JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 77, 612–625 (1980)

On the Switchback Term in the Asymptotic Expansion of a Model Singular Perturbation Problem

A. D. MACGILLIVRAY

Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14214

Submitted by K. L. Cooke

I. INTRODUCTION

In 1961, Lagerstrom introduced some model singular perturbation problems to illustrate the mathematical ideas underlying the analysis of low Reynolds number flow by means of asymptotic expansions. One of these model problems is the following:

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y\frac{dy}{dx} = 0, \tag{1a}$$

$$y(\varepsilon) = 0, \tag{1b}$$

$$y(\infty) = 1. \tag{1c}$$

This problem corresponds to three-dimensional, viscous, incompressible flow; ε plays the role of Reynolds number. The asymptotic expansion of the solution of this problem exhibits what is known as a switchback term, and it is this feature which we wish to consider.

Discussions of the above and related model problems are found in Lagerstrom (1961), Kaplun (1967), Cole (1968), Lagerstrom and Casten (1972), Hsiao (1973), Rosenblat and Shepherd (1975), MacGillivray (1978, 1979) and Cohen, Fokas, and Lagerstrom (1978). In this last paper there is proved an existence and uniqueness theorem for a very general class of problems which includes as a special case the problem (1a)–(1c).

Classical fluid mechanics problems in which switchback terms occur are the so-called Filon terms (see Chang [1]) and the $R^2 \ln R$ term in the low Reynolds number flow analysis by Proudman and Pearson [10].

For a comprehensive review of the Kaplun–Lagerstrom theory of *matched asymptotic expansions* the reader is referred to the review article of Lagerstrom and Casten. However, to make the present discussion reasonably self-contained, we indicate briefly the main ideas we shall be using.

0022-247X/80/100329–15$02.00/0

To begin with, we shall assume the first two terms of the *outer asymptotic expansion* have been constructed using the Kaplun–Lagerstrom theory. By this we mean that this theory produces an approximation of the following sort:

$$\begin{aligned} y(x;\varepsilon) &= f_0(x) + \varepsilon f_1(x) + o(\varepsilon) \\ &= 1 - \varepsilon \int_x^\infty e^{-t} t^{-2}\, dt + o(\varepsilon), \end{aligned} \tag{2}$$

uniformly on an interval of the form

$$x \in [\eta_0(\varepsilon), \infty), \tag{3}$$

where $\eta_0(\varepsilon)$ is known only to satisfy

$$\varepsilon < \eta_0(\varepsilon) = o(1). \tag{4}$$

Similarly, the theory produces the leading term of the *inner asymptotic expansion*. If we set

$$x^* = x/\varepsilon,$$

this means we have an approximation of the following sort:

$$\begin{aligned} y(x;\varepsilon) &= g_0(x^*) + o(1) \\ &= \left(1 - \frac{1}{x^*}\right) + o(1) \end{aligned} \tag{5}$$

uniformly on any interval of the form

$$x \in [\varepsilon, \eta_1(\varepsilon)] \tag{6}$$

with

$$\eta_1(\varepsilon) = o(1). \tag{7}$$

Notice that according to (3), the outer asymptotic expansion is expected to approximate $y(x;\varepsilon)$ away from $x=\varepsilon$, whereas, according to (6), the inner asymptotic expansion is expected to approximate $y(x;\varepsilon)$ at and near $x=\varepsilon$.

With these brief preliminaries, we proceed in somewhat more detail to find the next term in the inner expansion. The first step for accomplishing this is to write (1a)–(1c) in terms of the "inner" variable $x^* = x/\varepsilon$. This yields

$$\frac{d^2y}{dx^{*2}} + \frac{2}{x^*}\frac{dy}{dx^*} + \varepsilon y \frac{dy}{dx^*} = 0, \tag{8a}$$

$$y(1;\varepsilon) = 0, \tag{8b}$$

$$y(\infty;\varepsilon) = 1. \tag{8c}$$

If we now formally substitute

$$y(x;\varepsilon)=\left(1-\frac{1}{x^*}\right)+\beta(\varepsilon)\,g_1(x^*)+o(\beta(\varepsilon)) \tag{9}$$

($\beta(\varepsilon)$ to be determined, though of course $\beta(\varepsilon)=o(1)$) into (8a), we obtain, after dropping higher-order terms,

$$\beta\frac{d^2g_1}{dx^{*2}}+\beta\frac{2}{x^*}\frac{dg_1}{dx^*}=-\varepsilon\left(1-\frac{1}{x^*}\right)\left(\frac{1}{x^*}\right)^2. \tag{10}$$

