Applied Mathematics Letters 25 (2012) 1019–1024

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations[☆]

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ARTICLE INFO

Article history:
Received 28 August 2011
Accepted 19 September 2011

Keywords:
Monotone iterative technique
Systems of nonlinear fractional differential equations
Upper and lower solutions
Riemann–Liouville fractional derivatives

ABSTRACT

By establishing a comparison result and using the monotone iterative technique combined with the method of upper and lower solutions, we investigate the existence of solutions for systems of nonlinear fractional differential equations.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. (see [1–4] and references therein). Since, as cited in [2], a number of works have appeared, especially in the theory of viscoelasticity and in hereditary solid mechanics, where fractional derivatives are used to do a better description of material properties, some basic theory for fractional differential equations involving the Riemann–Liouville differential operator has been discussed by many authors [5–16]. On the other hand, the study of systems involving fractional differential equations is also important as such systems occur in various problems of applied nature, for example, see [17–25].

In this paper, we discuss some existence results for systems of nonlinear fractional differential equations. In order to obtain the solutions of systems of nonlinear fractional differential equations, we also develop the monotone iterative technique. It is well known that the method of upper and lower solutions coupled with its associated monotone iteration scheme is an interesting and powerful mechanism that offers theoretical as well constructive existence results for nonlinear problems in a closed set, generated by the lower and upper solutions, for instance, see [26,27]. To the best of our knowledge, this technique has not been applied yet to the systems of nonlinear fractional differential equations.

Consider the following system of nonlinear fractional differential equations

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), v(t)), & t \in (0, T], \\ D^{\alpha}v(t) = g(t, v(t), u(t)), & t \in (0, T], \\ t^{1-\alpha}u(t)|_{t=0} = x_0, \qquad t^{1-\alpha}v(t)|_{t=0} = y_0, \end{cases} \tag{1.1}$$

☆ The second and third authors are partially supported by the FEDER and the Ministerio de Educación y Ciencia, Spain, project MTM2010-15314.
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0893-9659/$ – see front matter
doi:10.1016/j.aml.2011.09.078

where $0 < T < \infty, f, g \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ x_0, y_0 \in \mathbb{R}$ and $x_0 \le y_0$, D^α is the standard Riemann–Liouville fractional derivative of order $0 < \alpha \le 1$, see [1]. It is worthwhile to indicate that the nonlinear terms in the system involve the unknown functions $u(t)$ and $v(t)$.

We organize the rest of this paper as follows. In Section 2, the existence and uniqueness of solutions for a linear problem for systems of differential equations is discussed and a differential inequality as a comparison principle is established. In Section 3, by use of the monotone iterative technique and the method of upper and lower solutions, we prove the existence of extremal solutions of system (1.1). Finally, an example is given to illustrate our results.

2. Preliminaries

In this section, we deduce some preliminary results that will be used in the next section to attain existence results for the nonlinear system (1.1).

First, consider the set $C_{1-\alpha}([0, T]) = \{u \in C(0, T]; t^{1-\alpha}u \in C([0, T])\}$. Since we look for solutions that belong to this set, we need to present some existence and uniqueness results for linear problems together with comparison results for functions in this space.

Now we enunciate the following existence and uniqueness results for initial linear equations.

Lemma 2.1 (*[1]*). *Let $0 < \alpha \le 1$ be fixed, then the linear initial value problem*

$$\begin{cases} D^\alpha u(t) + Mu(t) = \sigma(t), & t \in (0, T], \\ t^{1-\alpha}u(t)|_{t=0} = u_0, \end{cases} \tag{2.1}$$

where M is a real constant and $\sigma \in C_{1-\alpha}([0, T])$, has a unique solution which is given by the following integral representation of solution

$$u(t) = \Gamma(\alpha)u_0 t^{\alpha-1}E_{\alpha,\alpha}(-Mt^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-M(t-s)^\alpha)\sigma(s)ds, \tag{2.2}$$

where $E_{\alpha,\alpha}(\cdot)$ is the Mittag-Leffler function [1], defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0,\ z \in \mathbb{R}.$$

