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Jacobi polynomial solutions of first kind integral equations for numerical conformal mapping

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Abstract: A numerical method is described for the conformal mapping of simply connected domains whose boundaries contains sharp corner points. The method is based on a first kind integral equation formulation due to Wendland [15]. On each component analytic arc of the boundary, the dominant singularities of the unknown source density ν are incorporated in a Jacobi weight function w, so that ν/w may be approximated by a finite Jacobi polynomial series. The coefficients in these series are determined by collocation. Numerical examples illustrate the accuracy of the proposed technique.

Keywords: Numerical conformal mapping, first kind integral equations, Jacobi polynomials.

1. Introduction

This paper describes a particular technique for computing approximations to the function f which maps conformally a given simply connected domain Ω onto an appropriate canonical domain Δ. The technique consists essentially of a weighted orthogonal polynomial scheme for the numerical solution of the integral equation formulation of Wendland [15], this formulation being a modification of equations due originally to Symm [13] and Gaier [1]. The proposed method is specifically directed towards dealing with singularities in f arising from the presence of corner points on the boundary $\partial\Omega$. The present report should be regarded as a preliminary study, in so far as our chief concern is to investigate the potential accuracy of the proposed method without, at this stage, being unduly concerned with computational efficiency.

In this introductory section we define the mapping problem, explain how a boundary integral representation for f may be obtained and derive the integral equation system of Wendland [15]. At the end of this section we explain our approach to the treatment of corner singularities and outline the contents of the rest of the paper. Let $\partial\Omega \equiv \bigcup_{k=1}^{N}\Gamma_k$ be a piecewise analytic Jordan curve in the complex z-plane whose component anaytic arcs $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ are defined by

$$\Gamma_k \equiv \{z\colon z = \zeta_k(t),\ -1 \leqslant t \leqslant 1\}, \quad k = 1(1)N, \tag{1.1}$$

where ζ_k is analytic on a domain containing $[-1, 1]$ and satisfies

$$\zeta_k'(t) \neq 0, \qquad -1 \leqslant t \leqslant 1, \quad k = 1(1)N. \tag{1.2}$$

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We assume that these arcs are numbered consecutively, so that $\zeta_k(1)=\zeta_{k+1}(-1)$, and that the positive direction on each Γ_k keeps $\Omega_{\mathrm{I}} \equiv \mathrm{Int}(\partial\Omega)$ on the left.

Let Ω denote one of the two simply connected domains whose boundary is $\partial\Omega$, i.e. either $\Omega \equiv \Omega_{\mathrm{I}}$ or $\Omega \equiv \Omega_{\mathrm{E}} \equiv \mathrm{Ext}(\partial\Omega)$, and let f denote the function which maps conformally Ω onto the domain Δ in the complex w-plane, where

$$\Delta \equiv \begin{cases} \{w:\ |w|<1\}, & \Omega \equiv \Omega_{\mathrm{I}}, \\ \{w:\ |w|>1\}, & \Omega \equiv \Omega_{\mathrm{E}}. \end{cases} \tag{1.3}$$

Also let g: $\Delta \to \Omega$ be the inverse function to f.

We assume without loss of generality that the origin lies in Ω_{I}. In the case $\Omega \equiv \Omega_{\mathrm{I}}$ we assume that

$$f(0)=0, \quad f'(0)=b^{-1}>0 \tag{1.4}$$

whilst for the case $\Omega \equiv \Omega_{\mathrm{E}}$ we assume

$$\lim_{z\to\infty} f(z)/z = c^{-1} > 0. \tag{1.5}$$

The restrictions $b>0$, $c>0$ impose particular orientations on the domain Δ. Of course, b and c are domain constants whose actual values are initially unknown. In particular, c is a practically important constant known variously as the capacity of $\partial\Omega$, the transfinite diameter of $\overline{\Omega}_{\mathrm{I}}$ or the outer radius of Ω_{I}. Similarly, b is known as the conformal or inner radius of Ω_{I}.

The above mapping function f exists uniquely, is analytic almost everywhere in $\overline{\Omega}$ and is continuous on $\partial\Omega$; see Henrici [4, Section 5.10–5.11]. The only finite points in $\overline{\Omega}$ at which f may fail to be analytic are sharp corner points on $\partial\Omega$, at which points f generally has a branch point singularity. In particular $|f'(z)|$ is unbounded at any corner point on $\partial\Omega$ where the angle interior to Ω is re-entrant.

