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Discrete Applied Mathematics 143 (2004) 204–212

DISCRETE APPLIED MATHEMATICS

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Minimum average distance of strong orientations of graphs

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Received 16 December 2002; received in revised form 18 December 2003; accepted 31 January 2004

Abstract

The average distance of a graph (strong digraph) G, denoted by $\mu(G)$ is the average, among the distances between all pairs (ordered pairs) of vertices of G. If G is a 2-edge-connected graph, then $\vec{\mu}_{\min}(G)$ is the minimum average distance taken over all strong orientations of G. A lower bound for $\vec{\mu}_{\min}(G)$ in terms of the order, size, girth and average distance of G is established and shown to be sharp for several complete multipartite graphs. It is shown that there is no upper bound for $\vec{\mu}_{\min}(G)$ in terms of $\mu(G)$. However, if every edge of G lies on 3-cycle, then it is shown that $\vec{\mu}_{\min}(G) \leqslant \frac{7}{4}\mu(G)$. This bound is improved for maximal planar graphs to $\frac{5}{3}\mu(G)$ and even further to $\frac{3}{2}\mu(G)$ for eulerian maximal planar graphs and for outerplanar graphs with the property that every edge lies on 3-cycle. In the last case the bound is shown to be sharp.

MSC: 05C12; 05C20

Keywords: Minimum average distance; Oriented graphs; Bounds

1. Introduction

Let $G=(V,E)$ be a graph (digraph) on n vertices. Then the *average distance* of G, $\mu(G)$, is defined as the average among the distances between all pairs (ordered pairs) of vertices of G. Thus

$$\mu(G)=\sum_{u,v\in V} d_G(u,v)\Big/\binom{n}{2}$$

if G is a graph and

$$\mu(G)=\sum_{(u,v)\in V\times V} d_G(u,v)/n(n-1)$$

if G is a digraph, where $d_G(u,v)$ is the distance from u to v in G. Hence $\mu(G)$ is the expected distance between two randomly chosen distinct vertices of G. For a graph (digraph) G, $\sum_{u,v\in V} d(u,v)$ $(\sum_{(u,v)\in V\times V} d(u,v))$ is called the *total distance* of G and is denoted by $\sigma(G)$. If $d(u,v)=\infty$ for some pair u,v of vertices, $\sigma(u,v)=\infty$. We hence assume that

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[1] Research supported by an NSERC Canada Grant.

[2] This work was started while the third author was visiting the University of Winnipeg, and partially supported by the National Natural Science Foundation of China.

doi:10.1016/j.dam.2004.01.005

all graphs are connected and all digraphs strongly connected. An *orientation* of a graph G is an assignment of directions to the edges of G to produce a directed graph. An orientation of G is a *strong orientation* if for every pair u, v of vertices of G there is both a directed u–v and a directed v–u path. It is well-known that a graph G admits a strong orientation if and only if G is 2-edge-connected. However, the distances between vertices might be significantly greater in an orientation of a graph G than in G itself. Chvátal and Thomassen [3] proved the existence of a polynomial function f such that every 2-edge-connected graph of diameter d admits an orientation of diameter $f(d)$. They also proved that, given a 2-edge-connected graph G, the problem of finding an orientation of G of minimum diameter is NP-hard, even if G has diameter 2. Further results on the diameter of orientations of graphs can be found in [1].

In this paper we consider analogous questions for the average distance. For a 2-edge-connected graph G, $\vec{\mu}_{\min}(G)$ is the minimum of the average distances of strong orientations of G taken over all strong orientations of G. An orientation D of G such that $\mu(D) = \vec{\mu}_{\min}(G)$ is called an *optimal orientation* of G.

Unlike for the diameter, there is no function f such that every 2-edge-connected graph G of average distance μ has an orientation of average distance at most $f(\mu)$. To see this let n, k be positive integers and define $G_{n,k}$ to be the graph obtained from two disjoint cliques H_1 and H_2 of order n and a cycle C of length k by choosing two adjacent vertices v_1 and v_2 of the cycle and joining all vertices of H_i to v_i for $i = 1, 2$. Now assume that k is fixed and n tends to infinity. If we choose two vertices of $G_{n,k}$ randomly, then almost certainly they will be in the union of the two cliques. The probability that both are in the same clique (and also the probability that they are in different cliques) tends to $\frac{1}{2}$. Hence the expected distance between the two vertices tends to $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3 = 2$. If D is any strong orientation of $G_{n,k}$, then the vertices of C induce a directed cycle of length k. Without loss of generality, we can assume that $\overrightarrow{v_1 v_2} \in E(D)$. Then we have $d_D(v_1, v_2) = 1$ and $d_D(v_2, v_1) = k - 1$. Hence the expected value of $d_D(u, v) + d_D(v, u)$, if u and v are in distinct cliques, is at least $(k + 1) + 3 = k + 4$ and it is at least 3 if u and v are in the same clique. So the expected distance between two randomly chosen vertices in D is at least $(k + 7)/4$. Hence $\vec{\mu}_{\min}(G_{n,k})$ can be arbitrarily large.

