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Moduli of smoothness and approximation on the unit sphere and the unit ball ☆

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Abstract

A new modulus of smoothness based on the Euler angles is introduced on the unit sphere and is shown to satisfy all the usual characteristic properties of moduli of smoothness, including direct and inverse theorem for the best approximation by polynomials and its equivalence to a K-functional, defined via partial derivatives in Euler angles. The set of results on the moduli on the sphere serves as a basis for defining new moduli of smoothness and their corresponding K-functionals on the unit ball, which are used to characterize the best approximation by polynomials on the ball.

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Keywords: Modulus of smoothness; K-functional; Best approximation; Sphere; Euler angles; Ball

Contents

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1. Introduction

A central problem for approximation theory is to characterize the best approximation by polynomials via moduli of smoothness, or via K-functionals. In this paper we consider the setting of best approximation by polynomials on the unit sphere and the unit ball

$$\mathbb{S}^{d-1} = \{x:\ \|x\| = 1\} \quad \text{and} \quad \mathbb{B}^d = \{x:\ \|x\| \leqslant 1\}$$

of $\mathbb{R}^d$, where $\|x\|$ denotes the usual Euclidean norm.

1.1. Approximation on the unit sphere

On the unit sphere, we consider the best approximation by polynomials in the space $L^p(\mathbb{S}^d)$, $1 \leqslant p < \infty$, or $C(\mathbb{S}^{d-1})$ for $p = \infty$, with norm denoted by $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{S}^{d-1})}$, $1 \leqslant p \leqslant \infty$, in Part 1. Let Π_n^d denote the space of polynomials of total degree n in d variables and $\Pi_n(\mathbb{S}^{d-1}) := \Pi_n^d|_{\mathbb{S}^{d-1}}$, the space of spherical polynomials, or equivalently polynomials restricted on the sphere. In the following we shall write Π_n^d for $\Pi_n^d(\mathbb{S}^{d-1})$ whenever it causes no confusing. The quantity of best approximation is defined by

$$E_n(f)_p := \inf_{g \in \Pi_{n-1}^d} \|f - g\|_p, \quad 1 \leqslant p \leqslant \infty. \tag{1.1}$$

The first modulus of smoothness that characterizes $E_n(f)_p$ on the sphere is defined via the spherical means

$$S_\theta f(x) := \frac{1}{\sigma_{d-1}(\sin\theta)^{d-2}} \int_{\langle x,y\rangle=\cos\theta} f(y)\, d\sigma_{x,\theta}(y), \tag{1.2}$$

where $d\sigma_{x,\theta}$ is the Lebesgue measure on $\{y \in \mathbb{S}^{d-1}\colon \langle x, y\rangle = \cos\theta\}$ and $\sigma_d = 2\pi^{d/2}/\Gamma(d/2)$ ([1, p. 216], [25, p. 475], [29, p. 288]). For $r > 0$ and $t > 0$, this modulus of smoothness is defined by

$$\omega_r^*(f,t)_p := \sup_{|\theta|\leqslant t} \left\|(I - S_\theta)^{r/2} f\right\|_p, \tag{1.3}$$

where $(I - S_\theta)^{r/2}$ is defined in terms of infinite series when $r/2$ is not an integer [33, p. 183]. After earlier studies by several authors (see, for example, [1,21,25]), Rustamov [30, p. 315] finally established, for $1 < p < \infty$, both direct and inverse theorems for the polynomial best approximation, as well as the equivalence of $\omega_r^*(f,t)_p$ to the K-functional

$$K_r^*(f,t)_p := \inf_g \left\{\|f - g\|_p + t^r \left\|(-\Delta_0)^{r/2} g\right\|_p\right\}, \tag{1.4}$$

where Δ_0, given in (2.5) below, is the Laplace–Beltrami operator on the sphere and the infimum is taken over all g for which $(-\Delta_0)^{r/2} g \in L^p$. The proofs of these results for the full range of $1 \leqslant p \leqslant \infty$ can be found in [33, pp. 195–216]. The study of $\omega_r^*(f,t)_p$ and $K_r^*(f,t)_p$ relies heavily on the fact that both $(I - S_\theta)^{r/2}$ and $(-\Delta_0)^{r/2}$ are multiplier operators of Fourier series in spherical harmonics. This approach has been extended in [35, p. 15] to the setting of weighted space $L^p(\mathbb{S}^{d-1}, h_\kappa^2)$, where h_κ is a weight function invariant under a finite reflection group.

The second modulus of smoothness on the sphere is defined via rotation,

$$T_Q f(x) := f(Qx), \quad Q \in SO(d), \tag{1.5}$$

where $SO(d)$ denotes the group of rotations on $\mathbb{R}^d$, or the group of orthogonal matrices of determinant 1 in $\mathbb{R}^d$. For $t > 0$, define

$$O_t := \left\{Q \in SO(d)\colon \max_{x \in \mathbb{S}^{d-1}} d(x, Qx) \leqslant t\right\},$$

where $d(x, y) := \arccos \langle x, y\rangle$ denotes the geodesic distance on $\mathbb{S}^{d-1}$. For $r > 0$ and $t > 0$ define

$$\tilde{\omega}_r(f, t)_p := \sup_{Q\in O_t} \left\|\triangle_Q^r f\right\|_p, \quad \text{where } \triangle_Q^r := (I - T_Q)^r. \tag{1.6}$$

For $r = 1$ and $p = 1$ this modulus of smoothness was introduced and used by Calderòn, Weiss and Zygmund [4] and further studied in [19]. For other spaces, including $L^p(\mathbb{S}^{d-1})$, $p > 0$, these moduli were introduced and investigated in [13]. The direct and weak converse theorems for $L^p(\mathbb{S}^{d-1})$, $1 \leqslant p \leqslant \infty$ were given in [14, p. 23] and [13, p. 197], respectively. An easier proof of the direct result applicable to a more general class of spaces was given in [8]. In [9, (9.1)] it is shown that $\tilde{\omega}_r(f, t)_p$ is equivalent to $\omega_r^*(f, t)_p$ when $1 < p < \infty$, whereas the equivalence fails for $p = 1$ and $p = \infty$ [15].

The modulus of smoothness defined via multipliers, such as $\omega_r^*(f, t)_p$, allows an easy access to a neat theory, but it is hard to compute and more difficult to follow because of its dissimilarity to the traditional modulus of smoothness defined via differences of function evaluations. The modulus $\tilde{\omega}_r(f, t)_p$, on the other hand, is closer to the traditional form, as distance on the sphere is measured by geodesic distance; in fact, the authors in [4] considered it the most natural definition on the sphere. However, the supremum over O_t makes it difficult, if at all possible, to compute even for simple functions.

In the present paper we shall introduce another modulus of smoothness, denoted by $\omega_r(f, t)_p$, that can be expressed as forward differences in Euler angles. More precisely,

$$\omega_r(f, t)_p := \sup_{|\theta|\leqslant t} \max_{1\leqslant i<j\leqslant d} \left\|\triangle_{i,j,t}^r f\right\|_p, \tag{1.7}$$

where $\triangle_{i,j,t}^r$ denotes the r-th forward difference in the Euler angle $\theta_{i,j}$. These angles can be described (see next section and [32, Chapt. 9]) by rotations on two-dimensional planes, so that the new modulus of smoothness can be defined through a collection of two-dimensional forwarded differences, which are well understood and can be easily computed. Examples of functions will be given in Section 9, where $\omega_r(f, t)_p$ is computed whereas we do not see how to compute $\tilde{\omega}_r(f, t)_p$ or $\omega_r^*(f, t)_p$. Both direct and inverse theorems will be established in terms of this new modulus of smoothness. We will also define a new K-functional, using the derivatives with respect to the Euler angles, and show that it is equivalent to $\omega_r(f, t)_p$. Comparing to the other moduli of smoothness, we shall prove that $\omega_r(f, t)_p$ is bounded by both $\tilde{\omega}_r(f, t)_p$ and $\omega_r^*(f, t)_p$, and is equivalent to them for $1 < p < \infty$ and $r = 1, 2$. The strength of the new modulus of smoothness lies in its computability. We will give examples to show how the asymptotic order of $\omega_r(f, t)_p$ can be determined.

By taking the norm in a weighted L^p space, we can also define our modulus of smoothness for a doubling weight. Best approximation in a weighted space with respect to a doubling weight was first investigated, on an interval, in [23,24]. It was studied in [6] on the sphere in terms of weighted version of the modulus of smoothness $\omega_r(f, t)_p$. We shall show that the results in [6] can be established using our new modulus of smoothness.

1.2. Approximation on the unit ball

For the unit ball $\mathbb{B}^d$, a modulus of smoothness, denoted by $\omega_r^*(f, t)_{p,\mu}$, as it is in the spirit of $\omega_r^*(f, t)_p$ in (1.3), is introduced in [36, p. 503] for $L^p(\mathbb{B}^d, W_\mu)$, where

$$W_\mu(x) := \left(1 - \|x\|^2\right)^{\mu-1/2}, \quad \mu \geqslant 0, \tag{1.8}$$

in terms of the generalized translation operator of the orthogonal series, given explicitly in [36, p. 500], and used to characterize the best approximation by polynomials. There were also earlier results in [28, p. 164] of direct theorem given in terms of $\sup_{\|h\|\leqslant t} |f(x+h) - f(x)|$, which, however, does not take into account the boundary of $\mathbb{B}^d$ and, hence, does not have a matching inverse theorem. At the moment, the modulus $\omega_r^*(f,t)_{p,\mu}$ is the only one that gives both direct and inverse theorem for $d > 1$.

There is a close connection between analysis on the sphere and on the unit ball $\mathbb{B}^d = \{x\colon \|x\| \leqslant 1\}$ ([35, Sect. 4], [37, Sect. 4]). Our results on the sphere can be used to define a new modulus of smoothness and a new K-functional on the unit ball with weight function W_μ for μ being a half integer, and all results on the sphere can be carried over to the weighted approximation on the ball. It is worth to mention that our results appear to be new even in the case of $d = 1$, which corresponds to the extensively studied case of best approximation in $L^p([-1,1], W_\mu)$, and our new modulus of smoothness takes the form

$$\omega_r(f,t)_{p,\mu} = \sup_{|\theta|\leqslant t} \left(\int_{\mathbb{B}^2} \left|\overrightarrow{\triangle}_\theta^r f\left(x\cos(\cdot) + y\sin(\cdot)\right)\right|^p \left(1 - x^2 - y^2\right)^{\mu-1} dx\,dy \right)^p.$$

There are several well-studied moduli of smoothness in this setting of one variable. Among others the most established one is due to Ditzian and Totik [16, p. 11], defined in the unweighted case ($\mu = 1/2$ in (1.8)) by

$$\omega_\varphi^r(f,t)_p \equiv \hat{\omega}_r(f,t)_p := \sup_{0<\theta\leqslant t} \left\|\widehat{\triangle}_{\theta\varphi}^r f\right\|_{L^p[-1,1]}, \quad 1 \leqslant p \leqslant \infty, \tag{1.9}$$

where $\varphi(x) = \sqrt{1-x^2}$, $\widehat{\triangle}_h^r f(x)$ is the r-th central difference which equals 0 when $x \pm \frac{h}{2} \notin [-1,1]$ (see [16, p. 11] or (5.27) below for details) and we have dropped $\mu = 1/2$ in the notation of the norm. It turns out that

$$\omega_r(f,t)_{p,1/2} \leqslant c\,\omega_\varphi^r(f,t)_p, \quad 1 \leqslant p \leqslant \infty.$$

The comparison between the last two moduli of smoothness and the K-functional equivalent to $\omega_\varphi^r(f,t)_{p,\mu}$ suggests yet another pair of modulus of smoothness and K-functional on the unit ball, which can be regarded as a natural extension to those defined by Ditzian and Totik. In the unweighted case, it is defined by

$$\omega_\varphi^r(f,t)_{L^p(\mathbb{B}^d)} \equiv \hat{\omega}_r(f,t)_p := \sup_{|\theta|\leqslant t} \left\{ \max_{1\leqslant i<j\leqslant d} \left\|\triangle_{i,j,\theta}^r f\right\|_p, \max_{1\leqslant i\leqslant d} \left\|\widehat{\triangle}_{\theta\varphi e_i}^r f\right\|_p \right\}, \tag{1.10}$$

where $\varphi(x) = \sqrt{1 - \|x\|^2}$, e_i denotes the i-th coordinator vector, and $\|\cdot\|_p$ is the L^p norm computed with respect to the Lebesgue measure on $\mathbb{B}^d$ (see the definition in Section 7.3 for details). A corresponding K-functional can also be defined. We are able to prove the main results normally associated with moduli of smoothness and K-functionals for this pair.

These new moduli of smoothness and K-functionals provide, we believe, a satisfactory solution for the problem of characterizing the best approximation on the unit ball, and new tools for gauging the smoothness of functions on the unit ball. Our computation examples give the asymptotic order of the moduli of smoothness for several functions, which are not intuitively evident, and will be hard to distinguish without the new moduli of smoothness.

1.3. Organization of the paper

The paper is naturally divided into three parts. Part 1 deals with approximation on the sphere, whereas Part 2 deals with approximation on the ball. Part 3 contains examples of functions for which the asymptotic order of new moduli of smoothness and best approximation by polynomials are determined.

Throughout this paper we denote by $c, c_1, c_2, \dots$ generic constants that may depend on fixed parameters, whose value may vary from line to line. We write $A \lesssim B$ if $A \leqslant cB$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

Part 1. Approximation on the unit sphere

This part is organized as follows. The new modulus of smoothness and K-functional on the sphere are defined and studied in Section 2, their equivalence and the characterization of best approximation in terms of them are proved in Section 3. Finally, in Section 4, we discuss the weighted approximation with respect to a doubling weight.

2. A new modulus of smoothness and K-functional

2.1. Euler angles and Laplace–Beltrami operators

In the case of $d = 3$, the Euler angles are often used to describe motions in the Euclidean space and are well known to physicists and people working in computer graphics. We shall need the definition for $d \geqslant 3$, for which we follow [32, p. 438].

Let $e_1, \dots, e_d$ denote the standard orthogonal basis in $\mathbb{R}^d$. For $1 \leqslant i \neq j \leqslant d$ and $t \in \mathbb{R}$, we denote by $Q_{i,j,t}$ a rotation by the angle t in the (x_i, x_j)-plane, oriented such that the rotation from the vector e_i to the vector e_j is assumed to be positive. For example, the action of the rotation $Q_{1,2,t} \in SO(d)$ is given by

$$\begin{aligned} Q_{1,2,t}(x_1, \dots, x_d) &= (x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t, x_3, \dots, x_d) \\ &= \bigl(s \cos(\phi + t), s \sin(\phi + t), x_3, \dots, x_d\bigr), \end{aligned} \tag{2.1}$$

where $(x_1, x_2) = s(\cos\phi, \sin\phi)$, and other $Q_{i,j,t}$ are defined likewise. It is known [32, p. 438] that every rotation $Q \in SO(d)$ can be presented in the form

$$Q = Q_{d-1} Q_{d-2} \cdots Q_1, \quad \text{where } Q_k = Q_{1,2,\theta_1^k} Q_{2,3,\theta_2^k} \cdots Q_{k,k+1,\theta_k^k} \tag{2.2}$$

for some $\theta_1^k \in [0, 2\pi)$ and $\theta_2^k, \dots, \theta_k^k \in [0, \pi)$, and the representation (2.2) is unique for almost all elements Q of $SO(d)$. The numbers

$$\theta_j^k, \quad 1 \leqslant j \leqslant k,\ 1 \leqslant k \leqslant d-1$$

are called the Euler angles of the rotation Q. There are a total $d(d-1)/2$ Euler angles, which agrees with the dimension of $SO(d)$.

We note that an Euler angle comes from a two-dimensional rotation. The following simple fact is useful in our development below.

Lemma 2.1. *Suppose that $1 \leqslant i \neq j \leqslant d$ and $x, y \in \mathbb{S}^{d-1}$ differ only at their i-th and j-th components. Then $y = Q_{i,j,t}x$ with the angle t satisfying*

$$\cos t = (x_i y_i + x_j y_j)/s^2 \quad \textit{and} \quad t \sim \|x-y\|/s \quad \textit{with } s := \sqrt{x_i^2 + x_j^2}.$$

Proof. Since x and y differ at exactly two components, they differ by a two-dimensional rotation. Moreover, as $x_i^2 + x_j^2 = y_i^2 + y_j^2$, the formula for $\cos t$ is the classical formula for the angle between two vectors in $\mathbb{R}^2$. We also have

$$t^2 \sim 4\sin^2\frac{t}{2} = 2(1-\cos t) = \big\|(x_i, x_j) - (y_i, y_j)\big\|^2/s^2 = \|x-y\|^2/s^2,$$

where the first $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^2$ and the second one is of $\mathbb{R}^d$. $\square$

To each $Q \in SO(d)$ corresponds an operator $L(Q)$ in the space $L^2(\mathbb{S}^{d-1})$, defined by $L(Q)f(x) := f(Q^{-1}x)$ for $x \in \mathbb{S}^{d-1}$. Since $L(Q_1Q_2) = L(Q_1)L(Q_2)$, it is a group representation of $SO(d)$. In terms of Euler angles, the infinitesimal operator of $L(Q_{i,j,t})$ has the form

$$D_{i,j} := \frac{\partial}{\partial t}\big[L(Q_{i,j,t})\big]\big|_{t=0} = x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}, \quad 1 \leqslant i < j \leqslant d, \tag{2.3}$$

where the second equation follows from (2.1). For more details, see [32, Chapt. IX]. In particular, it is easy to verify that, taking $(i,j) = (1,2)$ as an example,

$$D_{1,2}^r f(x) = \left(-\frac{\partial}{\partial\phi}\right)^r f(s\cos\phi, s\sin\phi, x_3, \dots, x_d), \tag{2.4}$$

where $(x_1, x_2) = (s\cos\phi, s\sin\phi)$. It turns out that these operators are closely related to the Laplace–Beltrami operator on the sphere. Let $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ be the usual Laplace operator. The Laplace–Beltrami operator is defined by the relation

$$\Delta_0 f(x) = \Delta\left[f\left(\frac{y}{\|y\|}\right)\right](x), \quad x \in \mathbb{S}^{d-1}, \tag{2.5}$$

where the Laplace operator Δ acts on the variables y. The explicit formula of $\Delta_0 f(x)$, $x \in \mathbb{S}^{d-1}$, is often given in terms of differential operators in spherical coordinates as in [32, p. 494]. It turns out, however, that it also satisfies a decomposition,

$$\Delta_0 = \sum_{1 \leqslant i < j \leqslant d} D_{i,j}^2. \tag{2.6}$$

When applied to a function, the right-hand side of this decomposition is defined for all $x \in \mathbb{R}^d$, but the above identity holds for those x restricted to $\mathbb{S}^{d-1}$. The point is that each operator $D_{i,j}$ in this decomposition commutes with the Laplace–Beltrami operator Δ_0. This decomposition must be classical but we are not aware of a convenient reference. It can be easily verified, however. In fact, a straightforward computation of $\sum_{i=1}^d \partial_i^2 f(y/\|y\|)$ shows, by (2.5), that

$$\Delta_0 = \Delta - \sum_{i=1}^{d}\sum_{j=1}^{d} x_i x_j \partial_i \partial_j - (d-1)\sum_{i=1}^{d} x_i \partial_i, \tag{2.7}$$

where ∂_i denotes the i-th partial derivative, and the right-hand side of (2.6) gives the same formula as an other straightforward computation shows.

2.2. *New modulus of smoothness and K-functional*

For each $Q \in SO(d)$, we have defined $\triangle_Q^r f = (I - T_Q)^r f$ in (1.6). For the rotations $Q_{i,j,\theta}$ in the Euler angles, we shall denote

$$\triangle_{i,j,\theta}^r := \triangle_{Q_{i,j,\theta}}^r, \quad 1 \leqslant i \neq j \leqslant d$$

for convenience. Since $Q_{i,j,\theta} = Q_{j,i,-\theta}$, we have $\triangle_{i,j,\theta}^r = \triangle_{j,i,-\theta}^r$. Let $\overrightarrow{\triangle}_\theta^r$ denote the forward difference operator acting on $f : \mathbb{R} \mapsto \mathbb{R}$, defined by $\overrightarrow{\triangle}_\theta f(t) := f(t+\theta) - f(t)$ and $\overrightarrow{\triangle}_\theta^r := \overrightarrow{\triangle}_\theta^{r-1}\overrightarrow{\triangle}_\theta$; then

$$\overrightarrow{\triangle}_\theta^r f(t) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} f(t + \theta j).$$

Because of (2.1), it follows that $\triangle_{i,j,\theta}^r$ can be expressed in the forward difference. For instance, take $(i,j) = (1,2)$ as example,

$$\triangle_{1,2,\theta}^r f(x) = \overrightarrow{\triangle}_\theta^r f\big(x_1\cos(\cdot) - x_2\sin(\cdot), x_1\sin(\cdot) + x_2\cos(\cdot), x_3, \ldots, x_d\big), \tag{2.8}$$

where $\overrightarrow{\triangle}_\theta^r$ is acted on the variable $(\cdot)$, and is evaluated at $t = 0$.

Definition 2.2. For $r \in \mathbb{N}$, $t > 0$, and $f \in L^p(\mathbb{S}^{d-1})$, $1 \leqslant p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ for $p = \infty$, define

$$\omega_r(f,t)_p := \sup_{|\theta| \leqslant t} \max_{1 \leqslant i < j \leqslant d} \big\| \triangle_{i,j,\theta}^r f \big\|_p. \tag{2.9}$$

For $r = 1$ we write $\omega(f,t)_p := \omega_1(f,t)_p$.

Let us remark that this modulus of smoothness is not rotationally invariant, that is, if we define $f_Q(x) = f(Qx)$, then $\omega_r(f_Q,t)_p$ is in general different from $\omega_r(f,t)_p$, whereas both $\omega_r^*(f,t)_p$ and $\tilde{\omega}_r(f,t)_p$ are rotationally invariant. Moreover, $\tilde{\omega}_r(f,t)_p$ does not depend on the choice of the orthogonal basis of $\mathbb{R}^d$, whereas, on the face of it, the new modulus $\omega_r(f,t)_p$ relies on the standard basis $e_1, \dots, e_d$ of $\mathbb{R}^d$, but do not, we note, on the order of $e_1, \dots, e_d$. As will be shown later in Section 3.4, the three moduli are nevertheless closely related (see Corollary 3.11 for details) and the result below shows that $\omega_r(f,t)_p$ is smaller than $\tilde{\omega}_r(f,t)_p$.

Recall $O_t = \{Q \in SO(d)\colon \max_{x\in\mathbb{S}^{d-1}} d(x, Qx) \leqslant t\}$. For $d=2$, there is only one Euler angle θ; a rotation $Q \in SO(2)$ belongs to O_t if and only if its Euler angle θ satisfies $|\sin\theta| \leqslant \sin t$. Hence, for $d=2$, $\omega_r(f,t)_p$ agrees with $\tilde{\omega}_r(f,t)_p$ in (1.6). This, however, does not extend to $d \geqslant 3$. A rotation $Q \in SO(d)$ belonging to O_t may not be easily characterized by its Euler angles. For example, for $d=3$, the rotation $Q_{1,2,2\pi-\theta}Q_{2,3,t}Q_{1,2,\theta}$ is in O_t for all $\theta \in (0, 2\pi)$, as can be easily seen from (2.1). On the other hand, for $x \in \mathbb{S}^{d-1}$, a quick computation shows that

$$\langle Q_{i,j,\theta}x, x\rangle = \left(x_i^2 + x_j^2\right)\cos\theta + \sum_{k\neq i,j} x_k^2 = \cos\theta + \sum_{k\neq i,j} x_k^2(1-\cos\theta) \geqslant \cos\theta.$$

Consequently, since $\cos d(x,y) = \langle x, y\rangle$, we obtain

$$d(Q_{i,j,\theta}x, x) = \arccos\langle Q_{i,j,\theta}x, x\rangle \leqslant \theta$$

which shows that $Q_{i,j,\theta} \in O_t$ for $0 < \theta \leqslant t$. As a result, we immediately see that the following proposition holds.

Proposition 2.3. *For $f \in L^p(\mathbb{S}^{d-1})$ if $1 \leqslant p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$,*

$$\omega_r(f,t)_p \leqslant \tilde{\omega}_r(f,t)_p, \quad 1 \leqslant p \leqslant \infty,\ r \in \mathbb{N}.$$

The main advantage of the new modulus of smoothness is that it reduces to forward differences in Euler angles, which live on two-dimensional circles on the sphere, and many of its properties can be deduced from the corresponding results for trigonometric functions of one variable.

Our new K-functional is defined via the differential operators $D_{i,j}$ in (2.3), which can be regarded as derivatives with respect to the Euler angles.

Definition 2.4. For $r \in \mathbb{N}_0$ and $t \geqslant 0$,

$$K_r(f,t)_p := \inf_{g\in C^r(\mathbb{S}^{d-1})} \left\{\|f-g\|_p + t^r \max_{1\leqslant i<j\leqslant d} \left\|D_{i,j}^r g\right\|_p\right\}. \tag{2.10}$$

One usually defines the K-functional in L^p norm by taking the minimum over a Sobolev space, such as

$$W_r^p := \left\{g \in L^p\left(\mathbb{S}^{d-1}\right)\colon \left\|D_{i,j}^r g\right\|_p < \infty,\ 1 \leqslant i \neq j \leqslant d\right\}.$$

Since we will deal with several different K-functionals, it is more convenient to take the minimum over $C^r(\mathbb{S}^{d-1})$, the space of r-th continuous differentiable functions, which however is no less general by the density of $C^r(\mathbb{S}^{d-1})$ in the Sobolev spaces. In Section 3, we will show that $K_r(f,t)_p$ is equivalent to $\omega_r(f,t)_p$.

2.3. Properties of the modulus of smoothness

We will need the following elementary lemma (see Lemmas 3.8.9 and 3.6.1 in [17]), in which $d\sigma$ denotes the usual Lebesgue measure on $\mathbb{S}^{d-1}$ without normalization.

Lemma 2.5. *Let d and m be positive integers. If $m \geqslant 2$, then*

$$\int_{\mathbb{S}^{d+m-1}} f(y)\,d\sigma = \int_{\mathbb{B}^d} \left(1-\|x\|^2\right)^{\frac{m-2}{2}} \left[\int_{\mathbb{S}^{m-1}} f\left(x, \sqrt{1-\|x\|^2}\,\xi\right) d\sigma(\xi)\right] dx, \tag{2.11}$$

whereas if $m = 1$, then

$$\int_{\mathbb{S}^d} f(y)\,d\sigma = \int_{\mathbb{B}^d} \left[f\left(x, \sqrt{1-\|x\|^2}\right) + f\left(x, -\sqrt{1-\|x\|^2}\right)\right] \frac{dx}{\sqrt{1-\|x\|^2}}. \tag{2.12}$$

Because of the maximum in the definition of $\omega_r(f,t)_p$, it is more convenient, and often more useful, to state the properties on $\sup_{|\theta|\leqslant t} \|\triangle^r_{i,j,\theta} f\|_p$, some of which are collected in the lemma below.

Lemma 2.6. *Let $r \in \mathbb{N}$ and let $f \in L^p(\mathbb{S}^{d-1})$ with $1 \leqslant p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ when $p = \infty$.*

(i) *For any $\lambda > 0$, $t \in (0, 2\pi]$, and $1 \leqslant i < j \leqslant d$, we have*

$$\sup_{|\theta|\leqslant \lambda t} \left\|\triangle^r_{i,j,\theta} f\right\|_p \leqslant (\lambda+1)^r \sup_{|\theta|\leqslant t} \left\|\triangle^r_{i,j,\theta} f\right\|_p.$$

(ii) *For $1 \leqslant i \neq j \leqslant d$ and $\theta \in [-\pi, \pi]$,*

$$\left\|\triangle^r_{i,j,\theta} f\right\|_p \leqslant 2^r \|f\|_p \quad \text{and} \quad \left\|\triangle^r_{i,j,\theta} f\right\|_p \leqslant c|\theta|^r \left\|D^r_{i,j} f\right\|_p.$$

(iii) *If $f \in \Pi^d_n$ and $1 \leqslant i < j \leqslant d$, then*

$$\left\|\triangle^r_{i,j,n^{-1}} f\right\|_p \sim n^{-r} \left\|D^r_{i,j} f\right\|_p.$$

(iv) *For $1 \leqslant i < j \leqslant d$ and $t \in (0, 2\pi)$,*

$$\sup_{|\theta|\leqslant t} \left\|\triangle^r_{i,j,\theta} f\right\|_p^p \sim \frac{1}{t} \int_0^t \left\|\triangle^r_{i,j,\theta} f\right\|_p^p d\theta$$

with $\|\cdot\|_p^p$ replaced by $\|\cdot\|_\infty$ when $p = \infty$.

Proof. Clearly we only need to consider the case of $(i,j) = (1,2)$. For f defined on $\mathbb{S}^{d-1}$ we set $g_{s,y}(\phi) := f(s\cos\phi, s\sin\phi, \sqrt{1-s^2}\,y)$, where $y \in \mathbb{S}^{d-3}$, $s \in [0,1]$ and $\phi \in [0, 2\pi]$.

(i) For a positive integer n, the well-known identity

$$\overrightarrow{\triangle}_{n\theta}^r g(t) = \sum_{\nu_1=0}^{n-1} \cdots \sum_{\nu_r=0}^{n-1} \overrightarrow{\triangle}_\theta^r g(t + \nu_1\theta + \cdots + \nu_r\theta)$$

and the connection (2.8) imply immediately the inequality

$$\sup_{|\theta|\leqslant nt} \left\| \triangle_{i,j,\theta}^r f \right\|_p \leqslant n^r \sup_{|\theta|\leqslant t} \left\| \triangle_{i,j,\theta}^r f \right\|_p,$$

from which (i) follows from the monotonicity of $\sup_{|\theta|\leqslant t} \|\triangle_{i,j,\theta}^r f\|_p$ in t and $n = \lfloor \lambda \rfloor$.

(ii) By (2.11), for $d > 3$,

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} f(y)\,d\sigma(y) &= \int_{\mathbb{B}^2} \int_{\mathbb{S}^{d-3}} f\big(x_1, x_2, \sqrt{1-\|x\|^2}\,y\big)\,d\sigma(y)\,\big(1-\|x\|^2\big)^{\frac{d-4}{2}}\,dx \\ &= \int_0^1 s\big(1-s^2\big)^{\frac{d-4}{2}} \int_{\mathbb{S}^{d-3}} \int_0^{2\pi} g_{s,y}(\phi)\,d\phi\,d\sigma(y)\,ds, \end{aligned} \tag{2.13}$$

where the second equality follows from changing variables $(x_1, x_2) = s(\cos\phi, \sin\phi)$. Using (2.8), the identity (2.13) implies immediately

$$\left\| \triangle_{1,2,t}^r f \right\|_p^p = \int_0^1 s\big(1-s^2\big)^{\frac{d-4}{2}} \int_{\mathbb{S}^{d-3}} \left[\int_0^{2\pi} \left| \overrightarrow{\triangle}_t^r g_{s,y}(\phi) \right|^p d\phi \right] d\sigma(y)\,ds. \tag{2.14}$$

In the case of $d = 3$, the formula (2.13) degenerated to a form in which the integral over $\mathbb{S}^{d-3}$ is replaced by a sum of two terms, see (2.12). By (2.1) and (2.3), it is easy to see that

$$(-1)^r D_{1,2}^r f\big(s\cos\phi, s\sin\phi, \sqrt{1-s^2}\,y\big) = g_{s,y}^{(r)}(\phi) \tag{2.15}$$

and $g_{s,y}(\phi)$ is a 2π-periodic function. Hence, the desired result follows from the corresponding result for the trigonometric functions on $\mathbb{T}$.

(iii) If $f \in \Pi_n^d$, then $g_{s,y}(\phi)$ is a trigonometric polynomial of degree at most n in ϕ. The classical result of Stečkin [31] shows that for a trigonometric polynomial T_n of degree at most n,

$$\left\| T_n^{(r)} \right\|_{L^p(\mathbb{T})} \sim h^{-r} \left\| \overrightarrow{\triangle}_h^r T_n \right\|_{L^p(\mathbb{T})}, \quad 0 < h \leqslant \pi n^{-1}$$

with the constant of equivalence depending only on r. Thus, (iii) follows by (2.14) and (2.15).

(iv) This again follows from (2.14) and the corresponding result for trigonometric function. Indeed, by [26, p. 191, Lemma 7.2], we have for $0 < t \leqslant 2\pi$,

$$\sup_{|\theta|\leqslant t} \left(\int_0^{2\pi} \left| \overrightarrow{\triangle}_\theta^r g_{s,y}(\phi) \right|^p d\phi \right)^{\frac{1}{p}} \sim \left(\frac{1}{t} \int_0^t \int_0^{2\pi} \left| \overrightarrow{\triangle}_\theta^r g_{s,y}(\phi) \right|^p d\phi\,d\theta \right)^{\frac{1}{p}}$$

with the usual modification when $p = \infty$, from which (iv) follows from (2.14). □

Proposition 2.7. *The modulus of smoothness $\omega_r(f,t)_p$ satisfies*

(1) *For $s<r$, $\omega_r(f,t)_p \leqslant 2^{r-s}\omega_s(f,t)_p$.*
(2) *For $\lambda>0$, $\omega_r(f,\lambda t)_p \leqslant (\lambda+1)^r\omega_r(f,t)_p$.*
(3) *For $0<t<\frac{1}{2}$ and every $m>r$,*

$$\omega_r(f,t)_p \leqslant c_m t^r \int_t^1 \frac{\omega_m(f,u)_p}{u^{r+1}}\,du.$$

Proof. The first property follows from the identity

$$(I-T)^r = (I-T)^s \sum_{k=0}^{r-s} \binom{r-s}{k} (-1)^k T^k$$

and the triangle inequality. The second one follows immediately from (ii) of Lemma 2.6. The third one is the Marchaud type inequality and it follows, by (2.14), from Marchaud inequality for the trigonometric functions, in which the additional term $t^r\|f\|_p$ that usually appears in the right hand can be removed upon using $\inf_{c\in\mathbb{R}}\|f-c\|_p \lesssim \omega_r(f,\pi)$ that follows from Lemma 3.2 below. □

Recall the distance $d(x,y) := \arccos\langle x,y\rangle$ on $\mathbb{S}^{d-1}$. It follows that

$$\|x-y\| = \sqrt{2-2\cos d(x,y)} = 2\sin\frac{d(x,y)}{2} \sim d(x,y). \tag{2.16}$$

Lemma 2.8. *For $x,y\in\mathbb{S}^{d-1}$,*

$$\bigl|f(x)-f(y)\bigr| \leqslant c\omega\bigl(f,d(x,y)\bigr)_\infty,$$

where c depends only on dimension.

