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STEEL FORCING AND BARWISE COMPACTNESS*

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0. Introduction

In [9] John Steel developed the method of forcing with tagged trees, which he used to settle several important questions concerning models of analysis. The Steel partial ordering refines the (Lévy-type) partial-ordering for collapsing an admissible ordinal to ω in that it permits a careful computation of the complexity of the forcing relation when restricted to statements of a bounded ordinal rank. (It is this aspect of the forcing which Leo Harrington exploited in his proof that Π^1_1-determinateness implies $O^\#$ exists [6].) In addition Steel's forcing allows one to generically construct trees on ω with complete control over which paths appear in the generic extension. For, a condition assigns ordinal 'tags' to nodes on the tree which do not lie on the intended paths; longer nodes get smaller ordinal tags. Thus the generic tree is well-founded below any node which receives an ordinal tag.

Our work here began with a generalization of Steel's forcing which uses tags which are not necessarily ordinals but are sets from a given admissible set. The idea is to require that longer nodes receive tags which descend in the $\in$-relation, thereby coding sets below nodes on the generic tree. We developed this forcing to provide a characterization of those admissible sets which appear as the pure part of HYP($\mathcal{M}$) where $\mathcal{M}$ is a structure of finite similarity type on urelements. This answers a question posed by Mark Nadel and Jonathan Stavi who obtained partial results in [7].

Later a much simpler proof of the above characterization was found which dispenses of forcing in favor of Barwise compactness techniques, especially the Barwise Hard Core Theorem (see [7]). This led us to re-examine Steel's original applications of his forcing and to discover easier, model-theoretic proofs of them. However our work does not appear to simplify deeper applications of Steel forcing (as for example in [1]) nor supplant Harrington's techniques for establishing lightface versions of Steel's results.

In Section 1 we review the aspects of the theory of admissible sets and Barwise compactness which we will need and characterize the pure parts of HYP($\mathcal{M}$) as the resolvable admissible sets. Thus if A is resolvable we shall construct a tree $\mathcal{T}_A$ (on

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urelements) such that $A = \text{pure part HYP}(\mathcal{T}_A)$. In Section 2 we embellish this construction to replace $\mathcal{T}_A$ by $\mathcal{L}_A$, a linear-ordering. We also show that A satisfies the strong global well-ordering principle if and only if $A = \text{pure part HYP}(W, <, U)$ where U is unary and $\text{HYP}(W, <, U) \models (W, <)$ is well-ordering. Section 3 gives model-theoretic proofs of several of Steel's results, including $\mathbf{\Delta}^1_1$-CA $\leftrightarrow \mathbf{\Sigma}^1_1$-AC.

We are extremely grateful to both John Steel and Leo Harrington for helpful discussions and for providing most of the ideas in the simpler proof of our solution to the Nadel–Stavi problem.

1. The pure part of HYP($\mathcal{M}$)

Let A be an admissible set, i.e., A is a transitive set closed under pairing, union and satisfying $\mathbf{\Delta}_0$-separation, $\mathbf{\Delta}_0$-bounding. (All admissible sets are taken here to be without urelements unless explicitly stated otherwise.) We say that A is *resolvable* if there exists a function $f : \text{ORD}(A) \to A$ such that $A = \bigcup \text{Range}(f)$ *and* $\langle A, \in, f\rangle$ is an admissible structure. Thus A satisfies separation and bounding for formulas with only bounded quantifiers but where f may occur as a predicate in the matrix. Our definition of resolvable differs from that given in Barwise's book [2]; A is resolvable in Barwise's sense if there exists f as above which is Σ_1-definable over $\langle A, \in\rangle$.

The theory of admissible sets with urelements [2] provides a wealth of examples of resolvable admissible sets. For, as pointed out by Nadel and Stavi in [7], the pure part of HYP($\mathcal{M}$) is always resolvable for any structure $\mathcal{M}$ (for a finite language). If $o(\mathcal{M}) = \text{ORD} \cap \text{HYP}(\mathcal{M})$ is equal to ω, then this is clear as $\text{pp HYP}(\mathcal{M}) = \text{HF}$, the hereditarily finite sets. Otherwise $\text{HYP}(\mathcal{M}) = L_{o(\mathcal{M})}(\mathcal{M})$ and the function $f(\beta) = \text{pp } L_\beta(\mathcal{M})$ demonstrates the resolvability of pp HYP($\mathcal{M}$).

It is not difficult to produce a non-resolvable admissible set: Define A to be *locally countable* if $\langle A, \in\rangle \models$ Every set is countable. Clearly an uncountable, locally countable admissible set of countable height cannot be resolvable. An example of such an admissible set is

$$\bigcup \{L_{\omega_1^{ck}}[F] \mid F \subseteq S, F \text{ finite}\}$$

where S is an uncountable collection of reals mutually Cohen-generic over $L_{\omega_1^{ck}}$.

Countable nonresolvable admissible sets are more difficult to come by. The dependent choice axiom:

$$\mathbf{\Sigma}_1\text{-DC:} \;\; \forall x\, \exists y\, \varphi(x, y) \to \forall x\, \exists f\, [f(0) = x \wedge \forall n\, \varphi(f(n), f(n+1))],\; \varphi\mathbf{\Delta}_0$$

holds in any resolvable locally countable admissible set. Harvey Friedman proved the existence of a countable, locally countable admissible set in which $\mathbf{\Sigma}_1$-DC fails for reals using proof-theoretic methods [3]. We have recently discovered an explicit forcing construction of such an admissible set.

