Topology and its Applications 48 (1992) 91-116
North-Holland

Subsets of ${}^{\omega}\omega$ and the Fréchet-Urysohn and α_i-properties

Peter J. Nyikos

Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA

Received 13 December 1988
Revised 15 February 1991 and 18 June 1991

Dedication. This paper is dedicated to the memory of Eric van Douwen, whose unexpected death may have been the greatest single loss to general topology since the tragic death of Urysohn in 1924.

Abstract

Nyikos, P.J., Subsets of ${}^{\omega}\omega$ and the Fréchet-Urysohn and α_i-properties, Topology and its Applications 48 (1992) 91-116.

Arhangel'skiĭ defined a number of related properties called α_i $(i=1,2,3,4)$ having to do with amalgamating countably many sequences each converging to the same point. Here we use the set ${}^{\omega}\omega$ of functions to produce examples of Fréchet spaces in the various classes and to study the relationships between the classes. We also introduce an intermediate class $\alpha_{1.5}$. Under various set-theoretic hypotheses we produce a countable Fréchet α_1-space that is not first countable, and several that are α_2 but not α_1, including one which is $\alpha_{1.5}$ and another which is not. It is now known to be consistent that none of these kinds of spaces exist, but we also construct a countable Fréchet-Urysohn α_2-space that is not first countable using only ZFC.

The existence of an α_2-space which is not α_1 in any given model of set theory is reduced to the existence of a certain kind of space whose underlying set is $(\omega\times\omega)\cup\infty$, with neighborhoods of ∞ defined using graphs of partial functions. Alan Dow has recently shown that every α_2-space is α_1 in the Laver model. A proof using the reduction theorem is outlined here and the result is used to obtain other information about this model.

An example of a countable α_2-topological group that is not first countable is given, and it is shown to be Fréchet-Urysohn under the relatively mild assumption $\mathbf{p}=\mathbf{b}$, as is a related separable nonmetrizable topological vector space.

Keywords: Fréchet-Urysohn, sheaf at x, α_i-space, v-space, w-space, Ψ-like, almost disjoint, $<^*$-unbounded, $<^*$-cofinal, minimax ideal, ω-splitting, Stone-Čech remainder, pseudo-P_κ-point.

AMS (MOS) Subj. Class.: Primary 54D55, 54G20, 54A35; secondary 03E65, 04A20, 54D40, 54E65.

Correspondence to: Professor P.J. Nyikos, Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA.

1. Preliminaries

One of the oldest and most natural generalizations of first countability is the Fréchet–Urysohn property: given x in the closure of A, there is a sequence from A converging to x. For the sake of brevity we will call such spaces Fréchet, except in speaking about topological vector spaces, where "Fréchet" means something else.

There are a number of standard structures which shed light on some problems concerning Fréchet and related spaces. One is the Cantor tree [11, 28], another is $C(X)$ with pointwise topology [10, 11], another is the family of (κ, λ)-gaps [30, 31, 36], another is the Stone–Čech remainder of ω [19, 23–25, 36]. Another is the subject of this paper, the family of partial functions from ω to ω. Most of the time the domain will be all of ω, hence the simplification in the title. Applications to other classes of spaces will be given at the end of Section 2.

Besides first-countable spaces, examples of Fréchet spaces include the one-point compactifications of discrete spaces, and Σ-products [9, 22] of first-countable spaces. Once one sees that these spaces are countably tight, that is, the closure of each set is the union of the closures of its countable subsets, the Fréchet property follows immediately from the fact that countable subsets are first countable. This is still the case in all known "real" examples of Fréchet topological groups:

Problem 1. (a) Is there a Fréchet topological group with a countable subset that is not first countable?

Problem 1. (b) Is there a countable Fréchet topological group that is not metrizable?

In this paper we will be assuming the Hausdorff separation axiom (though non-Hausdorff groups are not entirely devoid of interest: see [26]) so that metrizability is equivalent to first countability in a topological group [17]. Also, since the subgroup generated by a countable subset is countable, and the Fréchet property is hereditary, Problems 1(a) and 1(b) are equivalent. Also, since nonfirst-countability is inherited by dense subspaces of regular spaces [8, Exercise 2.1(c)], they are also equivalent to:

Problem 1. (c) Is there a separable Fréchet topological group that is not first countable?

We will see that if either $\mathbf{p} > \omega_1$ or $\mathbf{p} = \mathbf{b}$, then there is even a separable Fréchet–Urysohn topological vector space that is not first countable. The cardinal $\mathbf{b}$ has to do with the relation $<^*$ of eventual domination on the set ${}^{\omega}\omega$ of all functions from ω to ω: $f <^* g$ iff there exists n such that $f(k) < g(k)$ for all $k \geqslant n$. The following notation is becoming standard [40]:

$$\mathbf{b} = \min\{|F|: F \text{ is a } <^*\text{-unbounded ("undominated") subset of } {}^{\omega}\omega\};$$

$$\mathbf{d} = \min\{|F|: F \text{ is } <^*\text{-cofinal ("dominating") in } {}^{\omega}\omega\}.$$

The cardinal $\mathbf{p}$ has to do with the similar relation $\subset^*$ on $\mathbf{p}(\omega)$: $A \subset^* B$ if $A \backslash B$ is finite and $B \backslash A$ is infinite.

$$\mathbf{p} = \{|\mathscr{B}|\colon \mathscr{B} \text{ is a subbase for a free filter on } \omega, \text{ and there is no infinite } A \text{ such that } A \subset^* B \text{ for all } B \in \mathscr{B}\}.$$

The following concepts are due to Arhangel'skiĭ [1, 2].

Definition 1.1. A *sheaf at* x in a space X is a family γ of sequences converging to x. For $i = 1, 2, 3, 4$ we call x an *α_i-point* if for each countable sheaf γ at x there is a sequence σ converging to x such that ran σ intersects:

- α_1: each ran τ, $\tau \in \gamma$, in a cofinite set;
- α_2: each ran τ, $\tau \in \gamma$, in an infinite set;
- α_3: infinitely many ran τ, $\tau \in \gamma$, in an infinite set;
- α_4: infinitely many ran τ, $\tau \in \gamma$ in a nonempty set.

A space is called an *α_i-space* if each point is an α_i-point.

These concepts are important in determining when the product of Fréchet spaces is Fréchet: [1, 2, 23–25]. Of course, they could be satisfied vacuously.

In [2], Arhangel'skiĭ uses a different numbering than with the α_i-properties. The "2" position is taken up by the following property: if $\langle \sigma_n\colon n \in \omega\rangle$ is a sheaf at x, then there is a sequence σ converging to x whose range meets infinitely many ran σ_n in a cofinite set. As stated, this is equivalent to α_1, because we can replace each σ_n by a τ_n whose range is $\bigcup_{i=0}^{n} \operatorname{ran} \sigma_i$. And, of course, $\langle \tau_n\colon n \in \omega\rangle$ is also a sheaf at x.

However, if we require that the α_n have disjoint ranges, we get a property which is strictly weaker than α_1 in many models. In this respect the property is different from the α_i-properties above, all of which are equivalent to their disjoint versions. The hardest one to see this for is α_2, and the following lemma takes care of that.

Lemma 1.2. *Let $\{B_n\}_{n=1}^{\infty}$ be a family of infinite sets. There is a family $\{A_n\}_{n=1}^{\infty}$ of disjoint infinite sets, such that $A_n \subset B_n$ for all n.*

Proof. Let $B_{nm} = B_n$ for every $m \in \omega$. Define an order on $\omega \times \omega$ as follows: $(n, m) > (n', m')$ if either $n + m > n' + m'$ or $n + m = n' + m'$ and $n > n'$. Choose $a_{nm} \in B_{nm} - \{a_{ij}\colon (i, j) < (n, m)\}$. Then $A_n = \{a_{nm}\colon m \in \omega\}$ is as required. $\square$

I am indebted to Nogura for the above short proof: its use of $\omega \times \omega$ as a tool is typical of much of this paper.

For the sake of convenience we will say that a countably infinite set A *converges to* a point x when any 1-1 listing of A converges to x. Also, of course, any sequence whose range is a subset of A, and which lists each element of A no more than finitely many times, converges to x.

Definition 1.3. A point x in a space X is an *$\alpha_{1.5}$-point* if whenever $\langle \alpha_n\colon n \in \omega\rangle$ is a sheaf at x with ranges disjoint, there is α converging to x such that ran $\alpha_n \subset^*$ ran σ for infinitely many n. A space X is an *$\alpha_{1.5}$-space* if every point is an $\alpha_{1.5}$-point.

This notation is motivated by the fact that every α_1-space is $\alpha_{1.5}$ and every $\alpha_{1.5}$-space is α_2: given a collection $\langle A_n \rangle_{n=1}^{\infty}$ of disjoint sets, each converging to x, let A_n be the disjoint union of infinite sets $\langle A_n^m \rangle_{m=n}^{\infty}$ and let $B_m = \bigcup_{n=1}^{m} A_n^m$. Each B_m converges to x, and if $B_m \subset^* B$ for infinitely many B_m, then B meets every A_n in an infinite set.

A rule of thumb in α_i-spaces is that if there is one that is not in one of the other classes, then there is a countable one: just look at the union of the ranges of the sheaf members plus the point that together witness the original space not being in the more demanding class.

Definition 1.4. A space is a *v-space* [respectively *v′-space*] [respectively *w-space*] if it is an α_1 [respectively $\alpha_{1.5}$] [respectively α_2] Fréchet space. A space is *countably bisequential* if it is an α_4 Fréchet space.

The term "w-space" is due to Gruenhage [13], who defined it in terms of a topological game. Sharma [38] showed a characterization similar to that in Definition 1.4, but with "nonempty" in place of "infinite" in α_2. The concepts are easily shown equivalent [23]. Countably bisequential spaces were studied in [21, 36]. In [26] it was shown that every Fréchet topological group is countably bisequential, and recently Shakhmatov has shown that in any model produced by adding uncountably many Cohen reals, there is a Fréchet topological group which is not α_3, "consistently" answering the main problem of [26]:

Problem 2. Is there a Fréchet topological group which is not a w-space?