The natural choice for $\beta(\varepsilon)$ is

$$\beta(\varepsilon)=\varepsilon, \tag{11}$$

and the general solution of (10) which vanishes at $x^*=1$ is

$$g_1(x^*)=-\frac{\ln x^*}{x^*}-\ln x^*+B_1\,g_0(x^*), \tag{12}$$

where B_1 is a constant, and $B_1\,g_0(x^*)$ is the general complementary solution of (10) which vanishes at $x^*=1$:

$$B_1\,g_0(x^*)=B_1\left(1-\frac{1}{x^*}\right). \tag{13}$$

We notice that $g_1(x^*)$ becomes unbounded as $x^*\to\infty$, and in any case we expect (9) to be valid only at and near the boundary $x=\varepsilon$. For this reason, B_1 must be determined by a *matching condition* involving (2). This requires an interval of the form $[\eta_0(\varepsilon),\eta_2(\varepsilon)]$, with $\eta_0(\varepsilon)=o(\eta_2(\varepsilon))$ and $\eta_2(\varepsilon)=o(1)$, on which both (2) and (12) approximate $y(x;\varepsilon)$ to $O(\varepsilon)$, and hence (2) and (12) approximate each other to $O(\varepsilon)$. By this we mean that

$$\lim_{\epsilon\to 0}\frac{1}{\varepsilon}\left|\,g_0(x^*)-f_0(x)=\varepsilon f_1(\varepsilon)\right|=0 \tag{14}$$

uniformly for $x\in[\eta_0(\varepsilon),\eta_2(\varepsilon)]$. The interval $[\eta_0(\varepsilon),\eta_2(\varepsilon)]$ is called an *overlap domain*. The Kaplun–Lagerstrom theory provides heuristically motivated methods for determining overlap domains. For our present purposes, however, we shall simply assume the overlap domain exists and describe how it is used in the matching procedure.

Remark on notation. In what follows it will be convenient to use the notation $\psi(\varepsilon)\ll\varphi(\varepsilon)$ to mean $\psi(\varepsilon)=o(\varphi(\varepsilon))$ as $\varepsilon\to 0$.

Continuing, the next step is to introduce $\eta(\varepsilon)$ satisfying

$$\eta_0(\varepsilon)\ll\eta(\varepsilon)\ll\eta_2(\varepsilon)\ll 1, \tag{15}$$

and define

$$x = \eta(\varepsilon) x_\eta . \tag{16}$$

Then, for all sufficiently small ε, $x \in (\eta_0(\varepsilon), \eta_2(\varepsilon))$ if x_η is fixed and positive. With this notation, and (2) and (12), we can rewrite condition (14) after some trivial simplification as

$$\lim_{\substack{\epsilon \to 0 \\ x_\eta \text{ fixed}}} \left| -\frac{\ln(\eta/\varepsilon)x_\eta}{(\eta/\varepsilon)x_\eta} + \ln\varepsilon + B_1 - B_1 \frac{\varepsilon}{\eta x_\eta} + (\gamma - 1) + O(\eta) \right| = 0, \tag{17}$$

where we have used the result (see Lagerstrom and Casten [7])

$$\int_x^\infty e^{-t} t^{-2}\, dt = \frac{1}{x} + \ln x + (\gamma - 1) + O(x).$$

Using the fact that $\varepsilon \ll \eta(\varepsilon)$, (17) further simplifies to

$$\lim_{\substack{\epsilon \to 0 \\ x_\eta \text{ fixed}}} \left| \ln\varepsilon + B_1 + (\gamma - 1) \right| = 0, \tag{18}$$

and we see immediately that no choice of B_1 independent of ε will work. However, (18) suggests we might achieve matching by setting

$$B_1 = -\ln\varepsilon - (\gamma - 1). \tag{19}$$

To verify this, we substitute (19) into (14) and this time find after some simplification that the matching condition is

$$\lim_{\substack{\epsilon \to 0 \\ x_\eta \text{ fixed}}} \left| O(\eta) + 2\frac{\varepsilon \ln\varepsilon}{x_\eta \eta} - \frac{\varepsilon \ln x_\eta \eta}{x_\eta \eta} \right| = 0. \tag{20}$$

The expectation from the Kaplun–Lagerstrom theory is that the inner expansion thus far constructed,

$$\begin{aligned} y(x;\varepsilon) = {} & \left(1 - \frac{1}{x^*}\right) - \varepsilon\ln\varepsilon\left(1 - \frac{1}{x^*}\right) \\ & + \varepsilon\left[-\frac{\ln x^*}{x^*} - \ln x^* - (\gamma - 1)\left(1 - \frac{1}{x^*}\right)\right] + o(\varepsilon), \end{aligned} \tag{21}$$

is valid for $x \in [\varepsilon, \eta_2(\varepsilon)]$, where the only restriction on $\eta_2(\varepsilon)$ is $\eta_2(\varepsilon) \ll 1$.

