ELSEVIER

Journal of Computational and Applied Mathematics 55 (1994) 51–67

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Positive quartic, monotone quintic C^2-spline interpolation in one and two dimensions

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Received 25 August 1992; revised 25 June 1993

Abstract

This paper is concerned with shape-preserving interpolation of discrete data by polynomial splines. We show that positivity can be always preserved by quartic C^2-splines and monotonicity by quintic C^2-splines. This is proved for one-dimensional interpolation as well as for two-dimensional interpolation on rectangular grids.

Keywords: Quartic C^2-splines; Quintic C^2-splines; Sufficient positivity and monotonicity conditions; Existence of shape-preserving splines; One- and two-dimensional interpolation

1. Introduction

Let a data set $D_n = \{(x_i, z_i)\colon i = 0(1)n\}$ be given on the one-dimensional grid

$$\Delta_n\colon \quad x_0 < x_1 < \cdots < x_n .$$

This set is called to be in positive position if

$$z_i \geqslant 0, \quad i = 0(1)n, \tag{1.1}$$

and in monotone position if

$$z_{i-1} \leqslant z_i, \quad i = 1(1)n. \tag{1.2}$$

Analogously, a data set $D_{n,m} = \{(x_i, y_j, z_{i,j})\colon i = 0(1)n, j = 0(1)m\}$ on the two-dimensional grid

$$\Delta_{n,m}\colon \quad x_0 < x_1 < \cdots < x_n, \quad y_0 < y_1 < \cdots < y_m$$

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SSDI 0377-0427(93)E0177-N

is said to be in positive position if

$$z_{i,j} \geqslant 0, \quad i = 0(1)n,\ j = 0(1)m, \tag{1.3}$$

and in monotone position if

$$\begin{aligned} z_{i-1,j} &\leqslant z_{i,j}, \quad i = 1(1)n,\ j = 0(1)m, \\ z_{i,j-1} &\leqslant z_{i,j}, \quad i = 0(1)n,\ j = 1(1)m. \end{aligned} \tag{1.4}$$

In this paper we are interested mainly in the following existence problem. Are there polynomial splines s defined on Δ_n or on $\Delta_{n,m}$ which interpolate the data set D_n or $D_{n,m}$ and which, in addition, preserve the shape of D_n or $D_{n,m}$. In one dimension, the first positive result concerning this topic is given in [4]. There it is shown that monotone interpolation is always possible with cubic C^1-splines. The same holds true for positive interpolation due to [13]; cf [3]. In contrast to this, convex interpolation may fail. In an earlier paper [7] a strict convex data set D_n, $n \geqslant 4$, is constructed such that all cubic C^1-interpolants are not convex on $[x_0, x_n]$. With quadratic C^1-splines also positive and monotone interpolation is in general not realizable; see [11, 12].

Now, in the present paper we are concerned with shape-preserving C^2-interpolation. It is shown that positive interpolation is always successful with quartic C^2-splines. Because positive interpolation may fail when applying cubic C^2-splines this result cannot be improved. Analogously, quintic C^2-splines are that of lowest degree for which monotone interpolation is always possible. This last result, however without optimality, can also be found in [3]. In addition, we are in a position to extend these properties to the two-dimensional C^2-interpolation on rectangular grids.

For convex interpolation we mention the highly negative result from [6]. For all spaces of polynomial C^1- (or C^2-) splines of fixed degree there exist convex data sets D_n, $n \geqslant 4$, such that all spline interpolants fail to be convex on $[x_0, x_n]$. Moreover, in [6] this result is shown to be valid even for convex interpolation on finite-dimensional linear subspaces of C^1-functions.

The splines used for proving the existence theorems from above are in general not the best ones from geometrical point of view. We get visually more pleasing interpolants, e.g., by minimizing the mean curvature subject to the shape preservation constraints. In the one-dimensional case this optimization approach is elaborated in detail for the types of shape-preserving interpolation of interest here, while in the two-dimensional case several of the arising questions are open until now.

For surveys on shape-preserving interpolation the interested reader is referred, e.g. to the papers [1, 5, 8] and to the books [15, 16].

2. Shape-preserving interpolation with quartic C^2-splines in one dimension

The problem here of interest is to consider C^2-splines s on Δ_n which satisfy the interpolation condition

$$s(x_i) = z_i, \quad i = 0(1)n, \tag{2.1}$$

and which are nonnegative, monotone, or convex on $I = [x_0, x_n]$.

2.1. C^2-continuity of splines

Here it is of advantage to define a spline s on Δ_n, not necessarily a quartic, by

$$s(x) = a_i(u)^{\mathrm{T}} S_i, \quad x \in I_i = [x_{i-1}, x_i],\ 0 \leqslant u \leqslant 1, \tag{2.2}$$

with the local variable $u = (x - x_{i-1})/h_i$, $h_i = x_i - x_{i-1}$, and with vectors a_i, S_i, $i = 1(1)n$. Obviously, s is C^0-continuous on I if and only if

$$a_i(1)^{\mathrm{T}} S_i = a_{i+1}(0)^{\mathrm{T}} S_{i+1}, \quad i = 1(1)n-1. \tag{2.3}$$

In the case (2.3) we have C^1-continuity on I if and only if

$$\frac{1}{h_i} a_i'(1)^{\mathrm{T}} S_i = \frac{1}{h_{i+1}} a_{i+1}'(0)^{\mathrm{T}} S_{i+1}, \quad i = 1(1)n-1, \tag{2.4}$$

and in the cases (2.3) and (2.4) the spline s is C^2-continuous on I if and only if

$$\frac{1}{h_i^2} a_i''(1)^{\mathrm{T}} S_i = \frac{1}{h_{i+1}^2} a_{i+1}''(0)^{\mathrm{T}} S_{i+1}, \quad i = 1(1)n-1. \tag{2.5}$$

2.2. Quartic C^2-splines

Quartic splines are obtained if in (2.2) the vector a_i is specified by

$$a_i(u) = c(u; h_i) \tag{2.6}$$

with

$$c(u;h) = \begin{bmatrix} 1-u \\ u \\ 0 \\ 0 \\ 0 \end{bmatrix} + u(1-u) \begin{bmatrix} 1 + u - 3u^2 \\ -1 - u + 3u^2 \\ h + hu - 2hu^2 \\ -hu^2 \\ \frac{1}{2}h^2 u(1-u) \end{bmatrix}, \tag{2.7}$$

and the vector S_i by

$$S_i = \begin{bmatrix} z_{i-1} \\ z_i \\ p_{i-1} \\ p_i \\ P_{i-1} \end{bmatrix}; \tag{2.8}$$

here p_i and P_i are parameters having the geometrical meaning given by (2.10), (2.13).