It is still not known whether there is such a group in every model of ZFC. Shakhmatov also showed that in the same models, there is a Fréchet w-group which is not a v-group. In this paper we show that such groups also exist if $\mathbf{p} = \mathbf{d}$.

Of course, first-countable spaces are v-spaces, as are countably tight spaces in which every countable subset is first countable. A remarkable recent result of Dow and Steprāns is that it is consistent that these are the *only* examples of v-spaces or even v'-spaces. In this paper we will show that there are other examples if either $\mathbf{b} > \omega_1$, or $\mathbf{b} = \mathbf{d}$. Since the Dow–Steprāns model has $\mathbf{c} = \omega_2$, the latter example shows that $\mathbf{b} = \omega_1$ and $\mathbf{d} = \omega_2$ in this model. It overlaps a general construction that can be done "in ZFC" to form a countable w-space that is not first countable. This is only the second such example (the first was constructed by Isbell [36], see [24, 30]) and the first with a compactification that is also a w-space. A third example, the "Cantor tree over a λ'-set", is described in [11, 27, 32].

A remarkable fact about all three examples is that it is independent of ZFC whether they can/must be v-spaces or v'-spaces. In the Dow–Steprāns model they cannot; but Dow has also shown [7] that in Laver's model [18] *every α_2-space is α_1*. In this paper I will give a number of "consistent examples" that distinguish between v-, v'-, and w-spaces and thereby deduce some facts about how things behave in Laver's model. The examples predate Dow's discovery by over four years in some cases, and fit under the following heading:

Definition 1.5. A space is *Ψ-like* [terminology due to van Douwen] if it has a countable dense set D of isolated points, is locally compact, and its nonisolated points form a closed discrete subspace.

Note that if one assigns to each nonisolated point z of a Ψ-like space a compact neighborhood V_z missing all other nonisolated points, then V_z is clopen and the $V_z - \{z\}$ are an ADF of subsets of D:

Definition 1.6. Two subsets of a countable set are *almost disjoint* if their intersection is finite. A collection of infinite subsets of a countable set is an *almost disjoint family* (ADF) if any two members are almost disjoint.

This gives a recipe for constructing all Ψ-like spaces, like that for the original Ψ [12, Exercise 5I] except that the ADF is not assumed to be maximal: let D be the discrete topology, let $\mathscr{A}$ be an ADF of subsets of D, and to each $A = \mathscr{A}$ attach a point z_A, decreasing N to be a nbhd of z_A iff $z_A \in N$ and $A \subset^* N$. Local compactness is obvious, while "almost disjoint" is equivalent to the Hausdorff property. It is also easy to show that every Ψ-like space is a Moore space. For more on Ψ-like spaces in general, see [40], whose results we will frequently cite without identifying their original discoverer.

Theorem 1.7. *Let* $\mathbf{a}$ *denote the least cardinality of an infinite maximal ADF. The one-point compactification of a Ψ-like space of cardinality $<\mathbf{a}$ is Fréchet.*

Proof. Let Z denote the set of nonisolated points of the Ψ-like space X, and ∞ the extra point of the one-point compactification. Since any countably infinite subset of Z converges to ∞, it is enough to consider what happens if ∞ is in the closure of $A \subset X - Z = D$. The V_z trace an ADF on A, and no finite subcollection of these traces covers A since ∞ is in its closure, so there is an infinite subset A' of A that is almost disjoint from all the ($<\mathbf{a}$-many) infinite traces, so that A' converges to ∞. $\square$

This even gives a characterization of $\mathbf{a}$: if one uses a *maximal* ADF, no sequence from $X - Z$ converges to ∞.

Notation. If X is a locally compact space, we let $X + \infty$ denote its one-point compactification.

All examples in this paper, except in the proof of Theorem 1.8, are built using one-point compactifications of Ψ-like spaces.

I have named another important technique after Rothberger [37] and Hechler [14, 15].

The RH transfer. *Variation* 1. Let $\{A_n\}_{n=0}^{\infty}$ be a collection of disjoint infinite subsets of a countable set A. Let $\psi: A \to \omega \times \omega$ be a bijection defined by distributing the elements of $A - \bigcup_{n=0}^{\infty} A_n$ into the bottom row $\omega \times \{0\}$, and then sending A_n bijectively to either the $(n+1)$st column $\{n\} \times \omega$ minus its first element, or to the whole column, depending on whether the column contains $\psi(a)$ for some $a \in A - \bigcup_{n=0}^{\infty} A_n$.

Variation 2. Let $\{B_n\}_{n=0}^{\infty}$ be a collection of subsets of an infinite set A, such that for each n there exists $m > n$ such that $B_m^{\#} = B_m \setminus \bigcup_{i=0}^{m-1} B_i$ is infinite. Let $\{A_n\}_{n=0}^{\infty}$ list all the infinite $B_m^{\#}$, and define ψ as above.

A key observation about the RH transfer is that if $S \subset A$ is almost disjoint from all the B_n, then its image meets each column in a finite set and hence is below the graph of some function from ω to ω; and conversely. An application is:

Theorem 1.8. *Every space of character* $<\mathbf{b}$ *is* α_1. *On the other hand, there is a space of character* $\mathbf{b}$ *that is not even* α_4.

Proof. Let $x \in X$ have a local base $\mathcal{V}$ of cardinality $<\mathbf{b}$. Let γ be a sheaf at x, and let $\{B_n\}_{n=0}^{\infty}$ list the ranges of the members of γ. If there are only finitely many *infinite* $B_m^{\#}$, let σ list their union in 1-1 fashion. Clearly, σ converges to x and its range meets each B_n in a cofinite set. If there are infinitely many infinite $B_m^{\#}$, apply the RH transfer. The complement of each $V \in \mathcal{V}$ is almost disjoint from each $B_m^{\#}$, hence there is a function f_V whose graph is above the ψ-images of all the points of $X - V$ in the domain of ψ. Since $\{f_V: V \in \mathcal{V}\}$ is $<^*$-bounded, there exists $f: \omega \to \omega$ such that all but finitely many $X - V$ images are below the graph of f, the only possible exceptions occurring in those columns where f is below f_V. Let σ be any 1-1 listing of $\psi^{\leftarrow} f^{\uparrow}$, the inverse image of $f^{\uparrow} = \{\langle i, n\rangle: n \geq f(i)\}$. Then σ converges to x and its range meets each B_n in a cofinite set.

Conversely, let $\{f_\alpha: \alpha < b\}$ be a $<^*$-unbounded family of increasing functions in ${}^{\omega}\omega$, and let X have underlying set $\omega \times \omega \cup \{p\}$, where a local base at p is all sets of the form $f_\alpha^{\uparrow} \cup \{p\}$. The columns converge to p, but any sequence whose range meets infinitely many columns will also be below the graph of some f_α in infinitely many terms and hence will fail to converge to p. Indeed, let $f(n)$ be the highest member of ran σ in $\{k\} \times \omega$, where k is the least integer $\geq n$ such that $(\operatorname{ran} \sigma) \cap (\{k\} \times \omega) \neq \emptyset$; since the f_α are increasing, one of them must dominate f on an infinite subset of $\pi_1(\operatorname{ran} \sigma)$. $\square$

The last sentence in the above proof is an important motif, often expressed by saying that a $<^*$-unbounded family of increasing functions is $<^*$-unbounded on every infinite subset of ω. This remains true if "increasing" is replaced by "nondecreasing", meaning that $f(n) \leq f(m)$ whenever $n < m$. Also, a $<^*$-dominating family of functions is $<^*$-dominating on every infinite set whether or not the functions are nondecreasing; and if a family of nondecreasing functions is $<^*$-dominating on some infinite set, it is $<^*$-dominating on every infinite set [40, proof of 3.6].

Besides the notation $f^{\uparrow}$ above for functions, we will also use the notations:

$$f^{\downarrow} = \{\langle i, n\rangle : n \leq f(i)\},$$

$$f^{\downarrow\downarrow} = \{\langle i, n\rangle : n < f(i)\}.$$

[*Aside*: The proof of Theorem 1.8 suggests a property even weaker than α_4: given a countable sheaf γ at x, with the ranges disjoint, there is an infinite set A meeting each sequence in at most one point, with x in the closure of A. An easy modification of the above proof shows that the least character of a space failing to have this property is $\mathbf{d}$. For the converse, the sequential fan, the quotient of $\omega \times (\omega + 1)$ formed by identifying the nonisolated points, is homeomorphic to the natural analogue of $(\omega \times \omega) \cup \{p\}$ above.]

Corollary 1.9. *If* $\mathbf{b} > \omega_1$, *there is a countable v-space which is not first countable.*

Proof. Use a Ψ-like space X of cardinality ω_1 and the fact that $\mathbf{b} \leq \mathbf{a}$ [40, Theorem 3.1]. Take the one-point compactification and remove all nonisolated points of X. $\square$

In the proof of Theorem 1.8, $(\omega \times \omega) \cup \{p\}$ is *not* Fréchet unless the family of functions is actually $<^*$-cofinal. Of course, this requires $\mathbf{b} = \mathbf{d}$, which is ZFC independent: it is implied by MA [40, 5.1] but fails in the original Cohen model [40, 5.2]. It is equivalent to the existence of a *scale*, a $<^*$-cofinal subset of ${}^{\omega}\omega$ which is $<^*$-well-ordered [40, 3.5]. Hechler [14, 15] used "scale" to mean any $<^*$-cofinal family, but this usage is out of favor.

Notation. We write $\mathcal{B} \perp \mathcal{D}$ to indicate that every member of $\mathcal{B}$ is almost disjoint from every member of $\mathcal{D}$.