The boundary correspondence function θ_k associated with the arc Γ_k is defined by

$$\theta_k(t)=\arg(f(\zeta_k(t))), \quad k=1(1)N, \tag{1.6}$$

where $\arg(\cdot)$ is a continuous argument. Hence, it follows that

$$f(\zeta_k(t))=\exp(\mathrm{i}\theta_k(t)), \quad k=1(1)N, \tag{1.7}$$

and

$$g(\exp(\mathrm{i}\theta_k(t)))=\zeta_k(t), \quad k=1(1)N. \tag{1.8}$$

In order to derive a boundary integral representation for f we follow an approach suggested by Henrici [5] and consider the function K defined by

$$K(z)=\frac{1}{2\pi\mathrm{i}}\int_{|w|=1}\frac{\log(z-g(w))}{w}\,\mathrm{d}w, \tag{1.9}$$

where $\log(z-\zeta)$ is a continuous function of $\zeta=g(w)\in\partial\Omega$. Parametric equations for each arc $[\theta_k(-1),\ \theta_k(1)]$, $k=1(1)N$, on $|w|=1$ are given by (1.7) and by substituting these, together with (1.8), into (1.9) we obtain the expression

$$K(z)=\frac{1}{2\pi}\sum_{k=1}^{N}\int_{-1}^{1}\theta_k'(t)\log(z-\zeta_k(t))\,\mathrm{d}t. \tag{1.10}$$

The integrals (1.9) may be evaluated explicitly by elementary complex integration techniques to obtain the results

$$K(z)=\begin{cases}\log(z/f(z))+\mathrm{i}\theta_N(1), & z\neq 0, \quad \Omega\equiv\Omega_\mathrm{I}, \quad z\in\bar{\Omega},\\ \ln b+\mathrm{i}\theta_N(1), & z=0, \quad \Omega\equiv\Omega_\mathrm{I},\\ \ln c+\log f(z), & \qquad\quad \Omega\equiv\Omega_\mathrm{E}, \quad z\in\bar{\Omega},\end{cases} \tag{1.11}$$

see [6]. Clearly, by combining (1.10) and (1.11) we produce a representation for f in terms of the boundary integrals (1.10); for later convenience we write this representation in the form

$$f(z)=\begin{cases} z\exp(-\gamma+\mathrm{i}\theta_N(1)-K(z)), & \Omega\equiv\Omega_\mathrm{I}, \quad z\in\bar{\Omega},\\ \exp(\gamma+K(z)), & \Omega\equiv\Omega_\mathrm{E}, \quad z\in\bar{\Omega},\end{cases} \tag{1.12}$$

where

$$K(z)=\sum_{k=1}^{N}\int_{-1}^{1}\nu_k(t)\log(z-\zeta_k(t))\,\mathrm{d}t \tag{1.13}$$

and

$$\nu_k(t)=\theta_k'(t)/2\pi, \qquad \gamma=\begin{cases}0, & \Omega\equiv\Omega_\mathrm{I},\\ -\ln c, & \Omega\equiv\Omega_\mathrm{E}.\end{cases} \tag{1.14}$$

Since $|f(\zeta_j(\tau))|=1$, $-1\leqslant\tau\leqslant 1$, $j=1(1)N$, it follows from (1.12)–(1.13) that $\nu_1, \nu_2,\ldots,\nu_N$, γ satisfy the Fredholm first kind integral equation system

$$\sum_{k=1}^{N}\int_{-1}^{1}\nu_k(t)\ln|\zeta_j(\tau)-\zeta_k(t)|\,\mathrm{d}t+\gamma=\begin{cases}\ln|\zeta_j(\tau)|, & \Omega\equiv\Omega_\mathrm{I},\\ 0, & \Omega\equiv\Omega_\mathrm{E},\end{cases} \tag{1.15a}$$

$$-1\leqslant\tau\leqslant 1, \quad j=1(1)N,$$

whilst (1.14) implies that the side condition

$$\sum_{k=1}^{N}\int_{-1}^{1}\nu_k(t)\,\mathrm{d}t=1 \tag{1.15b}$$

is also satisfied. As remarked earlier, the system (1.15) is due to Wendland [15] and is a modification of the earlier integral equation formulations of Symm [13] and Gaier [1]. The advantage of the system (1.15) is that it always has the unique solution (1.14) whereas the equations considered in [1,13] have a unique solution only if $c\neq 1$. Clearly, if (1.15) can be solved for $\nu_1, \nu_2,\ldots,\nu_N$, γ then equations (1.12)–(1.13) enable $f(z)$ to be computed for all $z\in\bar{\Omega}$.

