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Convergence Estimates for Multigrid Algorithms

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Abstract—To estimate convergence of the multigrid algorithms, we need some assumptions on smoothers. The assumptions for typical smoothers are well analyzed in the multigrid literature [1,2]. However, numerical evidence shows that Kaczmarz smoother does not satisfy above assumptions. Thus, we introduce a weaker condition which is satisfied by Kaczmarz smoother as well as Jacobi and Gauss-Seidel smoother. Under these weaker assumptions, we show that the convergence factor of V-cycle multigrid algorithm is $\delta = 1 - 1/(C(j-1))$. These assumptions for Kaczmarz smoother are verified by numerical experiment.

Keywords—Multigrid method, Smoothing assumptions, Kaczmarz smoothing.

1. INTRODUCTION

Multigrid methods have been used extensively as tools for obtaining approximations to the solutions of partial differential equations. In conjunction, there has been intensive research into the theoretical understanding of these methods. Many papers present various analyses of multigrid methods which are often based on certain assumptions concerning smoothing process. These assumptions are sometimes verified for specific examples in [1–3]. In this paper, we provide weaker assumptions under which multigrid methods converge.

Assumptions concerning the smoothing process described in [2] are satisfied by Jacobi and Gauss-Seidel smoothing, but not by Kaczmarz smoothing, because the constant C_R grows with $1/h^2$. In this paper, we modify these assumptions so that these new assumptions are satisfied by Kaczmarz smoother as well as Jacobi and Gausss-Seidel smoother. It turns out that these assumptions are weaker than the conventional ones, thus Jacobi and Gauss-Seidel smoother satisfy them trivially. The case for Kaczmarz smoother is supported by numerical computation. Under these weaker assumptions, we prove that convergence factor of V-cycle multigrid algorithms is $\delta = 1 - 1/(C(j-1))$.

The outline of the remainder of this paper is as follows. In Section 2, we describe the basic multigrid algorithm in an abstract setting which utilizes some conditions on the smoothers. In Section 3, we prove multigrid convergence under these weaker assumptions. Various smoothing procedures including Kaczmarz iteration are described and analyzed in Section 4. Finally, in Section 5, we discuss finite element applications.

2. THE MULTIGRID ALGORITHMS

We assume that there is given a sequence of nested finite-dimensional inner product spaces

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_j,$$

with inner product $(\cdot,\cdot)_k$. In addition, we assume that symmetric positive definite operators $A_k : \mathcal{M}_k \to \mathcal{M}_k$ are given for $k = 1, \ldots, j$. The multigrid algorithm gives rise to iterative procedures for the solution of the problem on $\mathcal{M}_j$, i.e., given $f \in \mathcal{M}_j$ find $u \in \mathcal{M}_j$ satisfying

$$A_j u = f. \tag{2.1}$$

We define the operators $P^0_{k-1} : \mathcal{M}_k \to \mathcal{M}_{k-1}$ by

$$\left(P^0_{k-1} v, \phi\right)_{k-1} = (v, \phi)_k, \qquad \text{for all } \phi \in \mathcal{M}_{k-1}, \tag{2.2}$$

and the projectors $P_{k-1} : \mathcal{M}_k \to \mathcal{M}_{k-1}$ by

$$A\left(P_{k-1} v, \phi\right) = A(v, \phi), \qquad \text{for all } \phi \in \mathcal{M}_{k-1}, \tag{2.3}$$

where $A(\cdot,\cdot)$ is the bilinear operator defined by $A(u,v) = (A_k u, v)_k$ for $u, v \in \mathcal{M}_k$.

Also, we require a sequence of linear smoothing operators $R_k : \mathcal{M}_k \to \mathcal{M}_k$ for $k = 2, \ldots, j$. We shall always take $R_1 = A_1^{-1}$. We set

$$R_k^{(l)} = \begin{cases} R_k, & \text{if } l \text{ is odd,} \\ R_k^T, & \text{if } l \text{ is even,} \end{cases}$$

and set $K_k = I - R_k A_k$ on $\mathcal{M}_k$. Then we note that $K_k^* = I - R_k^T A_k$. Here, 'T' and '$*$' denote adjoint with respect to $(\cdot,\cdot)_k$ and $A(\cdot,\cdot)$, respectively.