Proof. We may assume that $d(x,y)\leqslant \delta_d := 1/(2d^2)$. Otherwise we can select an integer m such that $d(x,y)\leqslant m\delta_d<1$, then m is finite and we can select points $x=z_0,z_1,\dots,z_m=y$ on the great circle connecting x and y on $\mathbb{S}^{d-1}$ such that $d(z_i,z_{i+1})=\frac{d(x,y)}{m}\leqslant\delta_d$ for $i=0,1,\dots,m-1$, and then use triangle inequality. Since $\|x\|=1$ implies that $|x_i|\geqslant 1/\sqrt{d}$ for at least one i, we can assume without losing generality, as $\omega_r(f,t)_p$ is independent of the order of $e_1,\dots,e_d$, that $x_d=\max_{1\leqslant j\leqslant d}|x_j|\geqslant\frac{1}{\sqrt{d}}$.

For $1\leqslant j\leqslant d-2$, let $u_j' := (x_1,\dots,x_j,y_{j+1},\dots,y_{d-1})$ and $v_j := \sqrt{1-\|u_j'\|^2}$, where by the choice of x_d and δ_d,

$$\begin{aligned}\|u_j'\|^2 &= 1-\bigl(x_{j+1}^2-y_{j+1}^2\bigr)-\cdots-\bigl(x_{d-1}^2-y_{d-1}^2\bigr)-x_d^2\\ &\leqslant 1-\frac{1}{\sqrt{d}}+2(d-j-2)d(x,y)\leqslant 1-\frac{1}{\sqrt{d}}+\frac{1}{d}<1.\end{aligned}$$

We then define $u_0 = y$, $u_j = (u'_j, v_j) \in \mathbb{S}^{d-1}$ for $1 \leqslant j \leqslant d-2$, and $u_{d-1} = x$. By definition, u_j and u_{j-1} differ at exactly j-th and d-th elements, so that we can write $u_{j-1} = Q_{j,d,t_j} u_j$, where the Euler angle t_j satisfies, by Lemma 2.1,

$$t_j \sim \|u_{j-1} - u_j\|/s_j, \quad \text{where } s_j^2 = x_j^2 + v_j^2.$$

Our assumption shows that

$$s_j^2 \geqslant v_j^2 = x_d^2 + \left(x_{j+1}^2 - y_{j+1}^2\right) + \cdots + \left(x_{d-1}^2 - y_{d-1}^2\right) \geqslant d^{-1} - \|x-y\|^2 \geqslant \frac{1}{2d},$$

and, on the other hand, by (2.16),

$$\|u_j - u_{j-1}\|^2 = |x_j - y_j|^2 + \frac{(x_j^2 - y_j^2)^2}{(v_{j-1} + v_j)^2} \leqslant (1+8d)|x_j - y_j|^2 \lesssim d(x,y)^2.$$

Together the last three displayed equations imply that $t_j \lesssim d(x,y)$. Hence,

$$\left|f(x) - f(y)\right| \leqslant \sum_{j=1}^{d-1} \left|f(Q_{j,d,t_j} u_j) - f(u_j)\right| \leqslant (d-1)\omega\big(f, cd(x,y)\big)_\infty \leqslant c\omega\big(f, d(x,y)\big)_\infty,$$

where the last step uses (2) of Proposition 2.7. □

3. Approximation on the unit sphere

In this section we show that our new modulus of smoothness and K-functional are equivalent and use them to establish direct and inverse theorem for

$$E_n(f)_p := \inf_{g \in \Pi_{n-1}^d} \|f - g\|_p, \quad n = 1, 2, \ldots, \ 1 \leqslant p \leqslant \infty. \tag{3.1}$$

3.1. Preliminaries

Recall that Π_n^d denote the spherical polynomials of degree at most n. Let $\mathcal{H}_n^d$ denote the space of spherical harmonics of degree n, which are the restriction of homogeneous harmonic polynomials on $\mathbb{S}^{d-1}$. It is well known that the reproducing kernel of the space $\mathcal{H}_n^d$ in $L^2(\mathbb{S}^{d-1})$ is given by the zonal harmonic

$$Z_{n,d}(x,y) := \frac{n+\lambda}{\lambda} C_n^\lambda\big(\langle x,y\rangle\big), \quad \lambda = \frac{d-2}{2}, \tag{3.2}$$

where C_n^λ is the Gegenbauer polynomial with index λ, normalized by $C_n^\lambda(1) = \binom{n+2\lambda-1}{n}$.

Let η be a C^∞-function on $[0,\infty)$ with the properties that $\eta(x) = 1$ for $0 \leqslant x \leqslant 1$ and $\eta(x) = 0$ for $x \geqslant 2$. We define

$$V_n f(x) = \int_{\mathbb{S}^{d-1}} f(y) K_n\big(\langle x,y\rangle\big)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \ n = 1, 2, \ldots \tag{3.3}$$

with

$$K_n(t) = \sum_{k=0}^{2n} \eta\left(\frac{k}{n}\right) \frac{k+\lambda}{\lambda} C_k^\lambda(t), \quad t \in [-1,1].$$

By now it is well known (cf. [30, p. 316]) that $V_n(f)$ satisfies the following properties:

Lemma 3.1. *Let $f \in L^p$ if $1 \leqslant p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. Then*

(1) *$V_n f \in \Pi_{2n}^d$ and $V_n f = f$ for $f \in \Pi_n^d$.*
(2) *For $n \in \mathbb{N}$, $\|V_n f\|_p \leqslant c\|f\|_p$.*
(3) *For $n \in \mathbb{N}$,*

$$\|f - V_n f\|_p \leqslant c E_n(f)_p.$$

More importantly, the kernel is highly localized [3, p. 409]; that is, for any positive integer ℓ, $K_n(t)$ satisfies

$$\left|K_n(\cos\theta)\right| \leqslant c_\ell n^{d-1}(1+n\theta)^{-\ell} =: G_n(\theta), \quad \theta \in [0,\pi]. \tag{3.4}$$

The following lemma plays an essential role in our study below.

Lemma 3.2. *Suppose that $f \in L^p(\mathbb{S}^{d-1})$ for $1 \leqslant p < \infty$, and $G_n(t) \equiv G_{n,\ell}(t)$ is given by* (3.4) *with $\ell > p + d$. Then*

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \left|f(x) - f(y)\right|^p G_n\big(d(x,y)\big)\, d\sigma(x)\, d\sigma(y) \leqslant c\omega\big(f, n^{-1}\big)_p^p.$$

Proof. Let $E_j^+ := \{x \in \mathbb{S}^{d-1}\colon x_j \geqslant \frac{1}{\sqrt{d}}\}$ and $E_j^- := \{x \in \mathbb{S}^{d-1}\colon x_j \leqslant -\frac{1}{\sqrt{d}}\}$ for $1 \leqslant j \leqslant d$. Then $\mathbb{S}^{d-1} = \bigcup_{j=1}^d (E_j^+ \cup E_j^-)$. Hence, it is enough to show that for each $1 \leqslant k \leqslant d$,

$$\int_{E_k^\pm} \int_{\mathbb{S}^{d-1}} \left|f(x) - f(y)\right|^p \left|G_n\big(d(x,y)\big)\right| d\sigma(y)\, d\sigma(x) \leqslant c\omega\big(f, n^{-1}\big)_p^p. \tag{3.5}$$

By symmetry, it is enough to consider E_d^+. For $0 < \delta < \pi$ and $x \in \mathbb{S}^{d-1}$, let $c(x,\delta)$ denote the spherical cap defined by

$$c(x,\delta) := \left\{y \in \mathbb{S}^{d-1}\colon d(x,y) \leqslant \delta\right\}.$$

We choose $\delta = 1/(100d)$ and split the integral in (3.5) into two parts:

$$\int_{E_d^+} \int_{c(x,\delta)} \cdots d\sigma(y)\, d\sigma(x) + \int_{E_d^+} \int_{\mathbb{S}^{d-1}\setminus c(x,\delta)} \cdots d\sigma(y)\, d\sigma(x) =: A + B.$$

We first estimate the integral B:

$$B = \int_{E_d^+} \int_{\{y \in \mathbb{S}^{d-1}:\, d(x,y) \geqslant \delta\}} \left|f(x) - f(y)\right|^p \left|G_n\big(d(x,y)\big)\right| d\sigma(y)\, d\sigma(x)$$

$$\leqslant cn^{d-1-\ell} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \left|f(x) - f(y)\right|^p d\sigma(x)\, d\sigma(y)$$

$$= cn^{d-1-\ell} \int_{SO(d)} \int_{\mathbb{S}^{d-1}} \left|f(x) - f(Qx)\right|^p d\sigma(x)\, dQ, \tag{3.6}$$

where the last step uses the standard realization of $S^{d-1} = SO(d)/SO(d-1)$. Using the decomposition of Q in terms of Euler angles as in (2.2), each $Q \in SO(d)$ can be decomposed as $Q = Q_1 Q_2 \cdots Q_{d(d-1)/2}$ with $Q_k = Q_{i_k, j_k, t_k}$ for some $1 \leqslant i_k < j_k \leqslant d$ and $t_k \in [0, 2\pi]$. It then follows that

$$\int_{\mathbb{S}^{d-1}} \left|f(x) - f(Qx)\right|^p d\sigma(x) \lesssim \int_{\mathbb{S}^{d-1}} \left|f(Q_{d(d-1)/2}x) - f(x)\right|^p d\sigma(x)$$

$$+ \sum_{k=1}^{\frac{d(d-1)}{2}-1} \int_{\mathbb{S}^{d-1}} \left|f(Q_k \cdots Q_{d(d-1)/2}x) - f(Q_{k+1} \cdots Q_{d(d-1)/2}x)\right|^p d\sigma(x)$$

$$\lesssim \max_{1 \leqslant i < j \leqslant d} \sup_{0 < \theta \leqslant 2\pi} \int_{\mathbb{S}^{d-1}} \left|f(Q_{i,j,\theta}x) - f(x)\right|^p d\sigma(x)$$

$$\lesssim \omega(f, 2\pi)_p^p \lesssim n^p \omega\big(f, n^{-1}\big)_p^p,$$

which, together with (3.6), gives the desired estimate $B \leqslant c\omega(f, n^{-1})_p^p$.

It remains to estimate the integral A. Setting $x = (x', x_d)$ with $x_d = \sqrt{1 - \|x'\|^2}$, we deduce from (2.12) that

$$A = \int_{E_d^+} \int_{c(x,\delta)} \left|f(x) - f(y)\right|^p G_n\big(d(x,y)\big)\, d\sigma(y)\, d\sigma(x)$$

$$= \int_{\|x'\| \leqslant d^*} \int_{c(x,\delta)} \left|f(x) - f(y)\right|^p G_n\big(d(x,y)\big)\, d\sigma(y) \frac{dx'}{\sqrt{1 - \|x'\|^2}},$$

where $d^* = \sqrt{1 - d^{-1}}$. Since $x_d \geqslant \frac{1}{\sqrt{d}}$ it follows that for any $y = (y', y_d) \in c(x, \delta)$, $y_d \geqslant x_d - |y_d - x_d| \geqslant x_d - d(x,y) \geqslant x_d - \delta \geqslant \frac{1}{2\sqrt{d}}$, which further implies, by a simple computation, that

$$\left\|x' - y'\right\| \leqslant \|x - y\| \leqslant \left\|x' - y'\right\| + |x_d - y_d| = \left\|x' - y'\right\| + \frac{|\|x'\|^2 - \|y'\|^2|}{x_d + y_d}$$

$$\leqslant (1+2\sqrt{d})\|x'-y'\|.$$

Consequently, setting $g(x') := f(x', \sqrt{1-\|x'\|^2})$, using (2.12) again and observing that $G_n(\theta_1) \sim G_n(\theta_2)$ whenever $\theta_1 \sim \theta_2$, we obtain

$$A \leqslant c \int\limits_{\|x'\|\leqslant d^*} \int\limits_{\|x'-y'\|\leqslant\delta} \left|g(x')-g(y')\right|^p G_n\left(\|x'-y'\|\right) dy'\,dx'$$
$$= c \int\limits_{\|x'\|\leqslant d^*} \int\limits_{\|u\|\leqslant\delta} \left|g(x')-g(u+x')\right|^p G_n\left(\|u\|\right) du\,dx'.$$

Let $b_0(u) := 0$ and $b_j(u) := u_1e_1 + \cdots + u_je_j$, $1 \leqslant j \leqslant d-1$. Since

$$g(x') - g(x'+u) = \sum_{j=1}^{d-1}\left(g\left(x'+b_{j-1}(u)\right) - g\left(x'+b_j(u)\right)\right),$$

by triangle inequality it suffices to estimate, for $1 \leqslant j \leqslant d-1$,

$$A_j := \int\limits_{\|x'\|\leqslant d^*} \int\limits_{\|u\|\leqslant\delta} \left|g\left(x'+b_{j-1}(u)\right)-g\left(x'+b_j(u)\right)\right|^p G_n\left(\|u\|\right) du\,dx'$$
$$\leqslant \int\limits_{\|x'\|\leqslant d^*+\delta} \int\limits_{\|u\|\leqslant\delta} \left|g(x')-g\left(x'+u_je_j\right)\right|^p G_n\left(\|u\|\right) du\,dx',$$

where the second line follows from a change of variables $x'+b_{j-1}(u) \mapsto x'$. By symmetry, it suffices to consider A_1.

Observe that for $u_1 \in \mathbb{R}$ and $u = (u_1, v) \in \mathbb{R}^{d-2}$,

$$G_n\left(\|u\|\right) = n^{d-1}\left(1+n\|u\|\right)^{-\ell} \leqslant H_n\left(|u_1|\right)n^{d-2}\left(1+n\|v\|\right)^{-d+1},$$

where $H_n(s) = n(1+ns)^{-\ell+d-1}$, and we have used the assumption $\ell > d-1$ in the last step. This implies

$$A_1 \lesssim \int\limits_{\|x'\|\leqslant d^*+\delta} \int_{-\delta}^{\delta} \left|g(x')-g\left(x'+u_1e_1\right)\right|^p$$
$$\times \left[\int\limits_{\{v\in\mathbb{R}^{d-2}:\ \|v\|\leqslant\sqrt{\delta^2-|u_1|^2}\}} G_n\left(\left\|(u_1,v)\right\|\right) dv\right] du_1\,dx'$$
$$\lesssim \int\limits_{\|x'\|\leqslant d^*+\delta} \int_{-\delta}^{\delta} \left|g(x')-g\left(x'+se_1\right)\right|^p H_n\left(|s|\right) ds\,dx'. \tag{3.7}$$

Set $v_1(t, x') = -x_1 + x_1 \cos t - \sqrt{1 - \|x'\|^2} \sin t$. A straightforward calculation shows that

$$\frac{1}{\pi\sqrt{d}} \leqslant \frac{-v_1(t, x')}{t} \leqslant 2 \quad \text{and} \quad \frac{1}{4\sqrt{d}} \leqslant -\frac{\partial}{\partial t} v_1(t, x') \leqslant 2 \tag{3.8}$$

whenever $|t| \leqslant \sqrt{d}\delta = \delta^*$ and $\|x'\| \leqslant d^* + \delta$. Thus, performing a change of variable $s = v_1(t, x')$ in (3.7) yields

$$\begin{aligned} A_1 &\leqslant c \int\limits_{\|x'\| \leqslant d^*+\delta} \int\limits_{-\delta^*}^{\delta^*} \left|g(x') - g(x' + v_1(x', t)e_1)\right|^p H_n\left(\left|v_1(x', t)\right|\right) \left|\frac{\partial v_1(t, x')}{\partial t}\right| dt\, dx' \\ &\leqslant c' \int\limits_{\|x'\| \leqslant \rho} \int\limits_{-\delta^*}^{\delta^*} \left|g(x') - g(x' + v_1(x', t)e_1)\right|^p H_n(|t|)\, dt\, dx', \end{aligned}$$

where $\rho := \sqrt{1 - (2d)^{-1}} \geqslant d^* + \delta$, and we used (3.8) and the monotonicity of H_n in the last step. Now observe that for $x = (x', \sqrt{1 - \|x'\|^2})$ with $\|x'\| \leqslant \rho$,

$$Q_{1,d,t}x = \left(x' + v_1(x', t)e_1, z_d\right), \quad \forall t \in [-\delta^*, \delta^*],$$

where, using the fact that $\sin t \leqslant t \leqslant 1/(8\sqrt{d})$,

$$z_d = \sqrt{1 - \left\|x' + v_1(x', t)e_1\right\|^2} = x_1 \sin t + \sqrt{1 - \left\|x'\right\|^2} \cos t \geqslant 1/(4\sqrt{d}) > 0.$$

Thus, using (2.12), we deduce that

$$\begin{aligned} A_1 &\lesssim \int\limits_{-\delta^*}^{\delta^*} \int\limits_{\{x \in \mathbb{S}^{d-1}:\, x_d \geqslant (2d)^{-1}\}} \left|f(x) - f(Q_{1,d,t}x)\right|^p d\sigma(x) H_n(|t|)\, dt \\ &\lesssim \int\limits_{-\delta^*}^{\delta^*} \omega(f, |t|)_p^p H_n(|t|)\, dt. \end{aligned}$$

Hence, by (2) of Proposition 2.7 and the definition of H_n,

$$\begin{aligned} A_1 &\lesssim \omega(f, n^{-1})_p^p \int\limits_0^{\delta^*} (1 + nt)^p H_n(t)\, dt = \omega(f, n^{-1})_p^p n \int\limits_0^{\delta^*} (1 + nt)^{-\ell+p+d-1}\, dt \\ &\lesssim \omega(f, n^{-1})_p^p \int\limits_0^{\infty} (1 + s)^{-\ell+p+d-1}\, ds \lesssim \omega(f, n^{-1})_p^p, \end{aligned}$$

since $\ell > p + d$. This completes the proof. □

The operator $V_n f$ plays an important role in our study. Our next lemma shows that it commutes with $\triangle_{i,j,t}$.

Lemma 3.3. *For $1 \leqslant p \leqslant \infty$ and $1 \leqslant i \neq j \leqslant d$,*

$$\triangle_{i,j,t}^r V_n f = V_n\big(\triangle_{i,j,t}^r f\big).$$

In particular, for $t > 0$,

$$\omega_r(f - V_n f, t)_p \leqslant c\omega_r(f, t)_p.$$

Proof. Recall that $T_Q f(x) = f(Qx)$ for $Q \in SO(d)$. By the definition of $V_n f$,

$$\begin{aligned} T_Q V_n f(x) &= \int_{\mathbb{S}^{d-1}} f(y) K_n\big(\langle Qx, y\rangle\big)\, d\sigma(y) = \int_{\mathbb{S}^{d-1}} f(y) K_n\big(\big\langle x, Q^{-1}y\big\rangle\big)\, d\sigma(y) \\ &= \int_{\mathbb{S}^{d-1}} f(Qy) K_n\big(\langle x, y\rangle\big)\, d\sigma(y) = V_n(T_Q f)(x) \end{aligned}$$

by the rotation invariance of $d\sigma(y)$, which gives the stated result as $\triangle_{i,j,t}^r = (I - T_{Q_{i,j,t}})^r$. By (2) of Lemma 3.1,

$$\big\|\triangle_{i,j,t}^r (f - V_n f)\big\|_p = \big\|\triangle_{i,j,t}^r f - V_n \triangle_{i,j,t}^r f\big\|_p \leqslant (1 + c)\big\|\triangle_{i,j,t}^r f\big\|_p,$$

from which the stated inequality follows. □

3.2. Direct and inverse theorems for best approximation

We start with direct and inverse theorem characterized by our new modulus of smoothness.

Theorem 3.4. *For $f \in L^p$ if $1 \leqslant p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$, we have*

$$E_n(f)_p \leqslant c\omega_r\big(f, n^{-1}\big)_p, \quad 1 \leqslant p \leqslant \infty. \tag{3.9}$$

On the other hand,

$$\omega_r\big(f, n^{-1}\big)_p \leqslant cn^{-r} \sum_{k=1}^{n} k^{r-1} E_{k-1}(f)_p, \quad 1 \leqslant p \leqslant \infty, \tag{3.10}$$

where $\omega_r(f, t)_p$ and $E_n(f)_p$ are defined in (2.9) *and* (3.1)*, respectively.*

Proof. When $r = 1$ and $1 \leqslant p < \infty$, we use Lemma 3.1, Hölder's inequality and the fact that $\int_{\mathbb{S}^{d-1}} |K_n(\langle x, y\rangle)|\, d\sigma(y) \leqslant c$ for all $x \in \mathbb{S}^{d-1}$ to obtain

$$E_n(f)_p \leqslant |f - V_{\lfloor \frac{n}{2} \rfloor} f\|_p$$
$$\lesssim \left(\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \left|f(x) - f(y)\right|^p \left|K_{\lfloor \frac{n}{2} \rfloor}\big(\langle x, y\rangle\big)\right| d\sigma(x)\, d\sigma(y) \right)^{\frac{1}{p}},$$

from which (3.9) for $r = 1$ follows from Lemma 3.2. For $r = 1$ and $p = \infty$, we use Lemma 2.8 and $V_n f$ to conclude

$$E_n(f)_\infty \leqslant \|f - V_{\lfloor \frac{n}{2} \rfloor} f\|_\infty \leqslant \int_{\mathbb{S}^{d-1}} \left|f(x) - f(y)\right| \left|K_{\lfloor \frac{n}{2} \rfloor}\big(\langle x, y\rangle\big)\right| d\sigma(y)$$
$$\lesssim \int_{\mathbb{S}^{d-1}} \omega\big(f, d(x, y)\big)_\infty \left|K_{\lfloor \frac{n}{2} \rfloor}\big(\langle x, y\rangle\big)\right| d\sigma(y)$$
$$\lesssim \omega\big(f, n^{-1}\big)_\infty \int_{\mathbb{S}^{d-1}} \big(1 + n d(x, y)\big) \left|K_{\lfloor \frac{n}{2} \rfloor}\big(\langle x, y\rangle\big)\right| d\sigma(y)$$
$$\lesssim \omega\big(f, n^{-1}\big)_\infty,$$

where the last inequality follows from (3.4) and the fact that $\langle x, y\rangle = \cos d(x, y)$, just like the estimate of A_j in the previous proof.

For $r > 1$, we follow the induction procedure on r ([6, pp. 106–107], [8, pp. 191–192]) using $V_n f$ in Lemmas 3.1 and 3.3. Assume that we have proven (3.9) for some positive integer $r \geqslant 1$. Let $g = f - V_{\lfloor \frac{n}{2} \rfloor} f$. It suffices to show that $\|g\|_p \leqslant c\omega_{r+1}(f, n^{-1})_p$. The definition of $V_n f$ implies that $V_{\lfloor \frac{n}{4} \rfloor} g = 0$, so that

$$\|g\|_p = \|g - V_{\lfloor \frac{n}{4} \rfloor} g\|_p \leqslant c E_{\lfloor \frac{n}{4} \rfloor}(g)_p \leqslant c_1 \omega_r\big(g, n^{-1}\big)_p.$$

On the other hand, using (2) and (3) of Proposition 2.7, we obtain, for any $m \in \mathbb{N}$,

$$\omega_r(g, t)_p \leqslant c_r t^r \int_t^{2^m t} \frac{\omega_{r+1}(g, u)_p}{u^{r+1}}\, du + c_r 2^{r+1} t^r \|g\|_p \int_{2^m t}^1 u^{-r-1}\, du$$
$$\leqslant c_1(m, r)\omega_{r+1}(g, t)_p + c_2(r) 2^{-mr} \|g\|_p,$$

where $c_2(r)$ is independent of m. Choosing m so that $4^{-1} \leqslant c_1 c_2(r) 2^{-mr} < 2^{-1}$, we deduce from these two equations that

$$\|g\|_p \leqslant c\omega_{r+1}\big(g, n^{-1}\big)_p \leqslant c\omega_{r+1}\big(f, n^{-1}\big)_p,$$

where the last step follows from Lemma 3.3. This completes the proof of (3.9).

The proof of (3.10) follows the standard approach for deriving it from the Bernstein inequality (see, for example, [11, p. 208]). Upon using (ii) of Lemma 2.6, it reduces to the Bernstein type inequality

$$\left\| D_{i,j}^r P \right\|_p \leqslant cn^r \|P\|_p, \quad P \in \Pi_n^d,\ 1 \leqslant i < j \leqslant d,$$

which, however, immediately follows from (ii) and (iii) of Lemma 2.6. □

As a corollary of Theorem 3.9, we have the following:

Corollary 3.5. *For* $0 < \alpha < r$ *and* $f \in L^p(\mathbb{S}^{d-1})$ *if* $1 \leqslant p < \infty$ *and* $f \in C(\mathbb{S}^{d-1})$ *if* $p = \infty$*,*

$$E_n(f)_p \sim n^{-\alpha} \quad \text{and} \quad \omega_r(f,t)_p \sim t^{-\alpha}$$

are equivalent.

3.3. Equivalence of modulus of smoothness and K-functional

Theorem 3.6. *Let* $r \in \mathbb{N}$ *and let* $f \in L^p$ *if* $1 \leqslant p < \infty$ *and* $f \in C(\mathbb{S}^{d-1})$ *if* $p = \infty$*. For* $0 < t < 1$*,*

$$\omega_r(f,t)_p \sim K_r(f,t)_p, \quad 1 \leqslant p \leqslant \infty.$$

Proof. By (ii) of Lemma 2.6 and the triangle inequality,

$$\left\| \triangle_{i,j,\theta}^r f \right\|_p \leqslant \left\| \triangle_{i,j,\theta}^r (f-g) \right\|_p + \left\| \triangle_{i,j,\theta}^r g \right\|_p \lesssim \|f-g\|_p + \theta^r \left\| D_{i,j}^r g \right\|_p$$

from which $\omega_r(f,t)_p \lesssim K_r(f,t)_p$ follows. On the other hand, for $t > 0$ set $n = \lfloor \frac{1}{t} \rfloor$, then by Lemma 3.1, (3.9) and (iii) of Lemma 2.8

$$\begin{aligned} K_r(f,t)_p &\leqslant \|f - V_n f\| + t^r \max_{1 \leqslant i < j \leqslant d} \left\| D_{i,j}^r V_n f \right\|_p \\ &\lesssim \omega_r\left(f, n^{-1}\right)_p + t^r n^r \max_{1 \leqslant i < j \leqslant d} \left\| \triangle_{i,j,n^{-1}}^r V_n f \right\|_p \\ &\lesssim \omega_r\left(f, n^{-1}\right)_p \lesssim \omega_r(f,t)_p \end{aligned}$$

where the last step follows from (i) of Lemma 2.8. □

The proof of the above theorem, together with Lemma 3.3 and (iii) of Lemma 2.6, yields a realization of the K-functional.

Corollary 3.7. *Under the assumption of Theorem* 3.6*,*

$$K_r\left(f, n^{-1}\right)_p \sim \|f - V_n f\|_p + n^{-r} \max_{1 \leqslant i < j \leqslant d} \left\| D_{i,j}^r V_n f \right\|_p.$$

3.4. Comparison with other moduli of smoothness

We want to compare our modulus of smoothness $\omega_r(f,t)_p$ with $\omega_r^*(f,t)_p$ defined in (1.3) and $\tilde{\omega}_r(f,t)_p$ defined in (1.6). By the equivalence of modulus of smoothness and K-functional, we can work with $K_r^*(f,t)_p$ given in (1.4) and $K_r(f,t)_p$ in (2.10). Thus, we need to deal with the equivalence

$$\max_{1\leqslant i<j\leqslant d}\left\| D_{i,j}^r g\right\|_p \sim \left\|\Delta_0^r g\right\|_p,$$

where Δ_0 is the Laplace–Beltrami operator.

Let $\mathcal{H}_n^d$ denote the space of spherical harmonics of degree n in d-variables. The operator Δ_0 has $\mathcal{H}_n^d$ as its space of eigenfunctions. For any $\alpha\in\mathbb{R}$, we have

$$(-\Delta_0)^\alpha Y = \big(n(n+d-1)\big)^\alpha Y, \quad Y\in\mathcal{H}_n^d. \tag{3.11}$$

Lemma 3.8. *For $1\leqslant i\neq j\leqslant d$ and $\alpha\in\mathbb{R}$*

$$D_{i,j}(-\Delta_0)^\alpha = (-\Delta_0)^\alpha D_{i,j}.$$

Proof. Because of the density of the polynomials, we only need to establish this commutativity for spherical polynomials, which can be further decomposed in terms of spherical harmonics. Thus, it suffices to work with spherical harmonics. By (3.11), we only need to show that $D_{i,j}\mathcal{H}_n^d\subset\mathcal{H}_n^d$. Let Δ be the usual Laplacian operator. A straightforward computation shows that

$$\partial_i^2 D_{i,j} = 2\partial_1\partial_2 + D_{i,j}\partial_i^2, \qquad \partial_k^2 D_{i,j} = D_{i,j}\partial_k^2, \quad k\neq i,\ k\neq j,$$

from which it follows readily that

$$\Delta D_{i,j} = D_{i,j}\Delta, \quad 1\leqslant i,j\leqslant d.$$

This implies that $D_{i,j}Y\in\ker\Delta$ if $Y\in\ker\Delta$, which shows $D_{i,j}\mathcal{H}_n^d\subset\mathcal{H}_n^d$. $\square$

Lemma 3.9. *Let $f\in\bigcup_{n=0}^\infty \Pi_n^d$. For $r\in\mathbb{N}$,*

$$\max_{1\leqslant i<j\leqslant d}\|D_{i,j}f\|_p \sim \left\|(-\Delta_0)^{1/2}f\right\|_p, \quad 1<p<\infty, \tag{3.12}$$

and, for $r\in\mathbb{N}$,

$$\max_{1\leqslant i<j\leqslant d}\left\|D_{i,j}^r f\right\|_p \leqslant c\left\|(-\Delta_0)^{r/2}f\right\|_p, \quad 1<p<\infty. \tag{3.13}$$

Proof. The key ingredient of the proof is the following result proved in [9]

$$\left\|(-\Delta_0)^{1/2}f\right\|_p \sim \|\operatorname{grad} f\|_p, \quad 1<p<\infty, \tag{3.14}$$

where $\operatorname{grad} f$ is defined by

$$\operatorname{grad} f := \nabla F|_{\mathbb{S}^{d-1}}, \quad \text{with } F(x) := f\left(\frac{x}{\|x\|}\right), \quad x \in \mathbb{R}^d \setminus \{0\},$$

in which $\nabla F := (\partial_1 F, \dots, \partial_d F)$ and $\partial_j := \frac{\partial}{\partial x_j}$. The norm $\|\operatorname{grad} f\|_p$ is taken over the Euclidean norm of $\operatorname{grad} f$; that is, $\|\operatorname{grad} f\|_p = \|\langle \operatorname{grad} f, \operatorname{grad} f\rangle^{1/2}\|_p$.

We relate $\|\operatorname{grad} f\|_p$ to $\|D_{i,j} f\|_p$. A straightforward computation shows that

$$\frac{\partial}{\partial x_j}\left[f\left(\frac{x}{\|x\|}\right)\right]_{\|x\|=1} = \partial_j f - x_j \sum_{i=1}^d x_i \partial_i f = -\sum_{\{i:\ 1\leqslant i\neq j\leqslant d\}} x_i D_{i,j} f, \tag{3.15}$$

where we have used $1 = \sum_{i=1}^d x_i^2$ in both equations. The first equation of (3.15) implies that

$$|D_{i,j} f| = \left|\langle x_i e_j - x_j e_i, \operatorname{grad} f\rangle\right| \leqslant \langle \operatorname{grad} f, \operatorname{grad} f\rangle^{1/2}$$

on $\mathbb{S}^{d-1}$, from which follows immediately that $\|D_{i,j} f\|_p \leqslant \|\operatorname{grad} f\|_p$. On the other hand, the second equation of (3.15) implies that

$$\langle \operatorname{grad} f, \operatorname{grad} f\rangle = \sum_{j=1}^d \left(\sum_{i\neq j} x_i D_{i,j} f\right)^2 \leqslant \sum_{j=1}^d \sum_{i\neq j} (D_{i,j} f)^2 \leqslant d^2 \max_{1\leqslant i<j\leqslant d} (D_{i,j} f)^2,$$

so that we have

$$\|D_{i,j} f\|_p \leqslant \|\operatorname{grad} f\|_p \leqslant d \max_{1\leqslant i<j\leqslant d} \|D_{i,j} f\|_p,$$

which proves, upon using (3.14), the equivalence in (3.12). Furthermore, by the commutativity of $D_{i,j}$ and $(-\Delta_0)^\alpha$ in Lemma 3.8, we have

$$\left\|D_{i,j}^r f\right\|_p \leqslant c\left\|(-\Delta_0)^{1/2} D_{i,j}^{r-1} f\right\|_p = c\left\|D_{i,j}^{r-1} (-\Delta_0)^{1/2} f\right\|_p$$

from which the inequality (3.13) follows from induction on r. □

Remark 3.1. The decomposition (2.6) implies immediately that

$$\left\|(-\Delta_0) f\right\|_p \leqslant \frac{d(d-1)}{2} \max_{1\leqslant i<j\leqslant d} \left\|D_{i,j}^2 f\right\|_p, \quad 1 \leqslant p \leqslant \infty. \tag{3.16}$$

Together with (3.12) and (3.13), we see that

$$\left\|(-\Delta_0)^{r/2} f\right\|_p \sim \max_{1\leqslant i<j\leqslant d} \left\|D_{i,j}^r f\right\|_p, \quad 1 < p < \infty, \tag{3.17}$$

holds for $r = 1$ and $r = 2$. However, we do not know if (3.17) is true for all r.