On the other hand, (20) is satisfied if

$$|\varepsilon \ln\varepsilon| \ll \eta(\varepsilon). \tag{22}$$

Consequently an overlap domain is

$$|\varepsilon \ln \varepsilon| \ll \eta(\varepsilon) \ll 1. \tag{23}$$

The term $-\varepsilon \ln \varepsilon(1 - 1/x^*)$ which appears in (21) is the *switchback term.* Its order of magnitude, $O(\varepsilon \ln \varepsilon)$, lies "between" the $O(1)$ leading term and the $O(\varepsilon)$ term that were constructed originally, but its existence becomes apparent in the course of the construction of the $O(\varepsilon)$ term in the expansion.

In the next section we shall put the results of the above formal analysis on a rigorous basis by proving statement (21). In so doing we of course shall be rigorously establishing the existence of the switchback term. This seems to be the first proof of such a result.

II. Analysis

As mentioned above, we shall prove the following:

PROPOSITION 1.

$$y(x; \varepsilon) = \left(1 - \frac{1}{x^*}\right) - \varepsilon \ln \varepsilon \left(1 - \frac{1}{x^*}\right) + \varepsilon\left(-\frac{\ln x^*}{x^*} - \ln x^* - (\gamma - 1)\left(1 - \frac{1}{x^*}\right)\right) + o(x) \tag{24}$$

uniformly for $x \in [\varepsilon, \eta_2(\varepsilon)]$, *where the only restriction on* $\eta_2(\varepsilon)$ *is*

$$\eta_2(\varepsilon) \ll 1. \tag{25}$$

We begin by defining $r(x; \varepsilon)$ as follows:

$$y(x; \varepsilon) = (1 - \varepsilon \ln \varepsilon + \varepsilon - \varepsilon\gamma)\left(1 - \frac{\varepsilon}{x}\right) - \frac{\varepsilon^2}{x} \ln \frac{x}{\varepsilon} - \varepsilon \ln \frac{x}{\varepsilon} + r(x; \varepsilon). \tag{26}$$

Substitution into (1a), (1b), (1c) yields

$$\frac{d^2r}{dx^2} + \frac{2}{x}\frac{dr}{dx} = \left(1 - \frac{\varepsilon}{x}\right)\left(\frac{\varepsilon}{x^2}\right) - y\left[\frac{\varepsilon}{x^2}(1 - \varepsilon \ln \varepsilon + \varepsilon - \varepsilon\gamma) + \frac{\varepsilon^2}{x^2}\ln\frac{x}{\varepsilon} - \frac{\varepsilon^2}{x^2} - \frac{\varepsilon}{x} + \frac{dr}{dx}\right], \tag{27a}$$

$$r(\varepsilon; \varepsilon) = 0, \tag{27b}$$

$$\lim_{x\to\infty} r(x; \varepsilon) - \varepsilon \ln x = \varepsilon(\gamma - 1). \tag{27c}$$

An outline of our procedure is as follows. The purpose of the analysis is to show $r(x;\varepsilon)$ is $o(\varepsilon)$ for $x \in [\varepsilon, \eta_2(\varepsilon)]$. To this end Proposition 2 establishes a rather weak estimate for $y(x;\varepsilon)$. Specifically, $y(x;\varepsilon)$ is shown to be within $O(\varepsilon \ln \varepsilon)$ of $(1-\varepsilon/x)$. This estimate is sufficiently sharp to enable its use in the differential equation for $r(x;\varepsilon)$ to show that positive and negative values of $r(x;\varepsilon)$ are at most $o(\varepsilon)$; this is the content of Propositions 3 and 4, respectively. Proposition 1 then follows immediately. The proofs involve tedious but straightforward calculations.

We now proceed with the details.

Preliminary remark. It follows from (1a)–(1c) that $dy/dx > 0$ and hence $y(x;\varepsilon) > 0$ for all $x > \varepsilon$.

PROPOSITION 2.

$$1 - \frac{\varepsilon}{x} \leqslant y(x;\varepsilon) < 1 - \frac{\varepsilon}{x} - \varepsilon \ln \varepsilon + O(\varepsilon), \tag{28}$$

where the $O(\varepsilon)$ estimate is uniform for $x \in [\varepsilon, \infty)$.

Proof. We define $\bar{r}(x;\varepsilon)$ by the equation

$$y(x;\varepsilon) = \left(1 - \frac{\varepsilon}{x}\right) + \bar{r}(x;\varepsilon), \tag{29}$$

and find from (1a) that $\bar{r}$ satisfies

$$\frac{d^2\bar{r}}{dx^2} + \frac{2}{x}\frac{d\bar{r}}{dx} = -y\left(\frac{d\bar{r}}{dx} + \frac{\varepsilon}{x^2}\right), \tag{30a}$$

$$\bar{r}(\varepsilon;\varepsilon) = 0, \tag{30b}$$

$$\bar{r}(\infty;\varepsilon) = 0. \tag{30c}$$

We first show that $\bar{r}(x;\varepsilon) < 0$ is impossible. For, if so, it must have a minimum and there $d\bar{r}/dx = 0$ and, from (30a), there also is $d^2\bar{r}/dx^2$ negative, a contradiction. Therefore

$$y(x;\varepsilon) = \left(1 - \frac{\varepsilon}{x}\right) + \bar{r}(x;\varepsilon) \geqslant \left(1 - \frac{\varepsilon}{x}\right) \tag{31}$$

and the first part of (28) is proved.

