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An Inequality for B_2-Sequences

BERNT LINDSTRÖM

Department of Mathematics, University of Stockholm, Sweden

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A sequence $a_1, a_2, ..., a_r$ of integers is called a B_2-sequence if all the sums $a_i + a_j$, $1 \leqslant i \leqslant j \leqslant r$, are different (cf. [2, *p.* 85, Def. 3]). Let $F_2(n)$ be the maximum number of elements that can be selected from the set $\{1, 2, ..., n\}$ so as to form a B_2-sequence. Erdös and Turan proved in [1] (cf. also [2, p. 86]):

$$F_2(n) < n^{1/2} + O(n^{1/4}). \tag{1}$$

Here I want to prove

$$F_2(n) < n^{1/2} + n^{1/4} + 1. \tag{2}$$

PROOF: Let $a_1 < a_2 < \cdots < a_r$ be a B_2-sequence from the set $\{1, 2, ..., n\}$. The differences $a_j - a_i$, $1 \leqslant i < j \leqslant r$, are different. $j - i$ is said to be the *order* of the difference $a_j - a_i$.

The differences of order $\nu > 0$ can be arranged in sequences of the type

$$a_\alpha - a_\beta, a_\beta - a_\gamma, a_\gamma - a_\delta, ..., \alpha - \beta = \beta - \gamma = \cdots = \nu.$$

As a result of cancellations it follows that the sum of all differences of the order ν is less than νn. Hence the sum of all positive differences of the order at most m is less than $\frac{1}{2}m(m + 1)n$. If $1 \leqslant m < r$, then the number of these differences is $mr - \frac{1}{2}m(m + 1) = ms$, where $s = r - \frac{1}{2}(m + 1)$. Since all differences are different, we find that the sum of all positive differences of order at most m is at least $\frac{1}{2}ms(ms + 1) > \frac{1}{2}m^2s^2$. It follows

$$\tfrac{1}{2}m^2s^2 < \tfrac{1}{2}m(m + 1)n,$$

and then, since $(1 + x)^{1/2} < 1 + \frac{1}{2}x$ for $x = m^{-1}$, we have

$$r < \tfrac{1}{2}(m + 1) + n^{1/2}(1 + \tfrac{1}{2}m^{-1}).$$

We choose $m = [n^{1/4}] + 1$ and (2) follows.

References

1. P. Erdös and P. Turan, On a Problem of Sidon in Additive Number Theory and Some Related Problems, *J. London Math. Soc.* **16** (1941), 212–215.
2. H. Halberstam and K. F. Roth, *Sequences*, vol. I, Oxford Univ. Press, Oxford, 1966.