We can now state and prove our main result in this subsection:

Theorem 3.10. *Let $f \in L^p(\mathbb{S}^{d-1})$, $1 < p < \infty$. For $r \in \mathbb{N}$ and $0 < t < 1$,*

$$K_r(f,t)_p \leqslant c K_r^*(f,t)_p, \quad 1 < p < \infty. \tag{3.18}$$

Furthermore, for $r = 1$ or 2,

$$K_r(f,t)_p \sim K_r^*(f,t)_p, \quad 1 < p < \infty. \tag{3.19}$$

Proof. We only need to prove the inequality with $t = n^{-1}$. Furthermore, by Corollary 3.7, we only need to work with polynomials $V_n f$, for which the stated results are immediate consequences of Lemma 3.9 and Remark 3.1. □

For $1 < p < \infty$, it is shown in [9] that $\tilde{\omega}_r(f,t)_p \sim K_r^*(f,t)_p$ so that $\tilde{\omega}_r(f,t)_p \sim \omega_r^*(f,t)_p$ for $1 < p < \infty$. As a corollary of Proposition 2.3, Theorems 3.6 and 3.10, we can state the following equivalence:

Corollary 3.11. *Let $f \in L^p(\mathbb{S}^{d-1})$ with $1 < p < \infty$. For $r \in \mathbb{N}$ and $0 < t < 1$,*

$$\omega_r(f,t)_p \leqslant \tilde{\omega}_r(f,t)_p \sim \omega_r^*(f,t)_p, \quad 1 < p < \infty. \tag{3.20}$$

Furthermore, for $r = 1$ or 2,

$$\omega_r(f,t)_p \sim \tilde{\omega}_r(f,t)_p \sim \omega_r^*(f,t)_p, \quad 1 < p < \infty. \tag{3.21}$$

According to the inequality (3.20), our new modulus of smoothness $\omega_r(f,t)_p$ is at least as good as $\omega_r^*(f,t)_p$ for $1 < p < \infty$ and $r \geqslant 1$, and they are equivalent when $r = 1, 2$. We do not know if the equivalence holds for $r \geqslant 3$. In the case of $r = 2$, the inequality (3.16) shows that $\omega_2^*(f,t)_p \leqslant c\omega_2(f,t)_p$ also holds for $p = 1$ and $p = \infty$. A recent example of [15] shows that the equivalence (3.21) fails at the endpoints $p = 1, \infty$.

4. Weighted approximation on the unit sphere

In this section we consider approximation in the weighted L^p space on the sphere. The main result establishes the analogue of the Jackson estimate for the doubling weight using our new modulus of smoothness. Such a result has been established in [6, p. 94] using the weighted version of $\tilde{\omega}_r(f,t)_p$, following the lead of [23,24] for weighted approximation on the interval. We shall follow the approach in [6] closely.

4.1. Definition of modulus of smoothness for doubling weight

A non-negative integrable function w on $\mathbb{S}^{d-1}$ is called a *doubling weight* if there exists a constant $L > 0$, called the doubling constant, such that for any $x \in \mathbb{S}^{d-1}$ and $t > 0$

$$\int_{c(x,2t)} w(y)\,d\sigma(y) \leqslant L \int_{c(x,t)} w(y)\,d\sigma(y).$$

Many of the weight functions on $\mathbb{S}^{d-1}$ that appear in analysis satisfy the doubling condition, including all weights of the form

$$h_{\alpha,\mathbf{v}}(x)=\prod_{j=1}^{m}|x\cdot v_j|^{\alpha_j},\quad \alpha_j>0,\ v_j\in\mathbb{S}^{d-1}, \tag{4.1}$$

as shown in [5, (5.3)], which contains reflection invariant weight functions introduced by Dunkl (see [17, Chapt. 5]). Throughout this section, we assume that w is a doubling weight with the doubling constant L. We further define

$$w_n(x):=n^{d-1}\int_{c(x,\frac{1}{n})}w(y)\,d\sigma(y),\quad n=1,2,\ldots,\ x\in\mathbb{S}^{d-1} \tag{4.2}$$

and set $w_0(x):=w_1(x)$. Then w_n is again a doubling weight with doubling constant comparable to L. Moreover, it satisfies the following inequality

$$w_n(x)\leqslant L\big(1+nd(x,y)\big)^s w_n(y),\quad s=\log L/\log 2,\ n=0,1,\ldots. \tag{4.3}$$

We denote by $L^p(w)$ the weighted Lebesgue space endowed with the norm

$$\|f\|_{p,w}:=\bigg(\int_{\mathbb{S}^{d-1}}\big|f(y)\big|^p w(y)\,d\sigma(y)\bigg)^{1/p} \tag{4.4}$$

with the usual change when $p=\infty$. For $f\in L^p(w)$, our weighted r-th moduli of smoothness are defined by

$$\omega_r(f,t)_{p,w_n}:=\max_{1\leqslant i<j\leqslant d}\sup_{|\theta|\leqslant t}\big\|\triangle^r_{i,j,\theta}f\big\|_{p,w_n},\quad 0<p\leqslant\infty, \tag{4.5}$$

and the corresponding weighted r-th order K-functional is defined by

$$K_r(f,t)_{p,w_n}:=\inf_{g\in C^r(\mathbb{S}^{d-1})}\Big\{\|f-g\|_{p,w_n}+t^r\max_{1\leqslant i<j\leqslant d}\big\|D^r_{i,j}g\big\|_{p,w_n}\Big\}. \tag{4.6}$$

These definitions are analogues of those defined in [24, p. 181] and [6, p. 91]. They are used to study the weighted best approximation defined by

$$E_k(f)_{p,w_n}:=\inf_{g\in\Pi^d_{k-1}}\|f-g\|_{p,w_n},\quad 1\leqslant p\leqslant\infty. \tag{4.7}$$

The direct and the inverse theorems for $E_k(f)_{p,w_n}$ were established in [6] using the weighted modulus of smoothness

$$\tilde{\omega}_r\big(f,n^{-1}\big)_{p,w_n}:=\sup_{Q\in O_{n^{-1}}}\big\|\triangle^r_Q f\big\|_{p,w_n}.$$

Our development is parallel and follows along the same line. After establishing the properties of the modulus of smoothness, most of our proof will be similar to the unweighted case in Section 3, so that we can be brief.

4.2. Properties of modulus of smoothness

It was shown in [5, Corollary 3.4] that if $f \in \Pi_n^d$ then $\|f\|_{p,w} \sim \|f\|_{p,w_n}$ for all $0 < p < \infty$, with the constant of equivalence depending only on L and d. An important tool for our study is the Marcinkiewicz–Zygmund inequality. Let $\beta > 0$. A subset Λ of $\mathbb{S}^{d-1}$ is said to be maximal β-separated if $\mathbb{S}^{d-1} = \bigcup_{\omega\in\Lambda} c(\omega,\beta)$ and $\min\{d(\omega,\omega')\colon \omega,\omega' \in \Lambda,\ \omega \neq \omega'\} \geqslant \beta$. The following result is a simple consequence of [5, Corollary 3.3].

Lemma 4.1. *There exist a positive number ε depending only on d and L such that for any maximal $\frac{\delta}{n}$-separated subset Λ of $\mathbb{S}^{d-1}$ with $0 < \delta \leqslant \varepsilon$, $f \in \Pi_n^d$ and $0 < p < \infty$, we have*

$$\|f\|_{p,w}^p \sim \sum_{\omega\in\Lambda} \lambda_\omega \min_{x\in c(\omega,\frac{\delta}{n})} \bigl|f(x)\bigr|^p \sim \sum_{\omega\in\Lambda} \lambda_\omega \max_{x\in c(\omega,\frac{\delta}{n})} \bigl|f(x)\bigr|^p,$$

where $\lambda_\omega = \int_{c(\omega,\delta n^{-1})} w(x)\,d\sigma(x)$, and the constants of equivalence depend only on L, d and p.

We start with the following analogue of Lemma 2.6:

Lemma 4.2. *Let $r \in \mathbb{N}$ and $f \in C^r(\mathbb{S}^{d-1})$.*

(i) *If $0 < |t| \leqslant cn^{-1}$ and $1 \leqslant p < \infty$ then*

$$\bigl\|\triangle_{i,j,t}^r f\bigr\|_{p,w_n} \lesssim |t|^r \bigl\|D_{i,j}^r f\bigr\|_{p,w_n}, \quad 1 \leqslant i < j \leqslant d.$$

(ii) *Let $f \in \Pi_n^d$, $1 \leqslant p < \infty$ and let ε be as in Lemma 4.1. If $0 < |t| \leqslant \frac{\varepsilon}{nr}$, then*

$$\bigl\|\triangle_{i,j,t}^r f\bigr\|_{p,w} \sim |t|^r \bigl\|D_{i,j}^r f\bigr\|_{p,w}, \quad 1 \leqslant i < j \leqslant d,$$

where w_n and $\|\cdot\|_{w_n}$ are defined in (4.2) and (4.4), respectively.

Proof. (i) Let $F_{i,j}(t,x) := f(Q_{i,j,t}x)$. Note that for any $t_1, t_2 > 0$ and $x \in \mathbb{S}^{d-1}$,

$$F_{i,j}(t_1+t_2,x) = f(Q_{i,j,t_1+t_2}x) = F_{i,j}(t_1, Q_{i,j,t_2}x),$$

it follows by the definition of the $\triangle_{i,j,t}$ that

$$\triangle_{i,j,t}^r f(x) = \int_0^t \cdots \int_0^t \frac{\partial^r}{\partial t^r} F_{i,j}(t_1+\cdots+t_r,x)\,dt_1\cdots dt_r \tag{4.8}$$

$$= \int_0^t \cdots \int_0^t \frac{\partial^r}{\partial t^r} F_{i,j}(0, Q_{i,j,t_1+\cdots+t_r}x)\,dt_1\cdots dt_r, \tag{4.9}$$

which implies, by Minkowski's inequality, that

$$\|\triangle_{i,j,t}^r f\|_{p,w_n} \leqslant \int_{-|t|}^{|t|} \cdots \int_{-|t|}^{|t|} \left\| \frac{\partial^r}{\partial t^r} F_{i,j}(0, Q_{i,j,t_1+\cdots+t_r} x) \right\|_{p,w_n} dt_1 \cdots dt_r.$$

Since, by (4.3), $w_n(y) \sim w_n(x)$ whenever $d(x, y) \leqslant cn^{-1}$ and $d(Q_{i,j,t}x, x) \leqslant t$, it follows from the rotation invariance of $d\sigma$ and (2.3) that for $0 < |t| \leqslant n^{-1}$,

$$\|\triangle_{i,j,t}^r f\|_{p,w_n} \lesssim \int_{-|t|}^{|t|} \cdots \int_{-|t|}^{|t|} \|D_{i,j}^r f\|_{p,w_n} dt_1 \cdots dt_r \lesssim |t|^r \|D_{i,j}^r f\|_{p,w_n}.$$

(ii) Let Λ be a maximal $\frac{\varepsilon}{n}$-separated subset of $\mathbb{S}^{d-1}$ with ε being the same constant as in Lemma 4.1. Using (4.8), for any $\omega \in \Lambda$ and $0 < |t| \leqslant \frac{\varepsilon}{nr}$, we have

$$\begin{aligned} \left|\triangle_{i,j,t}^r(f)(\omega)\right| &= \left| \int_0^t \cdots \int_0^t \frac{\partial^r}{\partial t^r} F_{i,j}(0, Q_{i,j,t_1+\cdots+t_r}\omega)\, dt_1 \cdots dt_r \right| \\ &\leqslant |t|^r \max_{0 \leqslant u \leqslant r|t|} \left| \frac{\partial^r}{\partial t^r} F_{i,j}(0, Q_{i,j,u}\omega) \right| \leqslant |t|^r \max_{y \in c(\omega, \varepsilon/n)} \left| D_{i,j}^r f(y) \right|. \end{aligned}$$

Thus using Lemma 4.1 and setting $\lambda_\omega = \int_{c(\omega, \varepsilon n^{-1})} w(x)\, d\sigma(x)$, we obtain

$$\|\triangle_{i,j,t}^r f\|_{p,w}^p \sim \sum_{\omega \in \Lambda} \lambda_\omega \left|\triangle_{i,j,t}^r f(\omega)\right|^p \leqslant c|t|^{rp} \sum_{\omega \in \Lambda} \lambda_\omega \max_{y \in c(\omega, \varepsilon/n)} \left| D_{i,j}^r f(y) \right|^p.$$

However, as shown in the proof of Lemma 3.8, $D_{i,j}^r f \in \Pi_n^d$, so that the right-hand side of the above expression is, by Lemma 4.1, equivalent to $|t|^{rp} \|D_{i,j}^r f\|_{p,w}^p$. Thus, we have established the desired upper estimate $\|\triangle_{i,j,t}^r f\|_{p,w} \lesssim |t|^r \|D_{i,j}^r f\|_{p,w}$.

The lower estimate can be carried out along the same line. In fact, using (4.8), for $\omega \in \Lambda$ and $|t| \leqslant \frac{\varepsilon}{nr}$, we have

$$\left|\triangle_{i,j,t}^r(f)(\omega)\right| \geqslant |t|^r \min_{y \in c(\omega, \varepsilon/n)} \left| \frac{\partial^r}{\partial t^r} F_{i,j}(0, x) \right| = |t|^r \min_{y \in c(\omega, \varepsilon/n)} \left| D_{i,j}^r f(y) \right|.$$

Since $\triangle_{i,j,t}^r(f) \in \Pi_n^d$ and $D_{i,j}^r f \in \Pi_n^d$, it follows by Lemma 4.1 that

$$\begin{aligned} \|\triangle_{i,j,t}^r f\|_{p,w}^p &\sim \sum_{\omega \in \Lambda} \lambda_\omega \left|\triangle_{i,j,t}^r f(\omega)\right|^p \\ &\geqslant c|t|^{rp} \sum_{\omega \in \Lambda} \lambda_\omega \min_{y \in c(\omega, n^{-1}\varepsilon)} \left| D_{i,j}^r f(y) \right|^p \sim |t|^{rp} \|D_{i,j}^r f\|_{p,w}^p. \end{aligned}$$

This gives the desired lower estimate and completes the proof. □

Lemma 4.3. *Let $r \in \mathbb{N}$ and let $f \in L^p(\mathbb{S}^{d-1})$ if $1 \leqslant p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$.*

(i) *For $\lambda \geqslant 1$ and $t > 0$,*

$$\omega_r(f, \lambda t)_{p,w_n} \leqslant c(p, r, w)(1 + n\lambda t)^{sr/p} \lambda^r \omega_r(f, t)_{p,w_n}.$$

(ii) *Let m, n be positive integers. For $0 < t \leqslant \frac{1}{2^m n}$,*

$$\omega_r(f, t)_{p,w_n} \leqslant c_1(m, r)\omega_{r+1}(f, t)_{p,w_n} + c_2(r)\delta^{mr} \|f\|_{p,w_n},$$

where $\delta = \frac{(L2^s)^{1/p}}{(L2^s)^{1/p}+1} \in (0, 1)$, $c_1(m, r) > 0$ depends only on m, r and L, and $c_2(r) > 0$ depends only on r and L.

The analogue of this lemma using the weighted version of the modulus $\tilde{\omega}_r(f, t)_{p,w_n}$ was proved in Lemmas 2.2 and 4.1 of [6]. The proof there carries over to our new modulus of smoothness with obvious modification.

4.3. Weighted approximation on the sphere

Our main result is the Jackson estimate in the following theorem:

Theorem 4.4. *Let $f \in L^p(\mathbb{S}^{d-1})$ when $1 \leqslant p < \infty$ or $f \in C(\mathbb{S}^{d-1})$ when $p = \infty$. Then*

$$E_n(f)_{p,w_n} \leqslant c\omega_r\big(f, n^{-1}\big)_{p,w_n},$$

where $E_n(f)_{p,w_n}$ and $\omega_r(f, n^{-1})_{p,w_n}$ and defined in (4.7) *and* (4.5)*, respectively.*

The inverse theorem in terms of $\omega_r(f, t)_{p,w_n}$ follows from the one given in terms of $\tilde{\omega}_r(f, t)_{p,w_n}$, the weighted version of the modulus given in (1.6), in [6, p. 94], since $\omega_r(f, t)_{p,w_n} \leqslant \tilde{\omega}_r(f, t)_{p,w_n}$ as shown in Proposition 2.3. We can also state the following theorem:

Theorem 4.5. *Let $f \in L^p(\mathbb{S}^{d-1})$ when $1 \leqslant p < \infty$ or $f \in C(\mathbb{S}^{d-1})$ when $p = \infty$. Then*

$$\omega_r\big(f, n^{-1}\big)_{p,w_n} \sim K_r\big(f, n^{-1}\big)_{p,w_n}.$$

Furthermore, a realization of the K-functional is given by

$$K_r\big(f, n^{-1}\big)_{p,w_n} \sim \|f - V_n f\|_{p,w_n} + n^{-r} \max_{1 \leqslant i < j \leqslant d} \big\|D_{i,j}^r(V_n f)\big\|_{p,w_n}.$$

These two theorems are analogues of results in Sections 3.2 and 3.3. Their proofs are also similar, using the properties of the modulus of smoothness given above and the following two lemmas. The first one, proved in Lemma 2.5 of [6, p. 97], is as follows.

Lemma 4.6. *For $0 \leqslant k \leqslant 4n$, $f \in L^p(\mathbb{S}^{d-1})$ if $1 \leqslant p \leqslant \infty$, and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$,*

$$\|V_n f\|_{p,w_k} \leqslant c\|f\|_{p,w_k}, \quad \text{and} \quad \|f - V_n f\|_{p,w_k} \leqslant c E_n(f)_{p,w_k},$$

where $c > 0$ depends only on d and L.

The second lemma is the analogue of Lemma 3.2. Let

$$G_n(\theta) \equiv G_{n,\ell}(\theta) := n^{d-1}\bigl(1 + n|\theta|\bigr)^{-\ell} \quad \text{with } \ell > d + s + \frac{s}{p}.$$

Lemma 4.7. *Suppose $f \in L^p(\mathbb{S}^{d-1})$ for $1 \leqslant p < \infty$. Then*

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \bigl|f(x) - f(y)\bigr|^p G_n\bigl(d(x,y)\bigr) w_n(x)\, d\sigma(x)\, d\sigma(y) \leqslant c\omega\bigl(f, n^{-1}\bigr)^p_{p,w_n}.$$

Proof. The proof is similar to that of Lemma 3.2. We only list the necessary modification for the weighted cases. First we need to replace $d\sigma(x)$ and dx' by $w_n(x)\, d\sigma(x)$ and $\bar{w}_n(x')\, dx'$, where $\bar{w}_n(x') = w_n(x', \sqrt{1 - \|x'\|^2})$. Second, when making the change of variable $x' + b_{j-1}(u) \mapsto x'$, we need to use the estimate $\bar{w}_n(x' - b_{j-1}(u)) \lesssim (1 + n\|u\|)^s \bar{w}_n(x')$, which follows from (4.3). Third, Lemma 4.3(i) and (4.3) have to be used several times in the proof, and it is often necessary to replace $G_{n,\ell}$ by $G_{n,\ell-s}$ when (4.3) is used.

We also refer to the proof of Lemma 3.1 in [6] for details. □

Part 2. Approximation on the unit ball

This part is organized as follows. In Section 6 we derive a pair of new modulus of smoothness and K-functional on the ball from the results on the sphere. In Section 7, we study another pair of new modulus of smoothness and K-functional, which are extensions of those defined by Ditzian and Totik on $[-1, 1]$ to the ball. Finally, in Section 8, we discuss extensions of our result on the unit ball to W_μ with μ being a non-negative real number.

5. Approximation on the unit ball, part I

We consider approximation on the unit ball $\mathbb{B}^d$ and we often deal with the weighted L^p spaces $L^p(\mathbb{B}^d, W_\mu)$ for $1 \leqslant p < \infty$, where the weight function is defined by

$$W_\mu(x) := \bigl(1 - \|x\|^2\bigr)^{\mu - 1/2}, \quad \mu \geqslant 0. \tag{5.1}$$

For $1 \leqslant p < \infty$ we denote by $\|f\|_{p,\mu}$ the norm for $L^p(\mathbb{B}^d, W_\mu)$,

$$\|f\|_{p,\mu} := \biggl(\int_{\mathbb{B}^d} \bigl|f(x)\bigr|^p W_\mu(x)\, dx\biggr)^{1/p}, \tag{5.2}$$

and $\|f\|_{\infty,\mu} := \|f\|_\infty$ for $f \in C(\mathbb{B}^d)$. When we need to emphasis that the norm is taken over $\mathbb{B}^d$, we write $\|f\|_{p,\mu} = \|f\|_{L^p(\mathbb{B}^d, W_\mu)}$.

5.1. Preliminaries

There is a close relation between orthogonal structure on the sphere and on the ball, so much so that a satisfactory theory for the best approximation on the ball, including modulus of smoothness and its equivalent K-functional, can be established accordingly [35, Sect. 4] and [36, Sect. 3].

For $f \in L^p(\mathbb{B}^d, W_\mu)$, the modulus of smoothness in [36] is defined by

$$\omega_r^*(f,t)_{p,\mu} := \sup_{|\theta|\leqslant t} \left\| \left(I - T_\theta^\mu\right)^{r/2} f \right\|_{p,\mu}, \quad 1 \leqslant p \leqslant \infty, \tag{5.3}$$

where T_θ^μ is the generalized translation operator of the orthogonal expansion, which can be written explicitly as an integral operator [36, Theorem 3.6]. A K-functional $K_r^*(f,t)_{p,\mu}$ that is equivalent to this modulus of smoothness is defined by

$$K_r^*(f,t)_{p,\mu} := \inf_g \left\{ \|f-g\|_{p,\mu} + t^r \left\| (-\mathcal{D}_\mu)^{r/2} g \right\|_{p,\mu} \right\}, \quad 1 \leqslant p \leqslant \infty, \tag{5.4}$$

where $\mathcal{D}_\mu$ is the second order differential operator

$$\mathcal{D}_\mu := \sum_{i=1}^d \left(1-x_i^2\right)\partial_i^2 - 2 \sum_{1\leqslant i<j\leqslant d} x_i x_j \partial_i \partial_j - (d+2\mu) \sum_{i=1}^d x_i \partial_i, \tag{5.5}$$

which has orthogonal polynomials with respect to W_μ as eigenfunctions, see (5.9). Both $\omega_r^*(f;t)_{p,\mu}$ and $K_r^*(f;t)_{p,\mu}$ satisfy all the usual properties of moduli of smoothness and K-functionals, and they can be used to prove the direct and inverse theorems for

$$E_n(f)_{p,\mu} := \inf_{g\in\Pi_{n-1}^d} \|f-g\|_{p,\mu}. \tag{5.6}$$

The approach in [35] is based on treating both T_θ^μ and $\mathcal{D}_\mu$ as multiplier operators of the orthogonal expansions, and the results can be deduced from weighted counterparts on the unit sphere.

We shall define a new modulus of smoothness in the case of $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. The reason that we consider such values of μ lies in a close relation between the orthogonal structure on $\mathbb{S}^{d+m-1}$ and the one on $\mathbb{B}^d$, which was explored in [34].

Given a function f on $\mathbb{B}^d$, we will frequently need to regard it as a projection onto $\mathbb{B}^d$ of a function F, defined on $\mathbb{S}^{d+m-1}$ by

$$F\left(x,x'\right) := f(x), \qquad \left(x,x'\right) \in \mathbb{S}^{d+m-1}, \quad x \in \mathbb{B}^d,\ x' \in \mathbb{B}^m. \tag{5.7}$$

Under such an extension of f, Eqs. (2.11) and (2.12) become, for example,

$$\int_{\mathbb{S}^{d+m-1}} F(y)\, d\sigma(y) = \sigma_m \int_{\mathbb{B}^d} f(x)\left(1-\|x\|^2\right)^{\frac{m-2}{2}} dx, \tag{5.8}$$

where σ_m denotes the surface area of $\mathbb{S}^{m-1}$ for $m \geqslant 2$ and $\sigma_1 = 2$.

Let $\mathcal{V}_n^d(W_\mu)$ denote the space of orthogonal polynomials of degree n with respect to the weight function W_μ on $\mathbb{B}^d$. The elements of $\mathcal{V}_n^d(W_\mu)$ satisfy (cf. [17, p. 38])

$$\mathcal{D}_\mu P = -n(n+d+2\mu-1)P \quad \text{for all } P \in \mathcal{V}_n^d(W_\mu). \tag{5.9}$$

We denote by $P_n^\mu(x,y)$ the reproducing kernel of $\mathcal{V}_n^d(W_\mu)$ in $L^2(B^d, W_\mu)$. It is shown in [34, Theorem 2.6] that

$$P_n^\mu(x,y) = \int_{\mathbb{S}^{m-1}} Z_{n,d+m}\big(\langle x,y\rangle + \sqrt{1-\|y\|^2}\langle x',\xi\rangle\big)\,d\sigma(\xi) \tag{5.10}$$

for any $x, y \in \mathbb{B}^d$ and $(x,x') \in \mathbb{S}^{d+m-1}$, where $Z_{n,d}(t)$ is the zonal harmonic defined in (3.2) and $\mu = \frac{m-1}{2}$. For η being a C^∞-function on $[0,\infty)$ that satisfies the properties as defined in Section 3.1, we define an operator

$$V_n^\mu f(x) := a_\mu \int_{\mathbb{B}^d} f(y) K_n^\mu(x,y) W_\mu(y)\,dy, \quad x \in \mathbb{B}^d, \tag{5.11}$$

where a_μ is the normalization constant of W_μ and

$$K_n^\mu(x,y) := \sum_{k=0}^{2n} \eta\left(\frac{k}{n}\right) P_k^\mu(x,y). \tag{5.12}$$

The operator V_n^μ is an analogue of the operator V_n in (3.3), and it shares the same properties satisfied by $V_n f$. In particular, the kernel K_n^μ is highly localized [27, Theorem 4.2] and an analogue of Proposition 3.3 holds for $V_n^\mu f$ [35, p. 16]:

Lemma 5.1. *Let $f \in L^p(\mathbb{B}^d, W_\mu)$ if $1 \leqslant p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. Then*

(1) $V_n^\mu f \in \Pi_{2n}^d$ *and* $V_n^\mu f = f$ *for* $f \in \Pi_n^d$.
(2) *For* $n \in \mathbb{N}$, $\|V_n^\mu f\|_p \leqslant c\|f\|_p$.
(3) *For* $n \in \mathbb{N}$,

$$\big\|f - V_n^\mu f\big\|_{p,\mu} \leqslant c E_n(f)_{p,\mu}.$$

The following result shows a further connection between $V_n F$ and $V_n^\mu f$.

Lemma 5.2. *Let V_n denote the operator defined in* (3.3) *on $\mathbb{S}^{d+m-1}$. For $x \in \mathbb{B}^d$, $(x,x') \in \mathbb{S}^{d+m-1}$ and F in* (5.7),

$$(V_n F)\big(x,x'\big) = \big(V_n^\mu f\big)(x), \quad \text{where } \mu = \frac{m-1}{2}.$$

Proof. By the definition of V_n and (5.8),

$$(V_n F)(x, x') = \sigma_{d+m} \int_{\mathbb{S}^{d+m-1}} F(y) K_n\big(\langle (x, x'), y\rangle\big)\, d\sigma(y)$$

$$= \sigma_{d+m} \int_{\mathbb{B}^d} f(v) \int_{\mathbb{S}^{m-1}} K_n\big(\langle x, v\rangle + \sqrt{1-\|v\|^2}\langle x', \xi\rangle\big)\, d\sigma(\xi) W_\mu(v)\, dv$$

$$= \sigma_{d+m} \int_{\mathbb{B}^d} f(v) K_n^\mu(x, v) W_\mu(v)\, dv = \big(V_n^\mu f\big)(x),$$

where the third step follows from (5.10) and the definitions of K_n and K_n^μ. □

5.2. Modulus of smoothness and best approximation

For a given function $f \in L^p(B^d, W_{\frac{m-1}{2}})$, the extension F in (5.7) is an element of $L^p(\mathbb{S}^{d+m-1})$ according to (5.8). This relation can be used to define a modulus of smoothness on the unit ball.

We denote by $\tilde{f}$ the extension of f in (5.7) in the case of $m = 1$; that is,

$$\tilde{f}(x, x_{d+1}) = f(x), \qquad (x, x_{d+1}) \in \mathbb{R}^{d+1}, \quad x \in \mathbb{B}^d. \tag{5.13}$$

Recall that $\triangle_{i,j,\theta} = \triangle_{Q_{i,j,\theta}}$ and $Q_{i,j,\theta}$ is the rotation in angle θ in the (x_i, x_j)-plane.

Definition 5.3. Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Let $f \in L^p(\mathbb{B}^d, W_\mu)$ if $1 \leqslant p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. For $r \in \mathbb{N}$ and $t > 0$

$$\omega_r(f, t)_{p,\mu} := \sup_{|\theta| \leqslant t} \Big\{ \max_{1 \leqslant i < j \leqslant d} \big\| \triangle_{i,j,\theta}^r f \big\|_{L^p(\mathbb{B}^d, W_\mu)}, \max_{1 \leqslant i \leqslant d} \big\| \triangle_{i,d+1,\theta}^r \tilde{f} \big\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \Big\}, \tag{5.14}$$

where, for $m = 1$, $\|\triangle_{i,d+1,\theta}^r \tilde{f}\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})}$ is replaced by $\|\triangle_{i,d+1,\theta}^r \tilde{f}\|_{L^p(\mathbb{S}^d)}$.

Several remarks are in order. First, the second term in the right-hand side of (5.14) is necessary, as for any radial function f and $1 \leqslant i < j \leqslant d$, $\triangle_{i,j,\theta}^r f = 0$. The first term is also necessary, as will be shown in our examples in Section 10 (see the discussion after Example 10.4). Also, the second term in (5.14) can be made more explicit by, recalling (2.8),

$$\triangle_{i,d+1,\theta}^r \tilde{f}(x, x_{d+1}) = \overrightarrow{\triangle}_\theta^r f\big(x_1, \ldots, x_{i-1}, x_i \cos(\cdot) - x_{d+1} \sin(\cdot), x_{i+1}, \ldots, x_d\big),$$

with the forward difference in the right-hand side being evaluated at 0. Second, when $\mu = 1/2$, or $m = 2$, we have the unweighted case, whereas if $m = 1$ then $W_{\mu-1/2}$ becomes singular and the following limit holds:

$$\lim_{\mu \to 0+} \big\| \triangle_{i,d+1,\theta}^r \tilde{f} \big\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} = \big\| \triangle_{i,d+1,\theta}^r \tilde{f} \big\|_{L^p(\mathbb{S}^d)},$$

which follows from the limit relation that, for a generic function f,

$$\lim_{\mu\to 0+} c_\mu \int_{\mathbb{B}^{d+1}} f(x)\bigl(1-\|x\|^2\bigr)^{\mu-1}\,dx = \lim_{\mu\to 0+} c_\mu \int_0^1 \int_{\mathbb{S}^d} f\bigl(sx'\bigr)\,d\sigma\bigl(x'\bigr)s^d\bigl(1-s^2\bigr)^{\mu-1}\,ds = \int_{\mathbb{S}^d} f\bigl(x'\bigr)\,d\sigma\bigl(x'\bigr), \tag{5.15}$$

where $c_\mu = \sigma_{d+1}(\int_{\mathbb{B}^{d+1}}(1-\|x\|^2)^{\mu-1}\,dx)^{-1}$. Third, in (5.14), we have used the notation W_μ for both weight function on $\mathbb{B}^d$ and $\mathbb{B}^{d+1}$ which implies that x in the definition of W_μ is assumed to be in the appropriate set accordingly. Finally, just as we remarked after Definition 2.2, the modulus $\omega_r(f,t)_{p,\mu}$ is not rotationally invariant and it relies on the standard basis $e_1,\dots,e_d$ but independent of the order of $e_1,\dots,e_d$.

We can also define $\omega_r(f,t)_p$ in an equivalent but more compact form:

$$\omega_r(f,t)_{p,\mu} := \sup_{|\theta|\leqslant t}\ \max_{1\leqslant i<j\leqslant d+1} \bigl\|\triangle^r_{i,j,\theta}\tilde f\bigr\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})}. \tag{5.16}$$

Indeed, if $1\leqslant i<j\leqslant d$, then $\triangle^r_{i,j,\theta}\tilde f(x,x_{d+1}) = \triangle^r_{i,j,\theta} f(x)$ by the definition of $Q_{i,j,\theta}x$ in (2.1); consequently,

$$\bigl\|\triangle^r_{i,j,\theta}\tilde f\bigr\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})} = c\bigl\|\triangle^r_{i,j,\theta} f\bigr\|_{L^p(\mathbb{B}^d,W_\mu)},$$

which follows from, for a generic function f and $\lambda > -1$,

$$\int_{\mathbb{B}^{d+1}} \tilde f(y)\bigl(1-\|y\|^2\bigr)^\lambda\,dy = \int_{\mathbb{B}^d} f(x) \int_{-\sqrt{1-\|x\|^2}}^{\sqrt{1-\|x\|^2}} \bigl(1-\|x\|^2-u^2\bigr)^\lambda\,du\,dx = c\int_{\mathbb{B}^d} f(x)\bigl(1-\|x\|^2\bigr)^{\lambda+1/2}\,dx, \tag{5.17}$$

where $c = \int_{-1}^1 (1-t^2)^\lambda\,dt$. Thus, (5.14) and (5.16) are equivalent.