In [9] we proposed a technique for dealing with corner singularities in which each density function $\nu_k(t)$, $k=1(1)N$, is approximated either by a cubic spline, when $t\in[-l_k, u_k]$ (for certain numbers l_k, u_k satisfying $0<l_k$, $u_k<1$), or by appropriate singular functions, when $t\in[-1, -l_k]\cup[u_k, 1]$, these two types of approximation being blended together with C^1 continuity at the points $t=-l_k$, u_k. The technique of [9] has proven to be capable of achieving a high degree of accuracy in approximating f in all examples to which it has been applied. However, the use in [9] of two families of basis functions for the approximation of the densities leads to computer programs of some logical complexity.

In the present work we propose a somewhat simpler approach to the problem of corner singularities. In this approach the dominant end point singularities in $\nu_k(t)$ at $t = \pm 1$ are incorporated in a single weight function w_k and the smoother quotient function $\phi_k = \nu_k / w_k$ is approximated by a linear combination of the orthogonal polynomials associated with w_k. In Section 2 we identify the weight functions w_k, $k = 1(1)N$; these turn out to be classical Jacobi weight functions, In Section 3 we describe the proposed numerical method based on Jacobi polynomial expansions. In Section 4 we give two simple numerical examples which demonstrate that the proposed method gives accurate approximations to mapping functions f with serious corner singularities.

2. Corner singularities and Jacobi polynomial expansions

In this section let Γ denote a typical component analytic arc on $\partial\Omega$ and let ν, θ and ζ denote the associated density, boundary correspondence and parametric functions. The boundary correspondence function may be eliminated from (1.6) and (1.14) to obtain

$$\nu(t) = -\mathrm{i}\bar{f}(z) f'(z) \zeta'(t)/2\pi, \quad z = \zeta(t). \tag{2.1}$$

This equation demonstrates that corner singularities in $f(z)$ at $z = \zeta(\pm 1)$ induce end point singularities in $\nu(t)$ at $t = \pm 1$, the singularities in ν being one order stronger than those in f. It may also be noted from (2.1) that our assumption regarding the analyticity of $\zeta(t)$ ensures that $\nu \in C^\infty(-1, 1)$. Asymptotic expansions for $\nu(t)$ in the vicinity of $t = \pm 1$ are given in detail in [9]. For our present purposes we require only the dominant singular behaviour of ν near $t = \pm 1$, which is simply described as follows.

Let $\lambda\pi$, $\mu\pi$ denote respectively the angles interior to Ω at the corner points $\zeta(1)$, $\zeta(-1)$ and assume

$$0 < \lambda, \mu < 2. \tag{2.2}$$

Then it follows from results given in [9] that

$$\nu(t) = \begin{cases} (1-t)^\alpha (a + \mathrm{o}(1)), & \text{as } t \to 1 - 0, \\ (1+t)^\beta (b + \mathrm{o}(1)), & \text{as } t \to -1 + 0, \end{cases} \tag{2.3}$$

where a, b are strictly positive constants and

$$\alpha = -1 + \lambda^{-1}, \qquad \beta = -1 + \mu^{-1}. \tag{2.4}$$

We observe that the most serious singularity occurs at a re-entrant corner. For example, if $\lambda > 1$ then $\alpha < 0$ so that $\mathrm{Lim}_{t \to 1-0} \nu(t) = +\infty$.

The expression (2.3) suggests that we consider representing ν in the form

$$\nu(t) = (1-t)^\alpha (1+t)^\beta \phi(t), \quad -1 \leqslant t \leqslant 1, \tag{2.5}$$

where, from (2.1) and (2.3), $\phi \in C[-1, 1]$. The appearance of the Jacobi weight function in (2.5) further suggests that we consider expanding ϕ in a Jacobi polynomial series; i.e.

$$\phi(t) = \sum_{n=0}^{\infty} \phi_n p_n^{(\alpha,\beta)}(t) \tag{2.6}$$

where

$$\phi_n = \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta} p_n^{(\alpha,\beta)}(t)\phi(t)\,\mathrm{d}t \tag{2.7}$$

and $p_n^{(\alpha,\beta)}$, $n = 0, 1, 2, \ldots$, denote the orthonormal Jacobi polynomials; see Szegö [14, §4.3]. The above expansion has the following properties:

(i) The series (2.6) is uniformly convergent on

$[-1+\varepsilon, 1-\varepsilon]$ for any fixed ε, $0 < \varepsilon < 1$.