To prove the second inequality in (28) we begin by noting from (30a) that $\bar{r}(x;\varepsilon)$ cannot vanish identically. We showed above $\bar{r}(x;\varepsilon)$ cannot be negative. Hence $\bar{r}(x;\varepsilon)$ must have a positive maximum $\bar{r}_M$ at a point x_M. From the fact that $y(x;\varepsilon) < 1$ and the definition of $\bar{r}(x;\varepsilon)$ we conclude that

$\bar{r}_M \leqslant \varepsilon/x_M$. Thus, if $\bar{r}_M > 4\varepsilon$, say, then $x_M < 1/4$. (If $\bar{r}_M \leqslant 4\varepsilon$, there is nothing to prove.)

From (30a) we see that if $d\bar{r}/dx = 0$, $d^2\bar{r}/dx^2 = -y(\varepsilon/x^2) < 0$, from which we conclude that

$$\frac{d\bar{r}}{dx} < 0 \qquad \text{for all} \quad x > x_M. \tag{32}$$

Therefore, for all $x > x_M$, we have

$$\begin{aligned} \frac{d^2\bar{r}}{dx^2} + \frac{2}{x}\frac{d\bar{r}}{dx} &= -y\frac{d\bar{r}}{dx} - y\frac{\varepsilon}{x^2} \\ &> -y\frac{\varepsilon}{x^2} \\ &> -\frac{\varepsilon}{x^2}. \end{aligned} \tag{33}$$

Multiplying the inequality (33) by x^2 we obtain

$$\frac{d}{dx}\left(x^2\frac{d\bar{r}}{dx}\right) > -\varepsilon \tag{34}$$

which we integrate from x_M to $x > x_M$. This gives

$$\frac{d\bar{r}}{dx} > -\varepsilon\left(\frac{1}{x} - \frac{x_M}{x^2}\right). \tag{35}$$

Again we integrate from x_M to $x > x_M$, and find

$$\bar{r}(x) > \bar{r}_M - \varepsilon\left(\ln\frac{x}{x_M} + \frac{x_M}{x} - 1\right). \tag{36}$$

In particular, set $x = 1/2$. Then, since $\bar{r}(1/2) < 2\varepsilon$, (36) gives

$$2\varepsilon > \bar{r}_M - \varepsilon\left(\ln\frac{1}{2x_M} + 2x_M - 1\right) \tag{37}$$

or

$$\bar{r}_M < 2\varepsilon + \varepsilon\left(\ln\frac{1}{2x_M} + 2x_M - 1\right). \tag{38}$$

The expression in parentheses increases as x_M decreases, attaining its maximum possible value when $x_M = \varepsilon$. With this extreme value, (38) gives

$$\bar{r}_M < \varepsilon(1 - \ln 2) - \varepsilon\ln\varepsilon + 2\varepsilon^2. \tag{39}$$

Hence

$$y(x;\varepsilon)=\left(1-\frac{\varepsilon}{x}\right)+\bar{r}(x;\varepsilon)<\left(1-\frac{\varepsilon}{x}\right)-\varepsilon\ln\varepsilon+O(\varepsilon) \tag{40}$$

for $x\in[\varepsilon, 1/2]$. Since $\bar{r}\leqslant\varepsilon/x$ in any case, we see that inequality (40) is valid for all $x>\varepsilon$. This completes the proof of Proposition 2.

We now return to an estimation of $r(x;\varepsilon)$ as defined in (26).

PROPOSITION 3. *Let $\eta_2(\varepsilon)$ be such that $\varepsilon<\eta_2(\varepsilon)\ll 1$ as $\varepsilon\to 0$. Then positive values of $r(x;\varepsilon)$ on $[\varepsilon, \eta_2(\varepsilon)]$ are bounded above by $o(\varepsilon)$.*

Proof. Using the second inequality in (28) in (27a) yields

$$\begin{aligned}\frac{d^2r}{dx^2}+\frac{2}{x}\frac{dr}{dx}>\left(1-\frac{\varepsilon}{x}\right)\frac{\varepsilon}{x^2}-\left(1-\frac{\varepsilon}{x}-\varepsilon\ln\varepsilon+O(\varepsilon)\right)\\ \times\left[\frac{\varepsilon}{x^2}(1+O(\varepsilon\ln\varepsilon))-\frac{\varepsilon}{x}+\frac{dr}{dx}\right]\end{aligned} \tag{41}$$

uniformly on $[\varepsilon, 1/2]$. With trivial computation (41) simplifies to

$$\frac{d^2r}{dx^2}+\frac{dr}{dx}\left(\frac{2}{x}+O(1)\right)>\frac{O(\varepsilon^2\ln\varepsilon)}{x^2}+\frac{\varepsilon}{x}. \tag{42}$$