Because

$$a_i(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad a_i(1) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad a_i'(0) = \begin{bmatrix} 0 \\ 0 \\ h_i \\ 0 \\ 0 \end{bmatrix}, \quad a_i'(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ h_i \\ 0 \end{bmatrix}, \tag{2.9}$$

the conditions (2.3) and (2.4) are always satisfied, and

$$s(x_i) = z_i, \quad s'(x_i) = p_i, \quad i = 0(1)n. \tag{2.10}$$

Further, in view of (2.5) and

$$a_i''(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ h_i^2 \end{bmatrix}, \quad a_i''(1) = \begin{bmatrix} 12 \\ -12 \\ 6h_i \\ 6h_i \\ h_i^2 \end{bmatrix}, \tag{2.11}$$

a quartic spline s is C^2-continuous if and only if

$$12(z_{i-1} - z_i) + 6h_i(p_{i-1} + p_i) + h_i^2(P_{i-1} - P_i) = 0, \quad i = 1(1)n - 1. \tag{2.12}$$

In this case we have

$$s''(x_i) = P_i, \quad i = 0(1)n, \tag{2.13}$$

where P_n is defined by (2.12) for $i = n$.

2.3. *Positivity of quartic C^2-splines*

We are not in a position to give a criterion which is necessary and sufficient for the positivity of quartic splines. But we can derive a condition sufficient only, but sharp enough for our purposes. We substitute $u = \rho/(1 + \rho)$ implying $u \in [0, 1]$ if and only if $\rho \geqslant 0$. Thus, we get

$$(1 + \rho)^4 c(u; h) = e_0 + e_1(h)\rho + e_2(h)\rho^2 + e_3(h)\rho^3 + e_4\rho^4 \tag{2.14}$$

with

$$e_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_1(h) = \begin{bmatrix} 4 \\ 0 \\ h \\ 0 \\ 0 \end{bmatrix}, \quad e_2(h) = \begin{bmatrix} 6 \\ 0 \\ 3h \\ 0 \\ \frac{1}{2}h^2 \end{bmatrix}, \quad e_3(h) = \begin{bmatrix} 0 \\ 4 \\ 0 \\ -h \\ 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.15}$$

Hence, a sufficient condition for the positivity of s, i.e., for $s(x) \geqslant 0$, $x \in I$, reads

$$e_v(h_i)^T S_i \geqslant 0, \quad v = 0(1)4, \; i = 1(1)n. \tag{2.16}$$

These inequalities are equivalent to

$$\begin{aligned} &z_i \geqslant 0, \quad i = 0(1)n, \\ &4z_{i-1} + h_i p_{i-1} \geqslant 0, \qquad 4z_i - h_i p_i \geqslant 0, \\ &12z_{i-1} + 6h_i p_{i-1} + h_i^2 P_{i-1} \geqslant 0, \quad i = 1(1)n. \end{aligned} \tag{2.17}$$

Thus, an interpolating quartic spline (2.2), (2.6)–(2.8) is of C^2-continuity and positive on I if the parameters $p_i, P_i, i = 0(1)n$, satisfy the system (2.12), (2.17) of linear equalities and inequalities.

Now it can be shown that system (2.12), (2.17) is solvable if the data set is in positive position. This is done inductively. At the beginning, let p_0 and P_0 be such that

$$4z_0 + h_1 p_0 \geqslant 0, \qquad 12z_0 + 6h_1 p_0 + h_1^2 P_0 \geqslant 0.$$

The proof is complete if, under the assumption

$$12z_{i-1} + 6h_i p_{i-1} + h_i^2 P_{i-1} \geqslant 0,$$

there exist numbers p_i and P_i which satisfy

$$\begin{aligned} &4z_i + h_{i+1} p_i \geqslant 0, \qquad 4z_i - h_i p_i \geqslant 0, \\ &12z_i + 6h_{i+1} p_i + h_{i+1}^2 P_i \geqslant 0, \\ &12z_i - 6h_i p_i + h_i^2 P_i = 12z_{i-1} + 6h_i p_{i-1} + h_i^2 P_{i-1}. \end{aligned}$$

Obviously, such numbers are

$$p_i = \frac{4z_i}{h_i} \geqslant 0, \qquad P_i = \frac{1}{h_i^2}(12z_{i-1} + 6h_i p_{i-1} + h_i^2 P_{i-1}) + \frac{12z_i}{h_i^2} \geqslant 0.$$

We summarize these considerations in the following proposition.

Proposition 1. *For data sets in positive position the problem of positive one-dimensional interpolation is always solvable with quartic C^2-splines.*

This result does not hold for polynomial C^2-splines of degree lower than four. In this sense Proposition 1 is sharp. Indeed, when using cubic C^2-splines, for the data set $D_5 = \{(0,0),(1,0),(2,0),(3,1),(4,0),(5,0)\}$, e.g., which is in positive position, all interpolants are not nonnegative on the interval $[0,5]$. Further, the set of data sets D_n for which positive interpolation is successful is a closed set. Thus, the complementary set is open, and there exist data sets, in a neighbourhood of the above set D_5, which are even in strict positive position $z_0 > 0, \ldots, z_5 > 0$ such that the corresponding cubic C^2-spline interpolants are not nonnegative everywhere on $[0,5]$.