We shall make extensive use of a version of the Barwise Hard Core Theorem. Let $\langle A, \in, \ldots\rangle$ be a countable admissible structure and, for $L \in A$ a first-order language, let $\mathscr{L}_A$ be the corresponding fragment of $L_{\omega_1\omega}$ as defined in Barwise [2]. KP denotes Kripke–Platek set theory, a theory in the language of set theory whose only nonlogical symbol is the binary relation ϵ. The well-founded models of KP are precisely the admissible sets.

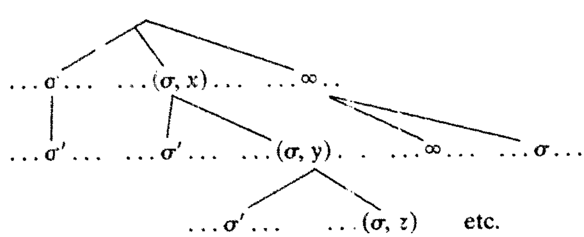
Hard Core Theorem. *Suppose T is a consistent theory in L_A which includes* KP *and is Σ_1-definable over $\langle A, \epsilon, \ldots \rangle$. If x belongs to the standard part of every model of T, then $x \in A$.*

The proof is essentially the same as that given in [2, Chapter IV, Section 1]. Note that an immediate corollary of this theorem is the fact that any consistent theory Σ_1 over $\langle A, \epsilon, \ldots \rangle$ which extends KP has a model whose standard ordinals all belong to A.

Fix for the remainder of this section a countable resolvable admissible set A with resolution $f : \mathrm{ORD}(A) \to A$. We make the harmless assumptions that $f(\sigma)$ is transitive for all $\sigma \in \mathrm{ORD}(A)$ and $f(0) = \emptyset$. Our goal is to construct a tree $\mathscr{T}_A$ such that $A = \mathrm{pp}\,\mathrm{HYP}(\mathscr{T}_A)$. It will be fairly clear from the construction that $A \subseteq \mathrm{pp}\,\mathrm{HYP}(\mathscr{T}_A)$. The reverse inclusion follows once we show that there is a consistent theory $T \supseteq \mathrm{KP}$ which is Σ_1 over $\langle A, \epsilon, f \rangle$, all of whose models contain an isomorphic copy of $\mathscr{T}_A$. For then any $x \in \mathrm{pp}\,\mathrm{HYP}(\mathscr{T}_A)$ must belong to the standard part of every model of T and hence must belong to A by the Hard Core Theorem.

We now describe $\mathscr{T}_A$ with the aid of a tagging function h_A. A *tag* is something in one of the forms ∞, σ, (σ, x) where $\sigma \in \mathrm{ORD}(A)$ and $x \in f(\sigma)$. Then $h_A : |\mathscr{T}_A| - \{\text{Top Node}\} \to \text{Tags}$ and h_A is used to control the growth of $\mathscr{T}_A$. The pair $(\mathscr{T}_A, h_A)$ is determined by the following prescription: There is a unique top node. It receives no tag. Infinitely many nodes at level 1 are tagged ∞; each other tag appears as $h_A(s)$ for exactly one node s at level 1. If $h_A(s) = \infty$, then s has infinitely many immediate extensions tagged with ∞, for each $\sigma \in \mathrm{ORD}(A)$ a unique immediate extension tagged with σ, and no immediate extensions tagged with (σ, x) for any σ, x. If $h_A(s) = \sigma \in \mathrm{ORD}(A)$, then for each $\sigma' < \sigma$, s has a unique immediate extension tagged with σ' and s has no immediate extension tagged with any other tag. Finally if $h_A(s) = (\sigma, x)$, then s has a unique immediate extension tagged with σ' for $\sigma' < \sigma$, a unique immediate extension tagged with (σ, y) for $y \in x$, and no immediate extension with any other tag.

Note that s is a terminal node of $\mathscr{T}_A$ if and only if $h_A(s) = 0$. Here is a picture of $\mathscr{T}_A$:



A node $s \in \mathcal{T}_A$ is in WF($\mathcal{T}_A$), the well-founded part of $\mathcal{T}_A$, if and only if $s \neq$ top node of $\mathcal{T}_A$ and $h_A(s) \neq \infty$. If $s \in$ WF($\mathcal{T}_A$), then the rank of s in $\mathcal{T}_A$ is an ordinal in A.

For any $\sigma \in$ ORD(A) let $\mathcal{T}_A(\sigma)$ consist of those nodes in WF($\mathcal{T}_A$) of rank less than σ.

Lemma 1. $A \subseteq$ pp HYP($\mathcal{T}_A$).

Proof. For $x \in A$ choose $\sigma \in$ ORD(A) and $s \in \mathcal{T}_A$ such that $h_A(s) = (\sigma, x)$. Now $\mathcal{T}_A - \mathcal{T}_A(\sigma)$ belongs to HYP($\mathcal{T}_A$) and $s \in \mathcal{T}_A - \mathcal{T}_A(\sigma)$.