To emphasise the dependence on the dimension, we shall write the modulus of smoothness on the sphere as $\omega_r(f,t)_p = \omega_r(f,t)_{L^p(\mathbb{S}^{d-1})}$ in the following.

Lemma 5.4. *Let $\mu = \frac{m-1}{2}$ and $m\in\mathbb{N}$. Let $f\in L^p(\mathbb{B}^d,W_\mu)$ if $1\leqslant p<\infty$ and $f\in C(\mathbb{B}^d)$ if $p=\infty$, and let F be defined as in* (5.7). *Then*

$$\omega_r(f,t)_{L^p(\mathbb{B}^d,W_\mu)} \sim \omega_r(F,t)_{L^p(\mathbb{S}^{d+m-1})}.$$

Proof. If $1\leqslant i<j\leqslant d$, then $\triangle^r_{i,j,\theta}F(x,x') = \triangle^r_{i,j,\theta}f(x)$ by (2.1) and, hence, for $m\geqslant 1$, it follows by (5.8) and (5.17) that

$$\int_{\mathbb{S}^{d+m-1}} \left|\triangle_{i,j,\theta}^r F(y)\right|^p d\sigma(y) = \sigma_m \int_{\mathbb{B}^d} \left|\triangle_{i,j,\theta}^r f(x)\right|^p \left(1-\|x\|^2\right)^{\frac{m-2}{2}} dx.$$

If $1 \leqslant i \leqslant d$ and $d+1 \leqslant j \leqslant d+m$, then it follows from (2.8) that $\triangle_{i,j,\theta}^r F(x,x') = \triangle_{i,d+1,\theta}^r \tilde{f}(x,x_j)$, where $x \in \mathbb{B}^d$, so that for $m \geqslant 2$,

$$\int_{\mathbb{S}^{d+m-1}} \left|\triangle_{i,j,\theta}^r F(y)\right|^p d\sigma(y) = \sigma_{m-1} \int_{\mathbb{B}^{d+1}} \left|\triangle_{i,d+1,\theta}^r \tilde{f}(x)\right|^p \left(1-\|x\|^2\right)^{\frac{m-3}{2}} dx$$

by (2.11) and (2.12), whereas there is nothing to prove for $m=1$ by the modification in the definition of $\omega_r(f,t)_{L^p(\mathbb{B}^d,W_\mu)}$ in that case. □

Theorem 5.5. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. For $f \in L^p(\mathbb{B}^d, W_\mu)$ if $1 \leqslant p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p=\infty$,*

$$E_n(f)_{p,\mu} \leqslant c\omega_r\left(f,n^{-1}\right)_{p,\mu}, \quad 1 \leqslant p \leqslant \infty; \tag{5.18}$$

on the other hand,

$$\omega_r\left(f,n^{-1}\right)_{p,\mu} \leqslant cn^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f)_{p,\mu} \tag{5.19}$$

where $E_n(f)_{p,\mu}$ and $\omega_r(f,t)_{p,\mu}$ are defined in (5.6) *and* (5.14)*, respectively.*

Proof. Let F be defined as in (5.7). By Lemma 5.2, $(V_n^\mu f)(x) = (V_n F)(x,x')$, so that by (5.8) and the Jackson estimate for F in (3.9),

$$\begin{aligned} \left\|V_n^\mu f - f\right\|_{p,\mu}^p &= \int_{\mathbb{B}^d} \left|V_n^\mu f(x) - f(x)\right|^p W_\mu(x)\,dx \\ &= c \int_{\mathbb{S}^{d+m-1}} \left|V_n F(y) - F(y)\right|^p d\sigma(y) \\ &\leqslant c\omega_r\left(F,n^{-1}\right)_{L^p(\mathbb{S}^{d+m-1})} \leqslant c\omega_r\left(f,n^{-1}\right)_{p,\mu}, \end{aligned}$$

which proves (5.18). The inverse theorem follows likewise from

$$E_n(F)_{L^p(\mathbb{S}^{d+m-1})} \leqslant c\|V_{\lfloor \frac{n}{2}\rfloor}F - F\|_{L^p(\mathbb{S}^{d+m-1})} = c\left\|V_{\lfloor \frac{n}{2}\rfloor}^\mu f - f\right\|_{p,\mu} \leqslant cE_{\lfloor \frac{n}{2}\rfloor}(f)_{p,\mu}$$

and the inverse theorem for F in (3.10). □

5.3. Equivalent K-functional and comparison

Recall the derivatives $D_{i,j}$ defined in (2.3). We use them to define a K-functional.

Definition 5.6. Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Let $f \in L^p(\mathbb{B}^d, W_\mu)$, if $1 \leqslant p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. For $r \in \mathbb{N}$ and $t > 0$,

$$K_r(f,t)_{p,\mu} := \inf_{g \in C^r(\mathbb{B}^d)} \Big\{ \|f - g\|_{L^p(\mathbb{B}^d, W_\mu)} + t^r \max_{1 \leqslant i < j \leqslant d} \big\| D_{i,j}^r g \big\|_{L^p(\mathbb{B}^d, W_\mu)} + t^r \max_{1 \leqslant i \leqslant d} \big\| D_{i,d+1}^r \tilde{g} \big\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \Big\}, \tag{5.20}$$

where if $m = 1$, then $\|D_{i,d+1}^r \tilde{g}\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})}$ is replaced by $\|D_{i,d+1}^r \tilde{g}\|_{L^p(\mathbb{S}^d)}$.

Although $\tilde{g}(x, x_{d+1}) = g(x)$ is a constant in x_{d+1} variable so that $\partial_{d+1}\tilde{g}(x, x_{d+1}) = 0$, we cannot replace $D_{i,d+1}^r$ by $(x_i \partial_i)^r$, since $D_{i,d+1}^r \neq (x_i \partial_i)^r$ if $r > 1$. Observe also, that if $f(x) = f_0(\|x\|)$ is a radial function and $1 \leqslant i < j \leqslant d$, then $D_{i,j}(fg)(x) = f_0(\|x\|) D_{i,j} g(x)$; in particular, this implies that $D_{i,j} f = 0$ for any radial function f.

In the case of $m \geqslant 2$, we can also define the K-functional in an equivalent but more compact form

$$K_r(f,t)_{p,\mu} = \inf_{g \in C^r(\mathbb{B}^d)} \Big\{ \|f - g\|_{L^p(\mathbb{B}^d, W_\mu)} + t^r \max_{1 \leqslant i < j \leqslant d+1} \big\| D_{i,j}^r \tilde{g} \big\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \Big\}.$$

The equivalence of the two definitions follows from (5.17). Just as in the case of modulus of smoothness, these K-functionals are related to the K-functional $K_r(f,t)_p = K_r(f,t)_{L^p(\mathbb{S}^{d-1})}$ defined on the sphere.

Lemma 5.7. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Let $f \in L^p(\mathbb{B}^d, W_\mu)$ if $1 \leqslant p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$, and let F be defined as in* (5.7). *Then*

$$K_r(f,t)_{p,\mu} \equiv K_r(f,t)_{L^p(\mathbb{B}^d, W_\mu)} \sim K_r(F,t)_{L^p(\mathbb{S}^{d+m-1})}.$$

Proof. The estimate $K_r(F,t)_{L^p(\mathbb{S}^{d+m-1})} \leqslant c K_r(f,t)_{L^p(\mathbb{B}^d, W_\mu)}$ follows directly from the definition, and the fact that for any $g \in C^r(B^d)$

$$\big\| D_{i,j}^r \tilde{g} \big\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} = c \big\| D_{i,j}^r G \big\|_{L^p(\mathbb{S}^{d+m-1})}, \quad 1 \leqslant i < j \leqslant d+1,$$

where $G(x, x') = g(x)$ for $x \in B^d$ and $(x, x') \in \mathbb{S}^{d+m-1}$.

To show the inverse inequality, we observe that on account of Lemma 5.2, (2.11), and (2.12), $\|f - V_n^\mu f\|_{L^p(\mathbb{B}^d, W_\mu)} = c\|F - V_n F\|_{L^p(\mathbb{S}^{d+m-1})}$, and

$$\big\| D_{i,j}^r \tilde{V}_n^\mu f \big\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} = c \big\| D_{i,j}^r V_n(F) \big\|_{L^p(\mathbb{S}^{d+m-1})}, \quad 1 \leqslant i < j \leqslant d+1.$$

The inverse inequality $K_r(f,t)_{L^p(\mathbb{B}^d, W_\mu)} \leqslant c K_r(F,t)_{L^p(\mathbb{S}^{d+m-1})}$ then follows by choosing $g = V_n^\mu f$ with $n \sim \frac{1}{t}$ in Definition 5.6, and using Corollary 3.7. □

By Lemmas 5.4 and 5.7 and the equivalence in Theorem 3.6, we further arrive at the following:

Theorem 5.8. *Let $r \in \mathbb{N}$ and let $f \in L^p(\mathbb{B}^d, W_\mu)$ if $1 \leqslant p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. Then, for $0 < t < 1$,*

$$\omega_r(f,t)_{p,\mu} \sim K_r(f,t)_{p,\mu}, \quad 1 \leqslant p \leqslant \infty.$$

Next we compare the moduli of smoothness $\omega_r(f,t)_{p,\mu}$ with $\omega_r^*(f,t)_{p,\mu}$ defined in (5.3). By Theorem 5.8 and [36, Theorem 3.11], it is enough to compare the K-functional $K_r(f,t)_{p,\mu}$ with $K_r^*(f,t)_{p,\mu}$ defined in (5.4). We start with an observation on the differential operator $\mathcal{D}_\mu$ defined in (5.5). To emphasize the dependence on the dimension, we shall use the notation $\Delta_{0,d} = \Delta_0$ for the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$.

Lemma 5.9. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Let F be defined as in (5.7). Then*

$$\Delta_{0,d+m} F\big(x, x'\big) = \mathcal{D}_\mu f(x), \qquad x \in \mathbb{B}^d,\ \big(x, x'\big) \in \mathbb{S}^{d+m-1}.$$

In fact, this follows immediately from comparing the expressions (2.7) and (5.5). Since both are multiplier operators, their fractional powers are also defined and equal. Thus, as a consequence of Lemma 5.7 and Corollary 3.7, we see that the comparison of the K-functionals, Theorem 3.10, and the comparison of the moduli of smoothness, Corollary 3.11, on the sphere carry over to the comparison on the ball.

Theorem 5.10. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$ and let $f \in L^p(\mathbb{S}^{d-1})$, $1 < p < \infty$. For $r \in \mathbb{N}$ and $0 < t < 1$,*

$$\omega_r(f,t)_{p,\mu} \leqslant c\,\omega_r^*(f,t)_{p,\mu}, \quad 1 < p < \infty. \tag{5.21}$$

Furthermore, for $r = 1$ or 2,

$$\omega_r(f,t)_{p,\mu} \sim \omega_r^*(f,t)_{p,\mu}, \quad 1 < p < \infty. \tag{5.22}$$

An equivalent result can be stated for K-functionals.

5.4. *The moduli of smoothness on* $[-1,1]$

When $d = 1$, the ball becomes the interval $B^1 = [-1,1]$. It turns out that our modulus of smoothness appears to be new even in this case. For $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$, the definition in (5.14) becomes, written out explicitly,

$$\omega_r(f,t)_{p,\mu} := \sup_{|\theta| \leqslant t} \left(c_\mu \int_{B^2} \big| \overrightarrow{\triangle}_\theta^r f\big(x_1 \cos(\cdot) + x_2 \sin(\cdot)\big)\big|^p W_{\mu-\frac{1}{2}}(x)\,dx \right)^{1/p} \tag{5.23}$$

for $1 \leqslant p < \infty$ with the usual modification for $p = \infty$, where $c_\mu^{-1} = \int_{B^2} W_{\mu-\frac{1}{2}}(x)\,dx$. The difference $\overrightarrow{\triangle}_\theta^r$ in this definition can be evaluated at any fixed point $t_0 \in [0, 2\pi]$. More precisely, $\overrightarrow{\triangle}_\theta^r f(x_1 \cos(\cdot) + x_2 \sin(\cdot)) = \overrightarrow{\triangle}_\theta^r g_{x_1,x_2}(t_0)$ for a fixed $t_0 \in [0, 2\pi]$, where $g_{x_1,x_2}(\theta) =$

$f(x_1\cos\theta + x_2\sin\theta)$. Clearly, the definition is independent of the choice of t_0, and makes sense for all real μ such that $\mu > 0$, whereas for $\mu = 0$ the integral is taken over $\mathbb{S}^1$ upon using the limit (5.15).

This modulus of smoothness is computable, as shown by the following example.

Example 5.11. For $g_\alpha(x) = (1-x)^\alpha$, $\alpha > 0$ and $x \in [-1, 1]$, $\mu > 0$ and $1 \leqslant p \leqslant \infty$,

$$\omega_2(g_\alpha, t)_{p,\mu} \sim \begin{cases} t^{2\alpha + \frac{2\mu+1}{p}}, & -\frac{2\mu+1}{2p} < \alpha < 1 - \frac{2\mu+1}{2p}, \\ t^2|\log t|^{1/p}, & \alpha = 1 - \frac{2\mu+1}{2p},\ p \neq \infty, \\ t^2, & \alpha > 1 - \frac{2\mu+1}{2p}. \end{cases} \tag{5.24}$$

This is proved later in Lemma 8.2. Notice that for $\mu = \frac{m-1}{2}$, this modulus of smoothness is the restriction of the modulus of smoothness from the sphere.

In this setting, several moduli of smoothness were defined and studied in the literature; we refer to the discussion in [16, Chapter 13]. In particular, $\omega_r^*(f,t)_{p,\mu}$ was studied by Butzer and his school and by Potapov. The most successful one has been the Ditzian–Totik modulus of smoothness [16], which we now recall.

Let $\varphi(x) = \sqrt{1-x^2}$ and let $w_\mu(x) := (1-x^2)^{\mu-1/2}$ on $[-1, 1]$. The Ditzian–Totik K-functional with respect to the weight w_μ is defined by

$$\widehat{K}_r(f,t)_{p,\mu} := \inf_g \bigl\{ \|f-g\|_{p,\mu} + t^r \bigl\|\varphi^r g^{(r)}\bigr\|_{p,\mu} \bigr\}. \tag{5.25}$$

This K-functional is equivalent to a modulus of smoothness $\hat{\omega}_r(f,t)_{p,\mu}$, called the Ditzian–Totik modulus of smoothness and usually denoted by $\omega_\varphi^r(f,t)_{p,\mu}$ (see [16]):

$$\widehat{K}_r(f,t)_{p,\mu} \sim \hat{\omega}_r(f,t)_{p,\mu}, \quad 1 \leqslant p \leqslant \infty,\ 0 < t < t_r. \tag{5.26}$$

In the unweighted case (i.e. in the case of $\mu = \frac{1}{2}$), the Ditzian–Totik modulus of smoothness is defined by, as in (1.9),

$$\hat{\omega}_r(f,t)_p := \sup_{0<h\leqslant t} \bigl\|\widehat{\Delta}_{h\varphi}^r f\bigr\|_p, \quad 1 \leqslant p \leqslant \infty, \tag{5.27}$$

where $\widehat{\Delta}_h^r$ is the r-th symmetric difference defined by

$$\widehat{\Delta}_{\theta\varphi}^r f(x) = \sum_{k=0}^r (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right)\theta\varphi(x)\right),$$

in which we define $\widehat{\Delta}_{\theta\varphi}^r f(x) = 0$ whenever $x + rh\varphi(x)/2$ or $x - rh\varphi(x)/2$ is not in $(-1, 1)$. In the weighted case with $\mu > 1/2$, the Ditzian–Totik modulus of smoothness is more complicated and defined by, for $1 \leqslant p < \infty$,

$$\hat{\omega}_r(f,t)_{p,\mu} := \sup_{0<h\leqslant t} \big\| \widehat{\triangle}^r_{h\varphi} f \big\|_{L^p(I_{rh};w_\mu)} + \sup_{0<h\leqslant 12r^2t^2} \big\| \overleftarrow{\triangle}^r_h f \big\|_{L^p(J_{1,rt};w_\mu)} + \sup_{0<h\leqslant 12r^2t^2} \big\| \overrightarrow{\triangle}^r_h f \big\|_{L^p(J_{-1,rt};w_\mu)}, \tag{5.28}$$

where the norms are taken over the intervals indicated with

$$I_t := \big[-1+2t^2, 1-2t^2\big], \qquad J_{1,t} := \big[1-12t^2, 1\big], \qquad J_{-1,t} := \big[-1, -1+12t^2\big],$$

and $\hat{\omega}_r(f,t)_{\infty,\mu} \equiv \hat{\omega}_r(f,t)_\infty$ is defined as $\sup_{x\in[-1,1]} |\widehat{\triangle}^r_{h\varphi} f(x)|$.

One important property of $\hat{\omega}_r(f,t)_p$ is the following equivalence established in [16, (2.1.4), (2.2.5)]:

$$\hat{\omega}_r(f,t)_p^p \sim \frac{1}{t}\int_0^t \big\| \widehat{\triangle}^r_{h\varphi} f \big\|_p^p \, dh, \quad 1 \leqslant p < \infty, \tag{5.29}$$

with the usual change when $p = \infty$. In the weighted case, the right-hand side needs to be replaced by a sum of three integrals on the respective intervals [16, (6.19)].

The success of the $\hat{\omega}_r(f,t)_p$ lies in the fact that it is computable and can be used to establish both the direct and inverse theorems for algebraic polynomial approximation on $[-1,1]$. The definition of $\hat{\omega}_r(f,t)_{p,\mu}$ for the weight w_μ is more complicated and will be discussed in Section 7.8. For even more general weight, see the book [16].

The connection between our modulus of smoothness and that of Ditzian–Totik is given in the following theorem.

Theorem 5.12. *Let $\mu = \frac{m-1}{2}$, $m \in \mathbb{N}$ and $r \in \mathbb{N}$. Let $f \in L^p([-1,1], w_\mu)$ if $1 \leqslant p < \infty$, and $f \in C[-1,1]$ if $p = \infty$. Assume further that r is odd if $p = \infty$. Then*

$$\omega_r(f,t)_{p,\mu} \leqslant c\hat{\omega}_r(f,t)_{p,\mu} + ct^r \|f\|_{p,\mu}, \quad 0 < t \leqslant t_r, \tag{5.30}$$

where the term $t^r\|f\|_{p,\mu}$ can be dropped when $r = 1$.

Proof. By Theorem 5.8 and the equivalence (5.26), it suffices to prove the inequality for the corresponding K-functionals:

$$K_r(f,t)_{p,\mu} \leqslant c\widehat{K}_r(f,t)_{p,\mu} + ct^r\|f\|_{p,\mu}, \quad 1 \leqslant p \leqslant \infty$$

with the additional assumption r is odd when $p = \infty$. This inequality, together with the equivalence $K_1(f,t)_{p,\mu} \sim \widehat{K}_1(f,t)_{p,\mu}$, is given in Theorem 6.2 of the next section. □

It is worth to point out that (5.29) and (5.30) imply that

$$\int_{\mathbb{B}^2} \big|\overrightarrow{\triangle}^r_\theta f\big(x_1\cos(\cdot) + x_2\sin(\cdot)\big)\big|^p \frac{dx_1\,dx_2}{\sqrt{1-x_1^2-x_2^2}} \leqslant c\frac{1}{t}\int_0^t \big\|\widehat{\triangle}^r_{h\varphi} f\big\|_p^p \, dh + ct^{rp}\|f\|_p^p, \tag{5.31}$$

with the usual modification when $p = \infty$. This highly non-trivial inequality will play a pivotal role in Section 6.3.

We do not know if the reversed inequality of (5.30) holds when $r \geqslant 2$. We note, however, that the order of $\omega_r(g_\alpha, t)_{p,\mu}$ given in (5.24) coincides, when $\mu = 1/2$, with the computation in [16, p. 34] for $\hat{\omega}_r(g_\alpha, t)_p$ and $1 \leqslant p \leqslant \infty$ except when $\alpha = 1$. For $\alpha = 1$, $g_\alpha(t)$ is a linear polynomial so that $\hat{\omega}_r(g_\alpha, t)_p = 0$, whereas $\omega_r(g_\alpha, t)_{p,\frac{1}{2}}$ is non-zero.

6. Approximation on the unit ball, part II

In this section, we introduce another pair of modulus of smoothness and K-functional on the ball that are in analogy with those of Ditzian and Totik on $[-1, 1]$, and utilize them to study best approximation on the unit ball. Both the direct and the inverse theorems are established.

6.1. A new K-functional and comparison

Let $\varphi(x) := \sqrt{1 - \|x\|^2}$ for $x \in \mathbb{B}^d$. Recall that the Ditzian–Totik K-functional $\widehat{K}_r(f, t)_{p,\mu}$ on $[-1, 1]$ is defined in (5.25). We now define its higher dimensional analogue on the ball $\mathbb{B}^d$.

Definition 6.1. Let $f \in L^p(\mathbb{B}^d, W_\mu)$ if $1 \leqslant p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. For $r \in \mathbb{N}$ and $t > 0$, define

$$\widehat{K}_r(f, t)_{p,\mu} := \inf_{g \in C^r(\mathbb{B}^d)} \Big\{ \|f - g\|_{p,\mu} + t^r \max_{1 \leqslant i < j \leqslant d} \big\| D^r_{i,j} g \big\|_{p,\mu} + t^r \max_{1 \leqslant i \leqslant d} \big\| \varphi^r \partial^r_i g \big\|_{p,\mu} \Big\}.$$

We establish a connection between $\widehat{K}_r(f, t)_{p,\mu}$ and the K-functionals $K_r(f, t)_{p,\mu}$ defined in (5.6). The result plays a crucial role in our development in this section. Recall, in particular, that the proof of Theorem 5.12 relies on the theorem below.

Theorem 6.2. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Let $f \in L^p(\mathbb{B}^d, W_\mu)$ if $1 \leqslant p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$. We further assume that r is odd when $p = \infty$. Then*

$$\widehat{K}_1(f, t)_{p,\mu} \sim K_1(f, t)_{p,\mu}, \tag{6.1}$$

and for $r > 1$, there is a $t_r > 0$ such that

$$K_r(f, t)_{p,\mu} \leqslant c \widehat{K}_r(f, t)_{p,\mu} + c t^r \|f\|_{p,\mu}, \quad 0 < t < t_r. \tag{6.2}$$

The proof of Theorem 6.2 relies on several lemmas. The first one contains two Landau type inequalities. In the case of no weight function and r is even, this lemma appeared in [16, p. 135] with $\|f\|_p$ in place of $\|\varphi^r f\|_p$ in the right-hand side of the inequalities. The proof of the general case follows along the same line, but there are enough modifications that we decide to include a proof.

Lemma 6.3. *Let $\mu \geqslant 0$ and $r \in \mathbb{N}$. Assume f defined on $[-1, 1]$ satisfies $\varphi^r f^{(r)} \in L^p[-1, 1]$.*

(i) *If $1 \leqslant p < \infty$ and $1 \leqslant i \leqslant \frac{r}{2}$ or $p = \infty$ and $1 \leqslant i < \frac{r}{2}$, then*

$$\|\varphi^{r-2i} f^{(r-i)}\|_{p,\mu} \leqslant c_1 \|\varphi^r f^{(r)}\|_{p,\mu} + c_2 \|\varphi^r f\|_{p,\mu}. \tag{6.3}$$

(ii) *If* r *is even, set* $\delta_r := 0$ *and assume* $1 \leqslant i \leqslant \frac{r}{2}$, $1 \leqslant p < \infty$*; If* r *is odd, set* $\delta_r := 1$ *and assume* $1 \leqslant i \leqslant \frac{r+1}{2}$, $1 \leqslant p \leqslant \infty$. *Then*

$$\|\varphi^{\delta_r} f^{(i)}\|_{p,\mu} \leqslant c_1 \|\varphi^r f^{(r)}\|_{p,\mu} + c_2 \|\varphi^r f\|_{p,\mu}. \tag{6.4}$$

Proof. First, we show that for $1 \leqslant p \leqslant \infty$ and $1 \leqslant i \leqslant r-1$,

$$\|f^{(i)}\|_{p,\mu} \leqslant c\big(\|f^{(r)}\|_{p,\mu} + \|f\|_{p,\mu}\big). \tag{6.5}$$

In the case when there is no weight function, this inequality is well known. We only need to establish it for $1 \leqslant p < \infty$. We derive it from the following result in [20, p. 109],

$$\int_0^\infty x^\alpha \big|g^{(i)}(x)\big|^p\,dx \leqslant c\left(\int_0^\infty x^\alpha \big|g(x)\big|^p\,dx\right)^{1-\frac{i}{r}} \left(\int_0^\infty x^\alpha \big|g^{(r)}(x)\big|^p\,dx\right)^{\frac{i}{r}}$$

for all $\alpha > -1$, which implies, by the elementary inequality $|ab| \leqslant \frac{|a|^p}{p} + \frac{|b|^q}{q}$ for $\frac{1}{p} + \frac{1}{q} = 1$, that for $0 \leqslant i \leqslant r$,

$$\int_0^\infty x^\alpha \big|g^{(i)}(x)\big|^p\,dx \leqslant c\int_0^\infty x^\alpha \big|g(x)\big|^p\,dx + c\int_0^\infty x^\alpha \big|g^{(r)}(x)\big|^p\,dx. \tag{6.6}$$

For f defined on $[-1,1]$, we write $f = f_1 + f_2 = f\psi + f(1-\psi)$, where ψ is a C^∞ function on $\mathbb{R}$ such that $\psi(x) = 1$ for $x \leqslant -1/2$ and $\psi(x) = 0$ for $x \geqslant 1/2$. It then follows by (6.6) that for $0 \leqslant i \leqslant r$

$$\|f_j^{(i)}\|_{p,\mu} \leqslant c\|f_j^{(r)}\|_{p,\mu} + c\|f_j\|_{p,\mu} \leqslant c\|f_j^{(r)}\|_{p,\mu} + c\|f\|_{p,\mu}, \quad j = 1,2.$$

Thus, the proof of (6.5) is reduced to showing

$$\|f_j^{(r)}\|_{p,\mu} \leqslant c\|f\|_{p,\mu} + c\|f^{(r)}\|_{p,\mu}, \quad j = 1,2. \tag{6.7}$$

To see this, we observe that ψ' is supported in $[-\frac{1}{2}, \frac{1}{2}]$. Thus, by the Leibnitz rule, we obtain

$$\begin{aligned}
\|f_j^{(r)}\|_{p,\mu} &\leqslant c\|f^{(r)}\|_{p,\mu} + c\max_{0\leqslant i\leqslant r-1}\|f^{(i)}\|_{L^p[-\frac{1}{2},\frac{1}{2}]} \\
&\leqslant c\|f^{(r)}\|_{p,\mu} + c\|f^{(r)}\|_{L^p[-\frac{1}{2},\frac{1}{2}]} + c\|f\|_{L^p[-\frac{1}{2},\frac{1}{2}]} \\
&\leqslant c\|f^{(r)}\|_{p,\mu} + c\|f\|_{p,\mu},
\end{aligned}$$

where the second step uses the unweighted version of (6.5). This proves the desired inequality (6.7), and hence completes the proof of (6.5).

Now we return to the proof of (6.3) and (6.4). For $f = f_1 + f_2$ decomposed as above, using (6.7) and the fact that $\varphi^{rp} w_\mu = w_{rp/2+\mu}$, we deduce

$$\left\|\varphi^r f_j^{(r)}\right\|_{p,\mu} \leqslant c\left(\left\|\varphi^r f^{(r)}\right\|_{p,\mu} + \left\|\varphi^r f\right\|_{p,\mu}\right), \quad j = 1, 2. \tag{6.8}$$

Thus, we can work with f_j, $j = 1, 2$ instead of f. However, by symmetry, we can assume, without loss of generality, that f is supported in $[-1, \frac{1}{2}]$. We claim that if $j \in \mathbb{Z}_+$ and $g \in C^j[-1, 1]$ is supported in $[-1, \frac{1}{2}]$ then

$$\left\|\varphi^a g\right\|_{p,\mu} \leqslant c\left\|\varphi^{a+2j} g^{(j)}\right\|_{p,\mu} \tag{6.9}$$

whenever $\mu + \frac{1}{2} + \frac{ap}{2} > 0$ and $1 \leqslant p < \infty$ or $a > 0$ and $p = \infty$. Clearly, once (6.9) is proved, then (6.3) follows by setting $g = f^{(r-i)}$, $a = r - 2i$ and $j = i$, whereas (6.4) follows by setting $g = f^{(i)}$, $a = \delta_r$ and $j = r - i$.

Clearly, for the proof of the claim (6.9), it suffices to consider the case of $j = 1$. For $p = \infty$, we use the inequality $|g(x)| \leqslant |\int_x^{\frac{1}{2}} g'(t)\,dt|$ to obtain, for $-1 \leqslant x \leqslant \frac{1}{2}$,

$$\begin{aligned}\left|\varphi^a(x) g(x)\right| &\leqslant c(1+x)^{\frac{a}{2}} \int_x^{\frac{1}{2}} \left|g'(t)\right| dt \\ &\leqslant c\left\|\varphi^{a+2} g'\right\|_\infty (1+x)^{\frac{a}{2}} \int_x^{\frac{1}{2}} (1+t)^{-\frac{a}{2}-1}\,dt \leqslant c\left\|\varphi^{a+2} g'\right\|_\infty,\end{aligned}$$

where we used the assumption $a > 0$ in the last step. This proves (6.9) for $p = \infty$.

Next, we show (6.9) for $1 \leqslant p < \infty$. Again, we only need to consider $j = 1$. Our main tool is the following Hardy inequality: for $1 \leqslant p < \infty$ and $\beta > 0$,

$$\left(\int_0^\infty \left(\int_x^\infty \left|f(y)\right| dy\right)^p x^{\beta-1}\,dx\right)^{1/p} \leqslant \frac{p}{\beta}\left(\int_0^\infty \left|yf(y)\right|^p y^{\beta-1}\,dy\right)^{1/p}. \tag{6.10}$$

Setting $G(y) = g(\frac{3}{2}y - 1)$ with $y \in [0, 1]$, we obtain

$$\begin{aligned}\left\|\varphi^a g\right\|_{p,\mu}^p &\leqslant c\int_{-1}^{\frac{1}{2}} \left|g(x)\right|^p (1+x)^{\mu-\frac{1}{2}+\frac{pa}{2}}\,dx = c\int_0^1 \left|G(y)\right|^p y^{\mu-\frac{1}{2}+\frac{pa}{2}}\,dy \\ &\leqslant c\int_0^1 \left(\int_y^1 \left|G'(x)\right| dx\right)^p y^{\mu-\frac{1}{2}+\frac{pa}{2}}\,dy \leqslant c\int_0^1 \left|yG'(y)\right|^p y^{\mu-\frac{1}{2}+\frac{pa}{2}}\,dy\end{aligned}$$

$$= c \int_{-1}^{\frac{1}{2}} \left|g'(x)\right|^p (x+1)^{\mu - \frac{1}{2} + \frac{pa}{2} + p}\, dx \leqslant c \left\|\varphi^{a+2} g'\right\|_{p,\mu}^p$$

proving the claim (6.9) for $j = 1$. This completes the proof. □

Lemma 6.4. *Let $\tilde{f}$ be defined as in* (5.13). *Then*

$$D_{1,d+1}^r \tilde{f}(x, x_{d+1}) = \sum_{j=1}^r p_{j,r}(x_1, x_{d+1}) \partial_1^j f(x), \quad x \in \mathbb{B}^d,\ (x, x_{d+1}) \in \mathbb{B}^{d+1},$$

where $p_{r,r}(x_1, x_{d+1}) = x_{d+1}^r$ and

$$p_{j,2r}(x_1, x_{d+1}) = \sum_{\max\{0, j-r\} \leqslant \nu \leqslant j/2} a_{\nu,j}^{(2r)} x_1^{j-2\nu} x_{d+1}^{2\nu}, \tag{6.11}$$

$$p_{j,2r-1}(x_1, x_{d+1}) = \sum_{\max\{0, j-r\} \leqslant \nu \leqslant (j-1)/2} a_{\nu,j}^{(2r-1)} x_1^{j-1-2\nu} x_{d+1}^{2\nu+1} \tag{6.12}$$

for $1 \leqslant j \leqslant 2r-1$ and $1 \leqslant j \leqslant 2r-2$, respectively, and $a_{\nu,j}^{(r)}$ are absolute constants.

Proof. Recall that $\tilde{f}(x, x_{d+1}) = f(x)$, so that $\partial_{d+1} \tilde{f}(x, x_{d+1}) = 0$. Starting from

$$D_{1,d+1}^{r+1} \tilde{f}(x_1, x_{d+1}) = (x_{d+1}\partial_1 - x_1 \partial_{d+1}) \sum_{j=1}^r p_{j,r}(x_1, x_{d+1}) \partial_1^j f(x),$$

a simple computation shows that $p_{j,r}$ satisfies the recurrence relation

$$p_{j,r+1} = x_{d+1} p_{j-1,r} + (x_{d+1}\partial_1 - x_1 \partial_{d+1}) p_{j,r}, \quad 1 \leqslant j \leqslant r, \tag{6.13}$$

where we define $p_{0,r} := 0$, and $p_{r+1,r+1} = x_{d+1} p_{r,r}$. Since $p_{1,1} = x_{d+1}$, we see that $p_{r,r} = x_{d+1}^r$ by induction. The general case also follows by induction: assuming $p_{j,r}$ takes the stated from, we apply (6.13) twice to get $p_{j,r+2}$ and verify that they are of the form (6.11) and (6.12). □

We will also need the following integral formula, which is a simple consequence of a change of variables.