The proof of the following theorem is virtually identical to that for Theorem 1.8 [40, 3.3].

Theorem 1.10. $\mathbf{b}$ *is the least cardinal* λ *for which there are families* $\mathcal{B}$ *and* $\mathcal{D}$ *of subsets of a countable set, with* $\mathcal{B}$ *countable and* $|\mathcal{D}| = \lambda$ *and* $\mathcal{B} \perp \mathcal{D}$, *such that if* $B \subset^* C$ *for all* $B \in \mathcal{B}$, *then* $C \cap D$ *is infinite for some* $D \in \mathcal{D}$.

It makes no difference if we confine our attention to the case where $\mathcal{D}$ and $\mathcal{B}$ are ADF's [*ibid.*].

The following lemma is similar in spirit to Theorems 1.7 and 1.8, but its proof is even more straightforward [40, proof of 6.2].

Lemma 1.11. $\mathbf{p}$ *is the least cardinal* κ *for which there is a countable space of character* κ *that is not Fréchet.*

Theorem 1.12 [3, comment following Question 26]. *If* $\mathbf{p} > \omega_1$, *there is a countable* v*-space which is also a topological group, but is not metrizable.*

Proof. In a product of ω_1 two-element groups, take a countable dense subgroup. The character is ω_1, so it is not first countable, but it is α_1 because of Theorem 1.8 and $\mathbf{p} \leq \mathbf{b}$, and is Fréchet by Lemma 1.11. $\square$

2. Column and graph examples

This section is concerned with a Ψ-like space constructed using the graphs of functions and the columns of $\omega \times \omega$ which always gives a countable w-space that is not first countable. In some models of ZFC, cases of it are v-spaces, others are not even v'-spaces. It and the similar Example 3.1 have been studied before, but for different reasons [40, 11.6 and 12.2].

Example 2.1. Let $\mathscr{F} = \{f_\alpha : \alpha < \mathbf{b}\}$ be a $<^*$-well-ordered, $<^*$-unbounded family of nondecreasing functions from ω to ω which we will identify with their graphs. We let X be the Ψ-like space that results from letting $\omega \times \omega$ be the set of isolated points and using the almost disjoint family $\mathscr{C} \cup \mathscr{F}$ where $\mathscr{C}$ is the set of all columns $C_n = \{n\} \times \omega$. In other words, to the product space $\omega \times (\omega + 1)$ we are adding points p_α ($\alpha < \mathbf{b}$) which we attach to the graphs f_α as their one-point compactification, in the manner outlined after Definition 1.5. Then the one-point compactification $X + \infty$ of X is obviously not first countable.

Theorem 2.2. $X + \infty$ *is a* w*-space.*

Proof. As with all Ψ-like spaces, it is enough to verify that $D \cup \{\infty\}$ (in this case, $(\omega \times \omega) \cup \{\infty\}$) is a w-space: see the proof of Theorem 1.7. Let ∞ be in the closure of $S \subset \omega \times \omega$. To show the Fréchet property, we will find an α such that $S \cap f_\alpha^{\downarrow\downarrow} = S'$ has ∞ in the closure. Then the only f_β which trace an infinite set on S' are those *fewer* than $\mathbf{b}$ ($\leq \mathbf{a}$) graphs which precede f_α, so that we can argue as in Theorem 1.7 to find an infinite subset of S' that converges to ∞.

Since the columns each converge to something other than ∞, S must meet infinitely many columns. By the comment following the proof of Theorem 1.8, $\{f_\alpha : \alpha < \mathbf{b}\}$ is $<^*$-unbounded on every infinite subset of ω, so there is some f_{α_1} such that $S \cap f_{\alpha_1}^{\downarrow\downarrow}$ is infinite. If this set does not have ∞ in its closure, then its closure in X is compact, which means that there is a *finite* set F_1 of ordinals $< \alpha_1$ such that $S \cap f_{\alpha_1}^{\downarrow\downarrow} \setminus \bigcup \{f_\beta : \beta \in F_1\}$ is finite. In general suppose $f_{\alpha_{n-1}}$ has been defined and $S \cap f_{\alpha_{n-1}}^{\downarrow\downarrow}$ does not have ∞ in its closure. Let α_n be such that $(S \cap f_{\alpha_n}^{\downarrow\downarrow}) \setminus f_{\alpha_{n-1}}^{\downarrow\downarrow}$ is infinite. If this set does not have ∞ in its closure, there is a finite set F_n of ordinals in the interval $[\alpha_{n-1}, \alpha_n)$ such that all but finitely many points of this set are in the union of the graphs indexed by F_n.

If this process must continue for ω steps, let $\alpha = \sup_n \alpha_n$. Now $S \cap f_\alpha^{\downarrow\downarrow}$ contains all but finitely many points of each $S \cap f_{\alpha_n}^{\downarrow\downarrow}$, so its closure in X is noncompact, hence it has ∞ in its closure, as desired.

To show that $(\omega \times \omega) \cup \{\infty\}$ is α_2, note that every sequence converging to ∞ from $\omega \times \omega$ must meet each column in a finite set. So if $\{\sigma_n : n \in \omega\}$ is a sheaf at ∞, we can find for each n an α_n such that $(\operatorname{ran} \sigma_n) \cap f_{\alpha_n}^{\downarrow\downarrow} = B_n$ is infinite. Let $\alpha = \sup_n \alpha_n$. By Theorem 1.10, there is a subset C of $f_\alpha^{\downarrow\downarrow}$ such that $B_n \subset^* C$ for all n, and $C \cap f_\beta$ is finite for all $\beta < \alpha$, hence for all β, and of course C meets each C_n in a finite set, so C converges to ∞. □

When is $X + \infty$ a v-space? Part of the answer is:

Theorem 2.3. *Let $\{f_\alpha : \alpha < \mathbf{b}\}$ be a scale, and define X as in Example* 2.1 [*the f_α do not have to be nondecreasing*]. *Then $X + \infty$ is a v-space.*

Proof. To show α_1, argue as for α_2 above, but choose α_n so that $\operatorname{ran} \sigma_n \subset^* f_{\alpha_n}^{\downarrow\downarrow}$. Then $\operatorname{ran} \sigma_n \subset^* C$ for each n. The proof of Fréchet is as before. □

Corollary 2.4. *If either $\omega_1 < \mathbf{b}$ or $\mathbf{b} = \mathbf{d}$, there is a countable v-space that is not first countable, and has a compactification that is the one-point compactification of a Ψ-like space and is also a v-space.*

On the other hand, in the Dow–Steprāns model, Example 2.1 is never a v-space. At the opposite extreme are models [5, 6, 18] where every $<^*$-unbounded $<^*$-well-ordered family of nondecreasing functions is a scale, so there Example 2.1 is always a v-space. The following construction shows that the mere existence of a scale is not enough to guarantee that it is a v-space or even a v'-space. This construction is the most complicated in the paper and will be used one more time, to establish the existence, under the given hypotheses, of Fréchet w-groups that are not v'-spaces.

Definition 2.5. A *descending complete tower* (which we will call simply a *tower*) on an infinite set D is a family $\{A_\alpha : \alpha < \tau\}$ of infinite subsets of D such that $A_\beta \mathbin{{}^*{\supset}} A_\alpha$ whenever $\beta < \alpha$, but if no infinite subset C of D can satisfy $C \subset^* A_\alpha$ for all $\alpha < \tau$. The least cardinality of a tower on a countably infinite set is denoted $\mathbf{t}$.

A standard result is that $\omega_1 \leqslant \mathbf{p} \leqslant \mathbf{t} \leqslant \mathbf{b} \leqslant \mathbf{d} \leqslant \mathbf{c}$. A diagonal argument shows the first inequality, and the others are trivial except $\mathbf{t} \leqslant \mathbf{b}$, whose proof may be found in [40] along with much information on all these cardinal numbers.

At this point, readers may skip to either Theorem 2.8, Section 3, or Section 5 without loss of continuity.

Example 2.6. (a) [$\mathbf{t}=\mathbf{d}$]. A version of Example 2.1 in which $(\omega\times\omega)\cup\{\infty\}$ is not α_1, and (b) [$\mathbf{t}=\mathbf{c}$] a version in which it is not $\alpha_{1.5}$.

Construction. For $k\in\omega$, let g_k be the function sending n to $(n+2)^{k+2}$. For the $\mathbf{t}=\mathbf{c}$ version, let $\langle Z_\alpha\colon \alpha<\mathbf{c}\rangle$ list all subsets Z of $\omega\times\omega$ such that $g_k\subset^* Z$ for infinitely many k. For the $\mathbf{t}=\mathbf{d}$ version, let $\langle Z_\alpha\colon \alpha<\mathbf{d}\rangle$ be a family of subsets of $\omega\times\omega$ such that $g_k\subset^* Z_\alpha$ for *all* k and, whenever $Z\subset\omega\times\omega$ is such that $g_k\subset^* Z$ for all k, we have $Z_\alpha\subset^* Z$ for some α. One of the basic characteristics of $\mathbf{d}$ is that such a family of Z_α exists for any countable collection of infinite subsets of a set, in this case the g_k; as usual, the proof is by RH transfer, similarly to the proofs of Theorems 1.8 and 1.10 [40, Theorem 3.3]. If $\mathbf{b}=\mathbf{d}$, it even is possible to choose $Z_\alpha\subset^* Z_\beta$ whenever $\alpha>\beta$, although this is not required for this example.

For either version, enlarge the set of g_k to a scale $\langle g_\alpha\colon \alpha<\mathbf{b}\rangle$ of nondecreasing functions. We will choose f_α so that it meets Z_α in an infinite set, is above the graph of each earlier f_β almost everywhere, is above the graph of g_α infinitely often, and is almost disjoint from the graph of each g_k. Once this is done for all $\alpha\leqslant\mathbf{b}$, the graphs of the g_k will converge to ∞, but no set Z satisfying $g_k\subset^* Z$ for all k can converge to ∞, because Z "almost" contains some Z_α and hence meets f_α in an infinite set. Of course in version (b), no set Z satisfying $g_k\subset^* Z$ for infinitely many k can converge to ∞.