Define T_x, the tree for x, by

$$T_x = \{(x_0, x_1, \ldots, x_n) \mid x_0 = x \text{ and } \forall\, i < n\; x_{i+1} \in x\}.$$

Now x belongs to any admissible set with urelements that contains an isomorphic copy of T_x. But T_x is isomorphic to the tree below s in $\mathcal{T}_A - \mathcal{T}_A(\sigma)$. So $x \in$ HYP($\mathcal{T}_A$).

It remains to describe a consistent theory $T \supseteq$ KP which is Σ_1 over $\langle A, \epsilon, f\rangle$ such that any model of T contains an isomorphic copy of $\mathcal{T}_A$. T is a theory in the language L_A where L is the language of $\langle A, \epsilon, a\rangle_{a \in A}$ augmented by a constant symbol $\underline{\infty}$, a unary function symbol f and a unary predicate symbol $\mathcal{T}$. Let KP($f, \mathcal{T}$) denote the extension of KP where f, $\mathcal{T}$ are allowed to appear in the matrix of the axioms for Δ_0-separation and Δ_0-bounding. Then the axioms for T are:

(1) KP($f, \mathcal{T}$) + Infinitary Diagram $\langle A, \epsilon\rangle$.

(2) (*a*) *f is a resolution of V*, i.e., Domain(f) = ORD and $\forall x\, \exists \sigma\, x \in f(\sigma)$.

(b) $f(\underline{\sigma}) = \underline{f(\sigma)}$, for each $\sigma \in$ ORD(A).

(c) $f(\sigma)$ is transitive for all σ.

(3) $\mathcal{T}$ is a tree (of ordinals) defined from ORD, f, $\underline{\infty}$ in exactly the way that $\mathcal{T}_A$ was defined from ORD(A), f, ∞.

(4) For each σ, $\mathcal{T}(\sigma) = \{s \in \mathcal{T} \mid \mathcal{T}\text{-rank}(s) < \sigma\}$ is a set.

Lemma 2. *If B is a model of T, then B contains an isomorphic copy of* $\mathcal{T}_A$.

Proof. If ORD(A) $\in$ Standard Part(B), then $A \in B$ as $A = \bigcup \{f^B(\sigma) \mid \sigma <$ ORD(A)$\}$. As $\mathcal{T}_A$ has an isomorphic copy definable over $\langle A, \epsilon, f\rangle$ we are done by Δ_0 separation.

Otherwise choose a (nonstandard) ordinal $\sigma \in B - A$ and consider $\mathcal{T}^B(\sigma) \in B$. The well-founded part of $\mathcal{T}^B(\sigma)$ consists of those nodes of standard ordinal rank and thus coincides with the well-founded part of $\mathcal{T}_A$. The remainder of $\mathcal{T}^B(\sigma)$ is isomorphic to the full tree $\omega^{<\omega}$, just as is the non well-founded part of $\mathcal{T}_A$. So $\mathcal{T}^B(\sigma)$ is isomorphic to $\mathcal{T}_A$.

T is clearly Σ_1 (in fact Δ_1) over $\langle A, \epsilon, f\rangle$ and is consistent since $\mathcal{T}_A$ has an isomorphic copy Δ_1 over $\langle A, \epsilon, f\rangle$. Thus we have completed the proof in the countable case of:

Theorem 3. *A is resolvable if and only if* $A = \text{pp HYP}(\mathcal{M})$ *for some* $\mathcal{M}$ *iff* $A = \text{pp HYP}(\mathcal{T}_A)$.

Proof. The only issue remaining concerns the case when A is uncountable. But '$A = \text{pp HYP}(\mathcal{T}_A)$' is a Π_1 property of (A, f):

$$A = \text{pp HYP}(\mathcal{T}_A) \leftrightarrow \forall B(B \text{ admissible, } A \in B \text{ implies } B \models A = \text{pp HYP}(\mathcal{T}_A)).$$

Thus by Lévy-absoluteness $A = \text{pp HYP}(\mathcal{T}_A)$ holds for uncountable resolvable admissible sets as well. We are grateful to John Steel for this observation.

2. Linear orderings and pseudo well-orderings

A tree can be converted into a linear ordering once a linear ordering is specified of the immediate extensions of each node.

Definition. A *linearized tree* $(\mathcal{T}, R)$ is a tree $\mathcal{T}$ of finite sequences together with a binary relation R such that:

(a) $R(s, t) \to s, t$ are immediate extensions of a common node in $\mathcal{T}$.

(b) For any $r \in \mathcal{T}$, R linearly orders the immediate extensions of r in $\mathcal{T}$.

If $(\mathcal{T}, R)$ is a linearized tree, then its associated linear ordering $\mathcal{L}$ of $|\mathcal{T}|$ is defined by:

$$s <_{\mathcal{L}} t \leftrightarrow s \text{ properly extends } t \text{ in } \mathcal{T} \text{ or}$$
$$\exists i(s \restriction i = t \restriction i \text{ and } R(s(i+1), t(i+1))).$$

This is simply a generalization of the usual Kleene–Brouwer linear ordering of finite sequences from ω.