We next define a multigrid process for iteratively computing the solution of (2.1).

ALGORITHM. Set $B_1^s = A_1^{-1}$. Assume that B_{k-1}^s has been defined and define $B_k^s g$ for $g \in \mathcal{M}_k$ as follows.

(1) Set $v^0 = 0$.
(2) Define v^i for $i = 1, 2, \ldots, m(k)$ by

$$v^i = v^{i-1} + R_k^{(i+m(k))}\left(g - A_k v^{i-1}\right).$$

(3) Define $w^{m(k)} = v^{m(k)} + B_{k-1}^s[P^0_{k-1}(g - A_k v^{m(k)})]$.
(4) Define w^i for $i = m(k)+1, \ldots, 2m(k)$ by

$$w^i = w^{i-1} + R_k^{(i+m(k))}\left(g - A_k w^{i-1}\right).$$

(5) Set $B_k^s g = w^{2m(k)}$.

In the above algorithm, by defining $B_k^n g = w^{m(k)}$, we get nonsymmetric multigrid algorithm B_k^n. From the above algorithm, fundamental recurrence relations for the nonsymmetric and the symmetric multigrid algorithm are

$$I - B_k^n A_k = \left[(I - P_{k-1}) + \left(I - B_{k-1}^n A_{k-1}\right)^p P_{k-1}\right] \bar{K}_k^{(m(k))}, \tag{2.4}$$

$$I - B_k^s A_k = \left(\bar{K}_k^{(m(k))}\right)^* \left[(I - P_{k-1}) + \left(I - B_{k-1}^s A_{k-1}\right)^p P_{k-1}\right] \bar{K}_k^{(m(k))} \tag{2.5}$$

on $\mathcal{M}_k$, where

$$\bar{K}_k^{(m(k))} = \begin{cases} \left(K_k^* K_k\right)^{m(k)/2}, & \text{if } m(k) \text{ is even,} \\ K_k \left(K_k^* K_k\right)^{(m(k)-1)/2}, & \text{if } m(k) \text{ is odd.} \end{cases}$$

To estimate the convergence of multigrid algorithm, we need some conditions concerning the smoothing operators. The conditions which were often assumed by many authors (see [1–7]) are the following.

CONDITION (C.1). There is a constant C_R which does not depend on k such that the smoothing procedure satisfies

$$\frac{\|u\|_k^2}{\lambda_k} \le C_R \left(\bar{R}_k u, u\right)_k, \qquad \text{for all } u \in \mathcal{M}_k. \tag{2.6}$$

Here, $\bar{R}_k$ is either $(I - K_k^* K_k)A_k^{-1}$ or $(I - K_k K_k^*)A_k^{-1}$. λ_k is the largest eigenvalue of A_k.

CONDITION (C.2). Let $T_k = R_k A_k$. There is a constant $\theta < 2$ not depending on k satisfying

$$A(T_k v, T_k v) \le \theta A(T_k v, v). \tag{2.7}$$

Kaczmarz smoothing, however, dose not satisfy (C.1). Thus, we modify (C.1) as follows.

CONDITION (SM.1). There is a constant C_R which does not depend on k such that the smoothing procedure satisfies

$$\frac{A(u,u)}{\lambda_k^2} \le C_R \left(\bar{R}_k u, u\right)_k, \qquad \text{for all } u \in \mathcal{M}_k. \tag{2.8}$$

3. CONVERGENCE ESTIMATES FOR MULTIGRID ALGORITHMS

To estimate the convergence of multigrid algorithm, we need some properties concerning the operator A_k and the subspaces. The following assumption will be verified in Section 5.

