1. Introduction. In this paper we continue our study of $F$-rings, begun in a previous paper [2], some of the results of which are used here. An $F$-ring is a $\sigma$-complete vector lattice [1, p. 238] which is also a commutative algebra with unit 1 in which $1 > 0$; $x > 0$, $y > 0 \Rightarrow xy > 0$; and $x \land 1 = 0 \Rightarrow x = 0$. Any ring $R$ is regular if for each $x \in R$ there is an $x^0 \in R$ such that $xx^0x = x$.

If $\Sigma$ is an abstract space and $\mathcal{Q}$ is a $\sigma$-algebra of subsets of $\Sigma$, then a real function $f(M)$ defined on $\Sigma$ is said to be $(\Sigma, \mathcal{Q})$-measurable if the set $\{M|f(M)<\lambda\} \in \mathcal{Q}$ for all real numbers $\lambda$. The main results of this paper are that a pair $(\Sigma, \mathcal{Q})$ exists for each regular $F$-ring $R$ such that $R$ is a uniquely defined $\sigma$-homomorphic image of the $F$-ring of all $(\Sigma, \mathcal{Q})$-measurable functions and that every $F$-ring is a sub-$F$-ring of a regular $F$-ring. A $\sigma$-homomorphism between $F$-rings preserves countable lattice operations as well as ring operations. In this paper, the term $\sigma$-homomorphism is also used to mean a homomorphism between $\sigma$-complete Boolean algebras which preserves countable lattice operations.

Our results are related to some results of OLMSTED. In [4] he considers for each $\sigma$-complete Boolean algebra $I$ the set $\Omega(I)$ of functions $f(\lambda)$ from the real line to $I$ which have the following properties:

(i) if $\mu < \lambda$, then $f(\mu) > f(\lambda)$,
(ii) $\lor_{\lambda} f(\lambda) = 1$,
(iii) $\land_{\lambda} f(\lambda) = 0$,
(iv) $\lor_{\mu > \lambda} f(\mu) = f(\lambda)$.

He defines ring and lattice operations in $\Omega(I)$ with respect to which $\Omega(I)$ forms what we call an $F$-ring. He then shows that every $\sigma$-complete vector lattice $F$ with weak unit [1, p. 223] can be imbedded $\sigma$-isomorphically in $\Omega(I)$ where $\mathcal{I} = \{e|e \in F, e \land (1 - e) = 0\}$. If $F$ is an $F$-ring, then $e \land (1 - e) = 0$ is equivalent to $e^2 = e$, so $I$ can be defined as the set of idempotents of $F$. In [2] it was shown that for each $F$-ring $R$ the set $I$ of its idempotents is a $\sigma$-complete Boolean algebra. $I$ is called the idempotent algebra of $R$. In Section 3 we show that if $R$ is a regular $F$-ring and $I$ is its idempotent algebra, then $R$ is $\sigma$-isomorphic to $\Omega(I)$.

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1) This paper was prepared while the author was a fellow at the Summer Research Institute of the Canadian Mathematical Congress.
An archimedian ring is (i) a commutative algebra with a unit over the real field, (ii) a partially ordered set which is a directed group with respect to addition, in which

(iii) \( 1 > 0 \)
(iv) \( a > 0, \ b > 0 \Rightarrow ab > 0 \)
(v) (archimedian condition) if \( a > 0 \), then \( \sum_{n=1}^{\infty} \frac{1}{n} \cdot a \) exists and equals zero.

It is shown, as a consequence of some results of Nakano [3], that an archimedian ring \( A \) can be imbedded in an \( F \)-ring \( R \) provided that \( A \) satisfies the following condition: (\( \pi \)) if \( \bigwedge_{\gamma \in \Gamma} a_{\gamma} = 0 \) for some subset \( \{a_{\gamma} \mid \gamma \in \Gamma \} \) of \( A \), and if \( b > 0 \), then \( \bigwedge_{\gamma \in \Gamma} a_{\gamma}b = 0 \). The imbedding isomorphism preserves sup’s and inf’s of sets of elements whose cardinality is \( \aleph_0 \) or less when these sup’s and inf’s exist in \( A \). Since \( R \) is a sub-\( F \)-ring of a \( \sigma \)-homomorphic image of the ring of all \((\Sigma, \xi)\)-measurable functions for some pair \((\Sigma, \xi)\), the ring \( A \) is a sub-archimedian ring of this \( \sigma \)-homomorphic image. This result is applied to show that in a regular archimedian ring which is a lattice and which satisfies condition (\( \pi \)), every element has a spectral integral representation.