If $\alpha=\beta+1$ and f_β has been defined, the construction of f_α will be done one coordinate at a time, setting ourselves ω tasks. An odd-numbered $(2k+1)$ task will be to get below g_0-k while increasing, staying above f_β, and avoiding all the g_j such that $j<k$. An even-numbered task $(2k+2)$ is to hit g_k in an element of Z_α while increasing, staying above f_β, and avoiding all the g_j such that $j<k$; and then on the next coordinate, to jump up above g_α. The point of the odd-numbered tasks is to be able to carry out the first part of each even-numbered task for later f_γ, in particular $f_{\alpha+1}$. As part of our induction hypothesis, we therefore assume f_β also got below g_0-k for each $k\in\omega$.

While performing task $2k+1$, we increase by at most two in going from one coordinate to the next, unless that causes us to go under or to coincide with f_β, in which case we go above f_β by at most two. In either case, we go up by only one unless this causes us to hit some g_j $(j<k)$, in which case we go up by two. [Of course, the g_j are spread far enough apart on each coordinate!] The gap between f_α and f_β can never grow by more than k since the beginning of the task, so eventually f_α gets below g_0 by at least k units, ending the task.

On task $2k+2$, we increase f_α exactly as on task $2k+1$, until we get to an i where $\langle i, g_k(i)\rangle\in Z_\alpha$ and $g_k(i)$ is above $f_\alpha(i-1)$ and also $f_\beta(i)$. Since all but finitely many points of (the graph of) g_k are in Z_α, and f_β gets below g_0 infinitely often while f_α never gets more than k higher above f_β than at the beginning of the task while the task is underway, this eventually does happen, and then we let $f_\alpha(i)=g_k(i)$, and then make $f_\alpha(i+1)>g_\alpha(i+1)$, ending the task.

If α is a limit ordinal, we first construct an auxiliary function h_α. Let A_α^k denote the set of coordinates on which f_β is below g_0 by k or more. For a fixed k, these form a $^*\supset$-well-ordered sequence and, using the fact that $\alpha<\mathbf{t}$, we take an infinite $A_\alpha^k\subset^* A_\beta^k$ for all $\beta<\alpha$; then we take an infinite $A_\alpha\subset^* A_\alpha^k$ for all k.

Let $\psi_\alpha(i)$ equal $g_0(i)$ whenever $i\in A_\alpha$, and be defined backwards on the line of slope 1 through $\langle i, g_0(i)\rangle$ from this point to the previous $i'\in A_\alpha$. Then $f_\beta <^* \psi_\alpha - k$ for all $k\in\omega$: it is enough to see this on the coordinates in $A_\alpha\cap A_\beta^{k+1}$, where it is obvious. By Theorem 1.10, there is a set $B\subset\omega\times\omega$ almost disjoint from all the $(\psi_\alpha-k)^\uparrow$ and almost containing each $f_\beta^\uparrow$. For each i, let $\langle i, h'_\alpha(i)\rangle$ be the least point of $\{i\}\times\omega$ not in B. The h'_α thereby defined satisfies $f_\beta <^* h'_\alpha <^* \psi_\alpha - k$ for all k, as does the least increasing function $h_\alpha\geq h'_\alpha$. [This is defined by induction, thus: $h_\alpha(0)=h'_\alpha(0)$, and $h_\alpha(i)=h'_\alpha(i)$ unless $h'_\alpha(i)\leq h_\alpha(i-1)$, in which case we let $h_\alpha(i)=h_\alpha(i-1)+1$.] Indeed, since h'_α is infinitely often above any given g_k, we have h_α coinciding with h'_α infinitely often. If N is one of these coordinates, and $\psi_\alpha(i)-k> h'_\alpha(i)$ for all $i\geq N$, then also "$\psi_\alpha(i)-k>h_\alpha(i)$ for all $i\geq N$" is true, because $\psi_\alpha(i)-k$ is increasing.

Now we proceed as in the case $\alpha=\beta+1$, using h_α in place of f_β.

Example 2.7. A simplified version of the above constructions gives a version under $\mathbf{t}=\mathbf{c}$ of Example 2.1 which is a v-space even though the f_α do not form a scale. Let $\{Z_\alpha\colon \alpha<\mathbf{c}\}$ list all infinite subsets of $\omega\times\omega$ which meet each column in at most finitely many points. Disregard all g_α except g_0. Otherwise odd-numbered tasks are as before, while on an even-numbered task we meet Z_α unless $Z_\alpha\subset^*(f_\beta+k)^\downarrow$ for some finite k if $\alpha=\beta+1$, with h_α replacing f_β if α is a limit ordinal. Every sequence converging to ∞ in the resulting space must be eventually below the graph of some f_α. Details are left to the reader.

I do not know whether it is consistent for there to be a version of Example 2.1 which is a v'-space without being a v-space. If not, then the $\mathbf{t}=\mathbf{c}$ construction in Example 2.6 becomes redundant.

Example 2.1 has another interesting property which is convenient to mention here:

Theorem 2.8. *Let X be as in Example 2.1 or the proof of Theorem 2.3. Then every pseudocompact subspace of $X+\infty$ is compact, yet X is not realcompact.*

Proof. In a compact scattered Fréchet space, every pseudocompact subspace is compact [44]. On the other hand, X is not wD: that is, it has a countable closed discrete subspace D (the nonisolated points of $\omega\times(\omega+1)$) such that, given any infinite $E\subset D$ and any family $\{U_e\colon e\in E\}$ of open sets such that $U_e\cap E=\{e\}$, the family must fail to be discrete. But every realcompact space satisfies wD [42]. $\square$

Thus $X+\infty$ provides a new answer to a question of E.K. van Douwen (Topology Proc. 8 (1983) 395): Does there exist a compact space Y such that every pseudocompact subspace is compact, yet Y is not hereditarily realcompact? The first "real" solution was $T^{+}+\infty$, where T^{+} is a Moore version of the space of positive tangent vectors over the long line [29]. A difference is that T^{+} is wD; in fact, it is pseudonormal, meaning that disjoint closed sets, one of which is countable, can be put into disjoint open sets.

3. Examples with graphs and partial graphs

A natural idea for modifying Example 2.1 to avoid getting a v'-space is to "leave the columns open", as in the next example. By not definining the functions for all integers, we even get an example which is "universal" in the sense of Theorem 3.9 below. We do not know whether this cutting down of domains is really needed for this (Remark 3.11).

Example 3.1. Let $\langle f_\alpha: \alpha<\mathbf{b}\rangle$ be a $<^*$-unbounded, $<^*$-well-ordered family of nondecreasing functions from ω to ω. Let X be the Ψ-like space for which $\omega\times\omega$ is the set of isolated points and the ADF is $\{f_\alpha: \alpha<\mathbf{b}\}$. Then $X+\infty$ is Fréchet, but not $\alpha_{1.5}$. Indeed, the columns converge to ∞, but a set which meets infinitely many columns in a cofinite set must also meet (the graph of) some f_α in an infinite set. The proof that $X+\infty$ is Fréchet is the same as for Example 2.1, except that a set can meet only finitely many columns in an infinite set and still have ∞ in its closure—but any infinite set that meets only finitely many columns will automatically converge to ∞.

A generalization is:

Example 3.2. Let X be defined as in Example 3.1, except that the domain of each f_α is merely required to be an infinite subset of ω. We call such functions "partial functions" and to each partial function f we associate the function $g:\omega\to\omega$ such that $g(n)=f(m)$ where $m=\min\{k\geqslant n: f(k) \text{ is defined}\}$. We then define $f^{\downarrow\downarrow}$ to be $g^{\downarrow\downarrow}$ and similarly for $f^{\downarrow}$ and $f^{\uparrow}$. Note that if f is nondecreasing, so is g. We also define $\langle f_\alpha: \alpha<\tau\rangle$ to be $<^*$-unbounded iff the family of associated functions from ω to ω is $<^*$-unbounded. The proof that $X+\infty$ is Fréchet is as before, and it is not α_1 because no set meeting all columns in a cofinite set can converge to ∞.

At the end of this section we construct, assuming the axiom $\mathbf{p}=\mathbf{c}$, a version of Example 3.2 which is a v'-space but not a v-space. Of course, some of the functions will have to be partial functions. Except for this example, we will only be concerned with the question of when Example 3.2 can be made α_2. This reduces to the question of whether, given a family of disjoint infinite sets, for each subset of some column

$\{k\}\times\omega$, there is a set converging to ∞ which meets each member of the family. Indeed, given a sheaf converging to ∞, those sequences whose ranges meet infinitely many columns can be handled together like the σ_n in the proof that Example 2.1 is α_2; while those sequences which meet only finitely many columns can first have their ranges cut down to inside a single column, and then replaced by subsequences with disjoint ranges, using Lemma 1.2.

Example 3.2 cannot be made α_2 in the Laver model, because it is not α_1. One hypothesis under which Example 3.1, hence 3.2, can be made α_2 is the existence of a tower [recall Definition 2.5] of cofinality **b**. This hypothesis can be weakened further, using the following concepts.

Definition 3.3. A collection $\mathcal{J}$ of subsets of a set X is called an *ideal* on X if it is closed under the taking of subsets and of finite unions. An ideal $\mathcal{J}$ is called a *P-ideal* if whenever $\mathscr{C}$ is a countable subcollection of $\mathcal{J}$, there exists $A\in\mathcal{J}$ such that $A\subset^* C$ for all $C\in\mathscr{C}$.