ASSUMPTION (A.1). *There exists a sequence of linear operators $Q_k : \mathcal{M}_j \to \mathcal{M}_k$ for $k = 1, \dots, j$, with $Q_j = I$ satisfying the following properties. There are constants C_1 and C_2 not depending on k for which*

$$\begin{aligned} \left(A_k^{-1}(Q_k - Q_{k-1})u, (Q_k - Q_{k-1})u\right)_k &\le C_1 \lambda_k^{-2} A(u,u), \qquad && \text{for } k = 2, \dots, j, \\ A(Q_k u, Q_k u) &\le C_2 A(u,u), && \text{for } k = 1, \dots, j-1. \end{aligned} \tag{3.1}$$

THEOREM 3.1. *Assume that Assumption (A.1) holds. Let R_k assume that (SM.1) and (C.2) hold. Let $B_j^s (B_j^n)$ be defined by symmtric (nonsymmetric) multigrid algorithm. Then,*

$$A\left(\left(I - B_j^s\right) v, v\right) \le \delta_j A(v,v), \qquad \text{for all } v \in \mathcal{M}_j \tag{3.2}$$

and

$$A\left(\left(I - B_j^n\right) v, \left(I - B_j^n\right) v\right) \le \delta_j A(v,v), \qquad \text{for all } v \in \mathcal{M}_j, \tag{3.3}$$

hold with

$$\delta_j = 1 - \frac{1}{C(j-1)},$$

where $C = [(1 + C_2^{1/2})(2\theta/(2-\theta))^{1/2} + (C_R C_1)^{1/2}]^2$.

PROOF. Since

$$I - B_j^s A_j = (I - B_j^n A_j)^* (I - B_j^n A_j),$$

holds by (2.4) and (2.5), it suffices to prove (3.3). If we let $T_k = (I - (\bar{K}_k^{(m(k))})^*)P_k$, we obtain the following recurrence relation:

$$\left(I - B_j^n A_j\right)^* = (I - T_j)(I - T_{j-1}) \cdots (I - T_1).$$

To use a product analysis, we set $E_0 = I$ and

$$E_k = (I - T_k)(I - T_{k-1}) \cdots (I - T_1) = (I - T_k) E_{k-1}.$$

Then, we have

$$
\begin{aligned}
A(u,u) - A(E_j u, E_j u) &= \sum_{k=1}^{j} \left[A\left(E_{k-1}u, E_{k-1}u\right) - A\left(E_k u, E_k u\right)\right] \\
&= \sum_{k=1}^{j} A\left(\left(2I - T_k\right) E_{k-1}u, T_k E_{k-1}u\right).
\end{aligned}
$$

Note that $I - B_j^n A_j = E_j^*$, and hence, the inequality (3.3) will follow if we can show that

$$
A(u,u) \le C(j-1) \sum_{k=1}^{j} A\left(\left(2I - T_k\right) E_{k-1}u, T_k E_{k-1}u\right). \tag{3.4}
$$

Since $Q_j = I$, we have

$$
\begin{aligned}
A(u,u) = &\sum_{k=2}^{j} A\left(E_{k-1}u, (Q_k - Q_{k-1})u\right) + A(u, Q_1 u) \\
&+ \sum_{k=2}^{j} A\left((I - E_{k-1})u, (Q_k - Q_{k-1})u\right).
\end{aligned} \tag{3.5}
$$

For the first sum on the right-hand side (3.5), from (3.1) and (2.8), we see that