The following special definitions and notations will be used: lower case Greek letters other than \( \gamma \) stand for real numbers; lower case Latin letters other than \( m \) and \( n \) stand for ring elements. The notations \( x^+ = x \lor 0 \), \( x^- = (-x) \lor 0 \), \( |x| = x^+ - x^- \), \( \varepsilon_x = \sqrt{\sum_{n=1}^{\infty} 1 \land n \cdot |x|} \), and \( e_x = 1 - \varepsilon_x \) will be used. It was shown in [2] that if \( R \) is a regular commutative ring with a unit, and if \( x \in R \), then there is an element \( a_x \in R \) such that \( a_x^2 = a_x \), \( xa_x = 0 \), and \((x + a_x)^{-1}\) exists, and if \( R \) is an \( F \)-ring, then \( a_x = e_x \).

A (ring) ideal \( J \) of an \( F \)-ring \( R \) is closed if \( a_n \in J \) for \( n > 1 \) and \( \bigvee_{n=1}^{\infty} a_n \subseteq R \), imply \( \bigvee_{n=1}^{\infty} a_n \subseteq J \).

2. **Ring homomorphisms of regular \( F \)-rings.** Let \( R \) be a regular \( F \)-ring. Every ideal of \( R \) is an \( \ell \)-ideal [1, p. 222]. This follows because first, the principal ideals \((x), (|x|), (\varepsilon_x), \) and \((\varepsilon_{|x|}) \) are all identical [2, proof of Theorem 2], and second \(|x| < |y| \) implies \( \varepsilon_x < \varepsilon_y \). This result is required to prove the following theorem.

**Theorem 1.** All ring-homomorphic images \( B \) of \( R \) are regular \( L \)-rings, that is, rings which satisfy all the axioms for regular \( F \)-rings except the axiom of conditional \( \sigma \)-completeness and the axiom: \( a \land 1 = 0 \Rightarrow a = 0 \).

**Proof:** Let \( J \) be the kernel of the homomorphism \( \varphi \) of \( R \) onto \( B \). From the remark at the beginning of this section, \( J \) is an \( \ell \)-ideal. Let “\( \geq \)” be defined in \( B \) as follows: \( a' \geq b' \) if there exists \( c \in R \) such that \( c > 0 \) and \( \varphi(c) = a' - b' \). It is a matter of simple but lengthy verification to show that, with respect to this relation, \( B \) is a regular \( L \)-ring. For example,
to show that \(a' \lor 0'\) exists in \(B\), consider \(b' > a', b' > 0'\) in \(B\). There exist \(e \in R\) and \(b \in R\) such that \(e > 0\), \(b > 0\), \(q(e) = b' - a'\), and \(q(b) = b'\). In \(R\), we find \(e = b - a + k\) where \(q(a) = a'\) and \(k \in J\). Hence \(b > a - k\), and since \(b > 0\), it follows that \(b > (a - k)^+\). Therefore there exists in \(R\), an element \(h > 0\) such that \(q(h) = b' - q[(a - k)^+]\), hence \(b' > q[(a - k)^+] > a'\) and \(q[(a - k)^+] > 0'\). The choice of \(k\), however, depends on \(b\), so it must be verified that \((a - k_1)^+ = (a - k_2)^+\) mod \(J\) for \(k_1 \neq k_2\) in \(J\), which follows from [1, Theorem 9, p. 222]. Therefore \(q[(a - k)^+] = a' \lor 0'\).

Theorem 2. Let \(q\) be a ring-homomorphism of \(R\) onto \(R'\). If the kernel \(J\) of \(q\) is a closed ideal, then \(R'\) is a regular \(F\)-ring and \(q\) preserves countable lattice operations.

Proof: From Theorem 1 it follows that \(R'\) is an \(L\)-ring. Therefore to prove Theorem 2 we must show first that every sequence \(\{a_n'\}\) of non-negative elements of \(R'\) has an inf and second that \(a' \land 1' = 0' \Rightarrow a' = 0'\) in \(R'\). Primed letters stand for elements of \(R'\) and unprimed letters for elements of \(R\).