(ii) If on each arc Γ an m-term partial sum of (2.6) is used to approximate ϕ, then (1.12), (1.13) and (2.5) generate an approximation f_m to f. It may be shown that $f_m(z) \to f(z)$ as $m \to \infty$ and that this convergence is either uniform on $\bar{\Omega}$ if $\Omega \equiv \Omega_{\mathrm{I}}$ or uniform on any compact subset of Ω if $\Omega \equiv \Omega_{\mathrm{E}}$.

These results are proved in [7].

We observe that the boundary correspondence function θ associated with Γ may be determined from (1.14), (2.5), (2.6) by using term-by-term integration. For this integration we need the result

$$(1-t)^{\alpha}(1+t)^{\beta} p_n^{(\alpha,\beta)}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(1-t)^{\alpha+1}(1+t)^{\beta+1} p_{n-1}^{(\alpha+1,\beta+1)}(t)}{(n(n+\alpha+\beta+1))^{1/2}}\right], \quad n \geqslant 1, \tag{2.8}$$

which is a generalisation of the Rodrigues' formula for the orthonormal Jacobi polynomials; see Szegö [14, eq. (4.10.1)]. We conclude that

$$\begin{aligned}\theta(\tau) = {} & \theta(-1) + 2\pi\phi_0 \int_{-1}^{\tau} (1-t)^{\alpha}(1+t)^{\beta} p_0^{(\alpha,\beta)}(t)\,\mathrm{d}t \\ & -(1-\tau)^{\alpha+1}(1+\tau)^{\beta+1} \sum_{n=1}^{\infty} \frac{\phi_n p_{n-1}^{(\alpha+1,\beta+1)}(\tau)}{(n(n+\alpha+\beta+1))^{1/2}}\end{aligned} \tag{2.9}$$

where the integral term in this expression is a multiple of the incomplete beta function. One consequence of (2.9), which also follows directly from (2.5)–(2.6) by the orthonormality of the $p_n^{(\alpha,\beta)}$, is that

$$\theta(1) = \theta(-1) + 2\pi\phi_0 h(\alpha, \beta), \tag{2.10}$$

where

$$h(\alpha, \beta) = \left(2^{\alpha+\beta+1} B(\alpha+1, \beta+1)\right)^{1/2} \tag{2.11}$$

and B denotes the usual complete beta function. Thus, the spacing on the unit circle of the images under f of the end points of the arc Γ depends only on the leading coefficient in the series (2.6).

3. The numerical method

In view of the observations made in the previous section we propose a numerical method for the solution of (1.15) whereby $\nu_1, \nu_2, \ldots, \nu_N$ are each represented in the form (2.5) and the corresponding quotient functions $\phi_1, \phi_2, \ldots, \phi_N$ are approximated by the truncated Jacobi polynomial series

$$\tilde{\phi}_k = \sum_{n=0}^{d_k} \tilde{\phi}_{k,n} p_n^{(\alpha_k,\beta_k)}(t), \quad k = 1(1)N. \tag{3.1}$$

Here α_k, β_k are the values obtained from (2.4) for each arc Γ_k, $k = 1(1)N$. We note that in the case where $N = 1$ and $\partial\Omega$ has no corner points, this scheme reduces to a Legendre series approximation for the single density function $\nu \equiv \nu_1$.

The approximate Jacobi coefficients $\tilde{\phi}_{k,n}$ and an approximation $\tilde{\gamma}$ to the domain constant γ are determined from (1.15) by collocation, as follows. Let the parameter values of the collocation points on Γ_k be τ_{kn} where

$$-1 < \tau_{kn} < 1, \qquad k = 1(1)N, \quad n = 0(1)d_k; \tag{3.2}$$

i.e. we do not collocate at corner points on $\partial\Omega$ and the number of collocation points on each arc Γ_k is the same as the number of unknown coefficients in the expansion (3.1) associated with Γ_k. Substituting (3.1) into (1.15) and collocating at the points (3.2) yields the approximating linear system

$$\begin{aligned} &\sum_{k=1}^{N} \sum_{n=0}^{d_k} A_{jkmn}\tilde{\phi}_{kn} + \tilde{\gamma} = b_{jm}, \qquad j = 1(1)N, \quad m = 0(1)d_j, \\ &\sum_{k=1}^{N} h_k\tilde{\phi}_{k0} = 1 \end{aligned} \tag{3.3}$$