Lemma 6.5. *For $1 \leqslant m \leqslant d-1$,*

$$\int_{\mathbb{B}^d} f(x)\, dx = \int_{\mathbb{B}^{d-m}} \left[\int_{\mathbb{B}^m} f\left(\sqrt{1 - \|v\|^2} u, v\right) du \right] \left(1 - \|v\|^2\right)^{\frac{m}{2}} dv. \tag{6.14}$$

We are now in a position to prove Theorem 6.2.

Proof of Theorem 6.2. We give the proof for the case $m \geqslant 2$ only. The proof for the case $m = 1$ follows along the same line. The only difference in this case is that we need to replace the integral over $\mathbb{B}^{d+1}$ by the one over $\mathbb{S}^d$ according to Definition 5.6, and use (2.12) instead of (6.14).

By definition, we need to compare $\|D^r_{i,d+1}\tilde{g}\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})}$ with $\|\varphi^r \partial_i^r g\|_{p,\mu}$, where $\|\cdot\|_{p,\mu} \equiv \|\cdot\|_{L^p(\mathbb{B}^d, W_\mu)}$. More precisely, we need to show

$$\|D_{i,d+1}\tilde{g}\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \sim \|\varphi\partial_i g\|_{p,\mu}, \quad 1 \leqslant i \leqslant d \tag{6.15}$$

and for $r \geqslant 2$

$$\big\|D^r_{i,d+1}\tilde{g}\big\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \leqslant c\big\|\varphi^r \partial_i^r g\big\|_{p,\mu} + c\|g\|_{p,\mu}, \quad 1 \leqslant i \leqslant d. \tag{6.16}$$

If $r = 1$, then by (2.8) $D_{1,d+1}\tilde{g}(x, x_{d+1}) = x_{d+1}\partial_1 g(x)$. Hence, by (6.14),

$$\begin{aligned}\|D_{i,d+1}\tilde{g}\|^p_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} &= \int_{\mathbb{B}^{d+1}} \big|x_{d+1}\partial_1 g(x)\big|^p \big(1 - \|x\|^2 - x_{d+1}^2\big)^{\mu-1}\, d(x, x_{d+1}) \\ &= c\int_{\mathbb{B}^d} \big|\varphi(x)\partial_1 g(x)\big|^p \big(1 - \|x\|^2\big)^{\mu-1/2}\, dx = c\|\varphi\partial_1 g\|^p_{p,\mu},\end{aligned}$$

where $c = \int_{-1}^1 |s|^p (1-s^2)^{\mu-1}\, ds$. The above argument with slight modification works equally well for $p = \infty$. This proves (6.15).

Next, we show (6.16) for $r \geqslant 2$. By symmetry, we only need to consider the case $i = 1$. We start with the case of even $r = 2\ell$ with $\ell \in \mathbb{N}$. In this case, $1 \leqslant p < \infty$, and by (6.11), we have

$$\big|D^{2\ell}_{1,d+1}\tilde{g}(x, x_{d+1})\big| \leqslant c\sum_{j=1}^{2\ell} \max_{\max\{0, j-\ell\} \leqslant \nu \leqslant j/2} \big|x_1^{j-2\nu} x_{d+1}^{2\nu} \partial_1^j g(x)\big|.$$

This implies

$$\big\|D^{2\ell}_{1,d+1}\tilde{g}\big\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \leqslant c\sum_{j=1}^{\ell} \max_{0 \leqslant \nu \leqslant j/2} I_{j,\nu} + c\sum_{j=\ell+1}^{2\ell} \max_{j-\ell \leqslant \nu \leqslant j/2} I_{j,\nu}, \tag{6.17}$$

with

$$\begin{aligned}I_{j,\nu} &:= \int_{\mathbb{B}^{d+1}} \big|x_1^{j-2\nu} x_{d+1}^{2\nu} \partial_1^j g(x)\big|^p \big(1 - \|x\|^2 - x_{d+1}^2\big)^{\mu-1}\, d(x, x_{d+1}) \\ &= c\int_{\mathbb{B}^d} \big|x_1^{j-2\nu} \varphi^{2\nu}(x) \partial_1^j g(x)\big|^p \big(1 - \|x\|^2\big)^{\mu-1/2}\, dx,\end{aligned}$$

where the last equation follows by (6.14). Let $x = (x_1, x') \in \mathbb{B}^d$. Using (6.14) again, and setting $g_{x'}(t) = g(t\varphi(x'), x')$, we see that

$$I_{j,\nu} = c \int_{\mathbb{B}^{d-1}} \int_{-1}^{1} \left|t^{j-2\nu}\varphi^j(x')\varphi^{2\nu}(t)\partial_1^j g(\varphi(x')t, x')\right|^p (1-t^2)^{\mu-1/2}\,dt\,(1-\|x'\|^2)^\mu\,dx'$$

$$\leqslant c \int_{\mathbb{B}^{d-1}} \left[\int_{-1}^{1} \left|\varphi^{2\nu}(t) g_{x'}^{(j)}(t)\right|^p (1-t^2)^{\mu-1/2}\,dt\right](1-\|x'\|^2)^\mu\,dx',$$

where the inequality is resulted from $|t^{j-2\nu}| \leqslant 1$.

If $1 \leqslant j \leqslant \ell = \frac{r}{2}$ and $\nu \geqslant 0$, then $\varphi^{2\nu}(t) \leqslant 1$, so that we can apply (6.4) in Lemma 6.3 to conclude that

$$I_{j,\nu} \leqslant c \int_{\mathbb{B}^{d-1}} \left[\int_{-1}^{1} \left|\varphi^{2\ell}(t)\frac{d^{2\ell}}{dt^{2\ell}}\left[g(\varphi(x')t, x')\right]\right|^p (1-t^2)^{\mu-1/2}\,dt\right](1-\|x'\|^2)^\mu\,dx'$$

$$+ c \int_{\mathbb{B}^{d-1}} \int_{-1}^{1} \left|g(\varphi(x')t, x')\right|^p (1-t^2)^{\mu-1/2}\,dt\,(1-\|x'\|^2)^\mu\,dx'$$

$$= c \int_{\mathbb{B}^d} \left|\varphi^{2\ell}(x)\partial_1^{2\ell} g(x)\right|^p (1-\|x\|^2)^{\mu-1/2}\,dx + c\|g\|_{p,\mu}^p.$$

If $\ell + 1 \leqslant j \leqslant 2\ell$, and $\nu \geqslant j - \ell$, then $\varphi^{2\nu}(t) \leqslant \varphi^{2j-2\ell}(t)$, so that we can apply (6.3) in Lemma 6.3 with $i = 2\ell - j$ and $r = 2\ell$ to the integral over t, which leads exactly as in the previous case to $I_{j,\nu} \leqslant c\|\varphi^{2\ell}\partial_1^{2\ell} g\|_{p,\mu}^p + c\|g\|_{p,\mu}^p$.

Putting these together, and using (6.17), we have established the desired result for the case of even $r = 2\ell$. The proof for the case of odd r follows along the same line. This completes the proof. □

Remark 6.1. In the case of $r = 2$ and $1 < p < \infty$, the reversed inequality (6.2) holds:

$$c_1\widehat{K}_2(f,t)_{p,\mu} \leqslant K_2(f,t)_{p,\mu} \leqslant c_2\widehat{K}_2(f,t)_{p,\mu} + ct^2\|f\|_{p,\mu}, \quad 1 < p < \infty. \tag{6.18}$$

This will be proved in a more general setting in Theorem 7.5.

We do not know if the first inequality in (6.18) holds for $r > 2$, but they have to be close as both direct and inverse theorems hold using either K-functional.

6.2. Direct and inverse theorem by K-functional

For the K-functional given in Definition 6.1, we establish both the direct and the inverse inequalities.

Theorem 6.6. *Let $\mu = \frac{m-1}{2}$, $m \in \mathbb{N}$ and $r \in \mathbb{N}$. Let $f \in L^p(\mathbb{B}^d, W_\mu)$ if $1 \leqslant p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$. Then*

$$E_n(f)_{p,\mu} \leqslant c\widehat{K}_r\big(f, n^{-1}\big)_{p,\mu} + cn^{-r}\|f\|_{p,\mu} \tag{6.19}$$

and

$$\widehat{K}_r\big(f, n^{-1}\big)_{p,\mu} \leqslant cn^{-r}\sum_{k=1}^{n} k^{r-1}E_k(f)_{p,\mu}. \tag{6.20}$$

Furthermore, the additional term $n^{-r}\|f\|_{p,\mu}$ on the right-hand side of (6.19) *can be dropped when $r = 1$.*

Proof. When $1 \leqslant p < \infty$ and $r \in \mathbb{N}$ or $p = \infty$ and r is odd, the Jackson type estimate (6.19) follows immediately from (5.18) and Theorem 6.2. Thus, it remains to prove (6.19) for even $r = 2\ell$ and $p = \infty$. Since we already proved (6.19) for $\widehat{K}_{2\ell+1}(f,t)_\infty$, it suffices to show the inequality

$$\widehat{K}_{2\ell+1}(f,t)_\infty \leqslant c\widehat{K}_{2\ell}(f,t)_\infty. \tag{6.21}$$

For $d = 1$, (6.21) has already been proved in [16, p. 38]; whereas in the case of $d \geqslant 2$ it is a consequence of the following inequalities

$$\big\|D_{i,j}^{r+1}g\big\|_\infty \leqslant c\big\|D_{i,j}^{r}g\big\|_\infty \quad \text{and} \quad \big\|\varphi^{r+1}\partial_i^{r+1}g\big\|_\infty \leqslant c\big\|\varphi^{r}\partial_i^{r}g\big\|_\infty$$

which can be deduced directly from the corresponding results for functions of one variable; see, for example, (6.9).

The inverse estimate (6.20) follows as usual from the Bernstein inequalities: for $1 \leqslant p \leqslant \infty$ and $P \in \Pi_n^d$,

$$\max_{1\leqslant i<j\leqslant d}\big\|D_{i,j}^{r}P\big\|_{p,\mu} \leqslant cn^r\|P\|_{p,\mu} \quad \text{and} \quad \max_{1\leqslant i\leqslant d}\big\|\varphi^{r}\partial_i^{r}P\big\|_{p,\mu} \leqslant cn^r\|P\|_{p,\mu}. \tag{6.22}$$

The second inequality in (6.22) has already been established in [5, Theorem 8.2], so we just need to show the first inequality. Without loss of generality, we may assume $(i,j) = (1,2)$. We then have, for $1 \leqslant p < \infty$,

$$\begin{aligned}\big\|D_{1,2}^{r}f\big\|_{p,\mu}^{p} &= \int_{\mathbb{B}^{d-2}}\Big[\int_{\mathbb{B}^2}\big|D_{1,2}^{r}f\big(\varphi(u)x_1, \varphi(u)x_2, u\big)\big|^p\big(1-x_1^2-x_2^2\big)^{\mu-\frac{1}{2}}\,dx_1\,dx_2\Big] \\ &\quad\times\big(1-\|u\|^2\big)^{\mu+\frac{1}{2}}\,du \\ &= \int_{\mathbb{B}^{d-2}}\int_0^1\Big[\int_0^{2\pi}\big|D_{1,2}^{r}f\big(\varphi(u)\rho\cos\theta, \varphi(u)\rho\sin\theta, u\big)\big|^p\,d\theta\Big] \\ &\quad\times\big(1-\rho^2\big)^{\mu-\frac{1}{2}}\rho\,d\rho\big(1-\|u\|^2\big)^{\mu+\frac{1}{2}}\,du\end{aligned}$$

$$= \int_{\mathbb{B}^{d-2}} \int_0^1 \left[\int_0^{2\pi} \left| f_{u,\rho}^{(r)}(\theta) \right|^p d\theta \right] \left(1-\rho^2\right)^{\mu-\frac{1}{2}} \rho\, d\rho \left(1-\|u\|^2\right)^{\mu+\frac{1}{2}} du$$

$$\leqslant cn^{rp} \int_{\mathbb{B}^{d-2}} \int_0^1 \left[\int_0^{2\pi} \left| f_{u,\rho}(\theta) \right|^p d\theta \right] \left(1-\rho^2\right)^{\mu-\frac{1}{2}} \rho\, d\rho \left(1-\|u\|^2\right)^{\mu+\frac{1}{2}} du$$

$$= cn^{rp} \|f\|_{p,\mu}^p, \tag{6.23}$$

where $f_{u,\rho}(\theta) = f(\varphi(u)\rho\cos\theta, \varphi(u)\rho\sin\theta, u)$, and the inequality step uses the usual Bernstein inequality for trigonometric polynomials. Using (2.4), the same argument works for $p = \infty$. This completes the proof of the inverse estimate. □

Remark 6.2. Since the Bernstein inequality (6.22) is proved for all $\mu > -\frac{1}{2}$, the inverse estimate (6.20) holds for all $\mu > -\frac{1}{2}$ as well.

6.3. Analogue of Ditzian–Totik modulus of smoothness on $\mathbb{B}^d$

Recall the definition of the Ditzian–Totik modulus of smoothness in (5.27). We define an analogue on the ball $\mathbb{B}^d$. Since the definition for the weighted space has an additional complication, we consider only the unweighted case, that is, the case $W_{1/2}(x)\,dx = dx$, in this section. Let e_i be the i-th coordinate vector of $\mathbb{R}^d$ and let $\widehat{\Delta}_{he_i}^r$ be the r-th central difference in the direction of e_i, more precisely,

$$\widehat{\Delta}_{he_i}^r f(x) = \sum_{k=0}^r (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right) he_i\right).$$

As in the case of $[-1,1]$, we assume that $\widehat{\Delta}_{he_i}^r$ is zero if either of the points $x \pm r\frac{h}{2}e_i$ does not belong to $\mathbb{B}^d$. We write $L^p(\mathbb{B}^d)$, $\|f\|_p$ and $\widehat{K}_r(f,t)_p$ for $L^p(\mathbb{B}^d, W_{1/2})$, $\|f\|_{L^p(\mathbb{B}^d, W_{1/2})}$ and $\widehat{K}_r(f,t)_{p,1/2}$ respectively. The modulus of smoothness $\hat{\omega}_r(f,t)_p$ in (5.27) for the case $d=1$ suggests the following definition:

Definition 6.7. Let $f \in L^p(\mathbb{B}^d)$ if $1 \leqslant p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. For $r \in \mathbb{N}$ and $t > 0$,

$$\hat{\omega}_r(f,t)_p = \sup_{0<|h|\leqslant t} \left\{ \max_{1\leqslant i<j\leqslant d} \left\| \Delta_{i,j,h}^r f \right\|_p, \max_{1\leqslant i\leqslant d} \left\| \widehat{\Delta}_{h\varphi e_i}^r f \right\|_p \right\}. \tag{6.24}$$

As in the case of Definitions 2.2 and 5.3, the new moduli are not rotationally invariant, they depend on the standard basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$ but independent of the order of this basis. In the case of $d = 1$, there is no Euler angle and the definition becomes exactly the one in (5.27).

Much of the properties of the modulus of smoothness $\hat{\omega}_r(f,t)_p$ follows from the corresponding properties of the moduli of smoothness on the sphere and on $[-1,1]$. For example, we have the following lemma.

Lemma 6.8. *Let $f \in L^p(\mathbb{B}^d)$ for $1 \leqslant p < \infty$ and $f \in C(\mathbb{B}^d)$ for $p = \infty$.*

(1) *For $0 < t < t_0$, $\hat{\omega}_{r+1}(f,t)_p \leqslant c\hat{\omega}_r(f,t)_p$.*
(2) *For $\lambda > 0$, $\hat{\omega}_r(f,\lambda t)_p \leqslant c(\lambda+1)^r\hat{\omega}_r(f,t)_p$.*
(3) *For $0 < t < \frac{1}{2}$ and every $m > r$,*

$$\hat{\omega}_r(f,t)_p \leqslant c_m\left(t^r\int_t^1 \frac{\hat{\omega}_m(f,u)_p}{u^{r+1}}\,du + t^r\|f\|_p\right).$$

(4) *For $0 < t < t_0$, $\hat{\omega}_r(f,t)_p \leqslant c\|f\|_p$.*

Proof. For $1 \leqslant p < \infty$ and $\triangle^r_{i,j,\theta}f$, we use the integral formula

$$\left\|\triangle^r_{i,j,\theta}f\right\|_p^p = \int_0^1 s^{d-1}\int_{\mathbb{S}^{d-1}}\left|\triangle^r_{i,j,\theta}f\left(sx'\right)\right|^p d\sigma\left(x'\right)ds$$

and apply Proposition 2.5. For $\widehat{\triangle}^r_{\theta\varphi e_i}f$, we use (6.14) with $m = 1$ and the fact that if $x = (\varphi(u)s, u)$ then $\varphi(x) = \varphi(s)\varphi(u)$ to conclude that

$$\begin{aligned}
\left\|\varphi^r\widehat{\triangle}^r_{\theta\varphi e_i}f\right\|_p^p &= \int_{BB^d}\left|\varphi^r(x)\widehat{\triangle}^r_{\theta\varphi(x)e_i}f(x)\right|^p dx \\
&= \int_{\mathbb{B}^{d-1}}\int_{-1}^1\left|\varphi^r(u)\varphi^r(s)\widehat{\triangle}^r_{\theta\varphi(s)\varphi(u)e_i}f\left(\varphi(u)s,u\right)\right|^p ds\varphi(u)\,du \\
&= \int_{\mathbb{B}^{d-1}}\left(\varphi(u)\right)^{rp+1}\left[\int_{-1}^1\left|\varphi^r(s)\widehat{\triangle}^r_{\theta\varphi(s)e_i}f_u(s)\right|^p ds\right]du,
\end{aligned}$$

where $f_u(s) = f(\varphi(u)s, u)$ and apply the result of one variable in [16, p. 38, p. 43] to the inner integral as well as the equivalence (5.29). $\square$

Next we establish the direct and the inverse theorems in $\hat{\omega}_r(f,t)_p$, one of the central results in this section.

Theorem 6.9. *Let $f \in L^p(\mathbb{B}^d)$ if $1 \leqslant p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$. Then for $r \in \mathbb{N}$*

$$E_n(f)_p \leqslant c\hat{\omega}_r\left(f,n^{-1}\right)_p + n^{-r}\|f\|_p \tag{6.25}$$

and

$$\hat{\omega}_r\left(f,n^{-1}\right)_p \leqslant cn^{-r}\sum_{k=1}^n k^{r-1}E_k(f)_p. \tag{6.26}$$

Furthermore, the additional term $n^{-r}\|f\|_p$ *on the right-hand side of* (6.25) *can be dropped when* $r=1$.

Proof. We start with the proof of the Jackson type inequality (6.25). By (5.18), it suffices to show that for the modulus $\omega_r(f,t)_p$ given in Definition 5.3,

$$\omega_r\big(f,n^{-1}\big)_p \leqslant c\hat{\omega}_r\big(f,n^{-1}\big)_p + cn^{-r}\|f\|_p. \tag{6.27}$$

However, using Definitions 5.3 and 6.7, this amounts to showing that for $1 \leqslant i \leqslant d$

$$\sup_{|\theta|\leqslant t} \big\|\triangle^r_{i,d+1,\theta}\tilde{f}\big\|_{L^p(\mathbb{B}^{d+1},W_0)} \leqslant c\hat{\omega}_r(f,t)_p + ct^r\|f\|_p, \tag{6.28}$$

where $\tilde{f}(x,x_{d+1}) = f(x)$ for $x\in\mathbb{B}^d$ and $(x,x_{d+1})\in\mathbb{B}^{d+1}$. By symmetry, we only need to consider $i=1$. Set

$$f_v(s) = f\big(\varphi(v)s,v\big), \quad v\in\mathbb{B}^{d-1},\ s\in[-1,1],$$

where $\varphi(v)=\sqrt{1-\|v\|^2}$. We can then write, by (6.14),

$$\begin{aligned}
&\big\|\triangle^r_{1,d+1,\theta}\tilde{f}\big\|^p_{L^p(\mathbb{B}^{d+1},W_0)}\\
&\quad= \int_{\mathbb{B}^{d+1}} \big|\overrightarrow{\triangle}^r_\theta f\big(x_1\cos(\cdot)+x_{d+1}\sin(\cdot),x_2,\dots,x_d\big)\big|^p \frac{dx}{\sqrt{1-\|x\|^2}}\\
&\quad= \int_{\mathbb{B}^{d-1}} \Bigg[\int_{\mathbb{B}^2} \big|\overrightarrow{\triangle}^r_\theta f_v\big(x_1\cos(\cdot)+x_{d+1}\sin(\cdot)\big)\big|^p \frac{dx_1\,dx_{d+1}}{\sqrt{1-x_1^2-x_{d+1}^2}}\Bigg]\varphi(v)\,dv.
\end{aligned}$$

Applying (5.31) to the inner integral, the last expression is bounded by, for $|\theta|\leqslant t$,

$$\begin{aligned}
&c\frac{1}{t}\int_0^t\int_{\mathbb{B}^{d-1}}\Bigg[\int_{-1}^1\big|\widehat{\triangle}^r_{h\varphi(s)}f_v(s)\big|^p\,ds\Bigg]\varphi(v)\,dv\,dh + t^{rp}\int_{\mathbb{B}^{d-1}}\int_{-1}^1\big|f_v(s)\big|^p\,ds\,\varphi(v)\,dv\\
&\quad= c\frac{1}{t}\int_0^t\int_{\mathbb{B}^{d-1}}\int_{-1}^1\big|\widehat{\triangle}^r_{h\varphi(\varphi(v)s,v)e_1}f\big(\varphi(v)s,v\big)\big|^p\,ds\,\varphi(v)\,dv\,dh + ct^{rp}\|f\|_p^p\\
&\quad= c\frac{1}{t}\int_0^t\int_{\mathbb{B}^d}\big|\widehat{\triangle}^r_{h\varphi(x)e_1}f(x)\big|^p\,dx\,dh + ct^{rp}\|f\|_p^p \leqslant c\hat{\omega}_r(f,t)_p^p + ct^{rp}\|f\|_p^p.
\end{aligned}$$

For $r=1$ the additional term $t^{rp}\|f\|_p^p$ can be dropped because of Theorem 5.12. Obviously, the above argument with slight modification works equally well for the case $p=\infty$. This proves the Jackson inequality (6.28).

Finally, the inverse estimate (6.26) follows by (6.20) and the inequality $\hat{\omega}_r(f,t)_{p,\mu} \leqslant c\widehat{K}_r(f,t)_{p,\mu}$, which will be given in Theorem 6.10 in the next subsection. □

6.4. Equivalence of $\hat{\omega}_r(f,t)_p$ and $\widehat{K}_r(f,t)_p$

As a consequence of Theorem 6.9, we can deduce the equivalence of the modulus of smoothness $\hat{\omega}^r(f,t)_p$ and the K-functional $\widehat{K}_r(f,t)_p$:

Theorem 6.10. *Let $f \in L^p(\mathbb{B}^d)$ if $1 \leqslant p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$. Then for $r \in \mathbb{N}$ and $0 < t < t_r$,*

$$c^{-1}\hat{\omega}^r(f,t)_p \leqslant \widehat{K}_r(f,t)_p \leqslant c\hat{\omega}^r(f,t)_p + ct^r\|f\|_p.$$

Furthermore, the term $t^r\|f\|_p$ on the right side can be dropped when $r = 1$.

For the proof of Theorem 6.10, we need the following lemma.

Lemma 6.11. *For $1 \leqslant p \leqslant \infty$ and $f \in \Pi_n^d$, we have*

$$n^{-r}\left\|D_{i,j}^r f\right\|_{p,\mu} \sim \sup_{|\theta| \leqslant n^{-1}} \left\|\triangle_{i,j,\theta}^r f\right\|_{p,\mu}, \quad 1 \leqslant i < j \leqslant d, \tag{6.29}$$

and

$$n^{-rp}\left\|\varphi^r \partial_i^r f\right\|_p^p \sim n \int_0^{n^{-1}} \left\|\widehat{\triangle}_{h\varphi e_i}^r f\right\|_p^p dh, \quad 1 \leqslant i \leqslant d, \tag{6.30}$$

with the usual change when $p = \infty$.

Proof. The relation (6.29) follows directly from (6.23) and the corresponding inequality for trigonometric inequality (see, for instance, [31]). The relation (6.30) can be proved similarly. In fact, setting $i = 1$ and $f_u(s) = f(\varphi(u)s, u)$, we have

$$\begin{aligned} n^{-rp}\left\|\varphi^r \partial_1^r f\right\|_p^p &= n^{-rp} \int_{\mathbb{B}^{d-1}} \left[\int_{-1}^{1} \left|\varphi^r(s) f_u^{(r)}(s)\right|^p ds\right] \varphi(u)\,du \\ &\sim n \int_{\mathbb{B}^{d-1}} \left[\int_0^{n^{-1}} \int_{-1}^{1} \left|\widehat{\triangle}_{h\varphi(s)}^r f_u(s)\right|^p ds\,dh\right] \varphi(u)\,du \\ &= n \int_0^{n^{-1}} \left\|\widehat{\triangle}_{h\varphi e_1}^r f\right\|_p^p dh, \end{aligned} \tag{6.31}$$

where we have used the equivalence of one variable in [18, p. 191] and (5.29). $\square$

Proof of Theorem 6.10. We start with the proof of the inequality

$$\hat{\omega}_r(f,t)_p \leqslant c\widehat{K}_r(f,t)_p, \quad 0<t<t_r. \tag{6.32}$$

Let $g_t \in C^r(\mathbb{B}^d)$ be chosen such that

$$\|f-g_t\|_p \leqslant 2\widehat{K}_r(f,t)_p, \qquad t^r \max_{1\leqslant i<j\leqslant d}\left\|D_{i,j}^r g_t\right\|_p \leqslant 2\widehat{K}_r(f,t)_p,$$

and

$$t^r \max_{1\leqslant i\leqslant d}\left\|\varphi^r \partial_i^r g_t\right\|_p \leqslant 2\widehat{K}_r(f,t)_p.$$

From the definition of $\hat{\omega}_r(f,t)_p$ and (4) of Lemma 6.8 it follows that

$$\hat{\omega}_r(f,t)_p \leqslant \hat{\omega}_r(f-g_t,t)_p + \hat{\omega}_r(g_t,t)_p \leqslant c\widehat{K}_r(f,t)_p + \hat{\omega}_r(g_t,t)_p.$$

Consequently, for the proof of the inequality of (6.32), it suffices to show that for $g \in C^r(\mathbb{B}^d)$,

$$\left\|\triangle_{i,j,\theta}^r g\right\|_p \leqslant c\theta^r \left\|D_{i,j}^r g\right\|_p \quad \text{and} \quad \left\|\widehat{\triangle}_{\theta\varphi e_i}^r g\right\|_p \leqslant c\theta^r \left\|\varphi^r \partial_i^r g\right\|_p. \tag{6.33}$$

First we consider $\widehat{\Delta}_{\theta\varphi e_i}^r f$, for which we will need the corresponding result for $[-1,1]$. It is known [16, (2.4.4)] that there exists $t_r \in (0,1)$ such that for $0<h<t_r$,

$$\left\|\widehat{\triangle}_{h\varphi}^r g_t\right\|_{L^p[-1,1]} \leqslant ch^r \left\|\varphi^r g_t^{(r)}\right\|_{L^p[-1,1]}. \tag{6.34}$$

For $p=\infty$, the proof of (6.33) follows from the usual relation between forwarded differences and derivatives. For $1\leqslant p<\infty$, we only need to consider the case of $i=1$. Using (6.14) with d replaced by $d-1$, we obtain by (6.34) that

$$\begin{aligned}
\left\|\widehat{\triangle}_{\theta\varphi e_1}^r g\right\|_p^p &= \int_{\mathbb{B}^{d-1}} \int_{-1}^{1} \left|\widehat{\triangle}_{\theta\varphi(y)\varphi(s)e_1}^r g\big(\varphi(y)s,y\big)\right|^p ds\,\varphi(y)\,dy \\
&= \int_{\mathbb{B}^{d-1}} \int_{-1}^{1} \left|\widehat{\triangle}_{\theta\varphi(s)}^r g_y(s)\right|^p ds\,\varphi(y)\,dy \\
&\leqslant c \int_{\mathbb{B}^{d-1}} \theta^{rp} \int_{-1}^{1} \left|\varphi^r(s)\frac{d^r}{ds^r}\big[g\big(\varphi(y)s,y\big)\big]\right|^p ds\,\varphi(y)\,dy \\
&= c\theta^{rp} \int_{\mathbb{B}^d} \left|\varphi^r(x)\partial_1^r g(x)\right|^p dx = c\theta^{rp}\left\|\varphi^r \partial_1^r g\right\|_p^p,
\end{aligned}$$

where $g_y(s)=g(\varphi(y)s,y)$. This proves the second inequality of (6.33).

Next, we consider $\triangle_{i,j,\theta}^r g$, for which we will need the corresponding result for trigonometric functions. Let h be a 2π periodic function in $L^p[0,2\pi]$ and let $\|h\|_p := (\int_0^{2\pi} |h(\theta)|^p\,d\theta)^{1/p}$ in the rest of this proof. Then it is known (see, for example, [11]) that

$$\big\|\overrightarrow{\triangle}_h^r h\big\|_p \leqslant ch^r \big\|h^{(r)}\big\|_p. \tag{6.35}$$

We consider only the case of $(i,j)=(1,2)$. By (2.8),

$$\begin{aligned}\big\|\triangle_{1,2,\theta}^r g\big\|_p^p &= \int_{\mathbb{B}^{d-2}}\int_{\mathbb{B}^2} \big|\triangle_{1,2,\theta}^r g\big(v,\varphi(v)u\big)\big|^p \varphi(v)^{d-2}\,dv\,du \\ &= \int_{\mathbb{B}^{d-2}}\int_0^1 \rho \int_0^{2\pi} \big|\overrightarrow{\triangle}_\theta^r g\big(\rho\cos t,\rho\sin t,\varphi(\rho)u\big)\big|^p\,dt\,\varphi(\rho)^{d-2}\,d\rho\,du.\end{aligned}$$

Setting $g_{\rho,u}(t)=g(\rho\cos t,\rho\sin t,\varphi(\rho)u)$, we deduce from (6.35) that

$$\begin{aligned}\big\|\triangle_{1,2,\theta}^r f\big\|_p^p &\leqslant c\theta^{rp} \int_{\mathbb{B}^{d-2}}\int_0^1 \rho\int_0^{2\pi} \big|g_{\rho,u}^{(r)}(t)\big|^p\,dt\,\varphi(\rho)^{d-2}\,d\rho\,du \\ &= c\theta^{rp}\int_{\mathbb{B}^d} \big|D_{1,2}^r g(x)\big|^p\,dx = c\theta^{rp}\big\|D_{1,2}^r f\big\|_p^p,\end{aligned}$$

which proves the first inequality of (6.33). Consequently, we have proved the inequality (6.32).

We now prove the reversed inequality

$$\widehat{K}_r(f,t)_p \leqslant c\hat{\omega}_r(f,t)_p + ct^r\|f\|_p. \tag{6.36}$$

Setting $n=\lfloor \frac{1}{t}\rfloor$, we have

$$\widehat{K}_r(f,t)_p \leqslant \big\|f - V_n^\mu f\big\|_p + t^r \max_{1\leqslant i<j\leqslant d}\big\|D_{i,j}^r V_n^\mu f\big\|_p + t^r \max_{1\leqslant i\leqslant d}\big\|\varphi^r\partial_i^r V_n^\mu f\big\|_p.$$

The first term is bounded by $c\hat{\omega}_r(f,t)_p + cn^{-r}\|f\|_p$ by (6.25). For the second term, we use (6.29) to obtain

$$\begin{aligned}t^r \max_{1\leqslant i<j\leqslant d}\big\|D_{i,j}^r V_n^\mu f\big\|_p &\leqslant c\hat{\omega}_r\big(V_n^\mu f, n^{-1}\big)_p \leqslant c\hat{\omega}_r\big(V_n^\mu f - f, n^{-1}\big)_p + c\hat{\omega}_r\big(f,n^{-1}\big)_p \\ &\leqslant c\big\|f - V_n^\mu f\big\|_p + c\hat{\omega}_r\big(f,n^{-1}\big)_p \leqslant c\hat{\omega}_r(f,t)_p + ct^r\|f\|_p.\end{aligned}$$

The third term can be treated similarly, using Lemma 6.11. This completes the proof of (6.36). □

6.5. Analogue of Ditzian–Totik modulus of smoothness with weight

For $w_\mu(x) = (1-t^2)^{\mu-1/2}$, the Ditzian–Totik modulus of smoothness $\hat{\omega}_r(f,t)_{p,\mu}$ is defined in (5.28), with two additional terms of forward and backward differences close to the boundary, which are shown to be necessary in [16, p. 56].