It was observed by Plesník [7] that the problem of finding an optimal orientation of a given 2-edge-connected graph is NP-hard. For that reason, bounds on $\vec{\mu}_{\min}(G)$ are of interest.

In Section 2 we present lower bounds on $\vec{\mu}_{\min}$ and show that these bounds are attained for several complete bipartite and multipartite graphs. In Section 3 we focus on orientations of graphs where every edge lies on a 3- or 4-cycle. Chvátal and Thomassen [3] showed that if G is any graph and an edge uv belongs to a cycle of length k, then G admits an orientation D such that $\overrightarrow{uv}$ or $\overrightarrow{vu}$ belongs to a cycle of length at most $(k-2)2^{(k-1)/2} + 2$. So if every edge of G belongs to a 3-cycle, then G admits an orientation D such that for every edge $\overrightarrow{uv}$ or $\overrightarrow{vu}$ lies on a directed cycle of length at most 4. An immediate consequence of this result is that for such graphs $\vec{\mu}_{\min}(G) \leqslant 2\mu(G)$. In Section 3 it is shown that if every edge of G lies on a 3-cycle this upper bound for $\vec{\mu}_{\min}(G)$ in terms of $\mu(G)$ can be improved to $\frac{7}{4}\mu(G)$. If the graph is maximal planar this upper bound can be improved further to $\frac{5}{3}\mu(G)$ and if it is an eulerian maximal planar graph or a maximal outerplanar graph it can be lowered even further to $\frac{3}{2}\mu(G)$. In the last two cases the bound is sharp.

Our notation follows [2]. If G is a graph, we use n to denote the order of G and m to denote the size of G. If v is a vertex of G, then the neighborhood of v, i.e., the set of all vertices adjacent to v in G is denoted by $N_G(v)$. The closed neighborhood of v is defined as $N_G[v] = N_G(v) \cup \{v\}$. For $A, B \subset V(G)$, the set of edges joining a vertex of A to a vertex of B is denoted by $E_G(A, B)$. If the graph is understood, we drop the subscript G.

2. Lower bounds

A lower bound on the average distance of a strongly connected digraph D of order n and size m,

$$\mu(D) \geqslant 2 - \frac{m}{n(n-1)},$$

was first observed by Ng and Teh [6]. It follows directly from the fact that for $u, v \in V(D)$, $d_D(u, v) = 1$ if $\overrightarrow{uv} \in E(D)$ and $d_D(u, v) \geqslant 2$ otherwise. Equality holds if and only if D has diameter 2. We summarize this in the following result.

Proposition 1. *If G is a 2-edge connected graph of order n and size m, then*

$$\vec{\mu}_{\min}(G) \geqslant 2 - \frac{m}{n(n-1)}$$

with equality if and only if G has an orientation of diameter 2.

Thus, for any complete graph G, $\vec{\mu}_{\min}(G) \geqslant \frac{3}{2}$. It was shown by Plesník [7] that this lower bound can be achieved for all complete graphs of order at least 5 and that $\vec{\mu}_{\min}(K_4) = \frac{19}{12}$.

The next result gives a lower bound on $\vec{\mu}_{\min}(G)$, which is also a lower bound on $\mu(D)$ for a strongly connected digraph with m edges and n vertices.

Lemma 2. *If G is a 2-edge-connected graph of girth at least g with n vertices and m edges, then*

$$\vec{\mu}_{\min}(G) \geqslant \frac{(g-2)m}{n(n-1)} + \mu(G).$$

Proof. Let D be an optimal orientation of G. Then for every pair u, v of vertices of G, we have

$$d_D(u,v) + d_D(v,u) \geqslant \begin{cases} g & \text{if } uv \in E(G), \\ 2d_G(u,v) & \text{otherwise.} \end{cases} \tag{1}$$

Hence

$$\sigma(D) \geqslant gm + 2 \sum_{uv \notin E(G)} d_G(u,v) = (g-2)m + 2 \sum_{u,v \in V(G)} d_G(u,v).$$

So

$$\mu(D) \geqslant \frac{(g-2)m}{n(n-1)} + \frac{2\sum_{u,v\in V(G)} d_G(u,v)}{n(n-1)} = \frac{(g-2)m}{n(n-1)} + \mu(G). \qquad \square$$

We now show that the bound in Lemma 2 is sharp for several complete bipartite graphs. We make use of a result by Šoltés [9].