Now let A be a countable resolvable admissible set. In order to obtain a linear ordering $\mathcal{L}$ such that $A = \text{pp HYP}(\mathcal{L})$ we shall 'linearize' a certain tree $\mathcal{T}^*_A$ which is very similar to $\mathcal{T}_A$ (in fact $A = \text{pp HYP}(\mathcal{T}^*_A)$). The associated linear ordering $\mathcal{L}^*_A$ will be dense and hence there is no hope of having $A = \text{pp HYP}(\mathcal{L}^*_A)$ as $\text{pp HYP}(\mathcal{L}^*_A) = \text{pp HYP}(\mathbb{Q}) = \text{HF}$. Instead we add certain points to $\mathcal{L}^*_A$ to obtain a linear ordering $\mathcal{L}_A$ such that $\mathcal{T}^*_A$ can be recovered from $\mathcal{L}_A$ and hence $\text{pp HYP}(\mathcal{L}_A) \supseteq A$. And, as in the previous section, there will be a consistent theory Σ_1 over (A, ϵ, f) all of whose models contain an isomorphic copy of $\mathcal{L}_A$. Thus $\text{pp HYP}(\mathcal{L}_A) = A$.

Choose a resolution $f : \text{ORD}(A) \to A$ such that $\langle A, \epsilon, f\rangle$ is an admissible structure. As before we assume that $f(0) = \emptyset$ and $f(\sigma)$ is transitive for all $\sigma \in \text{ORD}(A)$.

The tree $\mathcal{T}^*_A$ with tagging h^*_A is defined just like $(\mathcal{T}_A, h_A)$ except we require that each node (other than the top node) have infinitely many neighbors with the same tag. Thus we determine $(\mathcal{T}^*_A, h^*_A)$ by the following prescription: $\mathcal{T}^*_A$ has a unique top node which receives no tag. For each possible tag ∞, (σ, x), σ there are infinitely many nodes at level 1 receiving that tag. If $h^*_A(s) = \infty$, then s has infinitely many immediate extensions tagged with ∞, infinitely many immediate extensions tagged with σ (for each $\sigma \in \mathrm{ORD}(A)$) and no immediate extensions tagged with (σ, x) for any σ, x. If $h^*_A(s) = (\sigma, x)$, then s has infinitely many immediate extensions tagged with σ' (for each $\sigma' < \sigma$), infinitely many immediate extensions tagged with (σ, y) (for each $y \in x$) and no immediate extensions tagged with ∞. Finally, if $h^*_A(s) = \sigma$, then s has infinitely many extensions tagged with σ' (for each $\sigma' < \sigma$) and no immediate extension with any other tag.

Lemma 4. $A \subseteq \mathrm{pp}\ \mathrm{HYP}(\mathcal{T}^*_A)$.

Proof. By the proof of Lemma 1 it is enough to show that $\mathcal{T}_A(\sigma) \in \mathrm{HYP}(\mathcal{T}^*_A)$ for each $\sigma \in \mathrm{ORD}(A)$. But $\mathcal{T}_A(\sigma)$ is obtained from $\mathcal{T}^*_A(\sigma)$ by identifying adjacent nodes when they receive the same tag. As $h^*_A \restriction \mathcal{T}^*_A(\sigma)$ belongs to $\mathrm{HYP}(\mathcal{T}^*_A)$ we are done.

Now linearize $\mathcal{T}^*_A$ with a binary relation R which densely orders the immediate extensions of each node in such a way that any tag which appears as the tag of one of these immediate extensions actually appears as the tag of a dense set of them in the ordering R. This is easily done using a partition of the rationals into infinitely many dense subsets.

Lemma 5. $A \supseteq \mathrm{pp}\ \mathrm{HYP}(\mathcal{T}^*_A, R)$.

Proof. It is enough to describe a consistent theory T^* which is Σ_1 over $\langle A, \in, f\rangle$ and such that any model of T^* contains an isomorphic copy of $(\mathcal{T}^*_A, R)$. T^* is defined in a way similar to the definition of the theory T of Lemma 1. Its axioms are:

(1) $\mathrm{KP}(\boldsymbol{f}, \boldsymbol{\mathcal{T}}, \boldsymbol{R})$ + Infinitary Diagram $\langle A, \in\rangle$.

(2) Same as for T.

(3) $(\boldsymbol{\mathcal{T}}, \boldsymbol{R})$ is a linearized tree (of ordinals) defined from ORD, $\boldsymbol{f}$, ∞ in exactly the way that $(\mathcal{T}^*_A, R)$ was defined from $\mathrm{ORD}(A)$, f, ∞.

(4) For each $\sigma \in \mathrm{ORD}\ \boldsymbol{\mathcal{T}}(\sigma)$ is a set.

Now suppose B is a model of T^*. If $\mathrm{ORD}(A) \in \text{Standard Part}(B)$, then $A \in B$ and thus B contains an isomorphic copy of $(\mathcal{T}^*_A, R)$ since there is such a copy definable over $\langle A, \in, f\rangle$.