For the unit ball, we can define the modulus of smoothness with respect to W_μ for $\mu > 1/2$ in an analogous way. For this purpose, we first need to define the analogues of I_t and $J_{\pm 1,t}$. For $x \in \mathbb{B}^d$ and $1 \leqslant i \leqslant d$, we define $\hat{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)$. For $1 \leqslant i \leqslant d$, we define

$$\mathbb{I}_{i,t} := \left\{ x \in \mathbb{B}^d\colon \frac{x_i}{\sqrt{1-\|\hat{x}_i\|^2}} \in I_t \right\}, \qquad \mathbb{J}_{\pm 1,i,t} := \left\{ x \in \mathbb{B}^d\colon \frac{x_i}{\sqrt{1-\|\hat{x}_i\|^2}} \in J_{\pm 1,t} \right\}.$$

Definition 6.12. For $\mu > 1/2$ and $1 \leqslant p < \infty$, define

$$\begin{aligned} \hat{\omega}_r(f,t)_{p,\mu} := {} & \sup_{|\theta| \leqslant t} \max_{1 \leqslant i < j \leqslant d} \left\| \triangle^r_{i,j,\theta} f \right\|_{p,\mu} + \sup_{0<h \leqslant t} \max_{1 \leqslant i \leqslant d} \left\| \widehat{\triangle}^r_{h\varphi e_i} f \right\|_{L^p(\mathbb{I}_{i,rh}, W_\mu)} \\ & + \sup_{0<h \leqslant 12 r^2 t^2} \max_{1 \leqslant i \leqslant d} \left(\left\| \overleftarrow{\triangle}^r_{he_i} f \right\|_{L^p(\mathbb{J}_{1,i,rt}, W_\mu)} + \left\| \overrightarrow{\triangle}^r_{he_i} f \right\|_{L^p(\mathbb{J}_{-1,i,rt}, W_\mu)} \right), \end{aligned}$$

with the usual change when $p = \infty$.

The direct and the inverse theorems hold for this modulus of smoothness.

Theorem 6.13. *Let $\mu = \frac{m-1}{2}$ and $m > 2$, let $f \in L^p(\mathbb{B}^d, W_\mu)$ if $1 \leqslant p < \infty$. Then for $r \in \mathbb{N}$*

$$E_n(f)_{p,\mu} \leqslant c \hat{\omega}_r\left(f, n^{-1}\right)_{p,\mu} + n^{-r} \|f\|_{p,\mu} \tag{6.37}$$

and

$$\hat{\omega}_r\left(f, n^{-1}\right)_{p,\mu} \leqslant c n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f)_{p,\mu}. \tag{6.38}$$

Furthermore, the additional term $n^{-r}\|f\|_{p,\mu}$ on the right-hand side of (6.37) *can be dropped when $r = 1$.*

Proof. The proof of (6.37) follows along the line of Theorem 6.9 and we shall be brief. Since (5.18) is established for W_μ with $\mu = \frac{m-1}{2}$, we again come down to showing that for $1 \leqslant i \leqslant d$

$$\sup_{|\theta| \leqslant t} \left\| \triangle^r_{i,d+1,\theta} \tilde{f} \right\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \leqslant c \hat{\omega}_r(f,t)_{p,\mu} + c t^r \|f\|_{p,\mu}, \tag{6.39}$$

where $\tilde{f}(x, x_{d+1}) = f(x)$ for $x \in \mathbb{B}^d$ and $(x, x_{d+1}) \in \mathbb{B}^{d+1}$. The major difference is that instead of (5.31), we have

$$\int_{B^2} \left|\overrightarrow{\triangle}_\theta^r f\left(x_1\cos(\cdot)+x_2\sin(\cdot)\right)\right|^p \left(1-x_1^2-x_2^2\right)^{\mu-1}\,dx$$

$$\lesssim \frac{1}{t}\int_0^t \left\|\widehat{\triangle}_{h\varphi}^r f\right\|_{L^p(I_{rh},w_\mu)}^p dh + \frac{1}{t^2}\int_0^{t^2} \left\|\overleftarrow{\triangle}_h^r f\right\|_{L^p(J_{1,1,rt},w_\mu)}^p dh$$

$$+ \frac{1}{t^2}\int_0^{t^2} \left\|\overrightarrow{\triangle}_h^r f\right\|_{L^p(J_{-1,1,rt},w_\mu)}^p dh + t^{rp}\left\|\varphi^r f\right\|_{p,\mu}^p,$$

which follows from (5.30) and [16, p. 57]. Now, it is easy to see that if $x=(\varphi(v)s, v)$, then $s\in I_t$ is equivalent to $x\in \mathbb{I}_{1,t}$ and $s\in J_{\pm1,t}$ is equivalent to $x\in\mathbb{J}_{\pm1,1,t}$, from which we can carry out the computation and establish (6.39) exactly as in the proof of Theorem 6.9.

The proof of (6.38) follows again by (6.20) and Theorem 6.14 below. □

Two remarks are in order. First, it is worth to point out that, in the case of $d=1$, this theorem did not appear in [16], which gave the Jackson estimate for the weighted approximation in terms of the *main-part modulus of smoothness*. The result was later proved in [22, p. 556] and, for $1<p<\infty$, in [7, Corollary 7.3]. Second, in the case of $p=\infty$, the norm in $\hat{\omega}_r(f,t)_{\infty,\mu}$ is taken as $\|f\|_{\infty,\mu}=\|W_\mu f\|_\infty$, which is not what the norms in $\omega_r(f,t)_{\infty,\mu}$ or $\widehat{K}_r(f,t)_{\infty,\mu}$ are taken. Thus, we exclude the case of $p=\infty$ in the above theorem and the theorem below.

Theorem 6.14. *Let $\mu=\frac{m-1}{2}$, $m>1$, and let $f\in L^p(\mathbb{B}^d, W_\mu)$ if $1\leqslant p<\infty$. Then for $r\in\mathbb{N}$ and $0<t<t_r$,*

$$c^{-1}\hat{\omega}^r(f,t)_{p,\mu}\leqslant \widehat{K}_r(f,t)_{p,\mu}\leqslant c\hat{\omega}^r(f,t)_p+ct^r\|f\|_{p,\mu}. \tag{6.40}$$

Furthermore, the term $t^r\|f\|_p$ on the right side can be dropped when $r=1$.

Proof. The proof of this theorem follows along the same line as that of the proof of Theorem 6.10 and we only need to point out the difference. For the left hand inequality $\hat{\omega}^r(f,t)_{p,\mu}\leqslant c\widehat{K}_r(f,t)_{p,\mu}$ of (6.40), the counterpart on $\triangle_{i,j,\theta}^r f$ follows as in the unweighted case without further complication, so that the essential part is to show that

$$\left\|\widehat{\triangle}_{\theta\varphi e_i}^r g\right\|_{L^p(\mathbb{I}_{i,r\theta},W_\mu)}\leqslant c\|f\|_{p,\mu},\qquad \left\|\widehat{\triangle}_{\theta\varphi e_i}^r g\right\|_{L^p(\mathbb{I}_{i,r\theta},W_\mu)}\leqslant c\theta^r\left\|\varphi^r\partial_i^r g\right\|_{p,\mu},$$

$$\sup_{|\theta|\leqslant 12r^2t^2}\left\|\overleftarrow{\triangle}_{\theta e_i}^r g\right\|_{L^p(\mathbb{J}_{1,i,rt},W_\mu)}\leqslant c\|f\|_{p,\mu},$$

$$\sup_{|\theta|\leqslant 12r^2t^2}\left\|\overleftarrow{\triangle}_{\theta e_i}^r g\right\|_{L^p(\mathbb{J}_{1,i,rt},W_\mu)}\leqslant ct^r\left\|\varphi^r\partial_i^r g\right\|_{p,\mu}$$

and a similar inequality for $\overrightarrow{\triangle}_{\theta e_i}^r g$ with $g\in C^r(\mathbb{B}^d)$. As in the proof of Theorem 6.10, the proof of these inequalities reduces to the corresponding inequalities in one variable, and the weighted version of (6.34), which however follow from the results given in [16, p. 58].

For the right hand inequality of (6.40), we can follow the proof of Theorem 6.10 verbatim once we establish the relation, for $f\in\Pi_n^d$,

$$n^{-rp}\|\varphi^r\partial_i^r f\|_{p,\mu}^p \sim n\int_0^{n^{-1}} \|\widehat{\triangle}_{h\varphi e_i}^r f\|_{L^p(\mathbb{I}_{i,rh},W_\mu)}^p\,dh$$
$$+n^2\int_0^{n^{-2}} \|\overleftarrow{\triangle}_h^r f\|_{L^p(J_{1,i,rn^{-1}},W_\mu)}^p\,dh + n^2\int_0^{n^{-2}} \|\overrightarrow{\triangle}_h^r f\|_{L^p(J_{-1,i,rn^{-1}},W_\mu)}^p\,dh,$$

which is the analogue of (6.30). The proof of this relation follows as that of (6.30) from the corresponding result in one variable, and the equivalence in one variable follows from [16, p. 57] and [18, p. 193]. □

It should be mentioned that [16] considers far more general weight functions than w_μ in the case of $d = 1$, but we can only deal with W_μ as our results depend on Section 6.2, in which weight is W_μ with $\mu = \frac{m-1}{2}$. On the other hand, it is possible to consider doubling weights and establish the results as in Section 5.

We note that it is more involved to derive properties for the weighted modulus of smoothness, which requires us to verify that the corresponding results hold for the weighted L^p space on $[-1, 1]$. Such results are stated mostly for weighted main-part modulus of smoothness in [16] and a close look at the proof in [16] indicates that the weighted case requires caution and perhaps further work. Since the result is not needed in this paper, we shall not pursue it here.

7. The weighted $L^p(\mathbb{B}^d, W_\mu)$ space with $\mu \neq (m-1)/2$

The results that we obtained in the previous sections are established for the space $L^p(\mathbb{B}^d, W_\mu)$ with $\mu = \frac{m-1}{2}$. The definitions of the moduli of smoothness and the K-functionals, however, make sense for all $\mu \geqslant 0$. A natural question is if our results can be extended to the case of $L^p(\mathbb{B}^d, W_\mu)$ with $\mu \neq \frac{m-1}{2}$. This, however, appears to be a difficult problem. Below we give a positive result for the case of $r = 2$.

7.1. Decomposition of $\mathcal{D}_\mu$

Recall the second differential operator D_μ given in (5.5) and the operators $D_{i,j}^2$, $1 \leqslant i < j \leqslant d$ defined in (2.3). We further define

$$D_{i,i}^2 := [W_\mu(x)]^{-1}\partial_i[(1-\|x\|^2)W_\mu(x)]\partial_i, \quad 1 \leqslant i \leqslant d.$$

It turns out that D_μ can be decomposed as a sum of second order differential operators.

Proposition 7.1. *The differential operator $\mathcal{D}_\mu$ can be decomposed as*

$$\mathcal{D}_\mu = \sum_{i=1}^d D_{i,i}^2 + \sum_{1\leqslant i<j\leqslant d} D_{i,j}^2 = \sum_{1\leqslant i\leqslant j\leqslant d} D_{i,j}^2. \tag{7.1}$$

The proof is a straightforward computation. In the case of $d = 2$, $D_{1,2}^2$ is simply the second partial derivative with respect to θ in the polar coordinates. In this case, it is tempting to write the decomposition entirely in terms of polar coordinate (r,θ) but it does not seem to offer further structure.

The decomposition (7.1) implies immediately that $\|\mathcal{D}_\mu g\|_{p,\mu}$ is bounded by the sum of $\|D_{i,j}^2 g\|_{p,\mu}$ for all g for which the norms involved are finite. More importantly, however, the reversed inequality holds. For this, we relate $\mathcal{D}_\mu$ with a differential operator, $\mathcal{D}_{\mu,T}$ on the simplex $T^d := \{x \in \mathbb{R}^d\colon\ 1 - x_1 - \cdots - x_d \geqslant 0,\ x_i \geqslant 0,\ 1 \leqslant i \leqslant d\}$ and use a result for $\mathcal{D}_{\mu,T}$. Let $\mathbb{B}_+^d := \{x \in \mathbb{B}^d\colon\ x_i \geqslant 0,\ 1 \leqslant i \leqslant d\}$ and let

$$\psi : (u_1, \ldots, u_d) \in T^d \mapsto (\sqrt{u_1}, \ldots, \sqrt{u_d}) \in \mathbb{B}_+^d. \tag{7.2}$$

This change of variables leads immediately to the relation

$$\int_{\mathbb{B}_+^d} f(x_1, \ldots, x_d)\, dx = \frac{1}{2^d} \int_{T^d} f(\sqrt{u_1}, \ldots, \sqrt{u_d}) \frac{du}{\sqrt{u_1 \cdots u_d}}. \tag{7.3}$$

In particular, it maps the weight function W_μ to the weight function

$$W_\mu^T(x) = x_1^{-1/2} \ldots x_d^{-1/2} \bigl(1 - |x|_1\bigr)^{\mu - 1/2}, \quad x \in T^d, \tag{7.4}$$

where $|x|_1 = x_1 + \cdots + x_d$. Furthermore, the mapping (7.2) sends the differential operator $\mathcal{D}_\mu$ to

$$\mathcal{D}_{\mu,T} := \sum_{i=1}^d x_i(1 - x_i)\partial_i^2 - 2 \sum_{1 \leqslant i < j \leqslant d} x_i x_j \partial_i \partial_j + \sum_{i=1}^d \left(\frac{1}{2} - \left(\mu + \frac{d+1}{2}\right) x_i\right) \partial_i, \tag{7.5}$$

and $\mathcal{D}_{\mu,T}$ has orthogonal polynomials with respect to W_μ^T on T^d as eigenfunctions. Much of the analysis on $\mathbb{B}^d$ or T^d can be carried over to the other domain through this connection (see, for example, [37]). It is known that $\mathcal{D}_{\mu,T}$ satisfies a decomposition [2,12],

$$\mathcal{D}_{\mu,T} = \sum_{i=1}^d U_{i,i}^T + \sum_{1 \leqslant i < j \leqslant d} U_{i,j}^T = \sum_{1 \leqslant i \leqslant j \leqslant d} U_{i,j}^T,$$

where, with $\partial_{i,j} := \partial_i - \partial_j$,

$$U_{i,i}^T = \bigl[W_\mu^T(x)\bigr]^{-1} \partial_i \bigl[x_i\bigl(1 - |x|\bigr) W_\mu^T(x)\bigr] \partial_i, \quad 1 \leqslant i \leqslant d,$$
$$U_{i,j}^T = \bigl[W_\mu^T(x)\bigr]^{-1} \partial_{i,j} \bigl[x_i x_j W_\mu^T(x)\bigr] \partial_{i,j}, \quad 1 \leqslant i < j \leqslant d.$$

Let $\|\cdot\|_{p,\mu}^T$ denote the norm of $L^p(W_\mu^T) = L^p(T^d, W_\mu^T)$. In fact, the decomposition of $\mathcal{D}_\mu$ can also be derived from the mapping (7.2).

Lemma 7.2. *For $g \in C^2(\mathbb{B}^d)$, $1 \leqslant i \leqslant j \leqslant d$ and $u \in T^d$,*

$$\mathcal{D}_\mu g\bigl(\psi(u)\bigr) = 4\mathcal{D}_{\mu,T}(g \circ \psi)(u) \quad \text{and} \quad D_{i,j}^2 g\bigl(\psi(u)\bigr) = 4U_{i,j}^T(g \circ \psi)(u). \tag{7.6}$$

Proof. Under the change of variables $x_i = \sqrt{u_i}$, $1 \leqslant i \leqslant d$ so that $(g \circ \psi)(u) = g(\sqrt{u_1}, \dots, \sqrt{u_d}) = g(x)$, the relation for $\mathcal{D}_\mu$ follows from a straightforward computation and so is the case $D_{i,i}^2$, since $x_i = \sqrt{u_i}$ implies that $\partial_{x_i} = 2\sqrt{u_i}\partial_{u_i}$. We now consider the case of $D_{i,j}^2$ with $i < j$. First we note that

$$D_{i,j}^2 = \big[W_\mu(x)\big]^{-1}(x_i\partial_{x_j} - x_j\partial_{x_i})W_\mu(x)(x_i\partial_{x_j} - x_j\partial_{x_i}).$$

In fact, the above identity holds if any differentiable radial function is in place of W_μ. Setting $x_i = \sqrt{u_i}$, we see easily that $x_i\partial_{x_j} - x_j\partial_{x_i} = 2\sqrt{u_iu_j}(\partial_{u_j} - \partial_{u_i})$. Consequently, it follows that

$$\begin{aligned} D_{i,j}^2 g(x) &= 4\big(1-|u|_1\big)^{-\mu+\frac{1}{2}}\sqrt{u_iu_j}(\partial_{u_j} - \partial_{u_i})\sqrt{u_iu_j}\big(1-|u|_1\big)^{\mu-\frac{1}{2}}(\partial_{u_j} - \partial_{u_i})(g\circ\psi)(u) \\ &= 4\big[W_\mu^T(u)\big]^{-1}(\partial_{u_j} - \partial_{u_i})\big[u_iu_jW_\mu^T(u)\big](\partial_{u_j} - \partial_{u_i}) = 4U_{i,j}^T(g\circ\psi)(u), \end{aligned}$$

which verifies (7.6). □

7.2. Differential operators and K-functional

The following result was established in [10] recently: for $f \in C^2(T^d)$,

$$\|\mathcal{D}_{\mu,T} f\|_{L^p(T^d, W_\mu^T)} \sim \sum_{1\leqslant i\leqslant j\leqslant d} \big\|U_{i,j}^T f\big\|_{L^p(T^d,W_\mu^T)}, \quad 1 < p < \infty. \tag{7.7}$$

With the connection in the previous subsection, it immediately leads us to an analogous result for $\mathcal{D}_\mu$ on $\mathbb{B}^d$.

Theorem 7.3. *For $g \in C^2(\mathbb{B}^d)$,*

$$\|\mathcal{D}_\mu g\|_{p,\mu} \sim \sum_{1\leqslant i\leqslant j\leqslant d} \big\|D_{i,j}^2 g\big\|_{p,\mu}, \quad 1 < p < \infty.$$

Proof. By (7.1), it suffices to prove that

$$\big\|D_{i,j}^2 g\big\|_{p,\mu} \leqslant c\|\mathcal{D}_\mu g\|_{p,\mu} \tag{7.8}$$

for $1 \leqslant i, j \leqslant d$. This is the same as in the proof in [10], which amounts to showing that for $f \in C^2(T^d)$ and $1 \leqslant i < j \leqslant d$,

$$\big\|U_{i,j}^T f\big\|_{L^p(T^d,W_\mu^T)} \leqslant c\|\mathcal{D}_{\mu,T} f\|_{L^p(T^d,W_\mu^T)}, \quad 1 < p < \infty. \tag{7.9}$$

We shall deduce (7.8) from (7.9) and the change of variables (7.2). Now, for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{-1, 1\}^d$, we define g_ε by $g_\varepsilon(x) = g(x_1\varepsilon_1, \dots, x_d\varepsilon_d)$, $x \in \mathbb{B}^d$. We claim that

$$\big(D_{i,j}^2 g\big)_\varepsilon(x) = \big(D_{i,j}^2 g_\varepsilon\big)(x), \quad 1 \leqslant i \leqslant j \leqslant d,\ x \in \mathbb{B}^d. \tag{7.10}$$

Indeed, since $\partial_i g_\varepsilon(x) = \varepsilon_i(\partial_i g)_\varepsilon(x)$, this can be verified via a straightforward computation. Using (7.3), (7.6) and (7.10), we obtain

$$\int_{\mathbb{B}^d} \left|D_{i,j}^2 g(x)\right|^p W_\mu(x)\,dx = \sum_{\varepsilon\in\{-1,1\}^d} \int_{\mathbb{B}_+^d} \left|\left(D_{i,j}^2 g\right)_\varepsilon(x)\right|^p W_\mu(x)\,dx$$
$$= \sum_{\varepsilon\in\{-1,1\}^d} \int_{\mathbb{B}_+^d} \left|D_{i,j}^2 g_\varepsilon(x)\right|^p W_\mu(x)\,dx$$
$$= \frac{4^p}{2^d} \sum_{\varepsilon\in\{-1,1\}^d} \int_{T^d} \left|U_{i,j}^T(g_\varepsilon\circ\psi)(u)\right|^p W_\mu^T(u)\,du.$$

Consequently, using (7.9), followed by using (7.6) and (7.10) again, we conclude that

$$\int_{\mathbb{B}^d} \left|D_{i,j}^2 g(x)\right|^p W_\mu(x)\,dx \leqslant c\frac{4^p}{2^d} \sum_{\varepsilon\in\{-1,1\}^d} \int_{T^d} \left|\mathcal{D}_{\mu,T}(g_\varepsilon\circ\psi)(u)\right|^p W_\mu^T(u)\,du$$
$$\leqslant c\frac{1}{2^d} \sum_{\varepsilon\in\{-1,1\}^d} \int_{\mathbb{B}_+^d} \left|(\mathcal{D}_\mu g)_\varepsilon(x)\right|^p W_\mu(x)\,dx$$
$$= c\int_{\mathbb{B}^d} \left|\mathcal{D}_\mu g(x)\right|^p W_\mu(x)\,dx$$

for $1 < p < \infty$. This completes the proof. □

The differential operators $D_{i,j}^2$ for $i < j$ are second order derivatives with respect to the Euler angles, whereas $D_{i,i}^2$ does not have such simple interpretation. Our next result shows that $\|D_{i,i}^2 g\|_{p,\mu}$ can be further reduced. Recall $\varphi(x) = \sqrt{1-\|x\|^2}$.

Theorem 7.4. *For $1 < p < \infty$, $1 \leqslant i \leqslant d$ and $g \in C^2(\mathbb{B}^d)$,*

$$c_1\left\|\varphi^2\partial_i^2 g\right\|_{p,\mu} \leqslant \left\|D_{i,i}^2 g\right\|_{p,\mu} \leqslant c_2\left\|\varphi^2\partial_i^2 g\right\|_{p,\mu} + c_2\|g\|_{p,\mu}.$$

Proof. It is enough to consider $D_{1,1}^2$. We make a change of variables, $x \mapsto (s, y)$, where $y = (y_2, \dots, y_d)$, by setting $x_1 = \sqrt{1-\|y\|^2}s$ and $x_i = y_i$ for $i = 2, \dots, d$. It follows immediately that $\varphi(x) = \varphi(y)\varphi(s)$. Furthermore, a quick computation shows that

$$D_{1,1}^2 g(x) = \Lambda_s g_y(s), \quad \Lambda_s := \left(1-s^2\right)\frac{d^2}{ds^2} - (2\mu+1)s\frac{d}{ds},$$

where $g_y(s) = g(s\varphi(y), y)$. Let $w_\mu(s) = (1-s^2)^{\mu-1/2}$. It is easy to see then that Λ_s can be written as

$$\Lambda_s = w_\mu(s)^{-1}\frac{d}{ds}\left[\left(1-s^2\right)w_\mu(s)\right]\frac{d}{ds}.$$

It is known that the differential operator Λ_s satisfies [10]

$$c_1 \|\varphi^2 g''\|_{L^p(w_\mu)} \leqslant \|\Lambda_s g\|_{L^p(w_\mu)} \leqslant c_2 \|\varphi^2 g''\|_{L^p(w_\mu)} + c_2 \|g\|_{L^p(w_\mu)} \tag{7.11}$$

for $1 < p < \infty$, where the norm is taken over $[-1, 1]$. By (6.14), applying (7.11) we obtain

$$\begin{aligned}
\|D_{1,1}^2 g\|_{p,\mu}^p &= \int_{\mathbb{B}^{d-1}} \int_{-1}^{1} |\Lambda_s g_y(s)|^p w_\mu(s)\, ds\, W_{\mu+\frac{1}{2}}(y)\, dy \\
&\geqslant c_1^p \int_{\mathbb{B}^{d-1}} \int_{-1}^{1} \left|\varphi^2(s) \frac{d^2}{ds^2}\big[g\big(s\varphi(y), y\big)\big]\right|^p w_\mu(s)\, ds\, W_{\mu+\frac{1}{2}}(y)\, dy \\
&= c_1^p \int_{\mathbb{B}^{d-1}} \int_{-1}^{1} \big|\varphi^2(s)^2 \varphi^2(y) \big(\partial_1^2 g\big)\big(s\varphi(y), y\big)\big|^p w_\mu(s)\, ds W_{\mu+\frac{1}{2}}(y)\, dy \\
&= c_1^p \int_{\mathbb{B}^d} \big|\varphi^2(x) \partial_1^2 g(x)\big|^p W_\mu(x)\, dx = c_1^p \|\varphi^2 \partial_1^2 g\|_{p,\mu},
\end{aligned}$$

where in the last step we used (6.14) again. This proves the lower bound. The upper bound is proved likewise. □

As a consequence of this theorem, we can replace $\|\mathcal{D}_\mu g\|_{p,\mu}$ in K-functional $K_2^*(f;t)_{p,\mu}$, defined in (5.4) by the sum of $\|\varphi^2 \partial_i^2 g\|_{p,\mu}$ and $\|D_{i,j}^2 g\|_{p,\mu}$, at least for $1 < p < \infty$, which leads to a comparison between $K_r^*(f,t)_{p,\mu}$ and $\widehat{K}_r(f,t)_{p,\mu}$ defined in Definition 6.1. Indeed, from Theorems 7.3 and 7.4, and the triangle inequality, we obtain the following:

Theorem 7.5. *For* $f \in L^p(\mathbb{B}^d, W_\mu)$, $1 < p < \infty$, *and* $0 < t \leqslant 1$,

$$c_1 \widehat{K}_2(f;t)_{p,\mu} \leqslant K_2^*(f;t)_{p,\mu} \leqslant c_2 \widehat{K}_2(f;t)_{p,\mu} + c_2 t^2 \|f\|_{p,\mu}. \tag{7.12}$$

Recall that both direct and inverse theorems for best polynomial approximation are established for $K_r^*(f,t)_{p,\mu}$, the above shows that the same can be stated for $\widehat{K}_2(f,t)_{p,\mu}$.

In the case of $\mu = \frac{m-1}{2}$, $K_2^*(f,t)_{p,\mu}$ is equivalent to $K_2(f,t)_{p,\mu}$ for $1 < p < \infty$ by the K-functional counterpart of Theorem 5.10, which gives the inequality (6.18).

Part 3. Computational examples

In this part we give examples of functions for which the asymptotic orders of our new moduli of smoothness and best approximation by polynomials are explicitly determined. The first section contains a lemma, upon which most of the computations of our examples are based, and one of its applications. The examples for moduli of smoothness on the sphere are given in Section 9 and examples on the ball are given in Section 10.

8. Main lemma for computing moduli of smoothness

One of the advantages of our new moduli of smoothness lies in the fact that the divided difference in Euler angles can be reduced to the forwarded difference for trigonometric functions (cf. (2.8)), which are classical and well studied. Our claim that the new moduli of smoothness are computable is based on this fact. Below we present a lemma that gives the asymptotic order of the modulus of smoothness for a simple trigonometric function, upon which most of our examples in the following two sections are based.

Lemma 8.1. *Assume that $|a| < 1$, $1 \leqslant p \leqslant \infty$ and $\alpha \neq 0$. If $\alpha \neq \frac{1}{2}$, then there exists $\delta_\alpha \in (0,1)$ depending only on α such that for $|\theta| \leqslant \delta_\alpha$,*

$$\left(\int_0^{2\pi} \left|\overrightarrow{\Delta}_\theta^2 \big(1 - a\cos\big(\phi + \{\cdot\}\big)\big)^\alpha\right|^p d\phi\right)^{1/p} \sim |a|\theta^2 \begin{cases} (1-|a|+|a|\theta^2)^{(\alpha-1)+\frac{1}{2p}}, & \alpha < 1 - \frac{1}{2p}, \\ |\log(1-|a|+|a|\theta^2)|, & \alpha = 1-\frac{1}{2p}, \\ 1, & \alpha > 1 - \frac{1}{2p}, \end{cases} \tag{8.1}$$

with the usual change of the L^p-norm when $p = \infty$, where the constants of equivalence are independent of a. If $\alpha = \frac{1}{2}$, then the upper estimates in (8.1) *remain true, whereas the lower estimate holds under the additional condition $36\theta^2 \leqslant 1 - |a|$ when $p > 1$, and the lower bound becomes $c\theta^2$ when $p = 1$.*

Proof. Without loss of generality, we may assume $a > 0$, since $\cos(\pi + \phi) = -\cos\phi$. We shall prove the lemma for $p < \infty$ only. The case $p = \infty$ can be treated similarly, and in fact, is simpler. Let us set $h_\alpha(\phi) = (1 - a\cos\phi)^\alpha$. We will use the fact that $\overrightarrow{\Delta}_\theta^2 h_\alpha(\phi) = (1/2)\theta^2 h''_\alpha(\phi + \xi)$ for some ξ between 0 and 2θ, and

$$h''_\alpha(\phi+\xi) = \alpha(\alpha-1)\big(1 - a\cos(\phi+\xi)\big)^{\alpha-2} a^2 \sin^2(\phi+\xi) + \alpha\big(1 - a\cos(\phi+\xi)\big)^{\alpha-1} a\cos(\phi+\xi). \tag{8.2}$$

To show the upper estimates, we can restrict the integral in (8.1) over $[0,\pi]$ instead of $[0,2\pi]$, since we allow θ to take negative values, and $\overrightarrow{\Delta}_\theta^2 h_\alpha(-\phi + \{\cdot\}) = \overrightarrow{\Delta}_{-\theta}^2 h_\alpha(\phi + \{\cdot\})$. Using (8.2) and the identity $1 - a\cos\psi = 1 - a + 2a\sin^2\frac{\psi}{2}$, we have

$$\left|\overrightarrow{\Delta}_\theta^2 h_\alpha(\phi)\right| = \frac{1}{2}\theta^2\left|h''_\alpha(\phi+\xi)\right| \leqslant ca\theta^2\left(1 - a + 2a\sin^2\frac{\phi+\xi}{2}\right)^{\alpha-1}. \tag{8.3}$$

We break the integral of $|\overrightarrow{\Delta}_\theta^2 h_\alpha(\phi + \{\cdot\})|^p$ into two parts:

$$\int_0^\pi \left|\overrightarrow{\Delta}_\theta^2 h_\alpha\big(\phi + \{\cdot\}\big)\right|^p d\phi = \int_0^{3|\theta|} \cdots + \int_{3|\theta|}^\pi \cdots =: I_1 + I_2.$$

If $a\theta^2 \geqslant 1-a$, then $a \geqslant 1-\theta^2 \geqslant a_0 > 0$, and using the definition of $\overrightarrow{\triangle}_\theta^2$, we obtain $|\overrightarrow{\triangle}_\theta^2 h_\alpha(\phi+\{\cdot\})| \leqslant c(1-a+a\theta^2)^\alpha$ for $|\phi| \leqslant 3|\theta|$, which in turn implies

$$I_1 \leqslant C \int_0^{3|\theta|} \left(1-a+a\theta^2\right)^{\alpha p}\, d\phi \sim |\theta|^{2\alpha p+1} a^{\alpha p} \sim a^p |\theta|^{2p}\left(1-a+a\theta^2\right)^{(\alpha-1)p+\frac{1}{2}}.$$

On the other hand, if $a\theta^2 \leqslant 1-a$ then using (8.3), we have $|\overrightarrow{\triangle}_\theta^2 h_\alpha(\phi+\{\cdot\})| \leqslant ca\theta^2(1-a)^{\alpha-1}$ for $|\phi| \leqslant 3|\theta|$, and hence

$$I_1 \leqslant ca^p|\theta|^{2p+1}(1-a)^{(\alpha-1)p} \leqslant ca^p|\theta|^{2p}\left(1-a+a\theta^2\right)^{(\alpha-1)p+\frac{1}{2}},$$

where the last step uses the fact that $1-a \sim 1$ when $0 < a \leqslant \frac{1}{2}$. Finally, using (8.3), we deduce

$$I_2 \leqslant ca^p\theta^{2p} \int_{3\theta}^{\pi} \left(1-a+a\phi^2\right)^{(\alpha-1)p}\, d\phi \sim a^{p-\frac{1}{2}}\theta^{2p} \int_{3\sqrt{a}\theta}^{\sqrt{a}\pi} (\sqrt{1-a}+\phi)^{2(\alpha-1)p}\, d\phi,$$

which is estimated by the desired upper bounds. This completes the proof of the upper estimates.