where

$$A_{jkmn} = \int_{-1}^{1} (1-t)^{\alpha_k}(1+t)^{\beta_k} p_n^{(\alpha_k,\beta_k)}(t) \ln|\zeta_j(\tau_{jm}) - \zeta_k(t)| \, \mathrm{d}t, \tag{3.4}$$

$$b_{jm} = \begin{cases} \ln|\zeta_j(\tau_{jm})|, & \Omega \equiv \Omega_{\mathrm{I}}, \\ 0, & \Omega \equiv \Omega_{\mathrm{E}}, \end{cases} \tag{3.5}$$

$$h_k = h(\alpha_k, \beta_k) \tag{3.6}$$

and h is defined by (2.11). Equations (3.3) constitute a square linear system for the $(N + 1 + \sum_{k=1}^{N} d_k)$ unknowns $\tilde{\gamma}$, $\tilde{\phi}_{kn}$, $k = 1(1)N$, $n = 0(1)d_k$.

In general, the coefficients A_{jkmn} cannot be evaluated analytically so that approximate quadrature methods must be used. If $j \neq k$ then it follows from (3.2) that the logarithmic kernel in (3.4) is non-singular. Therefore we estimate A_{jkmn}, $j \neq k$, by using a single high order Gauss–Jacobi quadrature formula associated with the weight $(1-t)^{\alpha_k}(1+t)^{\beta_k}$.

For the evaluation of A_{jkmn} in the singular case $j = k$, we have to approximate integrals of the type

$$I_n(\tau) = \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta} p_n^{(\alpha,\beta)}(t) \ln|\zeta(\tau) - \zeta(t)| \, \mathrm{d}t. \tag{3.7}$$

For this, we introduce the decomposition

$$I_n(\tau) = R_n(\tau) + S_n(\tau), \tag{3.8}$$

where

$$R_n(\tau) = \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta} p_n^{(\alpha,\beta)}(t) \ln|\zeta[\tau, t]| \, \mathrm{d}t, \tag{3.9}$$

$$S_n(\tau) = \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta} p_n^{(\alpha,\beta)}(t) \ln|\tau - t| \, \mathrm{d}t, \tag{3.10}$$

and

$$\zeta[\tau, t] = \begin{cases} (\zeta(\tau) - \zeta(t))/(\tau - t), & \tau \neq t, \\ \zeta'(\tau), & \tau = t. \end{cases} \tag{3.11}$$

In view of (1.2) and the assumed analyticity of each parametric function ζ, it follows that the logarithmic kernel in (3.9) is infinitely differentiable with respect to t on $[-1, 1]$. Hence we estimate R_n by using an appropriate Gauss–Jacobi quadrature formula, as described above. It is of interest to note that if $\zeta(t)$ is linear in t then $\zeta[\tau, t]$ is a constant, say $\zeta'(0)$. Therefore, by the orthogonality of the Jacobi polynomials, (3.9) gives

$$R_n = \begin{cases} h(\alpha, \beta) \ln|\zeta'(0)|, & n = 0, \\ 0, & n \geq 1. \end{cases} \tag{3.12}$$

For example, if $\partial\Omega$ is a polygon then (3.12) may be used exclusively.

Although the singular integrals (3.10) are independent of the parametric function ζ, they cannot generally be evaluated analytically as a finite combination of elementary functions. However, these integrals are closely related to Jacobi functions of the second kind and may be shown to satisfy a three-term recurrence formula of the type

$$\begin{aligned} S_2(\tau) &= (A_1\tau - B_1)S_1(\tau) - C_1, \\ S_{n+1}(\tau) &= (A_n\tau - B_n)S_n(\tau) - C_nS_{n-1}(\tau), \quad n \geq 2. \end{aligned} \tag{3.13}$$

In order to initiate the recurrence sequence, $S_0(\tau)$ and $S_1(\tau)$ are computed to high accuracy directly from (3.10) by using the NAG library routine DØ1APF; see [12]. We note that the formula (3.13) is reasonably stable in that it does not possess a minimum solution in the sense described by Gautschi [2]; see [8] for a discussion of some properties of the integrals (3.10) and formula (3.13).