2.4. Curvature minimization

In general, there exist an infinite number of positive quartic C^2-interpolants. For selecting one of them a choice function is of interest. As usual, here the mean curvature is taken leading to the following program:

$$\text{minimize} \int_{x_0}^{x_n} s''(x)^2 \, dx$$
$$= \sum_{i=1}^{n} \left[\frac{6}{5h_i} \left(p_{i-1} - p_i + \frac{h_i}{12}(P_{i-1} + P_i) \right)^2 + \frac{h_i}{24}(3P_{i-1}^2 - 2P_{i-1}P_i + 3P_i^2) \right] \tag{2.18}$$

subject to (2.12), (2.17).

This is a quadratic program of partially separable structure. It is uniquely solvable, and can be solved effectively, e.g., via dualization. The general dual procedure described in [2] or [10] applies to program (2.18). The details are somewhat lengthy and will not be reproduced here. In the added test examples the splines, called there optsplines, are computed by means of this dual procedure.

2.5. Convexity of quartic splines

Substituting again $u = \rho/(1 + \rho)$ we find for $x \in I_i$,

$$(1+\rho)^2 s''(x) = P_{i-1} + \left(\frac{6}{h_i}(p_i - p_{i-1}) - 2(P_i + P_{i-1}) \right) \rho + P_i \rho^2 \tag{2.19}$$

if the C^2-condition (2.12) is taken into account. We remember a result from paper [13], namely that

$$\alpha + \beta\rho + \gamma\rho^2 \geqslant 0 \quad \text{for all } \rho \geqslant 0 \tag{2.20}$$

if and only if

$$\alpha \geqslant 0, \quad \gamma \geqslant 0, \quad \beta \geqslant -2\sqrt{\alpha\gamma}\,. \tag{2.21}$$

In this way we get the following proposition.

Proposition 2. *A quartic C^2-spline* (2.2), (2.6)–(2.8) *is convex on I if and only if*

$$P_i \geqslant 0,\ i = 0(1)n, \qquad P_{i-1} - \sqrt{P_{i-1}P_i} + P_i \leqslant \frac{3}{h_i}(p_i - p_{i-1}), \quad i = 1(1)n. \tag{2.22}$$

For $n \geqslant 4$ there exist data sets D_n being in strict convex position, i.e.,

$$\frac{z_i - z_{i-1}}{h_i} < \frac{z_{i+1} - z_i}{h_{i+1}}, \quad i = 1(1)n-1,$$

such that system (2.12), (2.22) is not solvable. In other words, for $n \geqslant 4$ the convex interpolation with quartic C^2-splines is not always successful. This holds true even for polynomial C^2-splines of arbitrary but fixed degree. For a proof we consider the data set $D_4 = \{(0,0),(1,0),(2,0),(3,1),(4,2)\}$, e.g., which is in convex position. It is seen straightforwardly that all polynomial C^2-spline interpolants to D_4 are not convex on the interval $[0,4]$. Furthermore, the set of data sets D_n not suitable for convex interpolation is open. Thus, there exist strictly convex data sets D_n, $n \geqslant 4$, which do not allow convex interpolation with polynomial C^2-splines of fixed degree.

2.6. Monotonicity of quartic splines

Because of

$$(1+\rho)^3 s'(x) = p_{i-1} + (3p_{i-1} + h_i P_{i-1})\rho + (3p_i - h_i P_i)\rho^2 + p_i \rho^3, \quad x \in I_i, \tag{2.23}$$

we get the following result by means of the criterion from [13] on the positivity of cubic polynomials. A quartic C^2-spline (2.2),(2.6)–(2.8) is monotone increasing on I if and only if

$$p_i \geqslant 0, \quad i = 0(1)n, \tag{2.24}$$

and

$$3p_{i-1} + h_i P_{i-1} \geqslant 0, \quad 3p_i - h_i P_i \geqslant 0, \qquad i = 1(1)n, \tag{2.25}$$

or

$$\begin{aligned} &36 p_{i-1} p_i \left(P_{i-1}^2 + P_{i-1} P_i + P_i^2 - \frac{3}{h_i}(p_i - p_{i-1})(P_{i-1} + P_i) + \frac{3}{h_i^2}(p_i - p_{i-1})^2 \right) \\ &\quad + 3(p_i P_{i-1} - p_{i-1} P_i)(2h_i P_{i-1} P_i - 3p_i P_{i-1} + 3p_{i-1} P_i) + 4h_i(p_i P_{i-1}^3 - p_{i-1} P_i^3) \\ &\quad - h_i^2 P_{i-1}^2 P_i^2 \geqslant 0, \quad i = 1(1)n. \end{aligned} \tag{2.26}$$

It can be shown that for data D_n, $n \geqslant 3$, even in strict monotone position $z_{i-1} < z_i$, $i = 1(1)n$, system (2.12),(2.24)–(2.26) is not always solvable, i.e., monotone interpolation with quartic C^2-splines is not always successful. Indeed, let $D_3 = \{(0,0),(1,0),(2,1),(3.1)\}$ be a data set which is in monotone position. It follows immediately that the corresponding interpolating quartic C^2-splines are not monotone on $[0,3]$, and in the same way as before we assure the existence of strictly monotone data sets being in a neighbourhood of the above set D_3 for which monotone interpolation with quartic C^2-splines fails.

3. Positive interpolation with quartic C^2-splines in two dimensions

The results from section 2 concerning the positivity now are extended from one-dimensional to two-dimensional interpolation. We are interested in C^2-splines s on the rectangular grid $\Delta_{n,m}$ which interpolate the given data set $D_{n,m}$, i.e.,

$$s(x_i, y_j) = z_{i,j}, \quad i = 0(1)n, \ j = 0(1)m, \tag{3.1}$$

and which are nonnegative on $J = [x_0, x_n] \times [y_0, y_m]$.