Suppose \(b' < a_n'\) for \(n > 1\). There is no loss in generality in assuming \(b' > 0'\). There exist \(b\) and \(a_n\) such that \(q(b) = b', q(a_n) = a_n', b > 0,\) and \(a_n > 0\). Now \((b \land a_n) - b \in J\) for \(n > 1\), and since \(J\) is closed, 
\[
\land_n b \land a_n - b = (b \land \land_1 a_n) - b \in J.
\]
Therefore \(b' = b' \land q(\land_n a_n)\) so \(q(\land_n a_n) = \land_n q(a_n)\).

If \(a' \land 1' = 0'\), then there exists \(a \in R\) such that \(a > 0\), \(q(a) = a'\) and \(a \land 1 = j \in J\). Since \((a \land 1) - j = (a - j) \land (1 - j) = 0\), it follows from [3, Theorems 29.5 and 29.9] that \((a - j) \cdot (1 - j) = 0\).

Therefore \(q(a - j) \cdot q(1 - j) = q(a) = 0'\).

3. Representations of \(F\)-rings. In this section it is shown that every regular \(F\)-ring is a uniquely defined \(\sigma\)-homomorphic image of the ring of all \((\Sigma, \Xi)\)-measurable functions where \(\Sigma\) is a Stone space and \(\Xi\) is a sub-\(\sigma\)-algebra of the Borel sets of \(\Sigma\). First, however, the following theorem is proved.

Theorem 3. If \(I\) is the idempotent algebra of an \(F\)-ring \(A\), then \(A\) can be \(\sigma\)-isomorphically imbedded in \(\Omega(I)\).

To prove this theorem the following lemmata are used.

Lemma 1. If \(d > \beta > 0\) and if \(a > 0\), then \(e_{ad} = e_a\).

Proof: Since there is an integer \(N > \beta^{-1}\), it follows that \(Nd > 1\) and that \(Na < d\). Hence \(e_{ad} = e_a\).

From \(a \land e_a = a \cdot e_a = 0\), we deduce that for all \(n\), \(nad \land 1 = nad \land (e_a \lor e_a) = nad \land e_a\). Thus \(e_{ad} \leq e_a\). Therefore \(e_a = e_{ad}\) and the lemma follows.

Lemma 2. If \(a > 0\) and \(b > 0\), then \(ab = 0 \Rightarrow a \cdot b = 0\).
Proof: Follows immediately from definition of \( \tilde{e}_y \).

Proof of Theorem 3: Olmsted [4, Theorem 2.2] shows that \( A \) can be imbedded in \( \Omega(I) \) so that the \( \sigma \)-lattice and vector space structures are preserved by the imbedding isomorphism. It remains to be shown that the images \( f_x, f_y, \) and \( f_{xy} \) in \( \Omega(I) \) of \( x, y, \) and \( xy \) in \( A \) combine as follows: \( f_x f_y = f_{xy} \). This is accomplished by first showing that \( f_x = (f_x)^2 \) for \( x > 0 \).

In Olmsted’s imbedding, \( f_x \) is the function \( \tilde{e}_{(x, \lambda)}^+ \) of \( \lambda \). In \( \Omega(I) \) Olmsted defines \( g = f^2 \) for \( f > 0 \) as follows: \( g(\lambda) = 1 \) for \( \lambda < 0 \) and \( g(\lambda) = f(\sqrt{\lambda}) \) for \( \lambda > 0 \). Therefore, in order to show \( (f_x)^2 = f_x \), we must show \( \tilde{e}_{(x^2, \lambda)}^+ = \tilde{e}_{(x, \sqrt{\lambda})}^+ \) for \( x > 0 \) and \( \lambda > 0 \).

\[
(x^2 - \lambda)^+ = (x - \sqrt{\lambda})^+ + (x + \sqrt{\lambda}).
\]

If \( \lambda > 0 \), then \( (x + \sqrt{\lambda}) > \sqrt{\lambda} \) and (by Lemma 1)

\[
3.1 \\
\tilde{e}_{(x^2, \lambda)}^+ = \tilde{e}_{(x, \sqrt{\lambda})}^+.
\]

The application of Lemma 2 to the valid equations \( x^2 \tilde{e}_x = 0 \) and \( x^2 \tilde{e}_x = 0 \) yields the results \( (\tilde{e}_x)^2 \cdot e_x = 0 \) and \( \tilde{e}_x \cdot e_x = 0 \). Therefore \( e_x = e_x \cdot e_x = e_x \), and since \( x = x^+ \) and \( x^2 = (x^2)^+ \), it follows that equation 3.1 is valid for all \( \lambda > 0 \). Therefore \( (f_x)^2 = f_x \).