Regarding the choice of collocation points, we observe that if the full infinite series expansions corresponding to (3.1) are substituted into (1.15), then the exact Jacobi coefficients ϕ_{kn}, $k = 1(1)N$, $n = 0, 1, 2, \ldots$, satisfy the linear system

$$\begin{aligned} &\sum_{k=1}^{N} \sum_{n=0}^{d_k} A_{jkmn}\phi_{kn} + \gamma + E_{jm} = b_{jm}, \qquad j = 1(1)N, \quad m = 0(1)d_j, \\ &\sum_{k=1}^{N} h_k\phi_{k0} = 1, \end{aligned} \tag{3.14}$$

where

$$E_{jm} = \sum_{k=1}^{N} \sum_{n=d_k+1}^{\infty} A_{jkmn}\phi_{kn}. \tag{3.15}$$

Comparing (3.3) and (3.14) we see that an optimum choice for the collocation parameters is one which minimises $\max_{j,m}|E_{jm}|$. The best we can suggest at the moment is to choose the collocation points so as to reduce the magnitude of the leading term

$$A_{jjm,d_j+1}\phi_{j,d_j+1} = \phi_{j,d_j+1}\int_{-1}^{1} (1-t)^{\alpha_j}(1+t)^{\beta_j} P_{d_j+1}^{(\alpha_j,\beta_j)}(t) \ln|\zeta_j(\tau_{jm}) - \zeta_j(t)| \, dt \tag{3.16}$$

of the series (3.15). If τ_{jm}, $m = 0(1)d_j$, are the zeros of $p_{d_j+1}^{(\alpha_j,\beta_j)}(t)$ then the integrand in (3.16) is continuous on $(-1, 1)$ and is actually zero at $t = \tau_{jm}$. This choice of collocation points will therefore have the effect of significantly reducing the magnitude of the leading contribution to E_{jm}.

We observe that the inner radius b and the capacity c are estimated by

$$\tilde{b} = \exp\left(\sum_{k=1}^{N} \sum_{h=0}^{d_k} \tilde{\phi}_{kn} \int_{-1}^{1} (1-t)^{\alpha_k}(1+t)^{\beta_k} p_n^{(\alpha_k,\beta_k)}(t) \ln|\zeta_k(t)| \, dt \right), \quad \Omega \equiv \Omega_1, \tag{3.17}$$

$$\tilde{c} = \exp(-\tilde{\gamma}), \quad \Omega \equiv \Omega_E,$$

where the integrals in (3.17) may be approximated by the same Gauss–Jacobi formulae which are used for the non-singular integrals of (3.4).

4. Numerical examples

In this section we present two examples to illustrate the application of the method described in Section 3. We consider only domains for which the exact mapping function is known and may be obtained from Kober [10]. The accuracy of the computed mapping is measured by the quantity

$$E_\theta = \max_{\substack{k=1(1)N \\ -1 \leqslant t \leqslant 1}} |\theta_k(t) - \tilde{\theta}_k(t)|, \tag{4.1}$$

where θ_k is the exact argument (1.6) and $\tilde{\theta}_k$ is the approximation obtained by using the truncated series (3.1) in (2.9). We estimate E_θ by means of

$$\tilde{E}_\theta = \max_{\substack{k=1(1)N \\ r=0(1)50}} |\theta_k(-1+0.04r) - \tilde{\theta}_k(-1+0.04r)|. \tag{4.2}$$

The values which are given to the degrees d_k of the approximating polynomials $\tilde{\phi}_k$ are, of course, of crucial importance in determining the accuracy of the resulting approximations. Our principal aim in this section is to indicate the level of accuracy that the proposed method is capable of achieving and, therefore, we go to some lengths to estimate the best values for d_k, $k = 1(1)N$, as follows. Given initial values for $d_1, d_2, \ldots, d_N$ we collocate at the zeros of $p_{d_k+1}^{(\alpha_k,\beta_k)}$ on each arc Γ_k, $k = 1(1)N$. From the resulting solution to (3.3) we compute $\tilde{E}_\theta$, which value occurs at some point on a certain arc Γ_{k^*} say. We then increase d_{k^*} by 1 and repeat the above process. In this way a sequence of $\tilde{E}_\theta$ values is generated. The optimum values for $d_1, d_2, \ldots, d_n$ are taken to be those for which $\tilde{E}_\theta$ achieves its minimum value, the investigation being terminated as soon as the sequence of $\tilde{E}_\theta$ values fails to be strictly monotonically decreasing.

In each example, in order to illustrate the effectiveness of collocating at Jacobi points, we also present results obtained by collocating at uniformly distributed points on each arc Γ_k. These results are indicated by the abbreviation J/U. Similarly, in order to demonstrate the effect of using the correct Jacobi weight function, we present results obtained by using Legendre polynomial approximations to the densities $\nu_1, \nu_2, \ldots, \nu_N$ with collocation at the corresponding Legendre points. These results are indicated by the abbreviation L/L.