Otherwise choose a (nonstandard) ordinal $\sigma \in B - A$ and consider $(\mathcal{T}^B(\sigma), R^B(\sigma))$ where by definition $R^B(\sigma) = R \restriction \mathcal{T}^B(\sigma) \times \mathcal{T}^B(\sigma)$. If each node $s \notin \mathrm{WF}(\mathcal{T}^B(\sigma))$ is retagged with ∞ (instead of τ or (τ, x) for some nonstandard τ)

we see that $(\mathcal{T}^B(\sigma), R^B(\sigma))$ obeys the prescription for $(\mathcal{T}^*_A, R)$. Thus $(\mathcal{T}^*_A, R) \simeq (\mathcal{T}^B(\sigma), R^B(\sigma)) \in B$ and we are done.

It remains to construct a linear ordering $\mathcal{L}_A$ such that $\mathrm{HYP}(\mathcal{T}^*_A, R) = \mathrm{HYP}(\mathcal{L}_A)$. Let $\mathcal{T}^*_A$ be the associated linear ordering (as defined at the beginning of this section) for the linearized tree $(\mathcal{T}^*_A, R)$. $\mathcal{L}^*_A$ is a dense linear ordering with a greatest element. Then $\mathcal{L}_A$ is obtained from $\mathcal{L}^*_A$ by adding a chain of length n immediately after each point in the $\mathcal{L}^*_A$ ordering which represents a node at level n in $\mathcal{L}^*_A$. More formally, $|\mathcal{L}_A| = \{(s, i) \mid s \in \mathcal{T}^*_A \text{ and } i \leq \text{level}(s) \text{ in } \mathcal{T}^*_A\}$ and $(s, i) < (s', i')$ in $\mathcal{L}_A$ iff $s < s'$ in $\mathcal{L}^*_A$ or ($s = s'$ and $i < i'$).

Lemma 6. $\mathrm{HYP}(\mathcal{T}^*_A, R) = \mathrm{HYP}(\mathcal{L}_A)$.

Proof. As $\mathcal{L}_A$ is simply defined in terms of $(\mathcal{T}^*_A, R)$ we easily get $\mathcal{L}_A \in \mathrm{HYP}(\mathcal{T}^*_A, R)$ and hence $\mathrm{HYP}(\mathcal{L}_A) \subseteq \mathrm{HYP}(\mathcal{T}^*_A, R)$. For the reverse inclusion begin by defining a tree $\mathcal{T}$ in $\mathrm{HYP}(\mathcal{L}_A)$ as follows: The top node of $\mathcal{T}$ is the greatest element of $\mathcal{L}_A$. The nodes of level n on $\mathcal{T}$ are those points on $\mathcal{L}_A$ with no immediate predecessor and which begin a succession of $n+1$ points, the last of which has no immediate successor. If s is a node of level n and t a node of level m, $n < m$, then t extends s in $\mathcal{T}$ iff t is less than s (in $\mathcal{L}_A$) and t is greater (in $\mathcal{L}_A$) than all points of level n which are less than s.

Clearly the tree $\mathcal{T}$ so defined is exactly $\mathcal{T}^*_A$. The relation R can be defined by $R(s, t)$ iff s, t are immediate extensions of a common node on $\mathcal{T}$ and $s < t$ in $\mathcal{L}_A$. Thus $(\mathcal{T}^*_A, R) \in \mathrm{HYP}(\mathcal{L}_A)$ and hence $\mathrm{HYP}(\mathcal{T}^*_A, R) \subseteq \mathrm{HYP}(\mathcal{L}_A)$.

As in the proof of Theorem 3 a Lévy-absoluteness argument now demonstrates:

Theorem 7. *A is resolvable if and only if* $A = \mathrm{pp}\,\mathrm{HYP}(\mathcal{L}_A)$.

Finally we shall characterize those admissible sets which can appear as $\mathrm{pp}\,\mathrm{HYP}(\mathcal{M})$ where $\mathrm{HYP}(\mathcal{M}) \models |\mathcal{M}| = \mathrm{Universe}(\mathcal{M})$ can be well-ordered. In this case there is a linear ordering $<$ of $\mathrm{HYP}(\mathcal{M})$ such that the function $p_<(x) =$ the $<$-predecessors of x is Σ_1 over $\mathrm{HYP}(\mathcal{M})$ and such that $\mathrm{HYP}(\mathcal{M}) \models <$ is a well-ordering. Thus if $A = \mathrm{pp}\,\mathrm{HYP}(\mathcal{M})$ then A obeys the property expressed in the next definition.

Definition. A satisfies the *Strong Global Well-Ordering Principle* (SGWOP) if for some linear ordering $<$ of A, $\langle A, p_< \rangle$ is admissible and $\langle A, p_< \rangle \models <$ is a well-ordering.

If A satisfies the SGWOP, then A is resolvable for one can define the associated resolution $f(\sigma) = p_<(\sigma)$ such that with this resolution $\langle A, f \rangle$ is admissible.