$$
\begin{aligned}
&\sum_{k=2}^{j} A\left(E_{k-1}u, (Q_k - Q_{k-1})u\right) = \sum_{k=2}^{j} \left(A_k^{-1} A_k^2 P_k E_{k-1}u, (Q_k - Q_{k-1})u\right)_k \\
&\le \sum_{k=2}^{j} A(A_k P_k E_{k-1}u, A_k P_k E_{k-1}u)^{1/2} \cdot \left(A_k^{-1}(Q_k - Q_{k-1})u, (Q_k - Q_{k-1})u\right)_k^{1/2} \\
&\le (C_1)^{1/2} A^{1/2}(u,u) \sum_{k=2}^{j} \frac{A^{1/2}(A_k P_k E_{k-1}u, A_k P_k E_{k-1}u)}{\lambda_k} \\
&\le (C_R C_1)^{1/2} A^{1/2}(u,u) \sum_{k=2}^{j} A^{1/2}\left(\left(I - K_k^* K_k\right) P_k E_{k-1}u, P_k E_{k-1}u\right) \\
&\le \left(C_R C_1\,(j-1)\right)^{1/2} A^{1/2}(u,u) \left(\sum_{k=2}^{j} A\left(\left(I - K_k^* K_k\right) P_k E_{k-1}u, P_k E_{k-1}u\right)\right)^{1/2}.
\end{aligned}
$$

The remainder of proof is the same with the proof of Theorem 4.3 in [2]. ∎

4. SMOOTHING PROCEDURES IN MULTIGRID ALGORITHMS

We define Gauss-Seidel smoother by the following algorithm.

ALGORITHM 4.1. Let $f \in \mathcal{M}_k$, we define $R_k f \in \mathcal{M}_k$ as follows.

(1) Set $v_0 = 0$.
(2) Define v_i for $i = 1, \ldots, l$ by

$$
v_i = v_{i-1} + A_{k,i}^{-1} Q_k^i (f - A_k v_{i-1}), \tag{4.1}
$$

where $A_{k,i}$ is an i^{th} diagonal element of A_k and Q_k^i is a projection onto span$\{e_i\}$ with respect to $(\cdot,\cdot)_k$.
(3) Set $R_k f = v_l$.

Let P_k^i be the projection onto span$\{e_i\}$ with respet to $(A_k\cdot,\cdot)_k$. It immediately follows from the identity $A_{k,i}P_k^i = Q_k^i A_k$ that

$$K_k = \left(I - P_k^l\right)\cdots\left(I - P_k^1\right), \tag{4.2}$$

for Gauss-Seidel smoother. If we let $R_k f = \gamma \sum_{i=1}^l A_{k,i}^{-1} Q_k^i f$, we obtain Jacobi smoother. It is well known that Jacobi and Gauss-Seidel smoother satisfy (C.1) (see [2]).

LEMMA 4.1. *Let R_k be a smoother which satisfy (C.1). Then R_k satisfy (SM.1).*

PROOF. Let $u = A_k w$, then (C.1) becomes

$$\frac{(A_k w, A_k w)_k}{\lambda_k} \le C_R\left[(A_k w, w)_k - (A_k K_k w, K_k w)_k\right].$$

Since λ_k is the largest eigenvalue of A_k, we know that

$$(A_k w, w)_k \le \lambda_k (w, w)_k, \qquad \text{for all } w \in \mathcal{M}_k.$$

Therefore,

$$\frac{(A_k^2 w, A_k w)_k}{\lambda_k^2} \le C_R\left[(A_k w, w)_k - (A_k K_k w, K_k w)_k\right].$$

∎

Now we consider Kaczmarz iteration. Let $A_k = (a_{ij})_{i,j=1}^l$. Then Kaczmarz smoother is defined by the following algorithm.

ALGORITHM 4.2. Let $f \in \mathcal{M}_k$. We define $R_k f \in \mathcal{M}_k$ as follows.

(1) Set $v_0 = 0$.

(2) Define v_i for $i = 1, \ldots, l$ by

$$v_i = v_i - \frac{a_i}{a_i^T a_i}\left(a_i^T v_{i-1} - f_i\right),$$

where $a_i^T = i^{\text{th}}$ row of A_k, i.e., $a_i^T = (a_{i1}, a_{i2}, \ldots, a_{il})$.

(3) Set $R_k f = v_l$.