3.1. C^2-continuity of bisplines

On the subrectangle $J_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, we define the spline s not necessary a biquartic one by

$$s(x,y) = a_i(u)^{\mathrm{T}} S_{i,j} b_j(v), \quad 0 \leqslant u, v \leqslant 1, \tag{3.2}$$

with the local variables $u = (x - x_{i-1})/h_i$, $h_i = x_i - x_{i-1}$, $v = (y - y_{j-1})/k_j$, $k_j = y_j - y_{j-1}$ and with vectors a_i, b_j and matrices $S_{i,j}$, $i = 1(1)n, j = 1(1)m$. The component functions of the vector a_i, respectively, b_j may be linearly independent.

The spline s is C^0-continuous on J if and only if

$$\begin{aligned} &a_i(1)^{\mathrm{T}} S_{i,j} = a_{i+1}(0)^{\mathrm{T}} S_{i+1,j}, \quad i = 1(1)n-1,\ j = 1(1)m, \\ &S_{i,j} b_j(1) = S_{i,j+1} b_{j+1}(0), \quad i = 1(1)n,\ j = 1(1)m-1. \end{aligned} \tag{3.3}$$

Further, if (3.3) holds we obtain C^1-continuity on J if and only if

$$\begin{aligned} &\frac{1}{h_i} a_i'(1)^{\mathrm{T}} S_{i,j} = \frac{1}{h_{i+1}} a_{i+1}'(0)^{\mathrm{T}} S_{i+1,j}, \quad i = 1(1)n-1,\ j = 1(1)m, \\ &\frac{1}{k_j} S_{i,j} b_j'(1) = \frac{1}{k_{j+1}} S_{i,j+1} b_{j+1}'(0), \quad i = 1(1)n,\ j = 1(1)m-1. \end{aligned} \tag{3.4}$$

In addition, the equalities (3.4) imply the continuity of the mixed derivative $\partial_1 \partial_2 s$ on J. Finally, if (3.3) and (3.4) are assumed, the C^2-continuity of s on J is equivalent to

$$\begin{aligned} &\frac{1}{h_i^2} a_i''(1)^{\mathrm{T}} S_{i,j} = \frac{1}{h_{i+1}^2} a_{i+1}''(0)^{\mathrm{T}} S_{i+1,j}, \quad i = 1(1)n-1,\ j = 1(1)m, \\ &\frac{1}{k_j^2} S_{i,j} b_j''(1) = \frac{1}{k_{j+1}^2} S_{i,j+1} b_{j+1}''(0), \quad i = 1(1)n,\ j = 1(1)m-1, \end{aligned} \tag{3.5}$$

and from (3.5) follows the continuity of the mixed derivatives $\partial_1^2 \partial_2 s$, $\partial_1 \partial_2^2 s$ and $\partial_1^2 \partial_2^2 s$ on J; see [9, 14].

3.2. Biquartic C^2-splines

In (3.2) we now define the vectors a_i and b_j by

$$a_i(u) = c(u; h_i), \qquad b_j(v) = c(v; k_j), \tag{3.6}$$

where c is given by (2.7). The matrix $S_{i,j}$ is set as

$$S_{i,j} = \begin{bmatrix} z_{i-1,j-1} & z_{i-1,j} & q_{i-1,j-1} & q_{i-1,j} & Q_{i-1,j-1} \\ z_{i,j-1} & z_{i,j} & q_{i,j-1} & q_{i,j} & Q_{i,j-1} \\ p_{i-1,j-1} & p_{i-1,j} & r_{i-1,j-1} & r_{i-1,j} & V_{i-1,j-1} \\ p_{i,j-1} & p_{i,j} & r_{i,j-1} & r_{i,j} & V_{i,j-1} \\ P_{i-1,j-1} & P_{i-1,j} & U_{i-1,j-1} & U_{i-1,j} & W_{i-1,j-1} \end{bmatrix}; \tag{3.7}$$

here $p_{i,j}, q_{i,j}, r_{i,j}, P_{i,j}, Q_{i,j}, U_{i,j}, V_{i,j}$ and $W_{i,j}$ are parameters representing derivatives; see (3.8) and (3.15).

In view of (2.9) we find the conditions (3.3) and (3.4) always satisfied, and

$$\begin{aligned} &s(x_i, y_j) = z_{i,j}, \quad \partial_1 s(x_i, y_j) = p_{i,j}, \quad \partial_2 s(x_i, y_j) = q_{i,j}, \\ &\partial_1\partial_2 s(x_i, y_j) = r_{i,j}, \quad i = 0(1)n,\ j = 0(1)m. \end{aligned} \tag{3.8}$$

Using (2.11), for biquartic splines the C^2-condition (3.5) is seen to be equivalent to

$$12(z_{i-1,j} - z_{i,j}) + 6h_i(p_{i-1,j} + p_{i,j}) + h_i^2(P_{i-1,j} - P_{i,j}) = 0, \quad i = 1(1)n-1,\ j = 0(1)m, \tag{3.9}$$

$$12(z_{i,j-1} - z_{i,j}) + 6k_j(q_{i,j-1} + q_{i,j}) + k_j^2(Q_{i,j-1} - Q_{i,j}) = 0, \quad i = 0(1)n,\ j = 1(1)m-1, \tag{3.10}$$

$$12(q_{i-1,j} - q_{i,j}) + 6h_i(r_{i-1,j} + r_{i,j}) + h_i^2(U_{i-1,j} - U_{i,j}) = 0, \quad i = 1(1)n-1,\ j = 0(1)m, \tag{3.11}$$

$$12(p_{i,j-1} - p_{i,j}) + 6k_j(r_{i,j-1} + r_{i,j}) + k_j^2(V_{i,j-1} - V_{i,j}) = 0, \quad i = 0(1)n,\ j = 1(1)m-1, \tag{3.12}$$

$$12(Q_{i-1,j} - Q_{i,j}) + 6h_i(V_{i-1,j} + V_{i,j}) + h_i^2(W_{i-1,j} - W_{i,j}) = 0, \quad i = 1(1)n-1,\ j = 0(1)m-1, \tag{3.13}$$

$$12(P_{i,j-1} - P_{i,j}) + 6k_j(U_{i,j-1} + U_{i,j}) + k_j^2(W_{i,j-1} - W_{i,j}) = 0, \quad i = 0(1)n-1,\ j = 1(1)m-1. \tag{3.14}$$