The dual concept of a [P-]ideal is a [P-]filter, i.e., $\mathcal{J}$ is a [P-]ideal iff its dual $\{X\backslash A\colon A\in\mathcal{J}\}$ is a [P-]filter. A well-known kind of P-filter is a P-point in $\omega^*=\beta\omega-\omega$. But there are examples not requiring special axioms beyond AC, including ones whose duals satisfy:

Definition 3.4. Let κ be a cardinal number. A *κ-minimax ideal* is an ideal $\mathcal{J}$ on ω with κ generators, such that:

$$\text{if } A\subset^* B \text{ for all } A\in\mathcal{J}, \quad \text{then } \omega\backslash B \text{ is finite,} \tag{$*$}$$

and such that no subideal with fewer than κ generators satisfies $(*)$. An ideal is called *minimax* if it is κ-minimax for some (obviously unique) κ.

The number **p** can be characterized as the least number of generators for an ideal $\mathcal{J}$ satisfying $(*)$ and containing no cofinite subsets of ω. It can also, clearly, be characterized as the least *infinite* κ for which there is a κ-minimax ideal.

An example of a minimax P-ideal is the dual of a filter whose base is a tower, so that **t**-minimax P-ideals exist. In Section 4 and [33] we give other constructions.

The following concept generalizes that of a P-ideal satisfying $(*)$.

Definition 3.5. A family $\mathscr{S}$ of subsets of a countable set X is *ω-hitting* [respectively *ω-splitting*] if, for every countable collection $\langle B_n\rangle_{n\in\omega}$ of infinite subsets of X, there is a member S of $\mathscr{S}$ that hits [respectively splits] them all, i.e., $B_n\cap S$ is infinite for each n [respectively and so is $B_n\backslash S$].

By Lemma 1.2, one obtains an equivalent definition if $\langle B_n\rangle$ is assumed to be disjoint. Parts (b) and (c) of the following lemma now follow easily. I am indebted to the referee for observing (c).

Lemma 3.6. (a) *Every ω-splitting family is ω-hitting.*

(b) *Every ω-hitting ideal is ω-splitting.*

(c) *The dual ideal of an ultrafilter is ω-hitting (hence ω-splitting).*

(d) *Every P-ideal satisfying* $(*)$ *of Definition* 3.4 *is ω-hitting (hence ω-splitting).*

Proof. (d) Let $\mathcal{J}$ be the ideal and for each B_n as in Definition 3.5, let $J_n \in \mathcal{J}$ be such that $J_n \cap B_n$ is infinite. Let $J \in \mathcal{J}$ satisfy $J_n \subset^* J$ for all n. Then J hits every B_n. □

Now we come to the main result of this section.

Theorem 3.7. *Each of the following statements implies the ones after it.*

(a) $\mathbf{b} < \mathbf{d}$.

(b) *There is a $<^*$-unbounded, $<^*$-well-ordered family of nondecreasing functions that is not $<^*$-cofinal in ${}^{\omega}\omega$.*

(c) *There is a tower of cofinality* $\mathbf{b}$.

(d) *There is a* $\mathbf{b}$*-minimax P-ideal.*

(e) *There is a* $\mathbf{b}$*-minimax ω-hitting ideal.*

(f) *There is a w-space that is not a v'-space.*

(g) *There is a w-space that is not a v-space.*

Proof. (a)→(b), (c)→(d)→(e) and (f)→(g) are all either obvious or have been established above.

(b)→(c): Let $(f_\alpha: \alpha < \lambda)$ be as in (b). If $\mathbf{b} < \mathbf{d}$, also let $\lambda = \mathbf{b}$. Let g be an increasing function which is not dominated by any of the f_α; then the sets $B_\alpha = \{n: g(n) > f_\alpha(n)\}$ are easily seen to form a tower. If $\mathbf{b} = \mathbf{d}$, then the cofinality of λ necessarily equals $\mathbf{b}$, because otherwise we could use a scale and a cofinality argument to show $\{f_\alpha: \alpha < \lambda\}$ is bounded. So again we can let $\lambda = \mathbf{b}$ and argue as before.

Finally, we show (e)→(f). We will construct a version of Example 3.1, which is not a v'-space, using (e) to make the space α_2.

Let $\{A_\alpha: \alpha < \mathbf{b}\}$ be a cofinal subset of a $\mathbf{b}$-minimax ω-hitting ideal, and let $\{g_\alpha: \alpha < \mathbf{b}\}$ be $<^*$-unbounded. For each $\alpha < \mathbf{b}$ let $f_\alpha: \omega \to \omega$ be an increasing function such that $g_\alpha <^* f_\alpha$ and $f_\xi <^* f_\alpha$ whenever $\xi < \alpha$, and such that (the graph of) f_α is almost disjoint from all $\omega \times A_\beta$, $\beta < \alpha$; this last feature can be insured by using minimaxity of $\{A_\alpha: \alpha < \mathbf{b}\}$. Our example is Example 3.1 with this choice of f_α.

We have already seen why $X + \infty$ is Fréchet and not $\alpha_{1.5}$. To show α_2, it is enough, by the reduction result after the description of Example 3.2, to take care of every sheaf at ∞ such that the range B_n of each member is a subset of some column $\{k_n\} \times \omega$, with $B_n \cap B_m = \emptyset$ whenever $n \neq m$, although we do allow $k_n = k_m$. Let A_α hit every $\pi_2 B_n$ (in an infinite set). Then $C = (\omega \times A_\alpha) \backslash f_\alpha^{\downarrow}$ meets each B_n in an infinite set and is almost disjoint from every f_β, hence converges to ∞. □

The set-theoretic hypothesis in (a) is already enough, by (a)→(g), to give us a nice "complement" to Corollary 2.4. That in (b) is very weak: besides the Laver model we have only two other published models where it fails [5, 6]. It holds, for example, if $\mathbf{t}=\mathbf{b}$: if $\mathbf{b}<\mathbf{d}$ this is obvious, while if $\mathbf{b}=\mathbf{d}$ we use Theorem 2.3 and Example 2.6.

Problem 3.8. In Theorem 3.7, which of (b)→(c)→(d)→(e)→(f)→(g) can be reversed?

We will show below that (e)-(g) are equivalent if $\mathbf{b}=\mathbf{c}$. The proof will hinge in part on the following theorem.

Theorem 3.9. *If there is an α_2-space that is not α_1, then there is one which is a version of Example* 3.2.

Proof. If $\mathbf{b}<\mathbf{d}$ this follows from the proof of Theorem 3.7. So we may assume the existence of a scale $\{f_\alpha\colon \alpha<\mathbf{b}\}$.

Let X be a space with an α_2-point x that is not α_1, let γ be a sheaf at x witnessing this, and let $\{B_n\}_{n=0}^{\infty}$ be the set of ranges. Then with $\{A_n\}_{n=0}^{\infty}$ as in variation 2 of the RH transfer, any sheaf with the A_n as its set of ranges still witnesses α_2 but not α_1. Let ψ be a transfer function and let $\mathcal{F}$ be the filter on $\omega\times\omega$ which is the image of the set of traces of the neighborhoods of x on $\bigcup_{n=0}^{\infty} B_n$. Of course, each $F\in\mathcal{F}$ meets each column in a cofinite set and hence its complement is in f_α for some $\alpha<\mathbf{b}$. Also, the image of each A_n is a column which converges to ∞.

Let $g_0=f_0=h_0$. If an increasing function $g_\beta:\omega\to\omega$ has been defined for each $\beta\leq\alpha$, let $A=g_\alpha^{\downarrow}$. By the way α_1 fails for X, there is a member F of $\mathcal{F}$ that *excludes* infinitely many elements of $(\omega\times\omega)\backslash A$, and only finitely many of these are in any one column, so there is a "partial function" h_α whose domain is an infinite subset of ω and whose graph consists of such excluded elements. Since g_α is increasing, we take h_α to be increasing (by cutting down its domain, if necessary). Let $g_{\alpha+1}$ be an increasing function that is everywhere greater than h_α and f_α. If α is a limit ordinal $<\mathbf{b}$, let g_α be an increasing function $<^*$-dominating all the earlier g_β.

We let our ADF be $\{h_\alpha\colon \alpha<\mathbf{b}\}$. The h_α form a "dominating family" in the sense that if $f\in{}^{\omega}\omega$, there is some h_α such that $f(n)<h_\alpha(n)$ for all but finitely many $n\in\operatorname{dom} h_\alpha$. By Example 3.2, the space is Fréchet and not α_1, and it is α_2 because the "reduced question" following Example 3.2 has an affirmative answer: the columns already converge to ∞ in the RH transfer of the original topology, which is α_2. $\square$

I am indebted to Alan Dow for the first paragraph of the following proof.

Theorem 3.10. *If* $\mathbf{b}=\mathbf{c}$, *the following are equivalent.*

(i) *There is a w-space that is not a v-space.*

(ii) *There is a* $\mathbf{b}$*-minimax* ω*-hitting ideal.*

Proof. Because of Theorem 3.7, it is enough to show (i)$\rightarrow$(ii). By Theorem 3.9, there is a w-space which is not a v-space and is a case of Example 3.2. Let Y be the subspace $(\omega \times \omega) \cup \{\infty\}$ and let $\langle f_\alpha : \alpha < \mathbf{b}\ (=\mathbf{c})\rangle$ be used to define the neighborhoods of ∞ as in Example 3.2. The following family of subsets of $\omega \times \omega$ is then ω-splitting:

$$\mathcal{S} = \{S \subset \omega \times \omega : S = T \cup W \text{ where } W \subset^* U \text{ for every neighborhood } U \text{ of } \infty \text{ and there exists } \alpha < \mathbf{b} \text{ such that } T \subset^* f_\alpha^\downarrow\}.$$

Indeed, if $\langle A_n \rangle_n$ is a family of infinite subsets of $\omega \times \omega$, then for each n, A_n either meets some column $\{k\} \times \omega$ in an infinite set B_n, or selse A_n meets some $f_\alpha^\downarrow$ in an infinite set C_n. Using Lemma 1.2, let $D_n \subset C_n$ be an infinite set such that $D_n \cap D_m = \emptyset$ if $n \neq m$. Let $D \subset \bigcup_n D_n$ split each D_n, hence each C_n. Using the α_2-property of Y, let B be the range of a sequence $\sigma \rightarrow \infty$ such that $B \cap B_n$ is infinite for all n, and using Lemma 1.2 again, pick $B' \subset B$ such that B' splits each B_n. Then $D \cup B'$ splits every A_n.