Lemma 2.2. *Let $0 < \alpha \le 1$ be fixed, $M, N \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in C_{1-\alpha}([0, T])$, then the problem*

$$\begin{cases} D^\alpha u(t) = \sigma_1(t) - Mu(t) - Nv(t), & t \in (0, T], \\ D^\alpha v(t) = \sigma_2(t) - Mv(t) - Nu(t), & t \in (0, T], \\ t^{1-\alpha}u(t)|_{t=0} = x_0, \qquad t^{1-\alpha}v(t)|_{t=0} = y_0, \end{cases} \tag{2.3}$$

has a unique system of solutions in $C_{1-\alpha}([0, T]) \times C_{1-\alpha}([0, T])$.

Proof. The proof follows from the fact that the pair (u, v) is a solution of problem (2.3) if and only if

$$u(t) = \frac{p(t) + q(t)}{2} \quad \text{and} \quad v(t) = \frac{p(t) - q(t)}{2}, \quad t \in [0, T],$$

where p and q solve the problems

$$\begin{cases} D^\alpha p(t) = (\sigma_1 + \sigma_2)(t) - (M + N)p(t), & t \in (0, T], \\ t^{1-\alpha}p(t)|_{t=0} = x_0 + y_0 \end{cases} \tag{2.4}$$

and

$$\begin{cases} D^\alpha q(t) = (\sigma_1 - \sigma_2)(t) - (M - N)q(t), & t \in (0, T], \\ t^{1-\alpha}q(t)|_{t=0} = x_0 - y_0. \end{cases} \tag{2.5}$$

By Lemma 2.1, we know that both problems (2.4) and (2.5) have a unique solution in $C_{1-\alpha}([0, T])$. In consequence, u and v are unique too. □

In the sequel, we prove a comparison result for the initial linear problem (2.1). The result is the following:

Lemma 2.3. *Let $0 < \alpha \le 1$ and $M \in \mathbb{R}$ be given. Then, if $w \in C_{1-\alpha}([0, T])$ satisfy the relations,*

$$\begin{cases} D^\alpha w(t) + Mw(t) \ge 0, & t \in (0, T], \\ t^{1-\alpha}w(t)|_{t=0} \ge 0. \end{cases}$$

Then $w(t) \ge 0,\ \forall t \in (0, T]$.

Proof. In [10, Lemma 2.1], it is shown that if $M > -\frac{\Gamma(1+\alpha)}{T^\alpha}$, then $w(t) \geq 0, \forall t \in (0, T]$. In particular, the result holds for all $M \geq 0$.

The case $M < 0$ follows from the fact that every function w that satisfies the assumptions of the enunciate is a solution of problem (2.1) for some nonnegative $\sigma \in C_{1-\alpha}([0, T])$ and a real number $u_0 \geq 0$. As a consequence, the expression of the function w is given by the equality (2.2), which is nonnegative. □

Now we are in a position to prove the following comparison result for system (2.3).

Lemma 2.4 (*Comparison Theorem*)**.** *Let $0 < \alpha \leq 1$, $M \in \mathbb{R}$ and $N \geq 0$ be given. Assume that $u, v \in C_{1-\alpha}([0, T])$ satisfy*

$$\begin{cases} D^\alpha u(t) \geq -Mu(t) + Nv(t), & t \in (0, T], \\ D^\alpha v(t) \geq -Mv(t) + Nu(t), & t \in (0, T], \\ t^{1-\alpha} u(t)|_{t=0} \geq 0, \\ t^{1-\alpha} v(t)|_{t=0} \geq 0. \end{cases} \tag{2.6}$$

Then $u(t) \geq 0$, $v(t) \geq 0$, $\forall t \in (0, T]$.