Moreover, if (3.9)–(3.14) are satisfied, also the mixed derivatives $\partial_1^2\partial_2 s$, $\partial_1\partial_2^2 s$ and $\partial_1^2\partial_2^2 s$ are continuous on J, and

$$\begin{aligned} &\partial_1^2 s(x_i, y_j) = P_{i,j}, \quad \partial_2^2 s(x_i, y_j) = Q_{i,j}, \quad \partial_1^2\partial_2 s(x_i, y_j) = U_{i,j}, \\ &\partial_1\partial_2^2 s(x_i, y_j) = V_{i,j}, \quad \partial_1^2\partial_2^2 s(x_i, y_j) = W_{i,j}, \quad i = 0(1)n,\ j = 0(1)m; \end{aligned} \tag{3.15}$$

here the quantities $P_{n,j}$ are defined by (3.9) for $i = n$, and so on.

The two systems (3.13) and (3.14) for determining the $W_{i,j}$ turn out to be not in contradiction. Indeed, assume $W_{i-1,j-1}$ to be given. At first compute $W_{i,j} = \tilde{W}_{i,j}$ via $W_{i-1,j}$ by (3.14) and (3.13) in this order. Secondly, $W_{i,j} = \tilde{\tilde{W}}_{i,j}$ is determined via $W_{i,j-1}$ by (3.13) and (3.14) in the opposite order. Then, using (3.9)–(3.12) it follows after some computations but straightforwardly that $\tilde{W}_{i,j} = \tilde{\tilde{W}}_{i,j}$. This property is immediately extended to a consistence proof for system (3.9)–(3.14).

3.3. *Positivity of biquartic C^2-splines*

Here we derive a condition for the positivity of the biquartic spline (3.2), (3.6), (3,7), i.e., for $s(x, y) \geqslant 0$, $(x, y) \in J$. Using (2.14) and the abbreviations (2.15) we obtain that

$$e_\nu(h_i)^{\mathrm{T}} S_{i,j} e_\mu(k_j) \geqslant 0, \quad \nu, \mu = 0(1)4,\ i = 1(1)n,\ j = 1(1)m, \tag{3.16}$$

is sufficient for the positivity of s on J. Though somewhat lengthy, the conditions (3.16) now are given explicitly in order to make the proof of the succeeding existence property readable:

$$z_{i,j} \geqslant 0, \quad i = 0(1)n,\ j = 0(1)m, \tag{3.17}$$

$$4z_{i,j} + h_{i+1}p_{i,j} \geqslant 0, \quad i = 0(1)n-1,\ j = 0(1)m, \tag{3.18}$$

$$4z_{i,j} + k_{j+1}q_{i,j} \geqslant 0, \quad i = 0(1)n,\ j = 0(1)m-1, \tag{3.19}$$

$$4z_{i,j} - h_i p_{i,j} \geqslant 0, \quad i = 1(1)n,\ j = 0(1)m, \tag{3.20}$$

$$4z_{i,j} - k_j q_{i,j} \geqslant 0, \quad i = 0(1)n,\ j = 1(1)m, \tag{3.21}$$

$$16z_{i,j} + 4h_{i+1}p_{i,j} + 4k_{j+1}q_{i,j} + h_{i+1}k_{j+1}r_{i,j} \geqslant 0, \quad i = 0(1)n-1,\ j = 0(1)m-1, \tag{3.22}$$

$$16z_{i,j} + 4h_{i+1}p_{i,j} - 4k_j q_{i,j} - h_{i+1}k_j r_{i,j} \geqslant 0, \quad i = 0(1)n-1,\ j = 1(1)m, \tag{3.23}$$

$$16z_{i,j} - 4h_i p_{i,j} + 4k_{j+1}q_{i,j} - h_i k_{j+1}r_{i,j} \geqslant 0, \quad i = 1(1)n,\ j = 0(1)m-1, \tag{3.24}$$

$$16z_{i,j} - 4h_i p_{i,j} - 4k_j q_{i,j} + h_i k_j r_{i,j} \geqslant 0, \quad i = 1(1)n,\ j = 1(1)m, \tag{3.25}$$

$$12z_{i,j} + 6h_{i+1}p_{i,j} + h_{i+1}^2 P_{i,j} \geqslant 0, \quad i = 0(1)n-1,\ j = 0(1)m, \tag{3.26}$$

$$12z_{i,j} + 6k_{j+1}q_{i,j} + k_{j+1}^2 Q_{i,j} \geqslant 0, \quad i = 0(1)n,\ j = 0(1)m-1, \tag{3.27}$$

$$\begin{gathered} 4(12z_{i,j} + 6h_{i+1}p_{i,j} + h_{i+1}^2 P_{i,j}) + k_{j+1}(12q_{i,j} + 6h_{i+1}r_{i,j} + h_{i+1}^2 U_{i,j}) \geqslant 0, \\ i = 0(1)n-1,\ j = 0(1)m-1, \end{gathered} \tag{3.28}$$

$$\begin{gathered} 4(12z_{i,j} + 6h_{i+1}p_{i,j} + h_{i+1}^2 P_{i,j}) - k_j(12q_{i,j} + 6h_{i+1}r_{i,j} + h_{i+1}^2 U_{i,j}) \geqslant 0, \\ i = 0(1)n-1,\ j = 1(1)m, \end{gathered} \tag{3.29}$$

$$\begin{gathered} 4(12z_{i,j} + 6k_{j+1}q_{i,j} + k_{j+1}^2 Q_{i,j}) + h_{i+1}(12p_{i,j} + 6k_{j+1}r_{i,j} + k_{j+1}^2 V_{i,j}) \geqslant 0, \\ i = 0(1)n-1,\ j = 0(1)m-1, \end{gathered} \tag{3.30}$$