From the above algorithm, we obtain

$$K_k = I - R_k A_k = \left(I - B_k^l\right)\cdots\left(I - B_k^1\right), \tag{4.3}$$

where

$$B_k^i = \frac{a_i a_i^T}{a_i^T a_i} = \frac{A_k^T Q_k^i A_k}{\left(A_k A_k^T\right)_{ii}}.$$

Let S_k^i be the projection with respect to $((A_k A_k^T)\cdot,\cdot)_k$. Then we have

$$\left(A_k A_k^T\right)_{ii} S_k^i = Q_k^i \left(A_k A_k^T\right).$$

From this we get $B_k^i = A_k^T S_k^i A_k^{-T}$ and (4.3) becomes

$$K_k = A_k^T\left(I - S_k^l\right)\cdots\left(I - S_k^1\right)A_k^{-T}. \tag{4.4}$$

The above presentation reflects that Kaczmarz iteration can be regarded as a Gauss-Seidel iteration applied to $A_k A_k^T v = f$ with $u = A_k^T v$. In fact, one can verify that

$$\left(I - S_k^l\right)\cdots\left(I - S_k^1\right) = I - (D + L)^{-1}\left(A_k A_k^T\right),$$

where $A_k A_k^T = D + L + L^T$, where L is a strictly lower trianglar matrix and D is a diagonal matrix. Therefore, $K_k = I - A_k^T (D + L)^{-1} A_k$.

Numerical Verification of (SM.1) and (C.2)

We consider an elliptic partial differential equation of the form

$$
\begin{aligned}
-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v}{\partial x_j}(x)\right) &= f(x), \qquad \text{in } \Omega, \\
v(x) &= 0, \qquad \text{on } \partial\Omega,
\end{aligned} \tag{4.5}
$$

where Ω is a unit rectangle and $(a_{ij})_{i,j=1}^2$ is a symmetric positive matrix.

To obtain A_k, we discretize Ω by uniform triangular element with mesh size $h_k = 2^{-k}$. Let S_0^h be the space of continuous piecewise linear functions with basis $(\phi_i)_{i=1}^l$. We define

$$
(A_k)_{rs} = \sum_{i,j=1}^{2} \int_\Omega a_{ij} \frac{\partial \phi_s}{\partial x_i} \frac{\partial \phi_r}{\partial x_j}\, dx,
$$

for $r, s = 1, \dots, l$.

First, for $(a_{ij})_{i,j=1}^2 = I$, we calculate C_R's in (C.1) and (SM.1) of damped Jacobi with $\omega = 0.8$, Gauss-Seidel, and Kaczmarz smoothing.

Table 1 shows that Kaczmarz smoother does not satisfy (C.1).

Table 1.

	Jacobi		Gauss-Seidel		Kaczmarz	
h	(C.1)	(SM.1)	(C.1)	(SM.1)	(C.1)	(SM.1)
$\frac{1}{8}$	1.409707	1.409707	1.118052	1.118052	2.281606	1.349255
$\frac{1}{16}$	1.519271	1.519271	1.123504	1.123504	8.349859	1.371804
$\frac{1}{32}$	1.551332	1.551332	1.124665	1.124665	32.667680	1.376693
$\frac{1}{64}$	1.559570	1.559776	1.124900	1.124900	129.937300	1.377789

Next, we calculate Table 2, C_R in (SM.1) and θ in (C.2) of Kaczmarz smoother for several $(a_{ij})_{i,j=1}^2$'s.

Table 2.