Since the product \( xy = \frac{1}{2}(x + y)^2 - x^2 - y^2 \), it follows from the linearity of the embedding isomorphism and from \( (f_x)^2 = f_x \) that \( f_{xy} = f_x f_y \) for \( x > 0 \) and \( y > 0 \). It is immediate that this result is true for arbitrary \( x \) and \( y \).

The following construction is used throughout the sequel. Let \( \Sigma \) be the Stone space corresponding to the \( \sigma \)-complete Boolean algebra \( I \), that is, the space of all prime ideals \( M \) of \( I \). By Stone’s Theorem [6] the collection \( D \) of sets of the from \( U(e) = \{ M \mid M \in \Sigma, e \notin M \} \) forms a Boolean algebra which is isomorphic to \( I \). If \( \Sigma \) is endowed with the topology generated by \( D \), then \( \Sigma \) is a compact Hausdorff space [7, p. 378]. Let \( \mathfrak{R} \) stand for the class of Borel subsets of \( \Sigma \) which are of first category and let \( \mathfrak{Q} \) be the class of all Borel subsets of the form \( (F \cup X) - Y \) where \( F \in \mathfrak{Q} \) and \( X, Y \in \mathfrak{R} \). Sikorski [5, pp. 255, 256] has shown first that \( \mathfrak{Q} \) is a \( \sigma \)-algebra, second that \( \mathfrak{R} \) is a \( \sigma \)-ideal (that is, \( \mathfrak{R} \) is closed with respect to countable sup’s), and third that the quotient space \( \mathfrak{Q}/\mathfrak{R} \) of \( \mathfrak{Q} \) by \( \mathfrak{R} \) is \( \sigma \)-isomorphic to \( I \).

Let \( \mathfrak{B}(I) \) stand for the set of all \((\Sigma, \mathfrak{Q})\)-measurable functions. \( \mathfrak{B}(I) \) is clearly a regular \( F \)-ring. Let \( \mathfrak{B}(I) \) be partitioned in the following manner: \( f = g \) if and only if \( f \) differs from \( g \) only on a set \( N \in \mathfrak{R} \). The class of functions \( h = 0 \) forms a closed ideal \( J \) of \( \mathfrak{B}(I) \). Therefore (Theorem 2) the quotient ring \( \mathfrak{B}(I)/J \) of \( \mathfrak{B}(I) \) by \( J \) is a regular \( F \)-ring and is a \( \sigma \)-homomorphic image of \( \mathfrak{B}(I) \). The idempotent algebra of \( \mathfrak{B}(I)/J \) is easily seen to be \( \sigma \)-isomorphic to \( \mathfrak{Q}/\mathfrak{R} \) and hence \( \sigma \)-isomorphic to \( I \). Therefore the following theorem is valid.
**Theorem 4.** Every σ-complete Boolean algebra \( I \) is \( \sigma \)-isomorphic to the idempotent algebra of some regular \( F \)-ring.

The following theorem relates Olmsted’s \( \Omega(I) \) to \( \mathfrak{A}(I) - J \).

**Theorem 5.** For each \( \sigma \)-complete Boolean algebra \( I \), the regular \( F \)-ring \( \mathfrak{A}(I) - J \) is isomorphic to \( \Omega(I) \).

**Proof:** By Theorem 3 the regular \( F \)-ring \( \mathfrak{A}(I) - J \) can be \( \sigma \)-isomorphically imbedded in \( \Omega(I) \).

To prove that \( \Omega(I) \) is actually isomorphic to \( \mathfrak{A}(I) - J \), suppose \( f(\lambda) \in \Omega(I) \). Then for each real \( \lambda \) the set \( U(\lambda) = \{ M \mid M \in \Sigma, f(\lambda) \not= \emptyset \} \) is the isomorphic copy of \( f(\lambda) \) under Stone’s representation. The sets \( U = \Sigma - \bigcup_{\mu \in \Lambda} U(\mu) \) and \( V = \bigcap_{\mu \in \Lambda} U(\mu) \) are sets of first category [5, p. 256]. If \( \Lambda \) is a countable dense subset of the real line, then \( U \) and \( V \in \mathfrak{R} \). Let \( H(\lambda) = U(\lambda) \cap (\Sigma - V) \) for \( \lambda < 0 \), and \( H(\lambda) = U(\lambda) \cup U \) for \( \lambda > 0 \). It is a consequence of [8, Theorem 3] that \( E(\lambda) = \bigcap_{\mu \in \Lambda} H(\mu) \) satisfies Olmsted’s conditions for a member of \( \Omega(\Sigma) \) and hence corresponds to \( (\Sigma, \Sigma) \)-measurable function \( f \). By [1, p. 251] \( f \) has a spectral representation