Of course, the ideal $\mathcal{J}$ generated by $\mathcal{S}$ is ω-hitting. Because $\mathbf{b} = \mathbf{c}$, it is generated by $\mathbf{b}$ sets. For each $S \in \mathcal{S}$, there exists an α such that the graph of f_γ is almost disjoint from S for all $\gamma > \alpha$. The same is true of each $J \in \mathcal{J}$, since each is contained in a finite union of members of $\mathcal{S}$. Thus if $|\mathcal{J}'| < \mathbf{b}$ and $\mathcal{J}' \subset \mathcal{J}$, then there is an f_γ whose graph is almost disjoint from each member of $\mathcal{J}'$. The complement of the graph is then a co-infinite subset B of $\omega \times \omega$ such that $J \subset^* B$ for all $J \in \mathcal{J}'$, so $\mathcal{J}'$ does not satisfy $(*)$ in Definition 3.4, and $\mathcal{J}$ is $\mathbf{b}$-minimax. □

Remark 3.11. Although we used a version of Example 3.2 to prove (i) implies (ii), we actually obtain a version of Example 3.1 if we now run through the proof of Theorem 3.7. This raises the question of whether there is a version of Example 3.1 which is a w-space in every model where there is a w-space that is not a v-space.

Remark 3.12. At the opposite extreme from the versions considered so far, Example 3.1 can be rigged so that it is not even α_3 if one assumes $\mathbf{b} = \mathbf{c}$: let $\{A_\alpha : \alpha < \mathbf{c}\}$ list all subsets of $\omega \times \omega$ that meet infinitely many columns in an infinite set, and have f_α meet A_α in an infinite set. On the other hand, Example 3.2 is always α_4, as is any compact Fréchet space [2]; a direct proof can be given for Example 3.2 similar to the proof of the Fréchet property.

Remark 3.13. Since Martin's axiom implies $\mathbf{p} = \mathbf{t} = \mathbf{b} = \mathbf{d} = \mathbf{c}$, it implies the existence of a w-space that is not a v'-space (Examples 2.6 or Theorem 3.7), a v'-space that is not a v-space (see the next example), a version of Example 3.1 that is not α_3, and also (Theorem 2.2) a countable v-space that is not first countable, all of which are subspaces of one-point compactifications of Ψ-like spaces.

We close this section with a construction, assuming the axiom $\mathbf{p} = \mathbf{c}$, of a version of Example 3.2 that is a v'-space but is not a v-space. This example will not be used later, except to underscore the significance of Theorem 5.3.

The cardinal $\mathbf{p}$ can be characterized as the fewest number of clopen subsets of ω^* $(=\beta\omega-\omega)$ with the finite intersection property having intersection with empty interior; or, taking complements, the smallest number of clopen sets whose union is dense without actually covering ω^*. We use $\omega\times\omega$ in place of ω and transfer to it the standard notation $A^*=\mathrm{cl}_{\beta(\omega\times\omega)}A-(\omega\times\omega)$. Recall that every nonempty clopen subset of $(\omega\times\omega)^*$ is of this form [43, p. 74].

Example 3.14 [$\mathbf{p}=\mathbf{c}$]. We will use the remainder $(\omega\times\omega)^*$. Let $\langle g_\alpha\colon \alpha<\mathbf{c}\rangle$ be a scale, with each g_α nondecreasing and unbounded, and let $C_\alpha=(g_\alpha^\downarrow)^*$ for each α; then the C_α form an ascending sequence of clopen sets, disjoint from each $B_n=(\{n\}\times\omega)^*$. [They also fill up the interior of $(\bigcup_{n=0}^\infty B_n)^{\mathrm{c}}$, but this is not needed here.] Let $\langle\mathcal{D}_\alpha\colon \alpha<\mathbf{c}\rangle$ list all families of infinitely many disjoint clopen sets, with $\mathcal{D}_0=\{B_n\}_{n=0}^\infty=\mathcal{D}_1$. These will be the candidates for remainders of ranges of sheaf members.

Let $V_0=\bigcup_{n=0}^\infty B_n$. Let $h_0=g_0$. Assume V_β and h_β have been defined for all $\beta<\alpha$ $(<\mathbf{c})$ with each V_β, $\beta>0$, a clopen set and no finite collection of V_β covering any set of the form $(C_\gamma)^{\mathrm{c}}$ [all complements are taken in $(\omega\times\omega)^*$]. Then by $\mathbf{p}=\mathbf{c}$, $\bigcup_{\beta<\alpha}V_\beta$ fails to be dense in any C_γ^{c}. Let $\delta\geq\alpha$ be such that $h_\beta<^* g_\delta$ for all $\beta<\alpha$. If $\bigcup\mathcal{D}_\alpha$ covers any (compact) set of the form $(\bigcup_{\beta<\alpha}V_\beta\cup C_\gamma)^{\mathrm{c}}$, then some finite subset covers it, and we let V_α be the complement of its union. Otherwise, let $\mathcal{D}_\alpha^0$ be an infinite, co-infinite subset of $\mathcal{D}_\alpha$ and let V_α^0 and V_α^1 be disjoint clopen sets containing $\bigcup\mathcal{D}_\alpha^0$ and $\bigcup(\mathcal{D}_\alpha-\mathcal{D}_\alpha^0)$, respectively [43, p. 64]. Obviously, at most one of these clopen sets can contain the interior of $(\bigcup_{\beta<\alpha}V_\beta)\cup C_\gamma$ for some γ, and we let V_α be one that does not. In either case, there is a nonempty clopen set H_α in the complement of $\bigcup_{\beta<\alpha}V_\beta\cup C_\delta$. Pick $A\subset\omega\times\omega$ such that $H_\alpha=A^*$. Now A is almost disjoint from every column and from $g_\delta^\downarrow$, so there is the graph of an increasing function h_α with infinite domain such that $h_\alpha\subset A$, and so $g_\delta<^* h_\alpha$.

Let X be the Ψ-like space whose ADF is $\langle h_\alpha\colon \alpha<\mathbf{c}\rangle$. The proof that $X+\infty$ is Fréchet and not α_1 is just as in Theorem 3.10. To see that it is $\alpha_{1.5}$, let $\langle A_n\rangle_{n=0}^\infty$ be a family of disjoint infinite subsets of $\omega\times\omega$, each converging to ∞. Then this family is $\mathcal{D}_\alpha$ for some $\alpha<\mathbf{c}$, and $\bigcup\{A_n^*\colon n\in S\}\subset V_\alpha$ for some infinite $S\subset\omega$. Let A be such that $A^*=V_\alpha$; then $A_n\subset^* A$ for all $n\in S$. If $\gamma\geq\alpha$, h_γ is almost disjoint from A. Let $B=A-g_\delta^\downarrow$, where $h_\beta<^* g_\delta$ for all $\beta<\alpha$. Then B clearly converges to ∞. By Theorem 1.10, there exists $C\leq A\cap g_\delta^\downarrow$ such that $A_n\cap g_\delta^\downarrow\subset^* C$ for all $n\in S$ while C is almost disjoint from all the h_β, $\beta<\alpha$. Then C converges to ∞ and so does $B\cup C$, and we have $A_n\subset^* B\cup C$ for all $n\in S$.

Example 3.14, together with the fact that v'-spaces coincide with v-spaces in the Laver model, allow us to answer Question 5.22.4 of [2] completely. That question can be interpreted as asking whether the classes of compact Fréchet α_1-, $\alpha_{1.5}$-, α_3-, and α_4-spaces coincide with each other. The answer is that they are all distinct except for α_1 and $\alpha_{1.5}$, whose distinctness is ZFC independent: examples of compact α_3-spaces which are not α_2, hence not $\alpha_{1.5}$, and compact α_4-spaces which are not α_3 were obtained earlier [10], [11], [27], [32], respectively [39].

4. Properties of the Laver model

With the machinery built up in Section 3, we can quickly outline a proof of Dow, different from that in [7] and included here with his permission, that every w-space is a v-space in the Laver model, and also extend some results of [33] about this model.

Three properties of the Laver model, shown in [7], will be used. The first is that it is produced by iteratively adding, with countable supports, special functions $f: \omega \to \omega$ known as "Laver reals", which dominate all functions appearing in earlier stages of the iteration, which is altogether of length ω_2. The second property is:

> Every ω-splitting family in an initial or intermediate model $\mathcal{M}_\alpha$ remains ω-splitting in each later model of the iteration. (**)

The third property is mentioned in the course of the following proof.

Theorem 4.1. *In the Laver model, every α_2-space is α_1.*

Proof. Recall that *if* there is an α_2-space that is not α_1, there is one of the form as in Example 3.2. These spaces *cannot* be α_1, as explained in Example 3.2. So we will show they cannot be α_2, either, in the Laver model.

Let $\mathcal{S}$ be as in the proof of Theorem 3.10. A standard reflection argument [7, Lemma 7] shows $\mathcal{S} \cap \mathcal{M}_\beta$ is ω-splitting in $\mathcal{M}_\beta$ for some $\beta < \omega_2$.

Now we use (**): $\mathcal{S} \cap \mathcal{M}_\beta$ remains ω-splitting in the Laver model $\mathcal{M}_{\omega_2}$. However, since $\langle f_\alpha : \alpha < \mathbf{b} \rangle$ is $<^*$-unbounded, there is an f_α that is not dominated by the Laver real $h_{\beta+1}$ added in passing from $\mathcal{M}_\beta$ to $\mathcal{M}_{\beta+1}$. Let A be the set of points $\langle n, f_\alpha(n) \rangle$ in the graph of f_α where $f_\alpha(n) \geq h_{\beta+1}(n)$. Now A is almost disjoint from each $f_\gamma^\downarrow \in \mathcal{M}_\beta$, and the complement of A is a neighborhood of ∞, so that no member of $\mathcal{S} \cap \mathcal{M}_\beta$ splits A. This contradicts the allegation that X is an α_2-space. $\square$

The following concept was introduced in [33].