From our earlier remarks we see that a necessary condition for $A = \text{pp}\,\text{HYP}(\mathcal{M})$, $\text{HYP}(\mathcal{M}) \models |\mathcal{M}|$ can be well-ordered, is that A satisfy the SGWOP. Our next result implies that this condition is also sufficient:

Theorem 8. *Let A be admissible. Then the following are equivalent:*

(a) *A satisfies SGWOP.*

(b) *$A = \text{pp}\,\text{HYP}(\mathcal{M})$ for some $\mathcal{M}$ such that $\text{HYP}(\mathcal{M}) \models |\mathcal{M}|$ can be well-ordered.*

(c) *$A = \text{pp}\,\text{HYP}(W, <, U)$ where U is unary and $\text{HYP}(W, <, U) \models (W, <)$ is a well-ordering.*

In case A is countable then the ordering $(W, <)$ in (c) can be explicitly chosen to be $(\alpha + \alpha \cdot \eta, <)$ where $\alpha = \text{ORD}(A)$ and $\eta =$ ordertype of the rationals.

The basic idea of the proof of Theorem 8 is similar to that used in the proof of Theorem 7: Given a countable, resolvable admissible set A which satisfies SGWOP one first constructs a special type of tree $\mathcal{T}_A^{**}$ such that $A = \text{pp}\,\text{HYP}(\mathcal{T}_A^{**})$. Then by use of an ordering $<$ as in the SGWOP $\mathcal{T}_A^{**}$ can be linearized by a binary relation R such that if $\mathcal{L}_A^{**}$ is the associated linear ordering then $\text{HYP}(\mathcal{T}_A^{**}, R) \models \mathcal{L}_A^{**}$ is a well-ordering. Moreover there is a consistent theory T^{**}, Σ_1 over $\langle A, \in, < \rangle$ such that any model of T^{**} contains an isomorphic copy of $(\mathcal{T}_A^{**}, R)$. Thus $A = \text{pp}\,\text{HYP}(\mathcal{T}_A^{**}, R)$. The proof is completed by adding certain points to $\mathcal{L}_A^{**}$, together with a unary predicate U distinguishing them, to obtain a linear ordering W_A such that $\text{HYP}(W_A, U) \models W_A$ is a well-ordering and in addition $\text{HYP}(W_A, U) = \text{HYP}(\mathcal{T}_A^{**}, R)$.

To demonstrate the key property of the theory T^{**} it will be necessary to construct $(\mathcal{T}_A^{**}, R)$ in a very canonical fashion (to guarantee that it looks the same when nonstandard ordinals are allowed). An important point which makes this possible is the following result which originates in [4]:

Lemma 9. *Suppose B is a countable model of KPU with standard ordinal α and $\langle B, \in \rangle \models L$ is a well-ordering. Then either ordertype(L) is an ordinal less than α or ordertype$(L) = \alpha + \alpha \cdot \eta + \sigma$ where $\eta =$ ordertype of the rationals and σ is an ordinal less than α.*

This lemma also makes it clear why the unary predicate U in Theorem 8(c) is necessary as if $\text{HYP}(W, <) \models (W, <)$ is a well-ordering, then $\text{pp}\,\text{HYP}(W, <) = L_\alpha$ where α is admissible. This follows from the lemma and results of Nadel and Stavi in [7].

Fix a countable admissible set A with linear ordering $<_A$ as in the SGWOP. We assume that $\{x \in A \mid x <_A \sigma\}$ is transitive for each σ. Let f be the resolution of A associated with $<_A$. We now define the tree $\mathcal{T}_A^{**}$ with its corresponding tagging function h_A^{**}. In this case tags have one of the three possible forms ∞, μ, (σ, x, τ) where $\mu < \alpha$, σ is an ordinal closed under ordinal addition and less than α, $\tau < \sigma$ and $x \in f(\sigma)$. $\mathcal{T}_A^{**}$ has a unique top node and it receives no tag. For any $x \in A$ let

σ_x be the least σ such that σ is closed under addition and $x \in f(\sigma)$. Then for each $x \in A$ there is a unique node on level 1 of $\mathcal{T}_A^{**}$ tagged with $(\sigma_x, x, 0)$ and there are infinitely many nodes on level 1 tagged with ∞. No other tag is the tag of a node of $\mathcal{T}_A^{**}$ on level 1. If $h_A^{**}(s) = \infty$, then s has a unique immediate successor tagged with σ, for each $\sigma < \alpha$, and s has infinitely many immediate successors tagged with ∞. No other tag is the tag of an immediate successor of s on $\mathcal{T}_A^{**}$. If $h_A^{**}(s) = (\sigma, x, \tau)$, then s has a unique immediate successor tagged with (σ, y, μ) for each $y \in x$, $\mu < \sigma$ (μ may be $\geq \tau$). Also s has a unique immediate successor tagged with μ, for each $\mu < \sigma$, and no immediate successors with any other tag.

Finally, if $h_A^{**}(s) = \sigma$, let $\hat{\sigma}$ be the greatest ordinal $\leq \sigma$ which is closed under addition. Then s has a unique immediate successor tagged with τ for $\tau < \hat{\sigma}$ and infinitely many immediate successors tagged with τ when $\hat{\sigma} \leq \tau < \sigma$. No other tag is the tag of an immediate successor to s in $\mathcal{T}_A^{**}$. This completes the description of $\mathcal{T}_A^{**}$.