\[
f = \int_{-\infty}^{\infty} \lambda d\chi[CE(\lambda)]
\]

where \( CE(\lambda) = \Sigma - E(\lambda) \). In addition \( E(\lambda) \equiv H(\lambda) \mod \mathfrak{R} \) and \( H(\lambda) \equiv U(\lambda) \mod \mathfrak{R} \), so \( E(\lambda) \equiv U(\lambda) \mod \mathfrak{R} \), and hence \( f(\lambda) \) is the image of both \( E(\lambda) \) and \( U(\lambda) \mod J \). Since \( \mathfrak{A}(I) - J \) is a \( \sigma \)-homomorph of \( \mathfrak{A}(I) \) it follows that there is an image \( f^* \) of \( f \in \mathfrak{A}(I) \) which belongs to \( \mathfrak{A}(I) - J \) and that

\[
f^* = \int_{-\infty}^{\infty} \lambda d(1 - f(\lambda)).
\]

\( f^* \) corresponds to \( f(\lambda) \) under Olmsted’s imbedding. Therefore \( \mathfrak{A}(I) - J \) contains all the elements of \( \Omega(I) \) and hence is \( \sigma \)-isomorphic to \( \Omega(I) \).

**Corollary 5.** \( \Omega(I) \) is a regular \( F \)-ring.

**Corollary 6.** Every \( F \)-ring with idempotent algebra \( I \) can be imbedded in a regular \( F \)-ring with idempotent algebra \( I \).

**Proof:** If \( A \) is the \( F \)-ring, then (Theorem 3) \( A \) can be imbedded in \( \Omega(I) \).

An algebra \( B \) over the real field with a partial order structure is defined to be \( \textit{real} \) if \( x > 0 \) implies \( (x + 1)^{-1} \in B \). In [2] it was shown that an algebra \( B \) is real if it simultaneously is contained in a regular \( F \)-ring \( R \) and contains every \( x \in R \) for which there exists a \( \lambda_x \) such that \( |x| < \lambda_x \). Corollary 6 yields the result that every \( F \)-ring is real.

**Theorem 7.** Every regular \( F \)-ring \( R \) with idempotent algebra \( I \) is \( \sigma \)-isomorphic to \( \mathfrak{A}(I) - J \) and hence to \( \Omega(I) \).

**Proof:** By Corollary 6, \( R \) can be imbedded in \( \mathfrak{A}(I) - J \). Suppose \( 0 \not< f \in \mathfrak{A}(I) - J \). Then \( f + 1 > 1 \) and \( e_{f+1} = 0 \). Therefore \( (f + 1)^{-1} \in \mathfrak{A}(I) - J \).
and \(0 < (f + 1)^{-1} < 1\). Since \(I \subseteq R\), all linear combinations of elements of \(I\) belong to \(R\) and all sup’s and inf’s of bounded sequences of these linear combinations belong to \(R\). Therefore, because \((f + 1)^{-1}\) has a spectral integral representation [1, p. 251], it belongs to \(R\). Hence \(f + 1 \in R\), because \(R\) is regular and \(e_{(f+1)^{-1}} = 0\). Therefore \(f \in R\). If \(f\) is an arbitrary element of \(\mathfrak{A}(I) - J\), then \(f^+\) and \(f^-\) belong to \(R\), whence \(f \in R\) as well.

**Corollary 8.** There is a one-to-one correspondence between the class of regular \(F\)-rings and the class of \(\sigma\)-complete Boolean algebras.