Definition 4.2. Let κ be an infinite cardinal. A point of $q \in \omega^*$ is a *pseudo-P_κ-point* if every intersection of fewer than κ (wlog clopen) neighborhoods of q has nonempty interior. [As usual, ω^* denotes the Stone–Čech remainder of ω.]

Note that q need not itself be in the interior: that would give the definition of a *P_κ-point.* A standard pair of facts about ω^* can be phrased: every point is a pseudo-P_{ω_1}-point [43, 3.27], but not every point is a P_{ω_1}-point [43, 4.31], that is, a P-point.

Elementary correspondences between infinite subsets of ω and clopen subsets of ω^* yield the fact that an ultrafilter q is a pseudo-P_κ-point iff whenever $\lambda < \kappa$ and $\{A_\alpha : \alpha < \lambda\} \subset q$, there is an infinite $A \subset \omega$ such that $A \subset^* A_\alpha$ for all $\alpha < \lambda$. From this it immediately follows that the dual ideal of a pseudo-P_κ-point with a base of cardinality κ is κ-minimax, and by Lemma 3.6(d) it is ω-splitting. Thus from Theorem 3.7 we obtain:

Lemma 4.3. *If there is a pseudo-$P_{\mathbf{b}}$-point q with a base of cardinality* $\mathbf{b}$, *then there is a w-space which is not a v-space.*

Corollary 4.4. *If* $\mathbf{b} = \mathbf{c}$, *and there is a pseudo-$P_{\mathbf{b}}$-point, then there is a w-space that is not a v-space.*

Proof. No free ultrafilter on ω has a base of cardinality $<\mathbf{b}$ [40], so the hypothesis of Lemma 4.3 is satisfied. □

Corollary 4.5. *In Laver's model, every free ultrafilter is a pseudo-P_{ω_1}-point but none is a pseudo-P_{ω_2}-point. In other words, every point of ω^* is in a nowhere dense set which is the intersection of a family of ω_1 clopen sets (but no fewer).*

The following theorem improves Corollary 3.8 of [33], where the P-sets involved were singletons.

Theorem 4.6. *In Laver's model, every nowhere dense P-set in ω^* is contained in a nowhere dense P-set which is the intersection of a chain of ω_1 clopen sets.*

Proof. Let N be a nowhere dense P-set. The ideal of all $A \subset \omega$ such that $A^* \cap N = \emptyset$ is a P-ideal satisfying $(*)$ in Definition 3.4. By Theorem 3.7 and Dow's theorem, it cannot be $\mathbf{b}$-minimax, so it must either have ω_1 generators or else have a subideal satisfying $(*)$ which has ω_1 generators. In either case, the generators correspond to a family $\{C_\alpha: \alpha < \omega_1\}$ of clopen subsets of ω^* whose intersection is a closed nowhere dense set M containing N. Now since N is a P-set, we can, for each $\alpha < \omega_1$, define clopen sets D_α by induction so that $C_\alpha \supset D_\alpha$ and $D_\beta \supset D_\alpha \supset N$ for all α. The D_α thus form a chain of clopen sets whose intersection is automatically a P-set, and it contains N and is contained in M, hence is nowhere dense. □

Problem 4.7. Can ω^* be covered by nowhere dense P-sets in the Laver model?

For information on other models, see [4; 35; 41, 1.9].

5. Topological groups

Given a filter $\mathscr{F}$, which we can assume without loss of generality to be the filter of neighborhoods of a point p in a space X, with all other points isolated, there is a very natural way of defining a topological group in which X can be embedded, with p sent to the identity element. We let $G = +\{\{0, 1\}_x: x \in X - \{p\}\}$, which we may as well identify with $[X-\{p\}]^{<\omega}$, each element identified with its set of nonzero coordinates. The group operation then becomes symmetric difference, with each element its own inverse and $\emptyset$ the identity. For the topology, we let the subgroups $[F-\{p\}]^{<\omega}=H_F$ be a base at $\emptyset$ and so their cosets will be a clopen base for the whole topology. Assuming the Hausdorff axiom on X means $\mathscr{F}|X-\{p\}$ is free and G is Hausdorff also. Now X can be embedded by taking p to $\emptyset$ and all other x to $\{x\}$.

It is natural to ask what properties of X carry over to $G=[X-\{p\}]^{<\omega}$. One is character (in particular, first countability or its lack). Another is the α_1-property.

Lemma 5.1. *An injective sequence $\langle a_n\colon n\in\omega\rangle$ converges to $\emptyset$ iff its range R is a point-finite collection of subsets of $X-\{p\}$ and $\bigcup R\subset^* F$ for all $F\in\mathscr{F}$.*

Proof. $\langle a_n\rangle$ converges to $\emptyset$ iff all but finitely many a_n are inside each H_F. This implies $\bigcup R\subset^* F$ for all $F\in\mathscr{F}$, and, since $\mathscr{F}$ is free, all but finitely many a_n must miss any given finite subset of X, and point-finiteness follows. The converse is just as easy. □

Theorem 5.2. *$[X-\{p\}]^{<\omega}$ is α_1 iff X is α_1.*

Proof. Since α_1 is a hereditary property, one implication is clear. Now suppose $\sigma_n=\langle a_i^n\colon i\in\omega\rangle$ converges to $\emptyset$ for each n, and let $A_n=\bigcup\operatorname{ran}\sigma_n$. Then A_n converges to p in X and if X is α_1, there exists A such that $A_n\subset^* A\subset^* F$ for each $n\in\omega$ and $F\in\mathscr{F}$. By point-finiteness of $\operatorname{ran}\sigma_n$, $a_i^n\subset A$ for all but finitely many i. It is now a routine matter to take a cofinite subset of each $\operatorname{ran}\sigma_n$ so that the chosen a_i^n are all in A and form a point-finite collection. □

I do not know whether the α_2-property carries over. It does in the case where $X=(\omega\times\omega)\cup\{\infty\}$ as in Example 2.1 (see Theorem 5.7 below).

The remaining α_i-properties do not carry over. Any X which is $\alpha_{1.5}$ but not α_1, and any X which is α_3 (hence α_4) but not α_2, will show this, in view of the following two theorems.

Theorem 5.3. *If $[X-\{p\}]^{<\omega}$ is $\alpha_{1.5}$, then it is α_1.*

Proof. By Theorem 5.2, it is enough to show X is α_1. Let $\langle\sigma_i\colon i\in\omega\rangle$ be a sheaf at p, with the ranges of the σ_i disjoint and each σ_i injective. Let τ_n be defined by $\tau_n(i)=\{\sigma_0(i),\dots,\sigma_n(i)\}$. Then each τ_n converges to $\emptyset$, and the ranges of the τ_n are disjoint. Let τ be a sequence whose range meets that of infinitely many τ_n in a cofinite set and converges to $\emptyset$. Then $\operatorname{ran}\sigma_i\subset^*\bigcup\operatorname{ran}\tau$ for all i, and $\bigcup\operatorname{ran}\tau$ converges to p by Lemma 5.1. □

Theorem 5.4. *If $[X-\{p\}]^{<\omega}$ is α_4, then it is α_2.*

Proof. Let σ_n be a sequence in $[X-\{p\}]^{<\omega}$ converging to $\emptyset$, and let $\tau_n(i)=\sigma_0(i)\cup\cdots\cup\sigma_n(i)$. From Lemma 5.1 it is easy to see that tn converges to $\emptyset$ for each n. Let τ be any sequence converging to $\emptyset$ whose range meets infinitely many tn. Then for each n, we can pick i_n such that $\sigma_n(i_n)$ is a subset of some $\tau(j_n)$, and so that the j_n are distinct. Another application of Lemma 5.1 shows that $\{\sigma_n(i_n)\colon n\in\omega\}$ converges to $\emptyset$; in particular, the sequence is point-finite because τ is. □

Corollary 5.5. *If $[X-\{p\}]^{<\omega}$ is Fréchet, then it is a w-space.*

Proof. Every Fréchet group is α_4 [26], so Theorem 5.4 gives α_2. □

Theorem 5.5 severely restricts the X that can be used to produce a countable Fréchet topological group that is not first countable, but there is one if either $\omega_1<\mathbf{p}$ or $\mathbf{p}=\mathbf{b}$. In the former case, $D\cup\{\infty\}$ works for any uncountable Ψ-like space of cardinality $<\mathbf{p}$ (Lemma 1.11). In the latter case, there is:

Theorem 5.6. *If $\mathbf{p}=\mathbf{b}$, and $X=(\omega\times\omega)\cup\{\infty\}$ as in Example 2.1, then $[X-\{p\}]^{<\omega}$ is a countable Fréchet topological group that is not first countable.*

Proof. Let $G=[X-\{p\}]^{<\omega}$, and let $A\subset G$ have $\emptyset$ in its closure. For each $\alpha<\mathbf{b}$ let $A_\alpha=\{a\in A\colon a\subset f_\alpha^{\downarrow\downarrow}\}$.

Claim. *For some α, A_α has $\emptyset$ in its closure.*

Once the claim is proved, let $A^\beta=\{a\in A_\alpha\colon a\cap f_\beta=\emptyset\}$ for each $\beta\leqslant\alpha$, and let $A_n^\beta=\{a\in A^\beta\colon a$ does not meet the first n columns of $\omega\times\omega\}$. Then the A_n^β form a subbase for a free filter on A_α, since each finite intersection has $\emptyset$ in its closure. Since $|\alpha|<\mathbf{p}$ [this is the only place where $\mathbf{p}=\mathbf{b}$ is used] there is an infinite $C\subset A_\alpha$ such that all but finitely many members of C are in each A_n^β, and C converges to $\emptyset$ because it is point-finite and only finitely many members meet any given f_γ, $\gamma>\alpha$.