Multiplying through by the integrating factor $x^2e^{O(1)x}$ gives

$$\frac{d}{dx}\left(x^2e^{O(1)x}\frac{dr}{dx}\right)>O(\varepsilon^2\ln\varepsilon)+\varepsilon x(1+o(1)), \tag{43}$$

uniformly on $[\varepsilon,\chi(\varepsilon)]$ where

$$\chi(\varepsilon)=\max\{\eta_2(\varepsilon),\varepsilon^{1/4}\}. \tag{44}$$

Now assume $r(x;\varepsilon)$ attains a positive maximum on $[\varepsilon,\eta_2(\varepsilon)]$ at the point x_0. Clearly $x_0>0$, since $r(\varepsilon;\varepsilon)=0$, and $dr/dx|_{x_0}\geqslant 0$, where the equal sign holds if $x_0<\eta_2(\varepsilon)$.

Integration of (43) from x_0 to x, $x_0\leqslant x\leqslant\chi(\varepsilon)$, yields, after some rearranging,

$$\frac{dr}{dx}\geqslant O(\varepsilon^2\ln\varepsilon)\left(\frac{1}{x}-\frac{1}{x_0}\right)+\frac{\varepsilon}{2}(1+o(1))-\frac{\varepsilon x_0^2}{2x^2}(1+o(1)) \tag{45}$$

uniformly for $x\in[\varepsilon,\chi(\varepsilon)]$.

We notice that when evaluated at $x = \chi(\varepsilon)$, (45) gives

$$\left.\frac{dr}{dx}\right|_{\chi(\varepsilon)} \geqslant O(\varepsilon). \tag{46}$$

Integrating (45), from x_0 to $\chi(\varepsilon)$, we find

$$r(\chi(\varepsilon); \varepsilon) \geqslant r(x_0; \varepsilon) + o(\varepsilon). \tag{47}$$

What we can conclude from (47) is that if positive values of $r(x; \varepsilon)$ on $[\varepsilon, \eta_2(\varepsilon)]$ are not bounded above by $o(\varepsilon)$, then the same is true of $r(\chi(\varepsilon); \varepsilon)$. Our strategy is to show $r(\chi(\varepsilon); \varepsilon)$ is in fact bounded above by $o(\varepsilon)$, and the conclusion will then follow. To do this we use the obvious estimate

$$y(x; \varepsilon) < 1, \qquad x > \varepsilon. \tag{48}$$

We then get from (27a)

$$\frac{d^2r}{dx^2} + \frac{2}{x}\frac{dr}{dx} > \left(1 - \frac{\varepsilon}{x}\right)\frac{\varepsilon}{x^2} - \left[\frac{\varepsilon}{x^2}(1 - \varepsilon \ln \varepsilon + \varepsilon - \varepsilon\gamma) + \frac{\varepsilon^2}{x^2}\ln\frac{x}{\varepsilon} - \frac{\varepsilon^2}{x^2} - \frac{\varepsilon}{x} + \frac{dr}{dx}\right] \tag{49}$$

which simplifies to

$$\frac{d^2r}{dx^2} + \left(\frac{2}{x} + 1\right)\frac{dr}{dx} > -\frac{\varepsilon^2}{x^3} + \frac{2\varepsilon^2 \ln \varepsilon}{x^2} + \frac{\varepsilon^2\gamma}{x^2} - \frac{\varepsilon^2}{x^2}\ln x + \frac{\varepsilon}{x}. \tag{50}$$

Multiplying by the integrating factor x^2e^x and integrating from $\chi(\varepsilon)$ to $x > \chi(\varepsilon)$ yields

$$x^2e^x\frac{dr}{dx} > \chi^2e^\chi\left.\frac{dr}{dx}\right|_\chi + (2\varepsilon^2 \ln \varepsilon + \varepsilon^2\gamma)e^x - (2\varepsilon^2 \ln \varepsilon + \varepsilon^2\gamma)e^\chi + \varepsilon x e^x - \varepsilon\chi e^\chi - \varepsilon e^x + \varepsilon e^\chi - \varepsilon^2\int_\chi^x e^s\left(\frac{1}{s} + \ln s\right) ds. \tag{51}$$