In each example we make use of symmetry of $\partial\Omega$ so as to reduce the volume of arithmetic and the size of the linear system which is to be solved. The number of equations actually solved is denoted by M. All results are based on the use of 40-point Gauss–Jacobi quadrature formulae for the approximate evaluation of the integrals (3.4), (3.9), (3.17). The weights and abscissae for these formulae are generated by the eigenvalue technique of Golub and Welsch [3]. All computations were performed on a Harris 800 computer using arithmetic correct to approximately 11 significant figures.

Example 1. Interior of a circular sector of radius 1 and angle $\frac{3}{2}\pi$:

$$\Omega \equiv \left\{ z:\ z = z_0 + \rho\, e^{i\psi},\ z_0 = \tfrac{1}{2}e^{-i\pi/4}, 0 < \rho < 1, 0 < \psi < \tfrac{3}{2}\pi \right\}.$$

This region is symmetrical on reflection about the line joining $z = 0$ and $z = z_0$ and has a re-entrant corner of angle $\frac{3}{2}\pi$ at $z = z_0$. The parametric functions and density components on the various arcs of $\partial\Omega$ are as follows:

$$\begin{aligned}
&\zeta_1(t) = z_0 + \tfrac{1}{2}(1+t), && \nu_1(t) = (1-t)(1+t)^{-1/3}\phi_1(t),\\
&\zeta_2(t) = z_0 + \exp(i\tfrac{3}{8}\pi(1+t)), && \nu_2(t) = (1+t)\phi_2(t),\\
&\zeta_3(t) = -i\bar{\zeta}_2(-t), && \nu_3(t) = \nu_2(-t),\\
&\zeta_4(t) = -i\bar{\zeta}_1(-t), && \nu_4(t) = \nu_1(-t).
\end{aligned} \tag{4.3}$$

The above expressions indicate the manner in which we treat symmetric regions. In particular, we adopt the convention that if an axis of symmetry cuts $\partial\Omega$ at a point which is not a sharp corner point, then we treat this point as if it is a 'corner' of interior angle π. This convention allows a simple logical structure to be given to the general computer program for the automatic treatment of symmetric domains.

The optimum degrees for $\tilde{\phi}_1$, $\tilde{\phi}_2$ are found to be $d_1 = 13$, $d_2 = 10$. A selection of numerical results is given in Table 1. The superiority of the proposed method over the J/U and L/L schemes is clear. By comparison we note that the cubic spline-singular function technique of [9] gives $E_\theta \simeq 5.5 \times 10^{-5}$ for the solution of 73 linear equations; see [6]. As a further comparison we note that this example is considered by Levin, Papamichael and Sideridis [11], who use a technique based on the theory of Bergman kernel functions, and their results give $E_\theta \simeq 1.7 \times 10^{-4}$. The exact value of the inner radius is $b = 1.5(1 - 2^{-4/3})/(1 + 2^{-4/3})$ and, of course, $\gamma = 0$.

Example 2. Region exterior to the union of two overlapping circles:

$$\Omega \equiv \{z: |z| > 1\} \cap \{z: |z + \sqrt{3}| > 2\}.$$

Table 1

d_1	d_2	M	$\tilde{E}_\theta$	$\lvert\tilde{\gamma}\rvert$	$\tilde{b}$ ($b = 0.64768904$)
4	4	11	5.4×10^{-3}	1.7×10^{-5}	0.64774
6	7	16	3.3×10^{-4}	9.8×10^{-8}	0.6476886
9	9	21	4.2×10^{-5}	1.6×10^{-10}	0.64768898
13	10	26	6.8×10^{-6}	8.7×10^{-9}	0.64768897
J/U: 13	10	26	1.5×10^{-4}	1.3×10^{-6}	0.64768912
L/L: 13	10	26	2.5×10^{-3}	2.6×10^{-7}	0.64775

Table 2

d_1	d_2	M	$\tilde{E}_\theta$	$\tilde{c}$ ($c = 2.121320344$)
4	4	11	1.7×10^{-3}	2.12134
4	10	17	2.3×10^{-5}	2.121320339
7	13	23	3.2×10^{-6}	2.121320345
9	17	29	5.9×10^{-7}	2.121320345
11	21	35	2.2×10^{-7}	2.121320347
J/U: 11	21	35	2.6×10^{-6}	2.121320336
L/L: 11	21	35	4.9×10^{-5}	2.12132031