Proof of Claim. Suppose not; then for each $\alpha<\mathbf{b}$, there is a finite set F_α of ordinals, all less than α, and $n_\alpha\in\omega$, so that the union of the graphs of the f_β, $\beta\in F_\alpha$, meets every member of A_α that does not meet $n_\alpha\times\omega$. Since the cofinality of $\mathbf{b}$ is uncountable [in fact, $\mathbf{b}$ is regular] there is a stationary subset S_0 of $\mathbf{b}$ such that $|F_\alpha|=N$ for all $\alpha\in S_0$. By the Pressing Down Lemma of $\mathbf{b}$, there is a stationary $S_1\subset S_0$ and a β_0 such that the least member of F_α is β_0 for all $\alpha\in S_1$. Repeating this argument N times, we arrive at a stationary subset S of $\mathbf{b}$ and a finite set E such that $F_\alpha=E$ for all $\alpha\in S$.

Let $B=\{a\in A\colon a\cap f_\beta=\emptyset$ for all $\beta\in E\}$. Since B is just the members of A in a basic clopen subgroup, it has $\emptyset$ in its closure. So, for each finite set of columns, there is a member of B that misses them; and so we can define $\{a_n\colon n\in\omega\}\subset B$ so that the least member $i(n)$ of the projection of a_n to the x-axis is greater than the greatest of a_{n-1}. Let h be a function whose domain is $\{i(n)\colon n \in \omega\}$ and is such that $h(i(n))$ exceeds the second coordinates of all the a_n. Now there exists $\alpha \in S$ such that $f_\alpha(i(n)) > h(i(n))$ for infinitely many $i(n)$ [see comment at the end of the proof of Theorem 1.8], and since f_α is nondecreasing, we will have $a_n \subset f_\alpha^{\downarrow\downarrow}$ for infinitely many n. But this contradicts $F_\alpha = E$. $\square$

It would be very interesting to know whether special axioms are really needed for making G Fréchet in the above proof. Is it possible to derive the existence of $C \subset A_\alpha$ just from $|\alpha| < \mathbf{b}$ and the special nature of the f_β? Note the lack of any special axioms in:

Theorem 5.7. *Let X be as in Example* 2.1. *Then* $[X-\{p\}]^{<\omega}$ *is* α_2.

Proof. Let σ^n converge to $\emptyset$ for each n. Then $\bigcup \operatorname{ran} \sigma_n$ meets each column in a finite set, so that, as in the last paragraph of the proof of Theorem 5.4, there is an α_n such that infinitely many terms of σ_n are subsets of $f_{\alpha_n}^{\downarrow\downarrow}$. Let $\alpha = \sup_n \alpha_n$. Only finitely many terms of each σ_n meet the columns where f_{α_n} is above f_α, and so there is a subsequence τ_n such that each term is a subset of $f_\alpha^{\downarrow\downarrow}$.

Let $A = \bigcup_{n=0}^{\infty} \operatorname{ran} \tau_n$. For each $\beta < \alpha$ let $B_\beta = \{a \in A\colon a \cap f_\beta \neq \emptyset\}$. For each $n \in \omega$ let $B^n = \{a \in A\colon a \cap (\{n\} \times n) \neq \emptyset\}$. Then each B_β and each B^n is almost disjoint from the range of each τ_n. So, by Theorem 1.10, there exists $C \subset A$ such that $\operatorname{ran} \tau_n \subset^* C$ for all n and C is almost disjoint from each B_β and each B^n. Thus C converges to $\emptyset$. $\square$

In [26], I showed that every sequential α_4-topological group is Fréchet, so we need "only" show $G = [X-\{p\}]^{<\omega}$ is sequential, but, at present, that seems no easier than trying to show it is Fréchet directly.

Whether G is α_1 seems to have no bearing on whether it is Fréchet. In the Dow–Steprāns model, where it cannot be α_1, it is always Fréchet because $\mathbf{p} = \mathbf{b}$. In the "dominating reals" models [5, 6, 18], where it is automatically α_1, we have $\omega_1 = \mathbf{p} < \mathbf{b}$ and we do not know whether G is ever Fréchet. In models where $\mathbf{p} = \mathbf{d}$, we can get G to be α_1, and it is always Fréchet. In these same models, however, it is also possible for G to fail to be α_1: use Example 2.6 for X in the construction of G.

6. Some topological vector spaces

A construction related to that of the preceding section is that of the topological vector space V whose Hamel basis is $X-\{p\}$ and whose base of neighborhoods of the identity is given by the sets

$$B_F^E = \{a_1 x_1 + \cdots + a_n x_n\colon |a_1| \leqslant \varepsilon \text{ whenever } x_i \in F\},$$

where F is in the filter $\mathscr{F}$ of neighborhoods of p, and $\varepsilon > 0$. In fact, the construction in the preceding section is the special case where the field of scalars is the two-element field. Here we are mostly concerned with when the field of scalars is $\mathbb{R}$ or $\mathbb{C}$ although our arguments work for any separable valuated field.

This may seem like an artificial construction, but V "is" actually the space of continuous scalar-valued functions whose support is a finite subspace of the discrete space $X - \{p\}$, and whose domain is the open subspace U of $\beta(X - \{p\})$ consisting of all ultrafilters which do *not* contain $\mathscr{F}|X - \{p\}$. The topology is then the relative topology inherited from the compact-open topology on the space $C(U)$. One of the keys to penetrating the disguise adopted above is the description of the compact-open topology given in [20]; another is that a subset of $X - \{p\}$ has compact closure in U precisely when it is disjoint from some member of F. In [29] this viewpoint is developed and exploited. In this section, however, we will only need the bare definition given in the first paragraph.

Lemma 6.1. *If $X - \{p\}$ is countable, then V is hereditarily separable and hereditarily Lindelöf.*

Proof. V is the union of subspaces generated by finite subsets of $X - \{p\}$, and each such subspace is second countable. $\square$

Corollary 6.2. *If $X - \{p\}$ is countable, then V is countably tight, i.e., if v is in the closure of A, then v is in the closure of a countable subset of A.*

Proof. Obviously, every hereditarily separable space is countably tight. $\square$

Corollary 6.3. *If $X - \{p\}$ is countable, and $\mathscr{F}$ has a base of cardinality $<\mathbf{p}$, then V is Fréchet-Urysohn.*

Proof. Every space of countable tightness and character $<\mathbf{p}$ is Fréchet-Urysohn; this is essentially a characterization of $\mathbf{p}$ and is essentially shown in [40, 6.2]. $\square$

Theorem 6.4. *If either $\omega_1 < \mathbf{p}$ or $\mathbf{p} = \mathbf{b}$, there is a separable version of V which is Fréchet-Urysohn, but not metrizable.*

Proof. The case of $\mathbf{p} > \omega_1$ is taken care of by Corollary 6.3 and a Ψ-like space Y of cardinality ω_1: we take the one-point compactification of Y, remove all nonisolated points of Y, and use that for X. For $\mathbf{p} = \mathbf{b}$, we let $X = (\omega \times \omega) \cup \{\infty\}$ of Example 2.1 just as in Theorem 5.6. The proof is a routine variation. By Corollary 6.2 and translation invariance, it is enough to take the case of a countable $A \subset V$ with the zero element in the closure. We now define, for each positive integer n,

$$A_\alpha^n = \left\{ a \in A \colon |\pi_x(a)| < \frac{1}{n} \text{ for all } x \in f_\alpha^{\downarrow\downarrow} \right\},$$

where $\pi_x(a)$ is the x-component of a. Now the claim is that for *each* n, there exists α_n such that $A^n_{\alpha_n}$ has 0 in its closure. Then if we let $\alpha = \sup_n \alpha_n$, this α will work similarly to the one in Theorem 5.6. We define:

$$A^\beta_n = \left\{ a \in A^n_\alpha : |\pi_x(a)| < \frac{1}{n} \text{ for all } x \in (n \times \omega) \cup f_\beta \right\},$$

and argue as before. To prove the claim, suppose it fails for some n, and argue as in the first paragraph of "Proof of Claim" that there is a stationary subset S of $\mathbf{b}$ and a finite set E such that, for all $\alpha \in S$, there is $n_\alpha \in \omega$ such that the union of the graphs of the f_β, $\beta \in E$, contains every $x \in n_\alpha \times \omega$ such that $|\pi_x(a)| \geq 1/n$ for some $a \in A^n_\alpha$. Then let B be the set of all $a \in A$ such that $\{x : |\pi_x(a)| \geq 1/n\} \cap f_\beta = \emptyset$ for all $\beta \in E$. In the rest of the argument, just as above, think of the coordinates of a_n and a_{n-1} on which it exceeds $1/n$ as "the only ones that count" and derive a contradiction as in Theorem 5.6. □

For a nontrivial use of the Fréchet-Urysohn property in topological vector spaces, see [9]. Unfortunately, completeness plays a key role, and V as defined above is not complete. It does have a natural completion, treated in [29], but the question of when that completion is Fréchet-Urysohn is a difficult one. The best positive result so far assumes $\mathrm{MA} + \mathfrak{c} \geq \omega_3$ to obtain a complete nonmetrizable topological vector space which is hereditarily separable, hereditarily Lindelöf, and Fréchet-Urysohn [29].

Afterword

The first draft for this paper, which was originally combined with [33], was completed just two days before van Douwen's death. I had been looking forward to showing it to him in Toronto and saying: "You should like this paper. It's got so many [40]'s in it, it looks almost like an advertisement for your Handbook article". In a more serious vein, I was also hoping he could tell me whether some elementary results such as Lemma 1.2 were already in the literature. He was always very good for such odd bits of information.

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