For the proof of the lower estimates, we shall use δ_α' or δ_α'' to denote a sufficiently small positive constant which depends only on α, and may vary at each occurrence. Note that if $1-a \geqslant \delta_\alpha' > 0$ then the desired lower estimates follow immediately since, by (8.2),

$$\left|\overrightarrow{\triangle}_\theta^2 h_\alpha(\phi)\right| = \frac{1}{2}\theta^2\left|h_\alpha''(\phi+\xi)\right| \geqslant c_\alpha a\theta^2$$

whenever $3|\theta| \leqslant \phi \leqslant \delta_\alpha'$. Thus, for the rest of the proof, we may assume that $|\theta| + \sqrt{1-a} \leqslant \delta_\alpha''$. We claim that for $\alpha \neq \frac{1}{2}$, there exists a constant $c_\alpha > 2$ such that

$$\left|\overrightarrow{\triangle}_\theta^2 h_\alpha(\phi)\right| = \frac{1}{2}\theta^2\left|h_\alpha''(\phi+\xi)\right| \geqslant c\theta^2\phi^{2\alpha-2} \tag{8.4}$$

whenever $c_\alpha(|\theta|+\sqrt{1-a}\,) \leqslant \phi \leqslant \delta_\alpha'$. Indeed, setting $\psi = \phi+\xi$, and using (8.2), we obtain, for $c_\alpha(|\theta|+\sqrt{1-a}\,) \leqslant \phi \leqslant \delta_\alpha'$,

$$\begin{aligned}\left|h_\alpha''(\psi)\right| &= a|\alpha|\left(1-a+a\sin^2\frac{\psi}{2}\right)^{\alpha-2}\left|\alpha a\sin^2\psi - a + \cos\psi\right| \\ &= a|\alpha|\left(1-a+a\sin^2\frac{\psi}{2}\right)^{\alpha-2}\left(\alpha-\frac{1}{2}\right)\psi^2 + O(1-a) + O\left(\psi^3\right) \\ &\geqslant c|\alpha|\left|\alpha-\frac{1}{2}\right|\phi^{2\alpha-2}\end{aligned}$$

provided that $(1-\frac{2}{c_\alpha}) \geqslant c\delta_\alpha'$. The assertion (8.4) then follows. Now raising (8.4) to the power p, and integrating it with respect to ϕ over $c_\alpha(|\theta|+\sqrt{1-a}\,) \leqslant \phi \leqslant \delta_\alpha'$ gives the desired lower estimates in (8.1) for $\alpha \neq \frac{1}{2}$. The lower estimate for $\alpha = \frac{1}{2}$ can be proved similarly. Indeed, setting $\psi = \phi + \xi$, and using (8.2), we obtain

$$h_{1/2}''(\psi) = \frac{1}{4}(1 - a\cos\psi)^{-\frac{3}{2}} a\bigl(2\cos\psi - a\cos^2\psi - a\bigr) \geqslant \frac{1}{4}(1 - a\cos\psi)^{-\frac{1}{2}} a\cos\psi,$$

provided that $1 - a \geqslant 2\sin^2\frac{\psi}{2}$. Thus, if $3|\theta| \leqslant \phi \leqslant \sqrt{(1-a)/2}$ then $1 - a \geqslant 2\phi^2 \geqslant \frac{1}{2}\psi^2 \geqslant 2\sin^2\frac{\psi}{2}$, and hence

$$\bigl|\overrightarrow{\triangle}_\theta^2 h_{1/2}(\phi)\bigr| = \frac{1}{2}\theta^2\bigl|h_{1/2}''(\phi + \xi)\bigr| \geqslant c|a|\theta^2(1-a)^{-\frac{1}{2}}.$$

Integrating the p-th power of this inequality with respect to ϕ over $3|\theta| \leqslant \phi \leqslant \sqrt{(1-a)/2}$ and using $1 - a + a|\theta| \sim 1 - a$, which holds for $c\theta^2 \leqslant 1 - a$, give the desired lower estimate for $\alpha = \frac{1}{2}$. □

As an application of Lemma 8.1, we prove the asymptotic of $\omega_2(g_\alpha, t)_{p,\mu}$ in Example 5.11. The proof also suggests what to come in the following two sections.

Lemma 8.2. *Let $h_\alpha(s,\phi) := (1 - s\cos\phi)^\alpha$, $\alpha \neq 0$, $\mu > 0$ and let*

$$\Omega_2(h_\alpha, \theta)_{p,\mu} := \left(\int_0^1 s\int_0^{2\pi} \bigl|\overrightarrow{\triangle}_\theta^2 h_\alpha(s,\phi)\bigr|^p \, d\phi\bigl(1 - s^2\bigr)^{\mu-1}\, ds\right)^{1/p}, \tag{8.5}$$

where for $p = \infty$ it is defined as maximum of $|\overrightarrow{\triangle}_\theta^2 h_\alpha(s,\phi)|$ over $0 \leqslant s \leqslant 1, 0 \leqslant \phi \leqslant 2\pi$. Then there is a $t_0 > 0$ such that for $0 < \theta < t_0$, $\mu \geqslant 0$ and $1 \leqslant p \leqslant \infty$,

$$\Omega_2(h_\alpha, \theta)_{p,\mu} \sim \begin{cases} |\theta|^{2\alpha + \frac{2\mu+1}{p}}, & -\frac{2\mu+1}{2p} < \alpha < 1 - \frac{2\mu+1}{2p}, \\ \theta^2|\log|\theta||^{1/p}, & \alpha = 1 - \frac{2\mu+1}{2p}, \quad p \neq \infty, \\ \theta^2, & \alpha > 1 - \frac{2\mu+1}{2p}. \end{cases} \tag{8.6}$$

Proof. Again we only consider $1 \leqslant p < \infty$. If $\alpha < 1 - \frac{1}{2p}$ and $\alpha \neq \frac{1}{2}$, then we apply (8.1) with $a = s$ to obtain that

$$\Omega_2(h_\alpha, \theta)_p^p \sim |\theta|^{2p}\int_0^1 s^{p+1}\bigl(1 - s + s\theta^2\bigr)^{(\alpha-1)p+\frac{1}{2}}\bigl(1 - s^2\bigr)^{\mu-1}\, ds$$

$$\sim |\theta|^{2p}\left[\int_0^{1-\theta^2} s^{p+1}(1-s)^{(\alpha-1)p+\mu-\frac{1}{2}}\, ds + \int_{1-\theta^2}^1 |\theta|^{(2\alpha-2)p+1}(1-s)^{\mu-1}\, ds\right],$$

which, when integrated out according to $(\alpha - 1)p + \mu - 1/2 < -1, = -1$, and > -1, is easily seen to be equivalent to the p-th power of the right-hand side of (8.6). Similarly, by (8.1) applied to $a = s$, we have, for $\alpha = 1 - \frac{1}{2p} \neq \frac{1}{2}$,

$$\begin{aligned}\Omega_2(h_\alpha,\theta)_p^p &\sim |\theta|^{2p}\int_0^1 s^{p+1}\bigl|\log\bigl(1-s+s\theta^2\bigr)\bigr|\bigl(1-s^2\bigr)^{\mu-1}\,ds\\ &\sim |\theta|^{2p}\left[\int_0^{1-\theta^2} s^{p+1}(1-s)^{\mu-1}\log\frac{1}{1-s}\,ds+\int_{1-\theta^2}^1 (1-s)^{\mu-1}\bigl|\log|\theta|\bigr|\,ds\right]\\ &\sim |\theta|^{2p},\end{aligned}$$

whereas for $\alpha > 1-\frac{1}{2p}$,

$$\Omega_2(h_\alpha,\theta)_p^p \sim |\theta|^{2p}\int_0^1 s^{p+1}\bigl(1-s^2\bigr)^{\mu-1}\,ds \sim |\theta|^{2p}.$$

Finally, in the case when $\alpha=\frac{1}{2}$, the desired upper estimate for $\Omega_2(h_{1/2},\theta)_p^p$ can be obtained exactly as above, while the lower estimate for $\Omega_2(h_{1/2},\theta)_p^p$ can be obtained using the second statement of Lemma 8.1:

$$\begin{aligned}\Omega_2(h_{1/2},\theta)_p^p &\geqslant \int_0^{1-36\theta^2} s\int_0^{2\pi}\bigl|\overrightarrow{\triangle}_\theta^2 h_{1/2}(s,\phi)\bigr|^p\,d\phi\bigl(1-s^2\bigr)^{\mu-1}\,ds\\ &\geqslant c|\theta|^{2p}\int_0^{1-36\theta^2} s^{p+1}(1-s)^{-\frac{p}{2}-\frac{1}{2}+\mu}\,ds,\end{aligned}$$

which, by an easy calculation, gives the desired lower estimate for the case of $\alpha=\frac{1}{2}$. $\square$

Remark 8.1. In the previous two lemmas we considered only the second order difference. Our proof can be adopted to give the upper estimates for $r>2$. For examples, let $r\geqslant 2$ and define

$$\Omega_r(h_\alpha,\theta)_{p,\mu} := \left(\int_0^1 s\int_0^{2\pi}\bigl|\overrightarrow{\triangle}_\theta^r h_\alpha(s,\phi)\bigr|^p\,d\phi\bigl(1-s^2\bigr)^{\mu-1}\,ds\right)^{1/p}$$

for $1\leqslant p<\infty$ and the usual convention for $p=\infty$. Then we can show that

$$\Omega_r(h_\alpha,\theta)_{p,\mu} \leqslant c\begin{cases}|\theta|^{2\alpha+\frac{2\mu+1}{p}}, & -\frac{2\mu+1}{2p}<\alpha<\frac{r}{2}-\frac{2\mu+1}{2p},\\ \theta^2|\log|\theta||^{1/p}, & \alpha=\frac{r}{2}-\frac{2\mu+1}{2p},\ p\neq\infty,\\ \theta^2, & \alpha>\frac{r}{2}-\frac{2\mu+1}{2p}.\end{cases}$$

Although we believe that the lower estimate should also hold, it is much more difficult to establish. For this reason, we only considered $r=2$.

For the same reason and because the computation for $r = 2$ is already rather involved, we shall consider only $r = 2$ in most of our examples in the next two sections. In all cases, our method can be adopted to establish the upper estimates for all $r \geqslant 1$.

9. Computational examples on the unit sphere

In this section we compute the modulus of smoothness $\omega_r(f,t)_p = \omega_r(f,t)_{L^p(\mathbb{S}^{d-1})}$ defined in (2.9) and the best approximation $E_n(f)_p := E_n(f)_{L^p(\mathbb{S}^{d-1})}$ of (3.1).

9.1. Computation of moduli of smoothness

We start with a simple example that follows directly from the modulus of smoothness for trigonometric functions.

Example 9.1. For $x \in \mathbb{S}^{d-1}$ and $d \geqslant 3$, let $f_\alpha(x) = x^\alpha$ with $\alpha = (\alpha_1, \dots, \alpha_d) \neq 0$. If $0 \leqslant \alpha_i < 1$ for $1 \leqslant i \leqslant d$, then for $r \geqslant 2$ and $1 \leqslant p \leqslant \infty$,

$$\omega_r(f,t)_{L^p(\mathbb{S}^{d-1})} \sim t^{\delta+1/p}, \qquad \delta = \min_{\alpha_i \neq 0}\{\alpha_1, \dots, \alpha_d\}. \tag{9.1}$$

Indeed, we only need to consider $\triangle^2_{1,2,\theta} f$, which, by (2.8), can be expressed as a forward difference

$$\triangle^r_{1,2,\theta} f_\alpha(x) = x_3^{\alpha_3} \cdots x_d^{\alpha_d} s^{\alpha_1+\alpha_2} \overrightarrow{\triangle}^r_\theta \big[(\cos\phi)^{\alpha_1}(\sin\phi)^{\alpha_2}\big]$$

where $(x_1, x_2) = (s\cos\phi, s\sin\phi)$. Hence, by (2.13) we obtain

$$\big\|\triangle^r_{1,2,\theta} f_\alpha\big\|_p = c\left(\int_0^{2\pi} \big|\overrightarrow{\triangle}^r_\theta\big[(\cos\phi)^{\alpha_1}(\sin\phi)^{\alpha_2}\big]\big|^p \, d\phi\right)^{1/p}.$$

Furthermore, using the well-known relation

$$\overrightarrow{\triangle}^r_\theta (fg)(\phi) = \sum_{k=0}^{r} \binom{n}{k} \overrightarrow{\triangle}^k_\theta f(\phi) \overrightarrow{\triangle}^{r-k}_\theta g(\phi + k\theta),$$

we can consider the differences for $\cos(\phi + \cdot)$ and $\sin(\phi + \cdot)$ separately. Since the sine and cosine functions cannot be both large or both small, we can divide the integral domain accordingly and estimate the integral in the L^p norm. Furthermore, in our range of α_i, we only need to consider the second difference ($r = 2$) upon using (1) of Proposition 2.7. Eq. (9.1) also holds for $r = 1$ and $p = \infty$.

Our second example is more interesting and appears to be non-trivial.

Example 9.2. For $d \geqslant 3$ and $\alpha \neq 0$, let $g_\alpha(x) = (1 - x_1)^\alpha$, $x = (x_1, \dots, x_d) \in \mathbb{S}^{d-1}$. Then for $1 \leqslant p \leqslant \infty$,

$$\omega_2(g_\alpha, t)_{L^p(\mathbb{S}^{d-1})} \sim \begin{cases} t^{2\alpha+\frac{d-1}{p}}, & -\frac{d-1}{2p} < \alpha < 1 - \frac{d-1}{2p}, \\ t^2|\log t|^{1/p}, & \alpha = 1 - \frac{d-1}{2p},\ p \neq \infty, \\ t^2, & \alpha > 1 - \frac{d-1}{2p}. \end{cases} \tag{9.2}$$

For $\alpha = 0$, $\omega_2(g_\alpha, t)_p = 0$.

If neither i nor j equals to 1, then $\triangle^2_{i,j,\theta} g_\alpha(x) = 0$. Thus, we only need to consider $\triangle^2_{1,j,\theta} g_\alpha$ and we can assume $j = 2$. Since $x \in \mathbb{S}^{d-1}$ and $d \geqslant 3$ imply that $(x_1, x_2) \in \mathbb{B}^2$, by (2.8),

$$\begin{aligned} \|\triangle^2_{1,2,\theta} g_\alpha\|_p^p &= c \int\limits_{B^2} |\triangle^2_{1,2,\theta} g_\alpha(x_1, x_2)|^p (1 - x_1^2 - x_2^2)^{\mu-1}\, dx \\ &= c \int\limits_0^1 s \int\limits_0^{2\pi} |\overrightarrow{\triangle}^2_\theta (1 - s\cos\phi)^\alpha|^p\, d\phi (1 - s^2)^{\mu-1}\, ds, \end{aligned}$$

where $\mu = \frac{d-2}{2}$ and the forward difference acts on ϕ; for $p = \infty$ the integral is replaced by the maximum taken over $0 \leqslant s \leqslant 1$ and $0 \leqslant \phi \leqslant 2\pi$. Hence, (9.2) follows from Lemma 8.2.

Our next example is more complicated and it should be compared with (9.2). In particular, we note that its asymptotic order is independent of d, in contrast to the order in (9.2).

Example 9.3. Let $d \geqslant 3$ and let $f(x) = (x_1^2 + x_2^2)^\alpha$ for $x \in \mathbb{S}^{d-1}$ and $\alpha \neq 0$. Then for $1 \leqslant p \leqslant \infty$,

$$\omega_2(f, t)_{L^p(\mathbb{S}^{d-1})} \sim \begin{cases} t^{2\alpha+\frac{2}{p}}, & \text{if } -\frac{1}{p} < \alpha < 1 - \frac{1}{p}; \\ t^2|\log t|^{\frac{1}{p}}, & \text{if } \alpha = 1 - \frac{1}{p}; \\ t^2, & \text{if } \alpha > 1 - \frac{1}{p}. \end{cases}$$

Proof. Since $\triangle^2_{i,j,\theta} f(x) = 0$ if $(i, j) = (1, 2)$ or $3 \leqslant i < j \leqslant d$, it suffices to consider $\triangle^2_{1,3,\theta} f(x)$. For a fixed $x \in \mathbb{S}^{d-1}$, let $g_x(t) = f(Q_{1,3,t}x)$. Clearly, $g_x(t) = (v(t)^2 + x_2^2)^\alpha$, where $v(t) \equiv v_x(t) = x_1 \cos t - x_3 \sin t$. A straightforward computation shows that

$$\begin{aligned} g_x''(t) &= 4\alpha(\alpha - 1)\big(v(t)^2 + x_2^2\big)^{\alpha-2}\big(v(t)v'(t)\big)^2 \\ &\quad + 2\alpha\big(v(t)^2 + x_2^2\big)^{\alpha-1}\big[\big(v'(t)\big)^2 + v(t)v''(t)\big]. \end{aligned} \tag{9.3}$$

Let us start with the proof of the lower estimate. Setting $c_\alpha = 8(1 + |\alpha|)$ and

$$E_\theta = \left\{x \in \mathbb{S}^{d-1}\colon \frac{1}{4} \geqslant |x_2| \geqslant 2\sqrt{c_\alpha}\,|x_1| \geqslant 4\sqrt{c_\alpha}\,|\theta|,\ \text{and } |x_3| \geqslant \frac{1}{2}\right\},$$

we assert that for $\alpha \neq 0$,

$$\big|g_x''(t)\big| \geqslant c|x_2|^{2\alpha-2}, \quad \text{whenever } |t| \leqslant 2|\theta| \text{ and } x \in E_\theta, \tag{9.4}$$

where c is a positive constant depending only on α. (9.4) together with the mean value theorem will imply that for $\theta \in (0, \delta_\alpha]$

$$\left|\triangle^2_{1,3,\theta} f(x)\right| \geqslant C|\theta|^2|x_2|^{2\alpha-2}, \qquad \text{whenever } x \in E_\theta.$$

Integrating the p-th power of the last inequality over E_θ will give the desired lower estimates. To show the assertion (9.4), we observe that if $x \in E_\theta$ and $|t| \leqslant 2|\theta|$, then $|v(t)| \leqslant |x_1| + |t| \leqslant 2|x_1| \leqslant |x_2|/\sqrt{c_\alpha}$, which implies

$$\begin{aligned} 4|1-\alpha|\big(v(t)^2+x_2^2\big)^{\alpha-2}\big(v(t)v'(t)\big)^2 &\leqslant 4\big(1+|\alpha|\big)\big(v(t)^2+x_2^2\big)^{\alpha-2}\frac{v(t)^2+x_2^2}{c_\alpha+1}\big(v'(t)\big)^2 \\ &\leqslant \big(v(t)^2+x_2^2\big)^{\alpha-1}\big(v'(t)\big)^2. \end{aligned}$$

Thus, using (9.3), we deduce that if $x \in E_\theta$ and $|t| \leqslant 2|\theta| \leqslant 2\delta_\alpha$, then

$$\begin{aligned} \left|g_x''(t)\right| &\geqslant 2|\alpha|\big(v(t)^2+x_2^2\big)^{\alpha-1}\big[\big(v'(t)\big)^2 - \left|v(t)v''(t)\right|\big] - |\alpha|\big(v(t)^2+x_2^2\big)^{\alpha-1}\big(v'(t)\big)^2 \\ &= |\alpha|\big(v(t)^2+x_2^2\big)^{\alpha-1}\big[v'(t)^2 - 2\left|v(t)v''(t)\right|\big] \\ &\geqslant c|\alpha|\big(v(t)^2+x_2^2\big)^{\alpha-1}\big[\big(|x_3|-|t|\big)^2 - 2\big(|x_1|+|t|\big)^2\big] \\ &\geqslant c|\alpha|\big(v(t)^2+x_2^2\big)^{\alpha-1} \sim \big(x_1^2+x_2^2\big)^{\alpha-1} \sim |x_2|^{2\alpha-2} \end{aligned}$$

proving the desired assertion (9.4).

For the upper estimate, it is easy to see by (9.3) that if $\sqrt{x_1^2+x_2^2} \geqslant 4|t|$ then

$$\left|g_x''(t)\right| \leqslant c\big(v(t)^2+x_2^2\big)^{\alpha-1} \sim \big(x_1^2+x_2^2\big)^{\alpha-1},$$

which, using the mean value theorem, implies that

$$\left|\triangle^2_{1,3,\theta} f(x)\right| \leqslant c|\theta|^2\big(x_1^2+x_2^2\big)^{\alpha-1} \tag{9.5}$$

whenever $\sqrt{x_1^2+x_2^2} \geqslant 8|\theta|$. On the other hand, however, if $\sqrt{x_1^2+x_2^2} \leqslant 8\theta$, then using the definition of $\triangle^2_{i,j,\theta}$, we have

$$\left|\triangle^2_{1,3,\theta} f(x)\right| \leqslant c|\theta|^{2\alpha} + c|x_2|^{2\alpha}. \tag{9.6}$$

Now we break the integral into two parts:

$$\begin{aligned} \int\limits_{\mathbb{S}^{d-1}} \left|\triangle^2_{1,3,\theta} f(x)\right|^p d\sigma(x) &= \int\limits_{\{x\in\mathbb{S}^{d-1}:\ x_1^2+x_2^2\leqslant 64\theta^2\}} \cdots + \int\limits_{\{x\in\mathbb{S}^{d-1}:\ x_1^2+x_2^2> 64\theta^2\}} \cdots \\ &\equiv I_1 + I_2. \end{aligned}$$

Using (9.6) and the condition $\alpha p + 1 > 0$, we have

$$I_1 \leqslant c \int\limits_{x_1^2+x_2^2\leqslant 64\theta^2} \left(\theta^{2\alpha p} + |x_2|^{2\alpha p}\right)\left(1 - x_1^2 - x_2^2\right)^{\frac{d-4}{2}} dx_1\, dx_2 \leqslant c|\theta|^{2\alpha p+2},$$

whereas using (9.5) gives

$$\begin{aligned} I_2 &\leqslant c|\theta|^{2p} \int\limits_{64\theta^2\leqslant x_1^2+x_2^2\leqslant 1} \left|x_1^2 + x_2^2\right|^{(\alpha-1)p}\left(1 - x_1^2 - x_2^2\right)^{\frac{d}{2}-2} dx_1\, dx_2 \\ &\leqslant c|\theta|^{2p} \int\limits_{8|\theta|}^{1} r^{(2\alpha-2)p+1}\left(1 - r^2\right)^{\frac{d}{2}-2} dr, \end{aligned}$$

which, by a simple calculation, leads to the desired upper estimates. □

Our last example includes a family of functions and will be useful in the next section. Note that the asymptotic orders in (9.7) and (9.8) below are different for $\|y_0\| = 1$ and $\|y_0\| < 1$, as can be expected.

Example 9.4. Let y_0 be a fixed point in $\mathbb{B}^d$, let $0 \neq \alpha > -\frac{d-1}{2p}$, and let $f_\alpha : \mathbb{S}^{d-1} \to \mathbb{R}$ be given by $f_\alpha(x) := \|x - y_0\|^{2\alpha}$. If $\alpha \neq 1 - \frac{d-1}{2p}$ then

$$\omega_2(f_\alpha, t)_{L^p(\mathbb{S}^{d-1})} \sim \|y_0\| t^2\left(t + 1 - \|y_0\|\right)^{2(\alpha-1)+\frac{d-1}{p}} + \|y_0\| t^2, \tag{9.7}$$

where the constants of equivalence are independent of t and y_0. Moreover, if $\alpha = 1 - \frac{d-1}{2p}$, then

$$c^{-1}\|y_0\| t^2 \leqslant \omega_2(f_\alpha, t)_{L^p(\mathbb{S}^{d-1})} \leqslant c\|y_0\| t^2 \left|\log\left(t + 1 - \|y_0\|\right)\right|^{\frac{1}{p}}, \tag{9.8}$$

where c is a positive constant independent of y_0 and t.

Proof. We start with the proof of the upper estimates

$$\left\|\triangle_{i,j,\theta}^2 f_\alpha\right\|_{L^p(\mathbb{S}^{d-1})} \leqslant c\Phi_\alpha\left(|\theta|\right), \quad 1 \leqslant i < j \leqslant d,\ |\theta| \leqslant \delta_\alpha,$$

where $\Phi_\alpha(t)$ denotes the desired upper bound of $\omega_2(f_\alpha, t)_{L^p(\mathbb{S}^{d-1})}$ defined by the right-hand side of either (9.7) or (9.8). By symmetry, it suffices to prove this inequality for $(i, j) = (1, 2)$. We write $y_0 = (y_{0,1}, y_{0,2}, \dots, y_{0,d}) = r y_0'$ with $0 \leqslant r \leqslant 1$ and $y_0' \in \mathbb{S}^{d-1}$. Let

$$E_1 := \left\{x \in \mathbb{S}^{d-1}\colon\ \|x - y_0\| \leqslant 4|\theta|\right\} \quad \text{and} \quad E_2 := \left\{x \in \mathbb{S}^{d-1}\colon\ \|x - y_0\| > 4|\theta|\right\}.$$

We break the integral of $|\triangle_{1,2,\theta}^2 f_\alpha(x)|^p$ into two parts:

$$\int\limits_{\mathbb{S}^{d-1}} \left|\triangle_{1,2,\theta}^2 f_\alpha(x)\right|^p d\sigma(x) = \int\limits_{E_1} \cdots + \int\limits_{E_2} \cdots =: I(E_1) + I(E_2).$$

To estimate $I(E_1)$, we assume, without loss of generality, that $r = \|y_0\| \geqslant 1 - 4|\theta| > \frac{1}{2}$ since otherwise $E_1 = \emptyset$. Then, for $x \in E_1$, $\|x - y_0'\| \leqslant \|x - y_0\| + (1 - r) \leqslant 8\theta$, which is equivalent to $1 - \langle x, y_0'\rangle \leqslant 32\theta^2$ or $\arccos\langle x, y_0'\rangle \leqslant c\theta$. Hence, we can deduce from the definition that

$$\begin{aligned} I(E_1) &\leqslant 4 \sup_{|t|\leqslant 2|\theta|} \int_{E_1} \left|f_\alpha(Q_{1,2,t}x)\right|^p d\sigma(x) \\ &= 4 \sup_{|t|\leqslant 2|\theta|} \int_{Q_{1,2,-t}(E_1)} \left|f_\alpha(x)\right|^p d\sigma(x) \leqslant 4 \int_{\|x-y_0'\|\leqslant 10|\theta|} \left|f_\alpha(x)\right|^p d\sigma(x) \\ &= c \int_{\|x-y_0'\|\leqslant 10|\theta|} \left|(1-r)^2 + 2r\big(1 - \langle x, y_0'\rangle\big)\right|^{\alpha p} d\sigma(x) \\ &\leqslant c \int_0^{c|\theta|} \big[|\theta|^{2\alpha p} + (1-\cos v)^{\alpha p}\big] \sin^{d-2} v\, dv \\ &\leqslant c|\theta|^{2\alpha p + d - 1} \sim r^p |\theta|^{2p} \big(1 - r + |\theta|\big)^{2(\alpha-1)p + d - 1}, \end{aligned}$$

where the last step uses the assumption $1 - r \leqslant 4|\theta| < \frac{1}{2}$. To estimate $I(E_2)$, we set $g_{x,y_0}(t) = f_\alpha(Q_{1,2,t}x)$ for a fixed $x \in E_2$. We then claim that

$$\left|g''_{x,y_0}(t)\right| \leqslant cr\|x - y_0\|^{2\alpha-2}, \quad \text{whenever } |t| \leqslant 2|\theta| \text{ and } x \in E_2. \tag{9.9}$$

To see this, recall that $Q_{1,2,t}x = (x_1(t), x_2(t), x_3, \dots, x_d)$, with $x_1(t) = x_1\cos t - x_2 \sin t$ and $x_2(t) = x_1 \sin t + x_2 \cos t$. Thus, a straightforward calculation shows that

$$\begin{aligned} g''_{x,y_0}(t) &= 4\alpha(\alpha-1)\|Q_{1,2,t}x - y_0\|^{2\alpha-4}\big(\big(x_1(t) - y_{0,1}\big)x_1'(t) + \big(x_2(t) - y_{0,2}\big)x_2'(t)\big)^2 \\ &\quad + 2\alpha\|Q_{1,2,t}x - y_0\|^{2\alpha-2}\big[\big(x_1'(t)\big)^2 + \big(x_1(t) - y_{0,1}\big)x_1''(t) + \big(x_2'(t)\big)^2 \\ &\quad + \big(x_2(t) - y_{0,2}\big)x_2''(t)\big]. \end{aligned} \tag{9.10}$$

For $x \in E_2$ and $|t| \leqslant 2|\theta|$, we have $\|Q_{1,2,t}x - x\| = 2\sqrt{x_1^2 + x_2^2}|\sin\frac{t}{2}| \leqslant 2|\theta| \leqslant \|x - y_0\|/2$, and hence, $\|Q_{1,2,t}x - y_0\| \sim \|x - y_0\|$. Therefore, using (9.10), we conclude that

$$\left|g''_{x,y_0}(t)\right| \leqslant c\|x - y_0\|^{2\alpha-2}, \quad \text{whenever } |t| \leqslant 2|\theta| \text{ and } x \in E_2,$$

which proves the claim (9.9) when $r = \|y_0\| \geqslant 1/2$. On the other hand, however, since the function $(x, y, t) \mapsto g''_{x,y}(t)$ is continuously differentiable on $\{(x, y, t)\colon x \in \mathbb{S}^{d-1},\ \|y\| \leqslant 1/2, |t| \leqslant \pi\}$, it follows that for $|t| \leqslant \pi$, $\|y_0\| \leqslant \delta_\alpha$ and $x \in \mathbb{S}^{d-1}$,

$$\left|g''_{x,y_0}(t)\right| = \left|g''_{x,y_0}(t) - g''_{x,0}(t)\right| \leqslant c\|y_0\| = cr,$$

where the second step uses the fact that $g_{x,0}(t) \equiv 1$ for $x \in \mathbb{S}^{d-1}$. This proves the claim (9.9) for the case of $r < 1/2$.

Now using (9.9) and the mean value theorem, we have, for some ξ_θ between 0 and 2θ,

$$\left|\triangle^2_{1,2,\theta} f(x)\right| = \frac{1}{2}\theta^2\left|g''_{x,y_0}(\xi_\theta)\right| \leqslant cr\theta^2\|x-y_0\|^{2\alpha-2}.$$

Integrating the p-th power of the last inequality over E_2 gives

$$I(E_2) \leqslant cr^p|\theta|^{2p}\int\limits_{E_2}\bigl((1-r)^2+2r\bigl(1-\langle x, y_0'\rangle\bigr)\bigr)^{(\alpha-1)p}\,d\sigma(x).$$

Thus, if $1-r \leqslant |\theta|$, then $3|\theta| \leqslant \|x-y_0'\| \leqslant \arccos\langle x, y_0'\rangle$, so that

$$I(E_2) \leqslant cr^p|\theta|^{2p}\int\limits_{\{x\in\mathbb{S}^{d-1}:\ \arccos\langle x,y_0'\rangle\geqslant 3|\theta|\}}\bigl((1-r)^2+2r\bigl(1-\langle x, y_0'\rangle\bigr)\bigr)^{(\alpha-1)p}\,d\sigma(x)$$

$$\sim r^p|\theta|^{2p}\int\limits_{3|\theta|}^{\pi}(1-r+s)^{2(\alpha-1)p}s^{d-2}\,ds \sim \bigl(\Phi_\alpha\bigl(|\theta|\bigr)\bigr)^p;$$

whereas if $1-r \geqslant |\theta|$, then $1-r+|\theta| \sim 1-r$ and considering $r \geqslant 1/2$ and $r \leqslant 1/2$, respectively, we get

$$I(E_2) \leqslant cr^p|\theta|^{2p}\int\limits_{\mathbb{S}^{d-1}}\bigl((1-r)^2+\bigl(1-\langle x, y_0'\rangle\bigr)\bigr)^{(\alpha-1)p}\,d\sigma(x) + cr^p|\theta|^{2p}$$

$$\sim r^p|\theta|^{2p}\int\limits_0^{\pi}(1-r+t)^{2(\alpha-1)p}\sin^{d-2}t\,dt + r^p|\theta|^{2p} \sim \bigl(\Phi_\alpha\bigl(|\theta|\bigr)\bigr)^p.$$

Putting the above together, we complete the proof of the upper estimates.

Next, we turn to the proof of the lower estimates. Without loss of generality, we may assume that $|y_{0,1}| \geqslant \|y_0\|/\sqrt{d} = r/\sqrt{d}$. We then claim that if $|\theta| \leqslant \delta_\alpha$ and $\alpha \neq 1-\frac{d-1}{2p}$,

$$\left\|\triangle^2_{1,j,\theta} f_\alpha\right\|_{L^p(\mathbb{S}^{d-1})} \geqslant c\Phi_\alpha\bigl(|\theta|\bigr) \quad \text{for } 2 \leqslant j \leqslant d \tag{9.11}$$

from which the desired lower estimate will follow. By symmetry, it is enough to consider $\triangle^2_{1,2,\theta} f_\alpha$. For $d > 3$, using the formula (2.13), we can write

$$\int\limits_{\mathbb{S}^{d-1}}\left|\triangle^2_{1,2,\theta} f_\alpha(x)\right|^p d\sigma(x)$$

$$= \int\limits_0^{\frac{\pi}{2}}\cos\beta(\sin\beta)^{d-3}\int\limits_{\mathbb{S}^{d-3}}\int\limits_0^{2\pi}\left|\overrightarrow{\triangle}^2_\theta g_{\beta,\xi}\bigl(\phi+\{\cdot\}\bigr)\right|^p d\phi\,d\sigma(\xi)\,d\beta, \tag{9.12}$$

where $g_{\beta,\xi}(\phi) = f_\alpha(\cos\beta\cos\phi, \cos\beta\sin\phi, \xi\sin\beta)$ for $\xi \in \mathbb{S}^{d-3}$, $\beta \in [0, \frac{\pi}{2}]$ and $\phi \in [0, 2\pi]$. For $d = 3$, we need to use (2.12), which is an easier case. We shall assume $d > 3$ in the rest of the proof.

We write $y_0 = ry_0' = r(\cos\gamma\cos\phi_0, \cos\gamma\sin\phi_0, v\sin\gamma)$, where $v \in \mathbb{S}^{d-3}$ and $0 \leqslant \gamma \leqslant \delta_d < \frac{\pi}{2}$, the latter follows from $|y_{0,1}| \geqslant r/\sqrt{d} \geqslant (1-\delta)/\sqrt{d}$. Then

$$g_{\beta,\xi}(\phi) = \big(1 + r^2 - 2r\cos\gamma\cos\beta\cos(\phi - \phi_0) - 2r\sin\beta\sin\gamma\langle\xi, v\rangle\big)^\alpha$$
$$= A^\alpha\big(1 - a\cos(\phi - \phi_0)\big)^\alpha,$$

where $A := 1 + r^2 - 2r(\sin\beta\sin\gamma)\langle\xi, v\rangle$ and $a = 2r\cos\gamma\cos\beta/A$. Since $0 \leqslant \sin\beta\sin\gamma \leqslant \sin\delta_d < 1$, we have

$$A = (1-r)^2 + 2r\big(1 - (\sin\beta\sin\gamma)\langle\xi, v\rangle\big) \sim 1,$$

and

$$1 - a = A^{-1}\left((1-r)^2 + 4r\sin^2\frac{\beta-\gamma}{2} + 2r\sin\beta\sin\gamma\big(1 - \langle\xi, v\rangle\big)\right)$$
$$\sim (1-r)^2 + r|\beta-\gamma|^2 + r\sin\beta\sin\gamma\big(1 - \langle\xi, v\rangle\big)$$
$$\sim (1-r)^2 + |\beta-\gamma|^2 + \beta\gamma\big(1 - \langle\xi, v\rangle\big).$$

In particular, there exists a constant $c_1 \geqslant 2$ so that $1 - r + |\beta - \gamma| \geqslant c_1|\theta|$ implies $1 - a \geqslant 36\theta^2$.