The above definition will appear somewhat less peculiar once the binary relation R for linearizing $\mathcal{T}_A^{**}$ is defined. If s and t are nodes of $\mathcal{T}_A^{**}$ at level 1 and $h_A^{**}(s) = (\sigma_x, x, 0)$, then $R(s, t)$ iff $h_A^{**}(t) = \infty$ or $(\sigma_y, y, 0)$ with $x <_A y$. In addition R orders $\{s \mid s$ is at level 1, $h_A^{**}(s) = \infty\}$ in ordertype $\alpha \cdot \eta$ where $\alpha = \mathrm{ORD}(A)$, η = ordertype of the rationals. Suppose $h_A^{**}(r) = \infty$. If s, t are immediate extensions of r, $h_A^{**}(s) = \sigma$, then $R(s, t)$ iff $h_A^{**}(t) = \infty$ or an ordinal $> \sigma$. In addition R orders $\{s \mid s$ immediately extends r, $h_A^{**}(s) = \infty\}$ in ordertype $\alpha \cdot \eta$. Next suppose $h_A^{**}(r) = (\sigma, x, \tau)$. If s, t are immediate extensions of r, then $R(s, t)$ iff either $h_A^{**}(s) = \mu$ and $h_A^{**}(t) = \mu' > \mu$; or $h_A^{**}(s) = \mu$ and $h_A^{**}(t) = (\sigma, y, \mu')$ some y, μ'; or $h_A^{**}(s) = (\sigma, y, \mu)$, $h_A^{**}(t) = (\sigma, z, \nu)$ with $\mu < \nu$ or with $\mu = \nu$ and $y <_A z$. Finally suppose $h_A^{**}(r) = \sigma$ and s, t are immediate extensions of r. If $h_A^{**}(s) < \hat{\sigma}$, then $R(s, t)$ iff $h_A^{**}(s) < h_A^{**}(t)$. Also if $\hat{\sigma} < \sigma$, then R orders $\{s \mid s$ immediately extends r, $h_A^{**}(s) \geq \hat{\sigma}\}$ in ordertype $\sigma \cdot \omega$ (an ordinal closed under addition). This completes the definition of R.

As in Lemma 4 it is easy to show that $A \subseteq \mathrm{pp\,HYP}(\mathcal{T}_A^{**})$. The above definition of $(\mathcal{T}_A^{**}, R)$ is carefully designed to enable us to show:

Lemma 10. $\mathrm{pp\,HYP}(\mathcal{T}_A^{**}, R) \subseteq A$.

Proof. Define T^{**} analogously to T^*. Thus the axioms for T^{**} are:

(1) $\mathrm{KP}(\leq, \mathcal{T}, \boldsymbol{R})$ + Infinitary Diagram $\langle A, \in \rangle$.

(2) (a) $\leq$ well-orders the universe; $\forall x, y\, (x \in y \to x \leq y)$.

(b) $\forall x\, (x \leq \mathbf{a} \leftrightarrow \bigvee_{b <_A a} x = \boldsymbol{b})$, for each $a \in A$.

(3) $(\mathcal{T}, \boldsymbol{R})$ is a linearized tree (of ordinals) defined from ORD, $\leq$, ∞ in exactly the way $(\mathcal{T}_A^{**}, R)$ was defined from ORD A), $<_A$, ∞.

(4) For all $\sigma \in \mathrm{ORD}$, $\mathcal{T}(\sigma)$ is a set.

Now suppose B is a model of T^{**}. If $\mathrm{ORD}(A) = \alpha \in \text{Standard Part}(B)$, then axioms (2) imply that $A \in B$ and thus B contains an isomorphic copy of $(\mathcal{T}_A^{**}, R)$ since there is such a copy definable over $\langle A, \in, <_A \rangle$.

Otherwise choose a (nonstandard) ordinal $\sigma \in B - A$ which is closed under multiplication and consider $(\mathcal{T}^B(\sigma), R^B(\sigma)) \in B$ where $R^B(\sigma) = R \restriction \mathcal{T}^B(\sigma) \times \mathcal{T}^B(\sigma)$. We claim that this linearized tree is isomorphic to $(\mathcal{T}_A^{**}, R)$.

To see this it suffices to show that, if each node in $\mathcal{T}^B(\sigma)$ which is tagged with a nonstandard tag is retagged with ∞, then the resulting linearized tree obeys the prescription for $(\mathcal{T}_A^{**}, R)$. (By a nonstandard tag we mean a tag τ or (τ, x, μ) where τ is not an element of A.) It is the demonstration of this fact that uses the details of our definition of $(\mathcal{T}_A^{**}, R)$.

Using Lemma 9 and the closure property of σ it is easy to check that the ordertype of $<^B \restriction (<^B$-predecessors of $\sigma)$ is $\alpha + \alpha \cdot \eta$. Thus on the first level of $\mathcal{T}^B(\sigma)$ the nodes with nonstandard tags follows the nodes with standard tags under R^B and are ordered by R^B in ordertype $\alpha \cdot \eta$. Thus if these nodes are retagged with ∞, then the prescription for $(\mathcal{T}_A^{**}, R)$ is met as far as nodes at level 1.