Lemma 3 (Šoltés [9]). *Let $2 \leqslant a \leqslant b$ be integers. Then $K_{a,b}$ has an orientation of diameter 3 if and only if $b \leqslant \binom{a}{\lfloor a/2 \rfloor}$.*

Theorem 4. *Let $2 \leqslant a \leqslant b$ be integers. Then*

$$\vec{\mu}_{\min}(K_{a,b}) \geqslant 2.$$

Equality holds if and only if $b \leqslant \binom{a}{\lfloor a/2 \rfloor}$.

Proof. Let D be a strong orientation of $K_{a,b}$. Let $n = a + b$. Then $\mu(K_{a,b}) = 2 - 2ab/(n(n-1))$ and, by Lemma 2,

$$\vec{\mu}_{\min}(K_{a,b}) \geqslant \frac{2ab}{n(n-1)} + \mu(K_{a,b}) = 2.$$

Equality holds if and only if we have equality in (1) of the proof of Lemma 2, i.e., if

$$d_D(u,v) + d_D(v,u) = 4 \quad \text{for all } u \neq v \in V,$$

which holds if and only if D has diameter 3. Hence, by Lemma 3, $\vec{\mu}_{\min}(K_{a,b}) = 2$ if and only if $b \leqslant \binom{a}{\lfloor a/2 \rfloor}$. $\square$

For the following theorem we need a result by Koh and Tan [5]. We call a pair a, b of integers a co-pair if $1 \leqslant a \leqslant b \leqslant \binom{a}{\lfloor a/2 \rfloor}$. We call a triple $a \leqslant b, c$ of integers a co-triple if a, b and a, c are co-pairs.

Lemma 5 (Koh and Tan [5]). *Let $a_1, a_2, \ldots, a_r$, $r \geqslant 3$, be positive integers and $n = \sum_{i=1}^{r} a_i$. Then $K_{a_1,a_2,\ldots,a_r}$ has an orientation of diameter 2 if $a_1, a_2, \ldots, a_r$ can be partitioned into co-pairs (if r is even) or into co-pairs and a co-triple (if r is odd).*

Theorem 6. *Let $a_1, a_2, \ldots, a_r$ be positive integers and $n = \sum_{i=1}^{r} a_i$. Then*

$$\vec{\mu}_{\min}(K_{a_1,a_2,\ldots,a_r}) \geqslant 2 - \frac{\sum_{i<j} a_i a_j}{n(n-1)}.$$

Equality holds if $a_1, a_2, \ldots, a_r$ can be partitioned into co-pairs (if r is even) or into co-pairs and a co-triple (if r is odd).

We note that the condition given in Lemma 5 and Theorem 6 is sufficient, but not necessary for the existence of an orientation of diameter 2. The problem of characterizing all complete multipartite graphs that admit an orientation of diameter 2 is still open.

The lower bound for $\vec{\mu}_{\min}(G)$ given in Lemma 2 is in terms the order, size, girth and average distance of G. Of course, $\mu(G)$ is also a lower bound for $\vec{\mu}_{\min}(G)$. That this bound is asymptotically sharp follows from a result by Füredi et al. [4]. They showed that there exist oriented graphs D of diameter 2 on n vertices and $m = n\log n + O(n\log\log n)$ edges. For these graphs the average distance equals $\mu(D) = 2 - m/(n(n-1)) = 2 - o(1)$. The underlying graph G, which also has diameter 2 has average distance $\mu(G) = 2 - 2m/(n(n-1)) = 2 - o(1)$. Hence $\vec{\mu}_{\min}(G)$ approaches $\mu(G)$ as n is made arbitrarily large.

If G and H are vertex-disjoint graphs, then the *join* $G \vee H$ of G and H is the graph consisting of $G \cup H$ and all edges between every vertex of G and every vertex of H.

Corollary 7. *If* $2 \leqslant |V(G)| \leqslant |V(H)| \leqslant \binom{|V(G)|}{\lfloor |V(G)|/2 \rfloor}$ *and* $|V(G)| + |V(H)| = N$, *then*

$$\vec{\mu}_{\min}(G \vee H) \leqslant 2 - \frac{|E(G)| + |E(H)|}{N(N-1)}.$$

Proof. Let $K = G \vee H - E(G) - E(H)$. Then K is isomorphic to a complete bipartite graph $K_{|V(G)|,|V(H)|}$. We orient the edges of K as in Theorem 4 and give any orientation to the edges in $E(G)$ and $E(H)$. The result follows from the proof of Theorem 4. □

If $G, H_1, H_2, \ldots, H_m$ are vertex-disjoint graphs, where $V(G) = \{v_1, v_2, \ldots, v_m\}$, then the *composition* of these graphs, denoted by $G[H_1, H_2, \ldots, H_m]$, is the graph consisting of $\bigcup_{i=1}^m H_i$ and all edges between every vertex of H_i and every vertex of H_j for any $v_i v_j \in E(G)$. Similar to Corollary 7, it is easy to establish the following result.