This region is symmetrical on reflection about the real axis and $\partial\Omega$ has corners of interior angle $\frac{2}{3}\pi$ at $z = \pm\mathrm{i}$. The parametric functions and density components on the various arcs of $\partial\Omega$ are as follows:

$$\begin{aligned}
\zeta_1(t) &= \exp(\mathrm{i}\tfrac{1}{4}\pi(1+t)), & \nu_1(t) &= (1-t)^{1/2}\phi_1(t),\\
\zeta_2(t) &= -\sqrt{3} + 2\exp(\mathrm{i}\tfrac{1}{12}\pi(5t+7)), & \nu_2(t) &= (1+t)^{1/2}\phi_2(t),\\
\zeta_3(t) &= \bar{\zeta}_2(-t), & \nu_3(t) &= \nu_2(-t),\\
\zeta_4(t) &= \bar{\zeta}_1(-t), & \nu_4(t) &= \nu_1(-t).
\end{aligned}$$

The optimum degrees for $\tilde{\phi}_1$, $\tilde{\phi}_2$ are found to be $d_1 = 11$, $d_2 = 21$ and a selection of numerical results is given in Table 2. Although the singularities in $\nu_1, \ldots, \nu_4$ are an order weaker than the singularity in Example 1, the superiority of the proposed method over the J/U and L/L schemes remains clear. The exact value of the capacity of $\partial\Omega$ is $c = 3/\sqrt{2}$.

5. Conclusions

The numerical results represented in the previous section demonstrate that the proposed technique is capable of achieving a high level of accuracy for domains with sharp corners. Furthermore, the computer implementation of the method is relatively straightforward.

We recognise that there are various features of the method as presented that need further development in order to produce a more efficient computational scheme. For example, the principal difficulty for the effective implementation of the method concerns the determination of the optimum degrees of the approximating Jacobi polynomials. In this connection, we note the orthogonal polynomial property that, if the degree of the approximating polynomial on a particular arc is increased and additional collocation points are chosen on this arc, then the coefficient matrix of the original collocation equation system remains as a submatrix in the new larger linear collocation system. It should be possible to exploit this property in an adaptive scheme for determining the optimum degrees of the approximating polynomials.

In conclusion, we believe that the method has the potentiality to be developed into an accurate and reasonably efficient algorithm for automatic conformal mapping, and this development is being actively pursued.

References

[1] D. Gaier, Integralgleichungen erster Art und konforme Abbildung, *Math. Z.* **147** (1976) 113–129.
[2] W. Gautschi, Computational aspects of three-term recurrence relations, *SIAM Rev.* **9** (1967) 24–82.
[3] G.H. Golub and J.H. Welsch, Calculation of Gauss quadrature rules, *Math. Comput.* **23** (1969) 221–230.
[4] P. Henrici, *Applied and Computational Complex Analysis* (Wiley, New York, 1974).
[5] P. Henrici, Lecture at the workshop on computational problems in complex analysis, Stanford University, 1981; Communicated by N. Papamichael.
[6] D.M. Hough, The use of splines and singular functions in an integral equation method for conformal mapping, Ph. D. Thesis, Brunel University, 1983.
[7] D.M. Hough, On the convergence of Jacobi polynomial expansions in conformal mapping, Technical Report, Department of Mathematics, Coventry Polytechnic, 1984.
[8] D.M. Hough, The evaluation of certain Jacobi polynomial integrals arising in potential theory, Technical Report, Department of Mathematics, Coventry Polytechnic, 1984.
[9] D.M. Hough and N. Papamichael, An integral equation method for the numerical conformal mapping of interior, exterior and doubly-connected domains, *Numer. Math.* **41** (1983) 287–307.
[10] H. Kober, *Dictionary of Conformal Representation* (Dover, New York, 1957).
[11] D. Levin, N. Papamichael and A. Sideridis, The Bergman kernel method for the numerical conformal mapping of simply connected domains, *J. Inst. Math. Appl.* **22** (1978) 171–187.
[12] *NAG Fortran Library Manual*, Mark 9, Volume 1.
[13] G.T. Symm, An integral equation method in conformal mapping, *Numer. Math.* **10** (1966) 250–258.
[14] G. Szegö, *Orthogonal Polynomials* (American Mathematical Society, New York, 1975).
[15] W.L. Wendland, On Galerkin collocation methods for integral equations of elliptic boundary value problems, in: J. Albrecht and L. Collatz, Eds., *Numerical Treatment of Integral Equations* (Birkhäuser Verlag, Basel, 1980).