$$\begin{gathered} 4(12z_{i,j} + 6k_{j+1}q_{i,j} + k_{j+1}^2 Q_{i,j}) - h_i(12p_{i,j} + 6k_{j+1}r_{i,j} + k_{j+1}^2 V_{i,j}) \geqslant 0, \\ i = 1(1)n,\ j = 0(1)m-1, \end{gathered} \tag{3.31}$$

$$\begin{gathered} 12(12z_{i,j} + 6h_{i+1}p_{i,j} + h_{i+1}^2 P_{i,j}) + 6k_{j+1}(12q_{i,j} + 6h_{i+1}r_{i,j} + h_{i+1}^2 U_{i,j}) \\ + k_{j+1}^2(12Q_{i,j} + 6h_{i+1}V_{i,j} + h_{i+1}^2 W_{i,j}) \geqslant 0, \quad i = 0(1)n-1,\ j = 0(1)m-1. \end{gathered} \tag{3.32}$$

Hence, the biquartic spline (3.2), (3.6), (3.7) is C^2-continuous and positive on J if the parameters $p_{i,j}, q_{i,j}, r_{i,j}, P_{i,j}, Q_{i,j}, U_{i,j}, V_{i,j}$ and $W_{i,j}$, $i = 0(1)n, j = 0(1)m$, satisfy the linear system (3.9)–(3.14), (3.18)–(3.32).

Now, we are in the position to construct inductively a solution of this system if $z_{i,j} \geqslant 0$, $i = 0(1)n$, $j = 0(1)m$. To this end we set

$$
\begin{aligned}
&p_{i,j} = \frac{4}{h_i} z_{i,j}, \quad i = 1(1)n,\ j = 0(1)m,\\
&p_{0,j} = 0, \quad j = 0(1)m,\\
&q_{i,j} = \frac{4}{k_j} z_{i,j}, \quad i = 0(1)n,\ j = 1(1)m,\\
&q_{i,0} = 0, \quad i = 0(1)n,\\
&r_{i,j} = \frac{16}{h_i k_j} z_{i,j}, \quad i = 1(1)n,\ j = 1(1)m,\\
&r_{i,j} = 0, \quad i = 0, j = 0(1)m, \text{ or } j = 0, i = 0(1)n.
\end{aligned}
\tag{3.33}
$$

In this case the inequalities (3.18)–(3.25) are immediately seen to be satisfied. Further let

$$
\begin{aligned}
&P_{0,j} = U_{0,j} = 0, \quad j = 0(1)m,\\
&Q_{i,0} = V_{i,0} = 0, \quad i = 0(1)n,\\
&W_{0,0} = 0.
\end{aligned}
\tag{3.34}
$$

Because of (3.13), (3.14) this implies $W_{0,j} = W_{i,0} = 0$, $i = 1(1)n, j = 1(1)m$.

Now, for $i = 0, j = 0$ the inequalities (3.26)–(3.28), (3.30), (3.32) are obviously valid. Next, if we assume

$$
\begin{aligned}
&P_{i-1,j} \geqslant 0, \qquad W_{i-1,j} \geqslant 0,\\
&Q_{i,j-1} \geqslant 0, \qquad W_{i,j-1} \geqslant 0,\\
&U_{i-1,j} = \frac{4}{k_j} P_{i-1,j}, \qquad V_{i,j-1} = \frac{4}{h_i} Q_{i,j-1}
\end{aligned}
\tag{3.35}
$$

for arbitrary but fixed $i \geqslant 1$ and $j \geqslant 1$, by means of (3.9)–(3.14) and (3.33) we find straightforwardly that

$$
\begin{aligned}
&P_{i,j} \geqslant 0, \quad Q_{i,j} \geqslant 0, \quad U_{i,j} \geqslant 0, \quad V_{i,j} \geqslant 0, \quad W_{i,j} \geqslant 0,\\
&U_{i,j} = \frac{4}{k_j} P_{i,j}, \qquad V_{i,j} = \frac{4}{h_i} Q_{i,j}
\end{aligned}
\tag{3.36}
$$

hold true, and that the inequalities (3.26)–(3.32) are satisfied. This property, which analogously holds for $i = 0, j \geqslant 1$ and $j = 0, i \geqslant 1$ is sufficient for determining recursively a solution of system (3.9)–(3.14), (3.18)–(3.32). Hence, we have obtained the following proposition.

Proposition 3. *For data sets in positive position the problem of positive two-dimensional interpolation is always solvable with biquartic C^2-splines.*

4. Monotone interpolation with quintic C^2-splines

In Section 3 we have seen that positive interpolation is always successful with quartic C^2-splines. On the other hand, convex interpolation fails in general when using polynomial C^2-splines. Thus, we are now interested in monotone interpolation only.

4.1. Monotonicity of quintic C^2-splines in one dimension

The spline (2.2) becomes a quintic one if the vector a_i is defined by

$$a_i(u) = d(u; h_i) \tag{4.1}$$

with

$$d(u;h) = \begin{bmatrix} 1-u \\ u \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u(1-u)\begin{bmatrix} 1+u-9u^2+6u^3 \\ -1-u+9u^2-6u^3 \\ h(1-u)^2(3u+1) \\ hu^2(3u-4) \\ \frac{1}{2}h^2u(1-u)^2 \\ \frac{1}{2}h^2u^2(1-u) \end{bmatrix} \tag{4.2}$$

and the vector S_i by

$$S_i = \begin{bmatrix} z_{i-1} \\ z_i \\ p_{i-1} \\ p_i \\ P_{i-1} \\ P_i \end{bmatrix}. \tag{4.3}$$

Now we get

$$a_i(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad a_i(1) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad a_i'(0) = \begin{bmatrix} 0 \\ 0 \\ h_i \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad a_i'(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ h_i \\ 0 \\ 0 \end{bmatrix}, \quad a_i''(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ h_i^2 \\ 0 \end{bmatrix}, \quad a_i''(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ h_i^2 \end{bmatrix}. \tag{4.4}$$

Thus, in view of (2.3)–(2.5) these quintic splines are C^2-continuous for all values of the parameters p_i, P_i, $i = 0(1)n$, and those have again the meaning (2.10), (2.13).