Dividing by x^2e^x and once again integrating from $\chi(\varepsilon)$ to x yields

$$r(x; \varepsilon) - r(\chi(\varepsilon); \varepsilon) > \left[\chi^2e^\chi\left.\frac{dr}{dx}\right|_\chi - (2\varepsilon^2 \ln \varepsilon + \varepsilon^2\gamma)e^\chi - \varepsilon\chi e^\chi + \varepsilon e^\chi\right] \times \int_\chi^x \frac{e^{-\xi}}{\xi^2}\,d\xi + (2\varepsilon^2 \ln \varepsilon + \varepsilon^2\gamma - \varepsilon)\left(-\frac{1}{x} + \frac{1}{\chi}\right) + \varepsilon \ln\frac{x}{\chi} - \varepsilon^2\int_\chi^x \frac{e^{-\xi}}{\xi^2}\int_\chi^\xi e^s\left(\frac{1}{s} + \ln s\right) ds. \tag{52}$$

We now write (26) out, using estimate (52). In this combined expression the $\varepsilon \ln x$ terms cancel each other, and we can let $x \to \infty$. Using the boundary condition (1c) and rearranging terms, these steps lead to

$$r(\chi(\varepsilon); \varepsilon) < -\varepsilon + \varepsilon\gamma - \left[\chi^2 e^\chi \frac{dr}{dx}\bigg|_\chi - (2\varepsilon^2 \ln \varepsilon + \varepsilon^2\gamma) e^\chi - \varepsilon\chi e^\chi + \varepsilon e^\chi\right] \int_\chi^\infty \frac{e^{-\xi}}{\xi^2} d\xi$$
$$- (2\varepsilon^2 \ln \varepsilon + \varepsilon^2 \gamma - \varepsilon) \frac{1}{\chi} + \varepsilon \ln \chi$$
$$+ \varepsilon^2 \int_\chi^\infty \frac{e^{-\xi}}{\xi^2} \int_\chi^\xi e^s \left(\frac{1}{s} + \ln s\right) ds\, d\xi. \tag{53}$$

Using the estimate in the formula following (17), we obtain from (53)

$$r(\chi(\varepsilon); \varepsilon) < O\left(\frac{\varepsilon^2 \ln \varepsilon}{\chi}\right) + \varepsilon^2 \int_\chi^\infty \frac{e^{-\xi}}{\xi^2} \int_\chi^\xi e^s \left(\frac{1}{s} + \ln s\right) ds\, d\xi, \tag{54}$$

and since $\chi(\varepsilon) \geqslant \varepsilon^{1/4}$,

$$r(\chi(\varepsilon); \varepsilon) < o(\varepsilon) + \varepsilon^2 \int_\chi^\infty \frac{e^{-\xi}}{\xi^2} \int_\chi^\xi e^s \left(\frac{1}{s} + \ln s\right) ds. \tag{55}$$

Our final task is to show the integral in (55) is $O(1/\chi^2)$. For $\chi \leqslant \xi \leqslant 1/\chi$, the integration with respect to s is estimated by

$$\int_\chi^\xi e^s \left(\frac{1}{s} + \ln s\right) ds \leqslant \int_\chi^\xi e^s \left(\frac{1}{\chi} + \ln \frac{1}{\chi}\right) ds$$
$$= (e^\xi - e^\chi)\left(\frac{1}{\chi} + \ln \frac{1}{\chi}\right). \tag{56}$$

For $1/\chi \leqslant \xi < \infty$, we have

$$\int_\chi^\xi e^s \left(\frac{1}{s} + \ln s\right) ds \leqslant \int_\chi^\xi e^s \left(\frac{1}{\chi} + \ln \xi\right) ds$$
$$= (e^\xi - e^\chi)\left(\frac{1}{\chi} + \ln \xi\right). \tag{57}$$

Using estimates (56) and (57), it is easy to arrive at

$$\int_\chi^\infty \frac{e^{-\xi}}{\xi^2} \int_\chi^\xi e^s \left(\frac{1}{s} + \ln s\right) ds\, d\xi = O\left(\frac{1}{\chi^2}\right). \tag{58}$$

Again reminding ourselves that $\chi(\varepsilon) \geqslant \varepsilon^{1/4}$, we find from (55) and (58) that

$$r(\chi(\varepsilon); \varepsilon) < o(\varepsilon). \tag{59}$$

Combining this with (47), we have

$$r(x_0; \varepsilon) < o(\varepsilon) \tag{60}$$

which completes the proof.

PROPOSITION 4. *Let $\eta_2(\varepsilon)$ be such that $\varepsilon < \eta_2(\varepsilon) \ll 1$. Then negative values of $r(x; \varepsilon)$ are bounded below on $[\varepsilon, \eta_2(\varepsilon)]$ by $o(\varepsilon)$.*