Corollary 8. *Let* $H_1, H_2, \ldots, H_r$ *be graphs*, $r \geqslant 3$. *Let* H_i *have order* n_i *and size* m_i *for* $i = 1, 2, \ldots, r$. *Let* $N = \sum n_i$ *and* $M = \sum m_i$. *If* $n_1, n_2, \ldots, n_r$ *can be partitioned into co-pairs* (*if* r *is even*) *or into co-pairs and a co-triple* (*if* r *is odd*), *then*

$$\vec{\mu}_{\min}(K_m[H_1, \ldots, H_r]) = 2 - \frac{M + \sum_{i<j} n_i n_j}{N(N-1)}.$$

3. Orientations of graphs in which every edge lies on a 3-cycle

Plesník [8] motivated the study of diameters of orientations of graphs in which every edge lies on a 3-cycle. In this section, we establish an upper bound for $\vec{\mu}_{\min}(G)$ in terms of $\mu(G)$ for this class of graphs. We improve this bound for maximal planar graphs and obtain a further refinement for eulerian maximal planar graphs and maximal outerplanar graphs. For the last two cases we show that the bound is asymptotically sharp. For a digraph D, let $t(D)$ denote the set of all arcs of D which are contained in a directed 3-cycle. We follow the convention that if f is a real valued function on a set A and $B \subset A$, then we write $f(B)$ for $\sum_{b\in B} f(b)$. We also adopt the convention that if f is a weight function on the edge set of a graph G and D is an orientation of G, then the definition of f is extended to the arc set of D such that an undirected edge of G and its corresponding directed edge of D have the same weight.

Lemma 9. *Let* $G = (V, E)$ *be a graph with the property that*

$$\text{every edge of } G \text{ is contained in a 3-cycle} \tag{2}$$

and let f *be a nonnegative weight function on the edges of* G. *Then there exists a strong orientation* D *of* G *such that*

$$f(t(D)) \geqslant \tfrac{1}{2} f(E). \tag{3}$$

Proof. We begin by proving by induction on $m = |E|$ that there is an orientation D of G that satisfies (3). If $m = 3$, then G consists of a triangle. Orienting the triangle such that it forms a directed 3-cycle yields $f(t(D)) = f(E)$; so Eq. (3) holds in this case.

Now assume that $m \geqslant 4$. Then G contains a vertex v which is neither isolated nor contained in every triangle. Let

$$E_v = \{e \in E \mid \text{each triangle containing } e \text{ contains } v\}.$$

So E_v contains all the edges incident with v as well as those edges not contained in a 3-cycle in $G-v$. The graph $G_1 = G - E_v$ has property (2) and smaller size than G. Hence there exists an orientation D_1 of G_1 such that

$$f(t(D_1)) \geqslant \tfrac{1}{2} f(E - E_v). \tag{4}$$

Let H be the graph induced by $N_G[v]$ in $G[E_v]$. Denote the set of non-isolates of $H-v$ by X. Let Y be the set of those isolates of $H-v$ which have a G-neighbor in $N_G(v)-X$, and let $Z = N_G(v) - X - Y$, respectively. Let $X' = X \cup \{v\}$. Consider the graph $H[X']$. It is easy to verify that $H[X']$ has property (2). Hence $H[X']$ has an orientation $D_{X'}$ with

$$f(t(D_{X'})) \geqslant \tfrac{1}{2} f(E(H[X'])). \tag{5}$$

Note that the converse orientation $\overline{D_{X'}}$ satisfies (5).

We now orient the edges in $E_G(\{v\}, Z)$. By (2) each $z \in Z$ has a $(G-E_v)$-neighbor x_z in X, for which either $\overrightarrow{zx_z}$ or $\overrightarrow{x_z z}$ is a directed edge of D_1. If now zx_z and $x_z v$ receive the orientations $\overrightarrow{zx_z}$ and $\overrightarrow{x_z v}$ (or $\overrightarrow{x_z z}$ and $\overrightarrow{vx_z}$), then zv can be oriented $\overrightarrow{vz}$ ($\overrightarrow{zv}$), so that it lies on a directed 3-cycle. Let D_Z be the following orientation of the edges in $E_G(\{v\}, Z)$. Orient vz as $\overrightarrow{vz}$ if $\overrightarrow{zx_z}$ is in D_1, and as $\overrightarrow{zv}$ if $\overrightarrow{x_z z}$ is in D_1. Then each directed edge of D_Z is contained in a directed 3-cycle in at least one of the orientations $D_1 \cup D_{X'} \cup D_Z$ and $D_1 \cup \overline{D_{X'}} \cup D_Z$. Therefore, at least one of these orientations, say, the first one, satisfies

$$f(E(D_Z) \cap t(D_1 \cup D_{X'} \cup D_Z)) \geqslant \tfrac{1}{2} f(E(\{v\}, Z)). \tag{6}$$