**4. Representations of Archimedean rings.** If \(A\) is an archimedean ring satisfying condition (\(\pi\)), then \(A\) can be imbedded [3, Theorem 31.3] in a universally continuous semi-ordered ring \(R\), that is, in a ring \(R\) which is a complete vector lattice where \(a > 0, b > 0\) imply \(ab > 0\). From [3, Theorem 30.1] it is clear (i) that the imbedding isomorphism preserves what sup’s and inf’s occur in \(B\), (ii) that \(B\) is “dense” in \(R\), that is, for each \(a \in R\) there are two systems \(\{a_{\gamma}\}\) and \(\{b_{\gamma}\}\), \(\gamma \in \Gamma\) of elements of \(B\) such that \(\hat{a} = \bigwedge_{\gamma \in \Gamma} \hat{a}_{\gamma} = \bigvee_{\gamma \in \Gamma} \hat{b}_{\gamma}\) where \(\hat{a}_{\gamma}\) and \(\hat{b}_{\gamma}\) are the images in \(R\) of \(a_{\gamma}\) and \(b_{\gamma}\), and (iii) that \(R\) has a unit. Nakano also shows [3, p. 148] that \(R\) satisfies condition (\(\pi\)); hence from [3, Theorems 29.9 and 29.10] it follows that \(R\) is an \(F\)-ring.

**Theorem 9.** If \(B\) is an archimedean ring satisfying property (\(\pi\)), then there exists a \(\sigma\)-complete Boolean algebra \(I\) such that \(B\) is isomorphic (this isomorphism preserves sup’s and inf’s of sets of elements of cardinality \(< \aleph_0\) when they exist) to a sub-archimedian-ring of \(\mathfrak{A}(I) - J\).

**Proof:** Since \(B\) can be imbedded in an \(F\)-ring, and since every \(F\)-ring can be \(\sigma\)-isomorphically imbedded in \(\mathfrak{A}(I) - J\) for some \(\sigma\)-complete Boolean algebra \(I\), the theorem follows.

As an application of Theorem 9, the following spectral theorem is proved.

**Theorem 10.** Let \(B\) be a regular archimedean ring which satisfies condition (\(\pi\)) and which forms a lattice with respect to its partial order. Every element \(f \in B\) has a spectral integral representation in \(B\).

**Proof:** Let \(\mathfrak{A}(I) - J\) be the regular \(F\)-ring in which \(B\) is imbedded. To every \(f \in B\) there corresponds a spectral resolution \(a_{\gamma}(\lambda)\) of \(1\) (for definition see [1, p. 251]) such that \(f\) has the spectral integral representation in \(\mathfrak{A}(I) - J\)

\[
f = \int_{-\infty}^{\infty} \lambda \, d \, a_{\gamma}(\lambda),
\]

and in addition \(a_{\gamma}(\lambda) \in B\) for all \(\lambda\). Indeed, by Theorem 9 every element of \(B\) can be thought of as a coset mod \(J\) of \(\mathfrak{A}(I)\). Therefore if \(f(M) \in f\), then \((f(M) - \lambda)^+ \in (f - \lambda)^+\). In addition the regularity of \(B\) implies the existence for each \(f \in B\) of \(a_{\gamma} \in B\) where \(a_{\gamma} = a_{\gamma}, \, fa_{\gamma} = 0,\) and \((f + a_{\gamma})^{-1} \in B\).
Whence it follows that the characteristic function $\chi\{M|f(M)\leq \lambda\}$ belongs to the coset $a_{(f,-\lambda)^+}$. By [1, Theorem 14, p. 251] the function $f(M)$ can be expressed by the spectral integral

$$f(M) = \int_{-\infty}^{\infty} \lambda \, d \, \chi\{M|f(M)\leq \lambda\},$$

hence in $\mathfrak{U}(I)-J$ we have

$$f = \int_{-\infty}^{\infty} \lambda \, d \, a_{(f,-\lambda)^+}.$$ 

Then $a_f(\lambda)=a_{(f,-\lambda)^+}$ is the required spectral resolution of 1.

If $f \in B$ and $f \geq 0$, then $f \cdot a_f(N) \leq N$ for each natural number $N$ and

$$f \cdot a_f(N) = \int_{0}^{N} \lambda \, d \, a_f(\lambda).$$

Each of the approximating partial sums for the integral in equation (4.1) belongs to $B$ as does the limit of these sums. Therefore the integral can be said to exist in $B$ and to equal $f \cdot a_f(N) \in B$. Similarly since $\sqrt[N-1]{f \cdot a_f(N)}=f$ and since $f \in B,$ it can be said that $\sqrt[N-1]{f \cdot a_f(N)}$ exists in $B.$ Therefore every $f \geq 0$ (and hence every $f$) in $B,$ has a spectral integral representation in $B.$

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