Proof. From the first inequality in (28), and (27a), we obtain

$$\frac{d^2r}{dx^2} + \frac{2}{x}\frac{dr}{dx} \leqslant \left(1 - \frac{\varepsilon}{x}\right)\frac{\varepsilon}{x^2} - \left(1 - \frac{\varepsilon}{x}\right)\left[\frac{\varepsilon}{x^2}(1 - \varepsilon \ln \varepsilon + \varepsilon - \varepsilon\gamma) + \frac{\varepsilon^2}{x^2}\ln\frac{x}{\varepsilon} - \frac{\varepsilon^2}{x^2} - \frac{\varepsilon}{x} + \frac{dr}{dx}\right] \tag{61}$$

which, upon rearranging, yields

$$\frac{d^2r}{dx^2} + \left(\frac{2 - \varepsilon}{x} + 1\right)\frac{dr}{dx} \leqslant (2\varepsilon^2 \ln \varepsilon - \varepsilon^2 + \varepsilon^2\gamma) \cdot \frac{1}{x^2} - \varepsilon^2 \frac{\ln x}{x^2} + \varepsilon^3 \frac{\ln x}{x^3} - (2\varepsilon^3 \ln \varepsilon + \varepsilon^2\gamma) \cdot \frac{1}{x^3} + \varepsilon \cdot \frac{1}{x}. \tag{62}$$

If $r(x; \varepsilon)$ has negative values on $[\varepsilon, \eta_2(\varepsilon)]$, let its minimum value occur at x_0. Then

$$\left.\frac{dr}{dx}\right|_{x_0} \leqslant 0. \tag{63}$$

Multiplying (62) by the integrating factor $e^x x^{2-\epsilon}$, and integrating from x_0 to $\xi > x_0$, yields

$$e^{\xi}\xi^{2-\epsilon}\left.\frac{dr}{dx}\right|_{\xi} \leqslant e^{x_0}x_0^{2-\epsilon}\left.\frac{dr}{dx}\right|_{x_0} + (2\varepsilon^2 \ln \varepsilon - \varepsilon^2 + \varepsilon^2\gamma)\int_{x_0}^{\xi} e^t t^{-\epsilon}\, dt - \varepsilon^2 \int_{x_0}^{\xi} e^t t^{-1-\epsilon} \ln t\, dt + \varepsilon^3 \int_{x_0}^{\xi} e^t t^{-1-\epsilon} \ln t\, dt - (2\varepsilon^3 \ln \varepsilon + \varepsilon^3\gamma)\int_{x_0}^{\xi} e^t t^{-1-\epsilon}\, dt + \varepsilon \int_{x_0}^{\xi} e^t t^{1-\epsilon}\, dt. \tag{64}$$

Dropping the nonpositive first term on the right side, then multiplying through by $e^{-\xi}\xi^{-2+\epsilon}$ and integrating from x_0 to $x > x_0$ yields the inequality

$$\begin{aligned} r(x) \leqslant r(x_0) &+ (2\varepsilon^2 \ln \varepsilon - \varepsilon^2 + \varepsilon^2\gamma) \int_{x_0}^{x} e^{-\xi}\xi^{-2+\epsilon} \int_{x_0}^{\xi} t^{-\epsilon} e^t \, dt \, d\xi \\ &- \varepsilon^2 \int_{x_0}^{x} e^{-\xi}\xi^{-2+\epsilon} \int_{x_0}^{\xi} t^{-\epsilon} e^t \ln t \, dt \, d\xi \\ &+ \varepsilon^3 \int_{x_0}^{x} e^{-\xi}\xi^{-2+\epsilon} \int_{x_0}^{\xi} e^t (\ln t)(t^{-1-\epsilon}) \, dt \, d\xi \\ &- (2\varepsilon^3 \ln \varepsilon + \varepsilon^3\gamma) \int_{x_0}^{x} e^{-\xi}\xi^{-2+\epsilon} \int_{x_0}^{\xi} e^t t^{-1-\epsilon} \, dt \, d\xi \\ &+ \varepsilon \int_{x_0}^{x} e^{-\xi}\xi^{-2+\epsilon} \int_{x_0}^{\xi} e^t t^{1-\epsilon} \, dt \, d\xi \end{aligned} \tag{65}$$

$$= r(x_0) + I(x; \varepsilon; x_0) + \varepsilon \int_{x_0}^{x} e^{-\xi}\xi^{-2+\epsilon} \int_{x_0}^{\xi} e^t t^{1-\epsilon} \, dt \, d\xi. \tag{66}$$

It is not difficult to show

$$\lim_{x\to\infty} I(x; \varepsilon; x_0) = o(\varepsilon). \tag{67}$$

It is also not difficult to estimate the last term in (77); we show the details. The first step involves an integration by parts:

$$\varepsilon \int_{x_0}^{x} e^{-\xi}\xi^{-2+\epsilon} \int_{x_0}^{\xi} e^t t^{1-\epsilon} \, dt \, d\xi$$

$$= \varepsilon \int_{x_0}^{x} \left\{ e^{-\xi}\xi^{-2+\epsilon} \left[t^{1-\epsilon} e^t \big|_{x_0}^{\xi} - (1-\varepsilon) \int_{x_0}^{\xi} e^t t^{-\epsilon} \, dt \right] \right\} d\xi \tag{68}$$