It remains to find a suitable orientation of the edges joining the vertices of Y to v. Let $Y' = Y \cup \{v\}$. We define a new weight function f' on the edge set f the graph $G[Y']$ as follows.

$$f'(e) = \begin{cases} f(e) & \text{if } e \text{ is incident with } v, \\ 0 & \text{if } e \text{ is not incident with } v. \end{cases}$$

By the definition of Y, the graph $G[Y]$ contains no isolates. Hence there exists a spanning forest F of $G[Y]$ with no isolates, which has a bipartition of the vertex set Y of F into two sets A and B such that each edge of F joins a vertex of A and a vertex of B. Let D_F be the orientation of F induced by the orientation D_1 of G_1. Now consider two different orientations, O_1 and O_2, of the edges joining v and Y in G. In O_1 let the edges be oriented from A to v and from v to B. In O_2 let the edges of O_1 be reversed. Since D_F contains no isolates, each G-edge joining v and Y has a corresponding directed edge, either in O_1 or O_2, which is contained in a directed 3-cycle. Hence

$$f'(t(D_F \cup O_1)) + f'(t(D_F \cup O_2)) \geqslant f(E_G(\{v\}, Y)).$$

Hence there exists an orientation $D_Y \in \{D_F \cup O_1, D_F \cup O_2\}$ with

$$f'(t(D_Y)) \geqslant \tfrac{1}{2} f(E_G(\{v\}, Y)). \tag{7}$$

We now show that the orientation $D = D_1 \cup D_{X'} \cup D_Y \cup D_Z$ has the desired property. It is easy to verify that D is indeed an orientation of G and that $t(D_1)$, $t(D_{X'})$, $t(D_Y)$, and $t(D) \cap E(D_Z)$ are disjoint. Since $t(D_1) \cup t(D_{X'}) \cup t(D_Y) \cup (t(D) \cap E(D_Z)) \subset t(D)$, we have by (4)–(7),

$$\begin{aligned} f(t(D)) &\geqslant f(t(D_1) \cup t(D_{X'}) \cup t(D_Y)) + f(t(D) \cap E(D_Z)) \\ &= f(t(D_1) + f(t(D_{X'})) + f'(t(D_Y)) + f(t(D) \cap E(D_Z)) \\ &\geqslant \tfrac{1}{2} f(E - E_v) + \tfrac{1}{2} f(E(H[X'])) + \tfrac{1}{2} f(E_G(\{v\}, Y)) + \tfrac{1}{2} f(E_G(\{v\}, Z)) \\ &= \tfrac{1}{2} f(E), \end{aligned}$$

as desired.

We now show that there is a strong orientation of G that satisfies (3). Among all orientations of G satisfying (3), let D be one for which the order of the largest strong component is maximum. We prove that D is strongly connected. Suppose not. Let C_1 be the largest strong component of D. Then G contains an edge joining a vertex in C_1 to a vertex not in C_1. By (2), this edge is contained in an undirected 3-cycle u, v, w, u.

If C_1 contains only one of u, v, w, say, u, and if the edge $vw \in E(G)$ has the orientation $\overrightarrow{vw}$ in D, then let D' be the orientation of G obtained from D by changing the orientation of the other two edges to $\overrightarrow{uv}$ and $\overrightarrow{wu}$. Then D' contains the directed triangle u, v, w, u and thus contains a strong component whose vertex set strictly contains $V(C_1)$. On the other hand, no directed triangle was destroyed by reorienting uv and wu. Therefore, $t(D') \supset t(D)$ and thus $f(t(D')) \geqslant f(t(D)) \geqslant \frac{1}{2} f(E)$, contradicting the choice of D.

If C_1 contains two of the vertices u, v, w, say, v, w, then reorienting the edge uv, uw as above yields a similar contradiction to the choice of D. Hence D is strongly connected. $\square$

Theorem 10. *Let $G=(V,E)$ be a connected graph with the property that every edge of G is on a 3-cycle. Then*

$$\vec{\mu}_{\min}(G) \leqslant \tfrac{7}{4}\mu(G).$$

Proof. Let $R=\{R(u,v)\,|\,u\neq v\in V\}$ be a routing of shortest paths in G. This routing defines an edge load function ξ_R. Let D be a strong orientation of G such that $\xi_R(t(D))$ is maximum. By the above lemma