In order to derive monotonicity conditions, we substitute $u = \rho/(1+\rho)$, and get with (4.2)

$$(1+\rho)^4 d'(u;h) = d_0(h) + d_1(h)\rho + d_2(h)\rho^2 + d_3(h)\rho^3 + d_4(h)\rho^4 \tag{4.5}$$

with

$$d_0(h) = \begin{bmatrix} 0 \\ 0 \\ h \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad d_1(h) = \begin{bmatrix} 0 \\ 0 \\ 4h \\ 0 \\ h^2 \\ 0 \end{bmatrix}, \quad d_2(h) = \begin{bmatrix} -30 \\ 30 \\ -12h \\ -12h \\ -\frac{3}{2}h^2 \\ \frac{3}{2}h^2 \end{bmatrix}, \quad d_3(h) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4h \\ 0 \\ -h^2 \end{bmatrix}, \quad d_4(h) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ h \\ 0 \\ 0 \end{bmatrix}. \tag{4.6}$$

Hence the conditions

$$d_\nu(h_i)^{\mathrm{T}} S_i \geqslant 0, \quad \nu = 0(1)4, \quad i = 1(1)n, \tag{4.7}$$

which obviously imply s to be monotone on I, are equivalent to

$$\begin{aligned} &p_i \geqslant 0, \quad i = 0(1)n, \qquad 4p_{i-1} + h_i P_{i-1} \geqslant 0, \qquad 4p_i - h_i P_i \geqslant 0, \\ &60(z_i - z_{i-1}) - 24h_i(p_i + p_{i-1}) + 3h_i^2(P_i - P_{i-1}) \geqslant 0, \quad i = 1(1)n. \end{aligned} \tag{4.8}$$

Consequently, an interpolating quintic spline (2.2), (4.1)–(4.3) is C^2-continuous and monotone on I if the parameters p_i, P_i, $i = 0(1)n$, satisfy system (4.8) of linear inequalities.

Now, if the data set D_n is in monotone position, we get immediately a solution of (4.8) by setting $p_i = P_i = 0$, $i = 0(1)n$. Thus, we have the following proposition.

Proposition 4. *For data sets in monotone position the problem of monotone one-dimensional interpolation is always solvable with quintic C^2-splines.*

4.2. Monotonicity of quintic C^2-splines in two dimensions

We get biquintic splines in (3.2) if the vectors a_i and b_j are set

$$a_i(u) = d(u; h_i), \qquad b_j(v) = d(v; k_j), \tag{4.9}$$

where d is given by (4.2). The matrix $S_{i,j}$ is now defined by

$$S_{i,j} = \begin{bmatrix} z_{i-1,j-1} & z_{i-1,j} & q_{i-1,j-1} & q_{i-1,j} & Q_{i-1,j-1} & Q_{i-1,j} \\ z_{i,j-1} & z_{i,j} & q_{i,j-1} & q_{i,j} & Q_{i,j-1} & Q_{i,j} \\ p_{i-1,j-1} & p_{i-1,j} & r_{i-1,j-1} & r_{i-1,j} & V_{i-1,j-1} & V_{i-1,j} \\ p_{i,j-1} & p_{i,j} & r_{i,j-1} & r_{i,j} & V_{i,j-1} & V_{i,j} \\ P_{i-1,j-1} & P_{i-1,j} & U_{i-1,j-1} & U_{i-1,j} & W_{i-1,j-1} & W_{i-1,j} \\ P_{i,j-1} & P_{i,j} & U_{i,j-1} & U_{i,j} & W_{i,j-1} & W_{i,j} \end{bmatrix}. \tag{4.10}$$

Because of (4.4), the smoothness conditions (3.3)–(3.5) are always satisfied. Thus, the biquintic splines (3.2), (4.2), (4.9), (4.10) are C^2-continuous for all values of the parameters $p_{i,j}$, $q_{i,j}, r_{i,j}, P_{i,j}, Q_{i,j}, U_{i,j}, V_{i,j}$ and $W_{i,j}$, $i = 0(1)n, j = 0(1)m$, and these represent the derivatives (3.8) and (3.15).

For the monotonicity of biquintic splines we need in addition to (4.5), (4.6) that

$$(1+\rho)^5 d(u;h) = e_0 + e_1(h)\rho + e_2(h)\rho^2 + e_3(h)\rho^3 + e_4(h)\rho^4 + e_5\rho^5 \tag{4.11}$$

with

$$e_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_1(h) = \begin{bmatrix} 5 \\ 0 \\ h \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_2(h) = \begin{bmatrix} 10 \\ 0 \\ 4h \\ 0 \\ \frac{1}{2}h^2 \\ 0 \end{bmatrix},$$

$$e_3(h) = \begin{bmatrix} 0 \\ 10 \\ 0 \\ -4h \\ 0 \\ \frac{1}{2}h^2 \end{bmatrix}, \quad e_4(h) = \begin{bmatrix} 0 \\ 5 \\ 0 \\ -h \\ 0 \\ 0 \end{bmatrix}, \quad e_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.12}$$

Then, a sufficient condition for the monotonicity of biquintic splines reads

$$\begin{aligned} &d_\nu(h_i)^{\mathrm{T}} S_{i,j} e_\mu(k_j) \geqslant 0, \\ &e_\mu(h_i)^{\mathrm{T}} S_{i,j} d_\nu(k_j) \geqslant 0, \quad \nu = 0(1)4,\ \mu = 0(1)5,\ i = 1(1)n,\ j = 1(1)m. \end{aligned} \tag{4.13}$$

We hesitate to give (4.13) explicitly. But for $p_{i,j} = q_{i,j} = r_{i,j} = P_{i,j} = Q_{i,j} = U_{i,j} = V_{i,j} = W_{i,j} = 0$, $i = 0(1)n$, $j = 0(1)m$, we find immediately that (4.13) reduces to

$$\begin{aligned} &z_{i-1,j} \leqslant z_{i,j}, \quad i = 1(1)n,\ j = 0(1)m, \\ &z_{i,j-1} \leqslant z_{i,j}, \quad i = 0(1)n,\ j = 1(1)m. \end{aligned}$$

This means that system (4.13) is always solvable if the data set $D_{n,m}$ is in monotone position. Thus, we have proved the following.