Proof. Put $p(t) = u(t) + v(t)$, $\forall t \in (0, T]$. Then, by (2.6), we have

$$\begin{cases} D^\alpha p(t) \geq -(M - N)p(t), & t \in (0, T], \\ t^{1-\alpha} p(t)|_{t=0} \geq 0. \end{cases} \tag{2.7}$$

Thus, by (2.7) and Lemma 2.3, we have that

$$p(t) \geq 0, \quad \forall t \in (0, T], \quad \text{i.e., } u(t) + v(t) \geq 0, \quad \forall t \in (0, T]. \tag{2.8}$$

Next, we show that $u(t) \geq 0$, $v(t) \geq 0$, $\forall\, t \in (0, T]$.
In fact, by (2.6) and (2.8), we have that

$$\begin{cases} D^\alpha u(t) \geq -(M + N)u(t), & t \in (0, T], \\ t^{1-\alpha} u(t)|_{t=0} \geq 0 \end{cases} \tag{2.9}$$

and

$$\begin{cases} D^\alpha v(t) \geq -(M + N)v(t), & t \in (0, T], \\ t^{1-\alpha} v(t)|_{t=0} \geq 0. \end{cases} \tag{2.10}$$

By (2.9) and (2.10), using the same method as above, it is easy to show that

$$u(t) \geq 0, \qquad v(t) \geq 0, \quad \forall\, t \in (0, T].$$

The proof of Lemma 2.4 is complete. □

Remark 2.1. Note that the previous result is not valid in general for $N < 0$. It is enough to consider problem (2.3) with $\sigma_2 = 0$ on $(0, T]$ and $y_0 = 0$. In consequence, for all $N < 0$, since $u(t) \geq -v(t)$ for all $t \in (0, T]$, we deduce that

$$D^\alpha v(t) \leq -(M + N)v(t), \quad t \in (0, T], \qquad t^{1-\alpha} v(t)|_{t=0} = 0,$$

which implies, from Lemma 2.3, that $v(t) \leq 0$ for all $t \in (0, T]$.

3. Main result

In this section, we prove the existence of extremal solutions of nonlinear system (1.1). We list the following assumptions for convenience.

(H_1) There exist $u_0, v_0 \in C_{1-\alpha}([0, T])$ and $u_0(t) \leq v_0(t)$, such that

$$\begin{cases} D^\alpha u_0(t) \leq f(t, u_0(t), v_0(t)), & t \in (0, T], \\ t^{1-\alpha} u_0(t)|_{t=0} \leq x_0, \\ D^\alpha v_0(t) \geq g(t, v_0(t), u_0(t)), & t \in (0, T], \\ t^{1-\alpha} v_0(t)|_{t=0} \geq y_0. \end{cases}$$

(H_2) There exist constants $M \in \mathbb{R}$ and $N \geq 0$, such that

$$\begin{cases} f(t, u, v) - f(t, \overline{u}, \overline{v}) \geq -M(u - \overline{u}) - N(v - \overline{v}), \\ g(t, u, v) - g(t, \overline{u}, \overline{v}) \geq -M(u - \overline{u}) - N(v - \overline{v}), \end{cases}$$

where $u_0(t) \leq \overline{u} \leq u \leq v_0(t)$, $u_0(t) \leq v \leq \overline{v} \leq v_0(t)$, and

$$g(t, v, u) - f(t, u, v) \geq M(u - v) + N(v - u), \quad \text{with } u_0(t) \leq u \leq v \leq v_0(t).$$

Theorem 3.1. *Suppose that conditions* (H_1) *and* (H_2) *hold. Then, there is* $(u^*, v^*) \in [u_0, v_0] \times [u_0, v_0]$ *an extremal solution of the nonlinear problem (1.1). Moreover, there exist monotone iterative sequences* $\{u_n\}, \{v_n\} \subset [u_0, v_0]$ *such that* $u_n \to u^*,\ v_n \to v^* (n \to \infty)$ *uniformly on* $t \in (0, T]$, *and*

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le u^* \le v^* \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{3.1}$$