$$\xi_R(t(D)) \geqslant \tfrac{1}{2}\xi_R(E).$$

We show that for every arc $\overrightarrow{uv}$ of D we have

$$d_D(v,u) \leqslant 3. \tag{8}$$

If $\overrightarrow{uv}$ is on a directed 3-cycle, then $d_D(v,u)=2$. Hence we can assume that $\overrightarrow{uv}$ is not on a directed 3-cycle in D while in G the edge uv is contained in a 3-cycle, say, u,v,w,u. The edges $uw, vw \in E(G)$ cannot have the orientation $\overrightarrow{wu}$ and $\overrightarrow{vw}$ since then $\overrightarrow{uv}$ would be on a directed 3-cycle. Also the orientation $\overrightarrow{uw}$ and $\overrightarrow{wv}$ cannot occur since otherwise reversal of the arc $\overrightarrow{uv}$ would create at least one new directed 3-cycle without destroying any directed 3-cycle, which contradicts the choice of the orientation D. Hence D contains either the arcs $\overrightarrow{uw}, \overrightarrow{vw}$ or the arcs $\overrightarrow{wu}, \overrightarrow{wv}$. If $\overrightarrow{uw}, \overrightarrow{vw} \in E(D)$, then $\overrightarrow{uw}$ is on a directed 3-cycle since otherwise reversal of this arc would create at least one new directed 3-cycle u,v,w,u without destroying any directed 3-cycles, thus contradicting the choice of D. Hence D contains a vertex x and arcs $\overrightarrow{wx}$ and $\overrightarrow{xu}$. The directed path v,w,x,u shows that $d_D(v,u)\leqslant 3$. The case $\overrightarrow{wu}, \overrightarrow{wv} \in E(D)$ is treated analogously. This proves (8).

We now define a routing S of D which is obtained by replacing the edges of the paths in R by directed paths as follows. For each edge $xy\in E(G)$ let $P(x,y)$ $(P(y,x))$ be a shortest directed $x-y$ directed path ($y-x$ directed path) in D. If for $u,v\in V$

$$R(u,v) = u, u_1, u_2, \ldots, u_k, v,$$

then let

$$S'(u,v) = P(u,u_1), P(u_1,u_2), \ldots, P(u_k,v),$$

$$S'(v,u) = P(v,u_k), P(u_k,u_{k-1}), \ldots, P(u_1,u).$$

Then $S'(u,v)$ is a directed u–v walk that contains a directed u–v path $S(u,v)$. Similarly, $S'(v,u)$ contains a directed v–u path $S(v,u)$. For each edge xy of G, exactly one of the directed paths $P(x,y)$ and $P(y,x)$ has length one, and by (8) the other one has length 2 or 3, depending on whether $\overrightarrow{xy}$ or $\overrightarrow{yx}$ is on a directed 3-cycle or not. If for a (directed) path P we denote the length of P by $l(P)$, then

$$\begin{aligned}
\sigma(D) &\leqslant \sum_{\{u,v\}\subset V} (l(S(u,v)) + l(S(v,u))) \\
&\leqslant \sum_{\{u,v\}\subset V} (l(S'(u,v)) + l(S'(v,u))) \\
&= \sum_{uv\in E(G)} \xi_R(uv)(l(P(u,v)) + l(P(v,u))) \\
&= \sum_{\overrightarrow{uv}\in t(D)} 3\xi_R(uv) + \sum_{\overrightarrow{uv}\in E(D)-t(D)} 4\xi_R(uv) \\
&= 4\xi_R(E(D)) - \xi_R(t(D)).
\end{aligned}$$

By the above remark we have $\xi_R(E(D)) = \xi_R(E) = \sigma(G)$, and, by Lemma 9, $\xi_R(t(D)) \geqslant \frac{1}{2}\xi_R(E)$. Hence we have

$$\sigma(D) \leqslant \tfrac{7}{2}\sigma(G).$$

Division by $n(n-1)$ yields the theorem. □

Corollary 11. *Let $G=(V,E)$ be a 2-edge-connected chordal graph. Then*

$$\vec{\mu}_{\min}(G) \leqslant \tfrac{7}{4}\mu(G).$$

Lemma 12. *Let $G=(V,E)$ be a maximal planar graph and let f be a nonnegative weight function on the edges of G. Then there exists a strong orientation D of G such that*

$$f(t(D)) \geqslant \tfrac{2}{3} f(E). \tag{9}$$

Proof. Let G be a maximal planar graph with a given embedding in the plane. If $G=K_4$, then, for each given edge e, there exists an orientation D such that $t(D)=E(G)-\{e\}$. The result therefore follows in this case. Hence we can assume that $G \neq K_4$.

Let $G^*=(V^*,E^*)$ be the dual of the graph G. Then G^* is 3-regular and not isomorphic to K_4. For each edge e of G, which is on the common boundary of faces, say f_1 and f_2, there exists a corresponding edge e^* of G^* joining the vertices f_1 and f_2. Since this defines a bijection between E and E^*, we can extend the definition of the weight function f to E^* by $f(e^*)=f(e)$.

