | 0.105 |
| | Fredlund and Xing (1994), $C(\psi) \equiv 1$ | 0.136 | 0.305 | 0.131 | 0.038 | 0.005 | 0.055 | 0.074 | 0.186 | 0.165 | 0.145 | 0.289 | 0.185 | 0.153 | 0.260 | 0.414 | 0.130 | 0.020 | 0.002 | 0.058 | 0.248 | 0.150 | 0.111 |
| | van Genuchten (1980) | 0.210 | 0.301 | 0.073 | 0.023 | 0.061 | 0.019 | 0.073 | 0.355 | 0.164 | 0.141 | 0.383 | 0.060 | 0.249 | 0.089 | 0.409 | 0.126 | 0.004 | 0.054 | 0.028 | 0.241 | 0.153 | 0.129 |
| | van Genuchten (1980), $m = 1 - 1/n$ | 0.321 | 0.365 | 0.229 | 0.067 | 0.051 | 0.054 | 0.103 | 0.219 | 0.252 | 0.177 | 0.348 | 0.253 | 0.365 | 3.220 | 3.377 | 0.174 | 0.103 | 0.080 | 0.094 | 0.125 | 0.499 | 0.963 |
| **k, Category** | Mualem | 0.938 | 0.619 | 1.000 | 1.678 | 0.931 | 2.236 | 3.529 | 1.656 | 0.221 | 0.390 | 0.132 | 0.725 | 1.557 | 3.519 | 1.058 | 0.067 | 1.201 | 1.337 | 5.232 | 0.372 | 1.420 | 1.323 |
| | Childs and Collis-George | 0.732 | 0.678 | 0.853 | 1.617 | 0.874 | 2.177 | 3.484 | 1.524 | 0.306 | 0.359 | 0.258 | 0.653 | 1.692 | 0.617 | 2.021 | 0.111 | 1.102 | 1.277 | 5.044 | 0.299 | 1.284 | 1.209 |
for Guelph loam soil (i.e., S1) for Mualem based models (i.e., FMM, FMM $C(\psi) = 1$, VG-1980 and VMM) are shown in Fig. 4. As shown in Fig. 4b, the four estimated relative permeability curves by Mualem based models varied although the same $k_r$ equation was used. The four best-fit SWCCs as shown in Fig. 4a had more or less the same shape until their last measured SWCC data point and started to vary significantly from each other after this point. It appears that, for soils whose best-fit SWCCs vary significantly, the estimated relative permeability curves would also vary significantly. This means that the SWCC equation plays an important role in the estimation of unsaturated permeability function. This behavior was observed for all of the soils in the $k_r$ category, as indicated by larger values of difference indices in Table 7.

In order to illustrate the effect of $k_r$ equations on the estimation of unsaturated permeability function, the results for Guelph loam soil (i.e., S1) using Fredlund and Xing (1994) based models (i.e., FCM, FMM) are shown in Fig. 5. As it can be seen from the figure, the estimated relative permeability curves had more or less the same shape and the variation between models was almost negligible even though two different relative permeability equations were used. It appears that relative permeability equation plays a less important role in the estimation of unsaturated permeability function of soil. This behavior was observed for all of the soils in the SWCC category, as indicated by the smaller values of difference indices in Table 7. Therefore, if different best-fit SWCC equations are used, the resulting relative permeability curves will have different shapes even if the same relative permeability equation is used in developing the estimation model. On the other hand, if the same best-fit SWCC equation is used (or if the SWCC curves are quite similar), the resulting relative permeability curves will have marginal variation even if different $k_r$ equations are used in the development of the estimation model.

The difference indices were computed at a suction value of one log cycle after the last measured SWCC data points (see Figs. 4 and 5) and the results are shown in Table 8. It can be seen from the table that the increase in the average value for the $k_r$ category was quite significant compared to that for the SWCC category. For instance, the average value for Fredlund and Xing (1994) based models increased from 0.146 to 0.162 in the SWCC category, while the average value for Mualem based models increased from 1.420 to 10.550 in the $k_r$ category (it should be noted that the average value for the Burdine based models increased from 0.880 to 5.339, which was consistent with the results of the study). From the results presented in

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**Fig. 4.** Typical effect of SWCC equations on estimation of permeability (a) Guelph Loam-best-fit SWCCs and (b) Guelph Loam-Mualem based models.

**Fig. 5.** Typical effect of relative permeability equation on estimation of permeability (a) Guelph Loam Fredlund and Xing (1994) SWCC and (b) Guelph Loam-Fredlund and Xing based models.
Table 8, it can be concluded that if the same SWCC equation and different \( k_r \) equations are used for the estimation of unsaturated permeability in seepage analyzes, the results of the analyzes will have small variation even at the extrapolated suction range. This means that a change of \( k_r \) model may not noticeably change the results, even if the permeability is estimated beyond the measured SWCC data. On the other hand, if different SWCC equations and the same \( k_r \) equation are used, the results of the analyzes will differ considerably, especially at the extrapolated region. This means that a change of SWCC model may significantly change the results, especially if the permeability is estimated beyond the measured SWCC data. Therefore, it can be concluded that the SWCC equation has a more significant effect on the estimation of unsaturated permeability function than the relative permeability equation.

Based on the study conducted by Rahimi et al. (2015), it was found that the effect of SWCC suction range is more significant than the effect of selected SWCC best-fit equation on the estimation of unsaturated permeability functions. Using SWCC over a full suction range would reduce the variation amongst all the models. A sensitivity analysis was performed and it was found that the models developed based on Fredlund and Xing (1994) best-fit SWCC equation were least sensitive to the SWCC suction range (Rahimi et al., 2015). Therefore, if the measured SWCC data were not available in the full suction range, it is best to use Fredlund and Xing (1994) based models (i.e., FMM and FCM) for the estimation of unsaturated permeability function.

5. Conclusions

The following conclusions can be made from the results of this study on the effect of SWCC and relative permeability equations on estimation of unsaturated permeability of unsaturated soils:

1. Twenty sets of published data, which included measured SWCC data and measured \( k(\psi) \) data, were collated from the literature. The root mean square error (RMSE) was computed for all of the soil in the database as a statistical measure to evaluate the fitness of the estimation models (i.e., 12 unsaturated permeability estimation models and the two existing models F&X-1994 and VG-1980). Comparisons between the average RMSE values for all estimation models on an overall basis suggest that VMM \( m = 1/n \), VCM \( m = 1 - 1/n \) and F&X-1994 models result in the lowest average and standard deviation RMSE values for all models for the selected soil database used in this study. On the other hand, the Burdine based estimation models (i.e., FBM, FBM \( C(\psi) = 1 \), VBM and VBM \( m = 1 - 1/n \)) gave the highest RMSE values for 16 of the soils.

2. The conclusion regarding which model offers the best or worst estimation, based only on RMSE values, is that it depends on the soil database and varies for different databases. Therefore, variation between all the estimation models was studied independently from the soil database.
for the controlling factors: SWCC and $k_r$ equations as identified in this study.

3. If different best-fit SWCC equations were used, the resulting relative permeability curves would have different shapes even if the same relative permeability equation was used in developing the estimation model especially at the extrapolated suction range. On the other hand, if the same best-fit SWCC equation was used (or the SWCC curves were quite similar to each other), the resulting relative permeability curves would have marginal variation in the entire suction range even if different $k_r$ equations were used in the development of the estimation model. In other words, the best-fit SWCC equation has more significant effect on the estimation of unsaturated permeability function as compared to the relative permeability equation.

References


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