$$< \varepsilon \ln x - \varepsilon \ln x_0 - \varepsilon x_0 e^{x_0} \int_{x_0}^{x} e^{-\xi}\xi^{-2} \, d\xi - \varepsilon(1-\varepsilon) \int_{x_0}^{x} e^{-\xi}\xi^{-2}(e^{\xi} - e^{x_0}) d\xi \tag{69}$$

$$\begin{aligned} &= \varepsilon \ln x - \varepsilon \ln x_0 - \varepsilon x_0 e^{x_0} \\ &\quad \times \left\{ \frac{1}{x_0} + \ln x_0 + (\gamma - 1) + o(1) - \int_{x}^{\infty} e^{-\xi}\xi^{-2} \, d\xi \right\} \\ &\quad + \varepsilon(1-\varepsilon)\left(\frac{1}{x} - \frac{1}{x_0}\right) + \varepsilon(1-\varepsilon) e^{x_0} \\ &\quad \times \left\{ \frac{1}{x_0} + \ln x_0 + (\gamma - 1) + o(1) - \int_{x}^{\infty} e^{-\xi}\xi^{-2} \, d\xi \right\} \end{aligned} \tag{70}$$

where once again we have used the formula following (17).

Remembering that $x_0 = o(1)$, we expand the exponentials and simplify to find

$$\varepsilon \int_{x_0}^{x} e^{-\xi} \xi^{-2+\varepsilon} \int_{x_0}^{\xi} e^{t} t^{1-\varepsilon}\, dt\, d\xi < \varepsilon \ln x + \{\varepsilon x_0 - \varepsilon(1-\varepsilon) e^{x_0}\} \int_{x}^{\infty} e^{-\xi} \xi^{-2} d\xi$$
$$+ \frac{\varepsilon}{x} - \frac{\varepsilon^2}{x} + \varepsilon(\gamma - 1) + o(\varepsilon). \qquad (71)$$

Using this estimate and the estimate (67) in (26), cancelling the $\varepsilon \ln x$ terms that appear, letting $x \to \infty$, and using the boundary condition (1c), we obtain

$$\lim_{x \to \infty} y(x; \varepsilon) = 1 < (1 + \varepsilon - \gamma\varepsilon) + r(x_0) + I(\infty; \varepsilon; x_0) + \varepsilon(\gamma - 1) + o(\varepsilon), \qquad (72)$$

which simplifies to

$$r(x_0) \geqslant o(\varepsilon), \qquad (73)$$

and our proof is complete.

Proof of Proposition 1. Combining Propositions 3 and 4 yields $r(x; \varepsilon) = o(\varepsilon)$ on $[\varepsilon, \eta_2(\varepsilon)]$. From the definition of $r(x; \varepsilon)$ given in (26), the conclusion of Proposition 1 follows immediately.

III. Conclusion

The correctness of a switchback term in a widely used, nontrivial model equation of Lagerstrom is established rigorously. Furthermore, the results prove the correctness of the domain of validity predicted by the Kaplun–Lagerstrom theory.

References

1. I. D. Chang, Navier–Stokes solutions at large distance froma finite body, *J. Math. Mech.* **10** (1961), 811–876.
2. D. S. Cohen, A. Fokas, and P. A. Lagerstrom, Proof of some asymptotic results for a model equation for low Reynolds number flow, *SIAM J. Appl. Math.* **35** (1978), 187–207.
3. J. D. Cole, "Perturbation Methods in Applied Mathematics," Blaisdell, Waltham, Mass., 1968.
4. G. C. Hsiao, Singular perturbations for a nonlinear differential equation with a small parameter, *SIAM J. Math. Anal.* **4** (1973), 283–301.
5. S. Kaplun, "Fluid Mechanics and Singular Perturbations" (P. A. Lagerstrom, L. N. Howard, and C. S. Liu, Eds.), Academic Press, New York, 1967.

6. P. A. LAGERSTROM, "Méthodes asymptotiques pour l'étude des équations de Navier–Stokes," Lecture Notes, Institut Henri Poincaré, Paris, 1961. [Translated by T. J. Tyson, California Institute of Technology, Pasadena, 1965.]
7. P. A. LAGERSTROM AND R. G. CASTEN, Basic concepts underlying singular perturbation techniques, *SIAM Rev.* **14** (1972), 63–120.
8. A. D. MACGILLIVRAY, On a model equation of Lagerstrom, *SIAM J. Appl. Math.* **34** (1978), 804–812.
9. A. D. MACGILLIVRAY, The existence of an overlap domain for a singular perturbation problem, *SIAM J. Appl. Math.* **36** (1979), 106–114.
10. I. PROUDMAN AND J. R. A. PEARSON, Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder, *J. Fluid Mech.* **2** (1957), 237.
11. S. ROSENBLAT AND J. SHEPHERD, On the asymptotic solution of the Lagerstrom model equation, *SIAM J. Appl. Math.* **29** (1975), 110–120.