By Brooks' Theorem, G^* has a vertex colouring with three colour classes, say, A^*, B^*, and C^*. Without loss of generality we can assume that $f(E_{G^*}(A^*,B^*)) \geqslant f(E_{G^*}(A^*,C^*)) \geqslant f(E_{G^*}(B^*,C^*))$. Hence the set A^* is an independent set in G^* such that for E_A^*, the set of edges of G^* incident with A^*,

$$f(E_A^*) \geqslant \tfrac{2}{3} f(E^*) = \tfrac{2}{3} f(E).$$

The independent set A^* is a set of faces of G, which are triangles, such that the edges forming their boundaries are disjoint. The set E_A^* corresponds to the set of edges $E_A \subset E(G)$ which form the union of the boundaries of the faces in A^*.

We now define an orientation of the edges of G. For each $f^* \in A^*$ orient the edges on the boundary of f^* in clockwise direction. This defines an orientation of the edges in E_A such that each edge of E_A is contained in a directed triangle. Orient the remaining edges of G arbitrarily. This yields an orientation D of G with $E_A \subset t(D)$. Thus

$$f(t(D)) \geqslant f(E_A) \geqslant \tfrac{2}{3} f(E).$$

We can now show as in Lemma 9 that there is a strong orientation D of G satisfying Eq. (9). □

Theorem 13. *Let $G=(V,E)$ be a maximal planar graph. Then,*

$$\vec{\mu}_{\min}(G) \leqslant \tfrac{5}{3}\mu(G).$$

Proof. Proceed as in the proof of Theorem 10 using Lemma 12. □

Corollary 14. *Let $G=(V,E)$ be an eulerian maximal planar graph. Then*

$$\vec{\mu}_{\min}(G) \leqslant \tfrac{3}{2}\mu(G).$$

Proof. Since the faces of a planar embedding of G can be 2-coloured, there is an orientation of G such that every edge lies on a directed 3-cycle. The result now follows as in the proof of Lemma 12. □

We now show that the factor $\frac{3}{2}$ in the bound given above is best possible. For every even $k \geqslant 4$ let G_k be the graph obtained from a cycle $C=u_0,u_1,\ldots,u_{k^2-1},u_0$ by joining a vertex v to each vertex of C, adding vertices $w_0,w_1,\ldots,w_{k-1},z$ and joining w_i to $u_{ik},u_{ik+1},\ldots,u_{ik+k}$ for $i=0,1,\ldots,k-1$ and z to u_{ik} and w_i for $i=0,1,\ldots,k-1$. It is easy to verify that G_k is a maximal planar and eulerian graph of order $n=k^2+k+2$. A planar embedding of G_4 is shown in Fig. 1.

We first show that, for large k,

$$\mu(G_k) = 2 + o(1). \tag{10}$$

For $i=0,1,\ldots,k-1$ define the segment S_i of the cycle C by $S_i=\{u_{ik+2},u_{ik+3},\ldots,u_{ik+k-2}\}$ and let A be the set of all unordered pairs of vertices of G_k belonging to distinct segments S_i and let B be the set of all remaining pairs of vertices of G_k. Simple counting shows that

$$|A| = \frac{k^4}{2} + O(k^3), \qquad |B| = O(k^3),$$

i.e., almost all pairs of vertices of G_k belong to A.

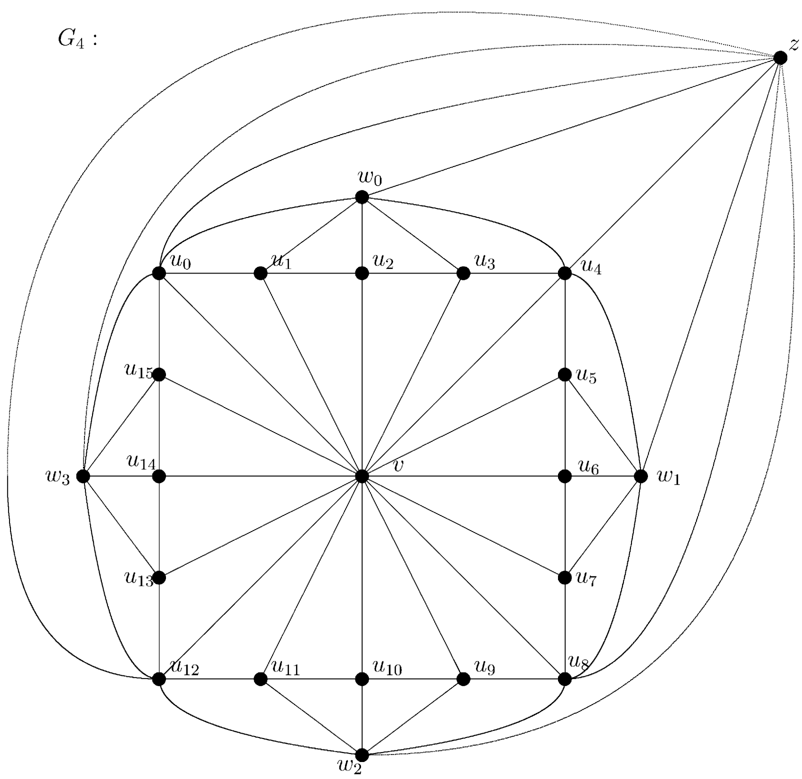

Fig. 1. A maximal planat eulerian graph illustrating sharpness for corollary 14.