$(a_{ij})_{i,j=1}^2$	I		$e^{0.2x+0.7y}I$		$\operatorname{diag}(e^{0.2x+0.3y}, x+0.5)$	
h	C_R	θ	C_R	θ	C_R	θ
$\frac{1}{8}$	1.349255	1.496815	1.321122	1.455358	1.322647	1.460811
$\frac{1}{16}$	1.371804	1.516820	1.342990	1.479987	1.341607	1.483321
$\frac{1}{32}$	1.376693	1.522054	1.356463	1.495506	1.353792	1.497318
$\frac{1}{64}$	1.377789	1.523371	1.364844	1.505585	1.361640	1.506282

5. FINITE ELEMENTS APPLICATIONS

We shall consider the problem of approximating the solution v of (4.5). The form A corresponding to the above operator is given by

$$
A(v, w) = \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j}\, dx. \tag{5.1}
$$

This form is defined for all v and w in the Sobolev space $H^1(\Omega)$. Clearly, $U \in H_0^1(\Omega)$ is the solution of

$$A(U,\theta) = (F,\theta), \qquad \text{for all } \theta \in H_0^1(\Omega),$$

where H_0^1 is the subspace of $H^1(\Omega)$ of functions which vanish in the appropriate sense on $\partial\Omega$ and $(\cdot,\cdot)$ denote the L^2 inner product on Ω.

By positive definiteness of $(a_{ij})_{i,j=1}^2$, $\|\cdot\|_A = A^{1/2}(\cdot,\cdot)$ is a norm on $H_0^1(\Omega)$ it is equivalent to $\|\cdot\|_1$ which denotes $H^1(\Omega)$-norm.

We assume that Ω has been triangulated with a sequence of quasi-uniform triangulations $\Omega = \cup_i \tau_k^i$ of size h_k for $k = 1,\ldots,j$, where quasi-uniformity constants are independent of k. We further assume that there is a constant c, independent of k, such that $h_{k-1} \le ch_k$. These triangulations should be nested in the sense that any triangle τ_{k-1}^l can be written as a union of triangles of $\{\tau_k^i\}$. We define $\mathcal{M}_k$ to be the set of piecewise linear functions with respect to the triangulation $\cup_i \tau_k^i$ which vanish on $\partial\Omega_k$.

Let $\{y_k^i\}$ be the collection of nodes corresponding to the triangulation for $\mathcal{M}_k$. Let

$$(u,v)_k = h_k^2 \sum_i u\left(y_k^i\right) v\left(y_k^i\right). \tag{5.2}$$

Note that the quasi-uniformity of the triangulations implies that the norm $\|\cdot\|_k$ is equivalent to the L^2 norm on the subspace $\mathcal{M}_k$. The operator A_k, $k = 1,\ldots,j$, are then defined by

$$(A_k v,\phi)_k = A(v,\phi), \qquad \text{for all } \phi \in \mathcal{M}_k. \tag{5.3}$$

Let Q_k denote the $L^2(\Omega)$ projection onto $\mathcal{M}_k$. We know that, since the triangulations are quasi-uniform and inverse property, for all $v \in H_0^1(\Omega)$,

$$\|(I - Q_k)v\| \le ch_k\|v\|_1 \tag{5.4}$$

and

$$\|Q_k v\|_1 \le C\|v\|_1. \tag{5.5}$$

From (5.4) and definition, we get

$$\begin{aligned}
&(A_k^{-1}(Q_k - Q_{k-1})v,(Q_k - Q_{k-1})v)_k \\
&= \sup_{u\in\mathcal{M}_k} \frac{((Q_k - Q_{k-1})v,u)^2}{(A_k u,u)_k} = \sup_{u\in\mathcal{M}_k} \frac{((Q_k - Q_{k-1})v,(Q_k - Q_{k-1})u)^2}{(A_k u,u)_k} \\
&\le \sup_{u\in\mathcal{M}_k} \frac{\|(Q_k - Q_{k-1})v\|^2 \|(Q_k - Q_{k-1})u\|^2}{(A_k u,u)_k} \\
&\le \frac{ch_k^2\|v\|_1^2 ch_k^2\|u\|_1^2}{\|u\|_1^2} \le Ch_k^4 A(v,v),
\end{aligned} \tag{5.6}$$

since $\|(Q_k - Q_{k-1})v\| \le \|(I - Q_k)v\| + \|(I - Q_{k-1})v\|$.

Combining (5.4)–(5.6) shows that (A.1) since $\lambda_k = O(h_k^{-2})$.

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