If $0 \leqslant r < 1 - \delta < 1$ for some small positive absolute constant δ, then applying Lemma 8.1, we deduce that for $|\theta| \leqslant \delta/c_1$,

$$\int_0^{2\pi} \big|\overrightarrow{\triangle}_\theta^2 g_{\beta,\xi}\big(\phi + \{\cdot\}\big)\big|^p\,d\phi \geqslant c|\theta|^{2p}A^{\alpha p}|a|^p(1-a)^{(\alpha-1)p+1/2}$$
$$\geqslant cr^p|\theta|^{2p}(\cos\beta)^p,$$

which, using (9.12), implies the desired lower estimate (9.11) in the case of $0 \leqslant r \leqslant 1 - \delta$.

For the rest of the proof, assume that $1 - \delta < r \leqslant 1$. We shall further assume that $\alpha < 1 - \frac{d-1}{2p}$, as the case $\alpha > 1 - \frac{d-1}{2p}$ can be treated similarly, and in fact, is much easier. Since $1 - \frac{d-1}{2p} < 1 - \frac{1}{2p}$, we can apply Lemma 8.1 to deduce that for $\beta \geqslant \gamma + c_1|\theta| + 1 - r$,

$$\int_0^{2\pi} \big|\overrightarrow{\triangle}_\theta^2 g_{s,\xi}\big(\phi + \{\cdot\}\big)\big|^p\,d\phi \geqslant c|\theta|^{2p}A^{\alpha p}|a|^p(1-a)^{(\alpha-1)p+1/2}$$
$$\sim |\theta|^{2p}(\cos\beta)^p\big[(1-r)^2 + |\beta-\gamma|^2 + \gamma\beta\big(1 - \langle\xi, v\rangle\big)\big]^{(\alpha-1)p+1/2},$$

where the last step uses the facts that $A \sim 1$, $\frac{1}{2} \leqslant r \leqslant 1$ and $0 \leqslant \gamma \leqslant \delta_d < \frac{\pi}{2}$. Moreover, we can further assume that $\beta < \beta_0 < \frac{\pi}{2}$ since $\gamma \leqslant \delta_d < \frac{\pi}{2}$. Thus, applying (9.12) and the formula

$$\int_{\mathbb{S}^{d-3}} \Psi\big(\langle \xi, u\rangle\big)\, d\sigma(\xi) = \sigma_{d-4} \int_{-1}^{1} \Psi\big(\|u\| z\big)\big(1-z^2\big)^{\frac{d-5}{2}}\, dz\, d\beta,$$

we conclude that

$$I := \int_{\mathbb{S}^{d-1}} \big|\triangle_{1,2,\theta}^2 f_\alpha(x)\big|^p\, d\sigma(x) \geqslant c|\theta|^{2p} \int_{\gamma+c_1|\theta|+1-r}^{\beta_0} \beta^{d-3}$$
$$\times \left[\int_{-1}^{1} \big(1-z^2\big)^{\frac{d-5}{2}} \big((1-r)^2 + |\beta-\gamma|^2 + \gamma\beta(1-z)\big)^{(\alpha-1)p+1/2}\, dz\right] d\beta. \quad (9.13)$$

To estimate I, we consider two cases.

Case 1. $0 \leqslant \gamma \leqslant 1 - r + c_1|\theta|$. Then, for β in the integral, $\gamma\beta(1-z) \leqslant \pi(1-r+c_1|\theta|) \leqslant \beta - \gamma$, and hence

$$I \geqslant c|\theta|^{2p} \int_{\gamma+c_1|\theta|+1-r}^{\beta_0} \beta^{d-3}(\beta-\gamma)^{2(\alpha-1)p+1}\, d\beta$$
$$\geqslant c|\theta|^{2p} \int_{2c_1|\theta|+2(1-r)}^{\beta_0} \beta^{2(\alpha-1)p+d-2}\, d\beta \geqslant c\Phi_\alpha\big(|\theta|\big)^p.$$

Case 2. $\gamma > 1 - r + c_1|\theta|$. If $\beta \leqslant \min\{3\gamma, \beta_0\} =: \beta_1$ then $\frac{(\beta-\gamma)^2}{4\gamma\beta} \leqslant \frac{\beta^2}{4\gamma\beta} \leqslant \frac{3}{4} < 1$. Thus, we can restrict the domain of the integral to

$$\gamma + c_1|\theta| + 1 - r \leqslant \beta \leqslant \beta_1, \qquad \frac{(\beta-\gamma)^2}{4\gamma\beta} \leqslant 1 - z \leqslant 1,$$

and then obtain from (9.13) that

$$I \geqslant c|\theta|^{2p} \int_{\gamma+c_1|\theta|+1-r}^{\beta_1} \beta^{d-3}(\gamma\beta)^{(\alpha-1)p+\frac{1}{2}} \int_{\frac{(\beta-\gamma)^2}{4\gamma\beta}}^{1} z^{\frac{d-4}{2}+(\alpha-1)p}\, dz\, d\beta$$
$$\sim |\theta|^{2p} \int_{\gamma+c_1|\theta|+1-r}^{\beta_1} (\beta/\gamma)^{-\frac{d-3}{2}} (\beta-\gamma)^{d-2+2(\alpha-1)p}\, d\beta$$
$$\geqslant |\theta|^{2p} \int_{\gamma+c_1|\theta|+1-r}^{\beta_1} (\beta-\gamma)^{d-2+2(\alpha-1)p}\, d\beta \sim |\theta|^{2p}\big(|\theta| + 1 - r\big)^{2(\alpha-1)p+d-1},$$

provided that $d - 1 + 2(\alpha-1)p < 0$. $\square$

9.2. Examples of best approximation on the sphere

Our computational examples, together with Theorem 3.4 and its corollary, immediately lead to the following examples on the asymptotic order of $E_n(f)_{L^p(\mathbb{S}^{d-1})}$.

Example 9.5. For $d \geqslant 3$ let $f_\alpha(x) = x^\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_d) \neq 0$. If $0 \leqslant \alpha_i < 1$ for $1 \leqslant i \leqslant d$, then for $n \in \mathbb{N}$,

$$E_n(f_\alpha)_{L^p(\mathbb{S}^{d-1})} \sim n^{-\delta-1/p}, \qquad \delta = \min_{\alpha_i \neq 0}\{\alpha_1, \ldots, \alpha_d\}.$$

Example 9.6. For $d \geqslant 3$ let $g_\alpha(x) = (1-x_1)^\alpha$, $x = (x_1, \ldots, x_d) \in \mathbb{S}^{d-1}$. Then for $-\frac{d-1}{2p} < \alpha < 1 - \frac{d-1}{2p}$ and $\alpha \neq 0$,

$$E_n(f)_{L^p(\mathbb{S}^{d-1})} \sim n^{-2\alpha - \frac{d-1}{p}}, \quad 1 \leqslant p \leqslant \infty.$$

It is interesting to compare the two examples. As functions defined on $\mathbb{R}^d$, the functions x_1^α and $(1-x_1)^\alpha$ have the same smoothness and a reasonable modulus of smoothness would confirm that. As functions on the sphere $\mathbb{S}^{d-1}$, however, they have different orders of smoothness as seen in Examples 9.1 and 9.2, and their errors of best approximation are also different as seen in Examples 9.5 and 9.6.

For $\alpha \geqslant 1 - \frac{d-1}{2p}$, the asymptotic order of $\omega_2(g_\alpha, t)_p$ in (9.2) does not lead to the asymptotic order of $E_n(f)_p$, since our inverse theorem in (3.10) is of weak type. This remark also applies to other examples below.

From our Examples 9.3 and 9.4, we also obtain the following results:

Example 9.7. For $d \geqslant 3$ let $f_\alpha(x) = (x_1^2 + x_2^2)^\alpha$, $x = (x_1, \ldots, x_d) \in \mathbb{S}^{d-1}$. Then for $-\frac{1}{p} < \alpha < 1 - \frac{1}{p}$ and $\alpha \neq 0$,

$$E_n(f)_{L^p(\mathbb{S}^{d-1})} \sim n^{-2\alpha - \frac{2}{p}}, \quad 1 \leqslant p \leqslant \infty.$$

Example 9.8. Let y_0 be a fixed point in $\mathbb{B}^d$, let $\alpha \neq 0$, and let $f_\alpha : \mathbb{S}^{d-1} \to \mathbb{R}$ be defined by $f_\alpha(x) := \|x - y_0\|^{2\alpha}$. If $-\frac{d-1}{2p} < \alpha < 1 - \frac{d-1}{2p}$, then

$$E_n(f)_{L^p(\mathbb{S}^{d-1})} \sim n^{-2}\|y_0\|\big(n^{-1} + 1 - \|y_0\|\big)^{2(\alpha-1)+\frac{d-1}{p}}.$$

In particular, if $\|y_0\| = 1$, then f_α has a singularity and the asymptotic order is $n^{-2\alpha - \frac{d-1}{p}}$ instead of $n^{-2\alpha}$.

10. Computational examples on the unit ball

In this section we compute the modulus of smoothness $\omega_r(f,t)_{L^p(\mathbb{B}^d)} := \omega_r(f,t)_{p,1/2}$ defined in (5.14) and the best approximation $E_n(f)_{L^p(\mathbb{B}^d)} := E_n(f)_{p,1/2}$ of (5.6), both with constant weight function.

10.1. Computation of moduli of smoothness

Since $\omega_r(f,t)_{L^p(\mathbb{B}^d)}$ is closely related to $\omega_r(f,t)_{L^p(\mathbb{S}^d)}$ according to Lemma 5.4, our first three examples are derived directly from those in the previous section.

Example 10.1. For $\alpha \neq 0$, define $f_\alpha : \mathbb{B}^d \to \mathbb{R}$ by $f_\alpha(x) = (1 - \|x\|^2 + \|x - y_0\|^2)^\alpha$, where y_0 is a fixed point on $\mathbb{B}^d$. If $\alpha \neq 1 - \frac{d+1}{2p}$, then

$$\omega_2(f_\alpha, t)_{L^p(\mathbb{B}^d)} \sim t^2 \|y_0\| \big(t + 1 - \|y_0\|\big)^{2(\alpha-1)+\frac{d+1}{p}} + t^2 \|y_0\|,$$

where the constants of equivalence are independent of y_0 and t. Moreover, if $\alpha = 1 - \frac{d+1}{2p}$, then

$$c_\alpha^{-1} t^2 \|y_0\| \leqslant \omega_2(f_\alpha, t)_{L^p(\mathbb{B}^d)} \leqslant c_\alpha t^2 \|y_0\| \big|\log\big(t + 1 - \|y_0\|\big)\big|^{\frac{1}{p}},$$

where c_α is independent of t and y_0.

Proof. Let $F_\alpha : \mathbb{S}^{d+1} \to \mathbb{R}$ be defined by

$$F_\alpha(x, x_{d+1}, x_{d+2}) = f_\alpha(x) = \big(\|x - y_0\|^2 + x_{d+1}^2 + x_{d+2}^2\big)^\alpha = \|X - Y_0\|^{2\alpha},$$

where $X = (x, x_{d+1}, x_{d+2}) \in \mathbb{S}^{d+1}$, $x \in \mathbb{B}^d$, and $Y_0 = (y_0, 0, 0) \in \mathbb{B}^{d+2}$. Since the moduli of smoothness of F_α on $\mathbb{S}^{d+1}$ were computed in Example 9.4, the stated result follows from Lemma 5.4. □

Similarly, we can deduce directly from Examples 9.1 and 9.3 the following results:

Example 10.2. For $\alpha \neq 0$, let $f_\alpha(x) = (1 - \|x\|^2)^\alpha$ for $x \in \mathbb{B}^d$. Then

$$\omega_2(f_\alpha, t)_{L^p(\mathbb{B}^d)} \sim \begin{cases} t^{2\alpha+\frac{2}{p}}, & \text{if } -\frac{1}{p} < \alpha < 1 - \frac{1}{p}; \\ t^2 |\log t|^{\frac{1}{p}}, & \text{if } \alpha = 1 - \frac{1}{p}; \\ t^2, & \text{if } \alpha > 1 - \frac{1}{p}. \end{cases}$$

Example 10.3. Let $f_\alpha(x) = x^\alpha$ for $x \in \mathbb{B}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \neq 0$. If $0 \leqslant \alpha_i < 1$ for all $1 \leqslant i \leqslant d$ then for $1 \leqslant p \leqslant \infty$,

$$\omega_2(f_\alpha, t)_{L^p(\mathbb{B}^d)} \sim t^{\delta+\frac{1}{p}}, \qquad \delta := \min_{\alpha_i \neq 0}\{\alpha_1, \ldots, \alpha_d\}.$$

Our next example is more complicated and requires a proof.

Example 10.4. Let $\alpha \neq 0$, $d \geqslant 2$ and let $f_\alpha : \mathbb{B}^d \to \mathbb{R}$ be given by $f_\alpha(x) = \|x - e_0\|^{2\alpha}$, where $e_0 = (1, 0, \ldots, 0) \in \mathbb{B}^d$. Then

$$\omega_2(f_\alpha,t)_{L^p(\mathbb{B}^d)} \sim \begin{cases} t^{2\alpha+\frac{d}{p}}, & -\frac{d}{2p}<\alpha<1-\frac{d}{2p}, \\ t^2|\log t|^{\frac{1}{p}}, & \alpha = 1-\frac{d}{2p},\ p\neq\infty, \\ t^2, & \alpha > 1-\frac{d}{2p}. \end{cases} \tag{10.1}$$

Before we give the proof of (10.1), several remarks are in order. First, it is interesting to compare these examples. We consider the function smoother when the asymptotic order of its modulus of smoothness is higher. Example 10.2 has a singularity at $\|x\|=1$, the boundary of $\mathbb{B}^d$, and is a radial function, for which the asymptotic order is independent of the dimension d. Example 10.4 has a singularity at $x=e_0$, also on the boundary of the ball, but it is smoother than the one in Example 10.2 for $d>2$ and $\alpha<1-\frac{d}{2p}$. Furthermore, Example 10.1 with $y=e_0$ also has a singularity at $x=e_0$ and its formulation is like the addition of the other two cases; it is, nevertheless, the smoothest one among the three functions. This does not seem to be intuitively evident. Second, the comparison of these cases shows the effect of the part of the modulus of smoothness in the Euler angles. In fact, as the proof below will show, the part defined via difference in Euler angles in the definition of $\omega_2(f,t)_{L^p(\mathbb{B}^d)}$ in (5.14) is dominating for Example 10.4. We also note that the asymptotic order of Example 10.2 is independent of the dimension. Finally, we should mention that the reason we restrict to e_0 in the last example is given after the proof in Remark 10.1.

Proof of Example 10.4. The proof of (10.1) proceeds in three steps. The first step deals with the difference in the Euler angles, which can be done, in fact, more generally.

Step 1. Let $x_0\in\mathbb{S}^{d-1}$ and $f_\alpha(x)=\|x-x_0\|^{2\alpha}$, which includes the case of $x_0=e_0$ as a special case. We prove that for $1\leqslant i<j\leqslant d$,

$$\big\|\triangle^2_{i,j,\theta} f_\alpha\big\|_{L^p(\mathbb{B}^d)} \sim \begin{cases} |\theta|^{2\alpha+\frac{d}{p}}, & -\frac{d}{2p}<\alpha<1-\frac{d}{2p}, \\ |\theta|^2|\log|\theta||^{\frac{1}{p}}, & \alpha = 1-\frac{d}{2p},\ p\neq\infty, \\ \theta^2, & \alpha > 1-\frac{d}{2p}. \end{cases} \tag{10.2}$$

For $x\in\mathbb{B}^d$, write $x=\|x\|x'$, $x'\in\mathbb{S}^{d-1}$. We then have

$$\|x-x_0\|^2 = 1+\|x\|^2-2\|x\|\langle x',x_0\rangle = \big\|x'-\|x\|x_0\big\|^2.$$

Let $g_{\alpha,r}(x')=\|x'-y_0\|^{2\alpha}$, $x'\in\mathbb{S}^{d-1}$ and $y_0=rx_0$. Then the above equation shows

$$f_\alpha(x)=g_{\alpha,\|x\|}\big(x'\big),\quad x'\in\mathbb{S}^{d-1}.$$

Since $\triangle^r_{i,j,\theta}$ commutes with $\|x\|$, by using polar coordinates and the proof of Example 2.1, we obtain for $\alpha\neq 0, 1-\frac{d-1}{2p}$ that

$$\int_{\mathbb{B}^d}\big|\triangle^2_{1,2,\theta} g_\alpha(x)\big|^p\,dx = \int_0^1 r^{d-1}\int_{\mathbb{S}^{d-1}}\big|\triangle^2_{1,2,\theta} f_{\alpha,r}\big(x'\big)\big|^p\,d\sigma\big(x'\big)\,dr$$

$$\leqslant c\int_0^1 r^{d-1+p}\big(|\theta|^{2p}(1+\theta-r)^{2(\alpha-1)p+d-1}+|\theta|^{2p}\big)\,dr$$
$$\leqslant c|\theta|^{2\alpha p+d},$$

if $2(\alpha-1)p+d<-1$ or $\alpha<1-\frac{d}{2p}$; moreover, the same computation also gives lower bound if we select a pair of i,j as in Example 2.1, the choice of which depends only on x_0 and is independent of $\|x\|$. The other cases of α can be handled similarly.

Step 2. Next we consider the term $\triangle^2_{1,d+1,\theta}\tilde{f}_\alpha$. We show that for $\alpha>-\frac{d}{2p}$

$$\left(\int_{\mathbb{B}^{d+1}}\big|\triangle^2_{1,d+1,\theta}\tilde{f}_\alpha(x)\big|^p\frac{dx}{\sqrt{1-\|x\|^2}}\right)^{\frac{1}{p}}\leqslant c|\theta|^{2\alpha+\frac{d+1}{p}}+c\theta^2, \tag{10.3}$$

where $\tilde{f}_\alpha(x)=f_\alpha(x')$ for $x=(x',x_{d+1})\in\mathbb{B}^{d+1}$. Let

$$E_{2,1}:=\big\{x\in\mathbb{B}^{d+1}:\ 1-c^2\theta^2\leqslant x_1\leqslant 1\big\},\qquad E_{2,2}:=\big\{x\in\mathbb{B}^{d+1}:\ -1\leqslant x_1\leqslant 1-c^2\theta^2\big\},$$

where $c>1$ is a sufficiently large absolute constant; we break the integral into two parts,

$$\int_{\mathbb{B}^{d+1}}\big|\triangle^2_{1,d+1,\theta}\tilde{f}_\alpha(x)\big|^p\frac{dx}{\sqrt{1-\|x\|^2}}=\int_{E_{2,1}}\cdots+\int_{E_{2,2}}\cdots\equiv I_2(E_{2,1})+I_2(E_{2,2}).$$

To estimate $I_2(E_{2,1})$, observe that for $x\in E_{2,1}$, $|x_j|\leqslant\sqrt{1-x_1^2}\leqslant c|\theta|$ for all $2\leqslant j\leqslant d+1$, which implies, upon using $(1-x_1\cos\psi-x_{d+1}\sin\psi)\leqslant c|\theta|$ for $\psi\leqslant 2|\theta|$, that $|f_\alpha(x_1\cos\psi+x_{d+1}\sin\psi,x_2,\dots,x_d)|\leqslant c(|\theta|^{2\alpha}+|x_2|^{2\alpha})$, so that, by the definition, $|\triangle^2_{1,d+1,\theta}\tilde{f}_\alpha(x)|\leqslant c(|\theta|^{2\alpha}+|x_2|^{2\alpha})$. Thus,

$$I_2(E_{2,1})\leqslant c\int_{1-c^2\theta^2}^1\big(1-x_1^2\big)^{\frac{d-1}{2}}\left[\int_{-1}^1\big(|\theta|^{2\alpha p}+\big|\sqrt{1-x_1^2}s\big|^{2\alpha p}\big)\big(1-s^2\big)^{\frac{d-2}{2}}\,ds\right]dx_1$$
$$\leqslant c|\theta|^{2\alpha p+d+1}.$$

For the estimate of $I_2(E_{2,2})$, we write $g_x(u)=\tilde{f}_\alpha(Q_{1,d+1,u}x)$ for a fixed $x\in\mathbb{B}^{d+1}$. That is, $g_x(u)=(t_x(u)^2+\sum_{j=2}^d x_j^2)^\alpha$ with $t_x(u)=1-x_1\cos u+x_{d+1}\sin u$. A straightforward calculation shows

$$g_x''(u)=4\alpha(\alpha-1)\left(t_x(u)^2+\sum_{j=2}^d x_j^2\right)^{\alpha-2}\big(t_x(u)t_x'(u)\big)^2$$
$$+2\alpha\left(t_x(u)^2+\sum_{j=2}^d x_j^2\right)^{\alpha-1}\big(\big(t_x'(u)\big)^2+t_x(u)t_x''(u)\big). \tag{10.4}$$

Observe that if $x \in E_{2,2}$ and $|\xi| \leqslant 2|\theta|$, then

$$\left|t_x'(\xi)\right| = |x_1 \sin\xi + x_{d+1}\cos\xi| \leqslant 2|\theta| + |x_{d+1}| \leqslant c'\sqrt{1-x_1},$$

and

$$\left|t_x(\xi) - (1-x_1)\right| \leqslant 2\theta^2 + 2\sqrt{1-x_1^2}|\theta| \leqslant \left(\frac{2}{c^2} + \frac{2\sqrt{2}}{c}\right)(1-x_1)$$

which, in particular, implies $|t_x(\xi)| \sim 1 - x_1$ provided that c is large enough. Thus, by (10.4) it follows that for $x \in E_{2,2}$ and $|\xi| \leqslant 2|\theta|$,

$$\left|g_x''(\xi)\right| \leqslant c\left((x_1-1)^2 + \sum_{j=2}^{d} x_j^2\right)^{\alpha-1}(1-x_1),$$

which, using the mean value theorem, implies that if $x \in E_{2,2}$, then for some ξ between 0 and 2θ,

$$\left|\triangle^2_{1,d+1,\theta}\tilde{f}_\alpha(x)\right| = \frac{1}{2}\theta^2\left|g_x''(\xi)\right| \leqslant c\theta^2(1-x_1)\left((1-x_1)^2 + \sum_{j=2}^{d} x_j^2\right)^{\alpha-1}. \tag{10.5}$$

If $\alpha \geqslant 1$, then we can drop the $(\ldots)^{\alpha-1}$ term and the estimate of $I_2(E_{2,2})$ follows trivially. For $\alpha < 1$, integrating the p-th power of the inequality (10.5) over $E_{2,2}$ yields

$$\begin{aligned} I_2(E_{2,2}) &\leqslant c|\theta|^{2p} \int_{-1}^{1-c\theta^2} \left(1-x_1^2\right)^{\alpha p + \frac{d-1}{2}} \int_{\mathbb{B}^d} \left((1-x_1)^2 + \sum_{j=1}^{d-1} u_j^2\right)^{(\alpha-1)p} \frac{du}{\sqrt{1-\|u\|^2}}\, dx_1 \\ &= c|\theta|^{2p} \int_{-1}^{1-c\theta^2} \left(1-x_1^2\right)^{\alpha p + \frac{d-1}{2}} \int_{\mathbb{B}^{d-1}} \left((1-x_1)^2 + \|v\|^2\right)^{(\alpha-1)p}\, dv\, dx_1\, dx_1 \end{aligned}$$

by (6.14). Hence, switching to spherical-polar coordinates, we obtain

$$\begin{aligned} I_2(E_{2,2}) &\leqslant c|\theta|^{2p} \int_{-1}^{1-c\theta^2} \left(1-x_1^2\right)^{\alpha p + \frac{d-1}{2}} \int_0^1 (\sqrt{1-x_1} + r)^{2(\alpha-1)p+d-2}\, dr\, dx_1 \\ &\leqslant c|\theta|^{2p} \int_{-1}^{1-c\theta^2} (1-x_1)^{(2\alpha-1)p+d-1}\, dx_1 \leqslant c|\theta|^{2\alpha p+d+1} + c|\theta|^{2p} \end{aligned}$$

where we have used, in the last step, that $2\alpha p + d \leqslant 4\alpha p + 2d$.

Step 3. Finally we consider $\triangle^2_{i,d+1,\theta}\tilde{f}_\alpha$ for $2 \leqslant j \leqslant d$. We prove that for $\alpha > -\frac{d}{2p}$,

$$\left(\int\limits_{\mathbb{B}^{d+1}} \left|\triangle^2_{j,d+1,\theta}\tilde{f}_\alpha(x)\right|^p \frac{dx}{\sqrt{1-\|x\|^2}}\right)^{\frac{1}{p}} \leqslant c|\theta|^{2\alpha+\frac{d+1}{p}} + c\theta^2. \tag{10.6}$$

As in the case $E_{2,2}$, the case of $\alpha \geqslant 1$ is easy and we assume $\alpha < 1$. Clearly, it suffices to consider $\triangle^2_{2,d+1,\theta}\tilde{f}(x)$. Let

$$E_{3,1} := \left\{x \in \mathbb{B}^{d+1}:\ 1 - x_1 \leqslant c\theta^2\right\},$$
$$E_{3,2} := \left\{x \in \mathbb{B}^{d+1}:\ 1 - x_1 \geqslant c\theta^2,\ |x_2| \geqslant 4\sqrt{1-x_1^2}|\theta|\right\},$$
$$E_{3,3} := \left\{x \in \mathbb{B}^{d+1}:\ 1 - x_1 \geqslant c\theta^2,\ \ |x_2| < 4\sqrt{1-x_1^2}|\theta|\right\},$$

where c is a sufficiently large absolute constant. We break the integral into three parts,

$$\int\limits_{\mathbb{B}^{d+1}} \left|\triangle^2_{2,d+1,\theta}\tilde{f}_\alpha(x)\right|^p \frac{dx}{\sqrt{1-\|x\|^2}} = \int\limits_{E_{3,1}} \cdots + \int\limits_{E_{3,2}} \cdots + \int\limits_{E_{3,3}} \cdots$$
$$\equiv I_3(E_{3,1}) + I_3(E_{3,2}) + I_3(E_{3,3}).$$

Clearly, $I_3(E_{3,1})$ can be estimated exactly as $I_2(E_{2,1})$ in Step 2.

To estimate $I_3(E_{3,2})$ and $I_3(E_{3,3})$, we set, as in Step 2, $g_x(u) = \tilde{f}_\alpha(Q_{2,d+1,u}x)$ for any fixed $x \in \mathbb{B}^{d+1}$. Since $|x_2|, |x_{d+1}| \leqslant \sqrt{1-x_1^2}$, it is easy to verify that

$$\left|g_x''(u)\right| \leqslant c\left((x_1-1)^2 + t_x(u)^2 + \sum_{j=3}^d x_j^2\right)^{\alpha-1}(1-x_1), \tag{10.7}$$

where $t_x(u) = x_2\cos u - x_{d+1}\sin u$. Observe that if $x \in E_{3,2}$ and $|u| \leqslant 2|\theta|$ then

$$\left|x_2 - t_x(u)\right| = \left|x_2(1-\cos u) + x_{d+1}\sin u\right|$$
$$\leqslant \sqrt{x_2^2 + x_{d+1}^2}\sqrt{(1-\cos u)^2 + \sin^2 u} \leqslant \sqrt{1-x_1^2}|u| \leqslant \frac{1}{2}|x_2|.$$

This implies $|t_x(u)| \sim |x_2|$ for $x \in E_{3,2}$. Thus, using (10.7), we have, for $x \in E_{3,2}$,

$$\left|\triangle^2_{2,d+1,\theta}\tilde{f}_\alpha(x)\right| = \frac{1}{2}\theta^2\left|g_x''(\xi)\right| \leqslant c\theta^2\left((x_1-1)^2 + \sum_{j=2}^d x_j^2\right)^{\alpha-1}(1-x_1), \tag{10.8}$$

where ξ is a number between 0 and 2θ. In particular, this allows us to estimate $I_3(E_{3,2})$ exactly as in Step 2.

It remains to estimate $I_3(E_{3,3})$. Using (10.7) and the mean value theorem, we have, for $x \in E_{3,3}$,

$$\left|\triangle^2_{2,d+1,\theta}\tilde{f}_\alpha(x)\right| = \frac{1}{2}\theta^2\left|g''_x(\xi)\right| \leqslant c\theta^2\left((x_1-1)^2+\sum_{j=3}^d x_j^2\right)^{\alpha-1}(1-x_1), \tag{10.9}$$

where ξ is a number between 0 and 2θ. For $m \geqslant 2$ let $E(m) := \{(x_1,\dots,x_m)\in\mathbb{R}^m\colon\ 1-x_1 \geqslant c\theta^2,\ |x_2| \leqslant 4\sqrt{1-x_1^2}|\theta|\}$. Integrating the p-th power of (10.9) over $E_{3,3}$ and using (6.14), we obtain

$$\begin{aligned} I_3(E_{3,3}) &\leqslant c|\theta|^{2p}\int\limits_{E(d)}(1-x_1)^p\left((1-x_1)^2+\sum_{j=3}^d x_j^2\right)^{(\alpha-1)p}dx \\ &\leqslant c|\theta|^{2p}\int\limits_{E(2)}(1-x_1)^p\left[\int\limits_{\mathbb{B}^{d-2}}\left((1-x_1)^2+\left(1-x_1^2-x_2^2\right)\|v\|^2\right)^{(\alpha-1)p}dv\right] \\ &\quad\times\left(1-x_1^2-x_2^2\right)^{\frac{d-2}{2}}dx_1\,dx_2+c|\theta|^{2p}. \end{aligned}$$

Note that if $0 \leqslant x_1 \leqslant 1-c\theta^2$ and $|x_2| \leqslant 4\sqrt{1-x_1^2}|\theta|$ with $c \geqslant 32$, then $|x_2| \leqslant 4|\theta|$ and $1-x_1^2 \geqslant 1-x_1 \geqslant c\theta^2 \geqslant 2x_2^2$, which implies $1-x_1^2-x_2^2 \sim 1-x_1^2$. Thus,

$$\begin{aligned} I_3(E_{3,3}) &\leqslant c|\theta|^{2p+1}\int\limits_0^{1-c\theta^2}(1-x_1)^{\alpha p+\frac{d-1}{2}}\left[\int\limits_0^1\left(\sqrt{1-x_1}+t\right)^{2(\alpha-1)p+d-3}dt\right]dx_1+c|\theta|^{2p} \\ &\leqslant c|\theta|^{4\alpha p+2d}+c|\theta|^p \leqslant c|\theta|^{2\alpha p+d+1}+c|\theta|^{2p}. \end{aligned}$$

Putting the above together, we have established (10.6). The proof is complete. □

Remark 10.1. Our proof in Step 1 works for the more general case of $f_\alpha(x) = \|x-x_0\|^{2\alpha}$, $x_0 \in \mathbb{S}^{d-1}$. We notice that Steps 2 and 3 have smaller estimate, so that the dominating term is in Step 1. We expect that (10.1) holds for $f_\alpha(x) = \|x-x_0\|^{2\alpha}$. For the case of $\alpha < 1-\frac{d}{2p}$, this is indeed the case, as can be derived from our direct and the inverse theorem, and the rotation invariance of $E_n(f)_{L^p(\mathbb{B}^d)}$, see (10.11) at the end of the next subsection.

10.2. Examples of best approximation on the ball

Our computational examples and Theorem 5.5 immediately lead to the following examples on the asymptotic order of $E_n(f)_{L^p(\mathbb{B}^d)}$. We give two examples, one corresponds to Example 10.1 and the other corresponds to Example 10.4.

Example 10.5. For $\alpha \neq 0$, let $f_\alpha(x) = (1-\|x\|^2)^\alpha$. Then for $-\frac{1}{p} < \alpha < 1-\frac{1}{p}$,

$$E_n(f_\alpha)_{L^p(\mathbb{B}^d)} \sim n^{-2\alpha-\frac{2}{p}}.$$

Example 10.6. For $\alpha \neq 0$, $d \geqslant 2$, let $f_\alpha(x) = \|x - e_0\|^{2\alpha}$, where $e_0 = (1, 0, \dots, 0)$. For $-\frac{d}{2p} < \alpha < 1 - \frac{d}{2p}$,

$$E_n(f_\alpha)_{L^p(\mathbb{B}^d)} \sim n^{-2\alpha - \frac{d}{p}}. \tag{10.10}$$

Although our moduli of smoothness on the ball are not rotationally invariant, the best approximation $E_n(f)_{L^p(\mathbb{B}^d)}$ is; that is, $E_n(f)_{L^p(\mathbb{B}^d)} = E_n(f(\rho\cdot))_{L^p(\mathbb{B}^d)}$ for $\rho \in O(d)$. This implies, since every point x_0 on $\mathbb{S}^{d-1}$ can be rotated to e_0, that (10.10) holds for $f_{\alpha,x_0}(x) := \|x - x_0\|^{2\alpha}$. In particular, Theorem 5.5 shows then

$$\omega_2(f_{\alpha,x_0}, t)_{L^p(\mathbb{B}^d)} \sim t^{2\alpha + \frac{d}{p}}, \quad -\frac{d}{2p} < \alpha < 1 - \frac{d}{2p}, \tag{10.11}$$

as we indicated in Remark 10.1.

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