Since $d_G(x, y) = 2$ for all $\{x, y\} \in A$ and $1 \leqslant d_G(x, y) \leqslant 3$ for all remaining pairs, we have

$$\sigma(G_k) = \sum_{\{x,y\}\in A} d_{G_k}(x, y) + \sum_{\{x,y\}\in B} d_{G_k}(x, y) = k^4 + \mathrm{O}(k^3).$$

Division by $\binom{n}{2} = \frac{1}{2}k^4 + \mathrm{O}(k^3)$ yields (10).

Now let D_k be a strong orientation of G_k. We show that

$$d_{D_k}(x, y) + d_{D_k}(y,x) \geqslant 6 \quad \text{for all } \{x, y\} \in A.$$

If $\overrightarrow{xv}, \overrightarrow{vy} \in E(D_k)$, then $d_{D_k}(x, y) = 2$ and $d_{D_k}(y,x) \geqslant 4$. If $\overrightarrow{xv}, \overrightarrow{yv} \in E(D_k)$, then $d_{D_k}(x, y) \geqslant 3$ and $d_{D_k}(y,x) \geqslant 3$. The same holds if $\overrightarrow{vx}, \overrightarrow{yv} \in E(D_k)$ or $\overrightarrow{vx}, \overrightarrow{vy} \in E(D_k)$. In each case $d_{D_k}(x, y) + d_{D_k}(y,x) \geqslant 6$.

As for G_k we have

$$\sigma(D_k) = \sum_{\{x,y\}\in A} (d_{D_k}(x, y) + d_{D_k}(y,x)) + \sum_{\{x,y\}\in B} (d_{D_k}(x, y) + d_{D_k}(y,x)) \geqslant 3k^4 + \mathrm{O}(k^3).$$

Division by $n(n-1) = k^4 + \mathrm{O}(k^3)$ yields

$$\mu(D_k) \geqslant 3 + \mathrm{o}(1).$$

Hence, for large k,

$$\mu(D_k)/\mu(G_k) \geqslant \tfrac{3}{2} + \mathrm{o}(1).$$

Theorem 15. *Let G be an outerplanar graph with the property that every edge lies on a 3-cycle. Then G has an orientation in which every arc lies on a directed 3-cycle.*

Proof. We proceed by induction on the order. The result holds for any outerplanar graph that has exactly three vertices and the property that every edge lies on a 3-cycle. Suppose now that G is an outerplanar graph with at least four vertices and the property that every edge lies on a 3-cycle. Then G has no bridges and has a vertex, v say, of degree 2. Let uv, vw be the edges incident with v. Since v has degree 2, and as the edges uv and vw lie in some 3-cycle, uw is an edge of G. Consider $G - v$. If uw lies in a 3-cycle in $G - v$, then $G - v$ is an outerplanar graph with the property that every edge lies on a 3-cycle. By the inductive hypothesis, $G - v$ has an orientation D with the property that every arc lies on a directed 3-cycle. We may assume uw has the direction $\overrightarrow{uw}$ in D. Assign wv and vu the direction $\overrightarrow{wv}$ and $\overrightarrow{vu}$, respectively. This produces an orientation of G in which every arc lies on a directed 3-cycle. Suppose now that uw does not lie in a 3-cycle in $G - v$. Then $G' = G - v - uw$ has the property that every edge lies on a 3-cycle. Again, by induction, G' has an orientation such that every edge lies on a directed 3-cycle. If in addition we orient the edges uv, uw, vw so that they form a directed 3-cycle, say u, v, w, u, then we obtain an orientation of G with the property that every arc lies on a directed 3-cycle. The result now follows. $\square$

Corollary 16. *Let $G = (V, E)$ be an outerplanar graph with the property that every edge lies on a 3-cycle. Then*

$$\vec{\mu}_{\min}(G) \leqslant \tfrac{3}{2}\mu(G).$$

The factor $\frac{3}{2}$ is best possible. This is shown by the graph G_n obtained from a cycle $v_1, v_2, \ldots, v_n, v_1$ by adding the edges $v_1 v_i$ for $i = 3, 4, \ldots, n-2$. It is easy to check that $\mu(G)$ approaches 2 as n tends to infinity. As before we can show that in every orientation D of G_n, for almost all pairs v_i, v_j,

$$d_{D_n}(v_i, v_j) + d_{D_n}(v_j, v_i) \geqslant 6.$$

Hence, for large n, $\mu(D_n) \geqslant 3 - o(1)$. Thus $\mu(D_n)/\mu(G_n) \geqslant \frac{3}{2} - o(1)$.

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