Proof. First, for any $u_{n-1}, v_{n-1} \in C_{1-\alpha}([0, T]),\ n \ge 1$, we consider the linear system

$$\begin{cases} D^\alpha u_n(t) = f(t, u_{n-1}(t), v_{n-1}(t)) + Mu_{n-1}(t) + Nv_{n-1}(t) - Mu_n(t) - Nv_n(t), & t \in (0, T], \\ D^\alpha v_n(t) = g(t, v_{n-1}(t), u_{n-1}(t)) + Mv_{n-1}(t) + Nu_{n-1}(t) - Mv_n(t) - Nu_n(t), & t \in (0, T], \\ t^{1-\alpha} u_n(t)|_{t=0} = x_0, \qquad t^{1-\alpha} v_n(t)|_{t=0} = y_0. \end{cases} \tag{3.2}$$

From Lemma 2.2, we know that (3.2) has a unique system of solutions in $C_{1-\alpha}([0, T]) \times C_{1-\alpha}([0, T])$.

Next, we show that $\{u_n(t)\}, \{v_n(t)\}$ satisfy the property

$$u_{n-1} \le u_n \le v_n \le v_{n-1}, \quad n = 1, 2, \ldots. \tag{3.3}$$

Let $p = u_1 - u_0, q = v_0 - v_1$. From (3.2) and (H_1), we have that

$$\begin{cases} D^\alpha p(t) = D^\alpha u_1(t) - D^\alpha u_0(t) \\ \quad \ge -Mp(t) + Nq(t), \\ D^\alpha q(t) = D^\alpha v_0(t) - D^\alpha v_1(t) \\ \quad \ge -Mq(t) + Np(t), \\ t^{1-\alpha} p(t)|_{t=0} \ge x_0 - x_0 = 0, \\ t^{1-\alpha} q(t)|_{t=0} \ge y_0 - y_0 = 0. \end{cases}$$

Thus, by Lemma 2.4, we have that $p(t) \ge 0, q(t) \ge 0,\ \forall t \in (0, T]$.

Let $w = v_1 - u_1$. By condition (H_2) and (3.2), we obtain

$$\begin{cases} D^\alpha w(t) = D^\alpha v_1(t) - D^\alpha u_1(t) \\ \quad = g(t, v_0(t), u_0(t)) + Mv_0(t) + Nu_0(t) - Mv_1(t) - Nu_1(t) \\ \qquad - f(t, u_0(t), v_0(t)) - Mu_0(t) - Nv_0(t) + Mu_1(t) + Nv_1(t) \\ \quad \ge -M(v_0 - u_0)(t) - N(u_0 - v_0)(t) + Mv_0(t) + Nu_0(t) - Mv_1(t) \\ \qquad - Nu_1(t) - Mu_0(t) - Nv_0(t) + Mu_1(t) + Nv_1(t) \\ \quad = -(M - N)w(t), \\ t^{1-\alpha} w(t)|_{t=0} = y_0 - x_0 \ge 0. \end{cases}$$

By Lemma 2.3, we obtain $w(t) \ge 0,\ \forall t \in (0, T]$. Hence, we have the relation $u_0 \le u_1 \le v_1 \le v_0$.

Now, we assume that $u_{k-1} \le u_k \le v_k \le v_{k-1}$, for some $k \ge 1$, and we prove that (3.3) is true for $k + 1$ too. Let $p = u_{k+1} - u_k, q = v_k - v_{k+1}, w = v_{k+1} - u_{k+1}$. By (H_2) and (3.2), we have that

$$\begin{cases} D^\alpha p(t) \ge -Mp(t) + Nq(t), \\ D^\alpha q(t) \ge -Mq(t) + Np(t), \\ t^{1-\alpha} p(t)|_{t=0} = 0, \\ t^{1-\alpha} q(t)|_{t=0} = 0 \end{cases}$$

and

$$\begin{cases} D^\alpha w(t) \ge -(M - N)w(t), \\ t^{1-\alpha} w(t)|_{t=0} \ge 0 \end{cases}$$

and so, by Lemmas 2.3 and 2.4, we have that $u_k \le u_{k+1} \le v_{k+1} \le v_k$.