Proposition 5. *For data sets in monotone position the problem of monotone two-dimensional interpolation is always solvable with biquintic C^2-splines.*

5. Concluding remarks

In this paper we have shown that in one as well as in two dimensions the problem of positive interpolation is always solvable with quartic C^2-splines, and so is that of monotone interpolation with quintic C^2-splines. For proving these existence properties the first and second derivatives in the nodes are constructed as simple as possible in order to meet the sufficient positivity and monotonicity constraints. We point out that these choices in general are not the most suitable ones for obtaining visually pleasing interpolants. In our experience, to this end special optimization algorithms which minimize the L_2-norm of the curvature subject to shape preservation constraints are more favourable; compare with Section 2.4. In the case of positive interpolation in one dimension, this is confirmed by the 1–3. Here, the positive splines named feasspline are computed via (2.2), (2.6)–(2.8) by means of the values used in Section 2.3 for the feasibility proof, i.e., with $p_0 = P_0 = 0$ and p_i, P_i for $i \geqslant 1$ from

$$p_i = 4z_i/h_i, \quad P_i = (12z_{i-1} + 12z_i + 6h_ip_{i-1} + h_i^2P_{i-1})/h_i^2.$$

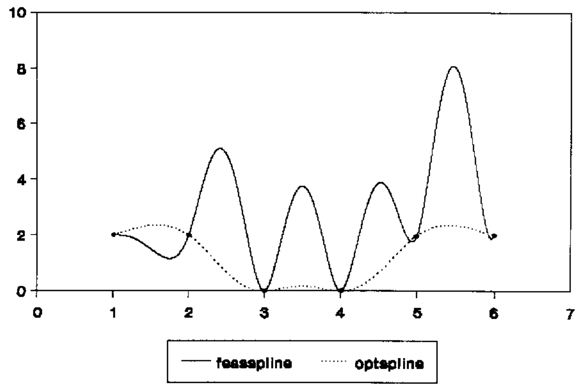

Fig. 1. Positive interpolation with quartic C^2-splines.

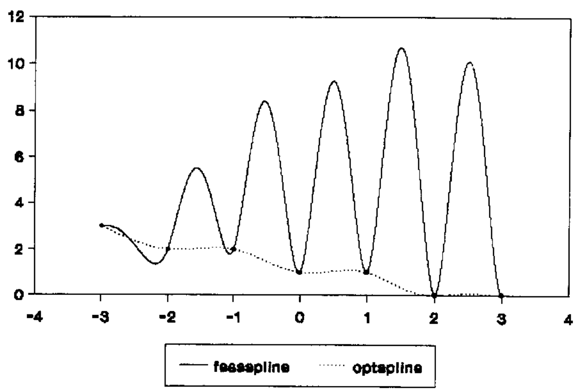

Fig. 2. Positive interpolation with quartic C^2-splines.

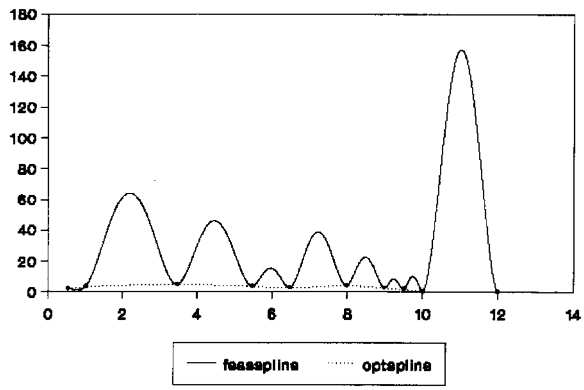

Fig. 3. Positive interpolation with quartic C^2-splines.

The positive interpolants optspline are optimal in the sense of program (2.18), which was solved numerically by dualization. Obviously, the positive splines called optspline are much more pleasant than the pictured feasible splines feasspline.

In the two-dimensional case, most of the questions arising in the optimization approach are still open until now. For instance, it is to find out which functionals are suitable as choice function.

In convex interpolation with polynomial splines or, more general, with C^1-functions from finite-dimensional linear spaces we have the negative result from [6]. But, when using nonlinear splines, convex interpolation in one dimension may be always successful. This is proven for some types of exponential, rational, and lacunary splines; see, e.g., [5, 9, 15]. In two dimensions there is only moderate progress in interpolation under convexity constraints. However, in S-convex interpolation positive results are received in [9, 14].

Finally we mention that the extension of the present results to the three-dimensional case being also of some practical interest seems to be possible.

6. Note added in proof

In the present paper, polynomial splines of lowest degrees are determined such that positivity and monotonicity are always preserved. To this end we have assumed that the spline grids Δ_n are built by the data sites. It should be mentioned that the received degrees can be reduced when splines on grids finer than Δ_n are admitted. In [22] it is shown that quadratic C^1-splines on grids with one additional knot in each subinterval allow monotonicity preservation. The same property for the positivity was recently observed in [18]. These results on the C^1-interpolation with quadratic splines are extended to the two-dimensional case; see [17, 19] for gridded data and [20] for scattered data. Finally, for the one-dimensional C^2-interpolation it was recently proved in [21] that positivity and monotonicity are always preserved by cubic splines on twofold refined grids.

Acknowledgements

The authors are grateful to cand. math. H. Pollex for performing the numerical experiments, and to Dr. B. Mulansky for stimulating discussions in constructing the counter examples.

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