From the above, by induction, it is not difficult to prove that

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{3.4}$$

Applying the standard arguments, we have

$$\lim_{n\to\infty} u_n(t) = u^*(t), \qquad \lim_{n\to\infty} v_n(t) = v^*(t)$$

uniformly on compact subsets of $(0, T]$, and the limit functions u^*, v^* satisfy (1.1). Moreover, $u^*, v^* \in [u_0, v_0]$. Taking the limits in (3.2), we know that (u^*, v^*) is a system of solutions of (1.1) in $[u_0, v_0] \times [u_0, v_0]$. Moreover, (3.1) is true.

Finally, we prove that (1.1) has an extremal solution. Assume that $(u, v) \in [u_0, v_0] \times [u_0, v_0]$ is any system of solutions of (1.1). That is

$$\begin{cases} D^\alpha u(t) = f(t, u(t), v(t)), & t \in (0, T], \\ D^\alpha v(t) = g(t, v(t), u(t)), & t \in (0, T], \\ t^{1-\alpha} u(t)|_{t=0} = x_0, \qquad t^{1-\alpha} v(t)|_{t=0} = y_0. \end{cases} \tag{3.5}$$

By (3.2), (3.5), (H_2) and Lemma 2.4, it is easy to prove that

$$u_n \leq u, \qquad v \leq v_n, \quad n = 1, 2, \ldots. \tag{3.6}$$

By taking the limits in (3.6) as $n \to \infty$, we have that $u^* \leq u$, $v \leq v^*$. That is, (u^*, v^*) is an extremal solution of system (1.1) in $[u_0, v_0] \times [u_0, v_0]$.

This completes the proof. □

4. Example

Consider the following problem:

$$\begin{cases} D^\alpha u(t) = 2t^3[t - u(t)]^3 - t^4 v^2(t), \\ D^\alpha v(t) = 2t^3[t - v(t)]^3 - t^4 u^2(t), \\ t^{1-\alpha} u(t)|_{t=0} = t^{1-\alpha} v(t)|_{t=0} = 0, \end{cases} \tag{4.1}$$

where $t \in J = [0, 1]$, D^α is the standard Riemann–Liouville fractional derivative of order $0 < \alpha \leq 1$.

Obviously,

$$\begin{cases} f(t, u, v) = 2t^3[t - u]^3 - t^4 v^2, \\ g(t, v, u) = 2t^3[t - v]^3 - t^4 u^2. \end{cases}$$

Take $u_0(t) = 0$, $v_0(t) = t$, then

$$D^\alpha u_0(t) = 0 \leq t^6 = f(t, u_0(t), v_0(t)),$$

$$D^\alpha v_0(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \geq 0 = g(t, v_0(t), u_0(t)),$$

$$t^{1-\alpha} u(t)|_{t=0} = t^{1-\alpha} v(t)|_{t=0} = 0.$$

It shows that condition (H_1) of Theorem 3.1 holds.

On the other hand, it is easy to verify that condition (H_2) holds for $M = 6$ and $N = 0$.

Thus, all conditions of Theorem 3.1 are satisfied. In consequence, the nonlinear system (4.1) has the extremal solution $(u^*, v^*) \in [u_0, v_0] \times [u_0, v_0]$, which can be obtained by taking limits from the iterative sequences:

$$u_n(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-6\,(t-s)^\alpha)(2s^3[s - u_{n-1}(s)]^3 + 6\,u_{n-1}(s) - s^4 v_{n-1}^2(s))ds, \quad n \geq 1,$$

and

$$v_n(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-6\,(t-s)^\alpha)(2s^3[s - v_{n-1}(s)]^3 + 6\,v_{n-1}(s) - s^4 u_{n-1}^2(s))ds, \quad n \geq 1.$$

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