Now suppose that $s \in \mathcal{T}^B(\sigma)$ receives a nonstandard tag. We must show that the immediate extensions of s are linearly ordered by R^B in ordertype $\alpha + \alpha \cdot \eta$ with initial segment consisting of node tagged with ordinals $\sigma < \alpha$ (in their natural order) followed by nodes with nonstandard tags. First consider the case in which s has an ordinal tag τ. If τ is closed under addition then the immediate extensions of s are ordered by R^B in the odertype of the ordinal predecessors of τ. As τ is closed under addition this ordertype must be $\alpha + \alpha \cdot \eta$. And clearly the nodes with standard ordinal tags form an initial segment (in the natural order) of ordertype α in this linear ordering. If τ is not closed under addition, then the immediate extensions of s are ordered in the ordertype of the ordinal predecessors to $\hat{\tau} + \tau \cdot \omega$ with standardly-tagged nodes forming an initial segment (in the natural order) of ordertype α. But of course $\hat{\tau} + \tau \cdot \omega$ is closed under addition so again this ordertype is what it should be, $\alpha + \alpha \cdot \eta$.

Now consider the case in which s receives a nonstandard tag (τ, x, μ). Let $L =$ ordertype $<^B \restriction \{y \mid y \in^B x\}$. Then the immediate extensions of s are ordered by R^B in ordertype $\tau + L \cdot \tau$. Note that τ is closed under addition and hence the ordinal predecessors of τ have ordertype $\alpha + \alpha \cdot \eta$. It is now easy to see (using Lemma 9) that $\tau + L \cdot \tau$ also has ordertype $\alpha + \alpha \cdot \eta$. Also the only immediate extensions of s with standard tags have ordinal tags and these nodes form an initial segment of ordertype of the immediate extensions of s (under R^B).

Of course $(\mathcal{T}^B(\sigma), R^B(\sigma))$ and $(\mathcal{T}_A^{**}, R)$ agree on nodes with standard tags. This completes the proof that these two structures are isomorphic and hence Lemma 10 is proved.

Thus we have $A = \mathrm{pp\,HYP}(\mathcal{T}_A^{**}, R)$. We let $\mathcal{L}_A^{**}$ denote the linear ordering associated to $(\mathcal{T}_A^{**}, R)$. Note that $\mathrm{HYP}(\mathcal{T}_A^{**}, R) \models \mathcal{L}_A^{**}$ is a well-ordering: For, let B be a model of T^{**} such that the well-founded part of B has height $\alpha = \mathrm{ORD}(A)$. The existence of such a B follows from the Hard Core Theorem. If $\sigma \in \mathrm{ORD}(B)$ let $\mathcal{L}(\sigma)$ be the linear ordering in B associated to $(\mathcal{T}^B(\sigma), R^B(\sigma))$.

Then $B \models \mathscr{L}(\sigma)$ is a well-ordering so $\mathrm{HYP}(\mathscr{T}^{**}_{\mathrm{A}}, R) \models \mathscr{L}(\sigma)$ is a well-ordering, since B contains an isomorphic copy of $(\mathscr{T}^{**}_{\mathrm{A}}, R)$. But if $\sigma \in B - A$ is chosen to be closed under multiplication then $\mathscr{L}(\sigma)$ is isomorphic to $\mathscr{L}^{**}_{\mathrm{A}}$.

Now enlarge $\mathscr{L}^{**}_{\mathrm{A}}$ to a linear ordering W_{A} as follows: To each $s \in |\mathscr{L}^{**}_{\mathrm{A}}|$ which represents a node at level n of $\mathscr{T}^{**}_{\mathrm{A}}$ add a chain of n points immediately after s. Thus $|W_{\mathrm{A}}| = \{(s, i) \mid s$ is a node of $\mathscr{T}^{**}_{\mathrm{A}}$ at level n, $0 \leq i \leq n\}$ and (s, i) is less than (s', i') in W_{A} iff s is less than s' in $\mathscr{L}^{**}_{\mathrm{A}}$ or ($s = s'$ and $i < i'$). In addition let $U \subseteq |W_{\mathrm{A}}|$ consist of those points of the form (s, i) where $i > 0$.

It is easy to see that $\mathrm{HYP}(\mathscr{T}^{**}_{\mathrm{A}}, R) = \mathrm{HYP}(W_{\mathrm{A}}, U)$. The argument is virtually identical to the earlier proof that $\mathrm{HYP}(\mathscr{T}^{*}_{\mathrm{A}}, R) = \mathrm{HYP}(L_{\mathrm{A}})$. Finally suppose $\mathrm{HYP}(W_{\mathrm{A}}, U) \models W_{\mathrm{A}}$ is not a well-ordering. Choose a sequence $(s_0, i_0), (s_1, i_1), \ldots,$ which descends through W_{A} and which belongs to $\mathrm{HYP}(W_{\mathrm{A}}, U)$. Then there must be an infinite $X \subseteq \omega$ such that $X \in \mathrm{HYP}(W_{\mathrm{A}}, U)$ and $n, m \in X$, $n < m \rightarrow s_m$ is less than s_n in W_{A}. Thus $\mathrm{HYP}(W_{\mathrm{A}}, U) = \mathrm{HYP}(\mathscr{T}^{**}_{\mathrm{A}}, R) \models W_{\mathrm{A}}$ is not a well-ordering, contradicting our earlier claim.

By invoking Lévy-absoluteness in the uncountable case, we now have completed the proof of Theorem 8.

Remarks. (a) In the proofs of Lemmas 2, 5, 10 the break into cases as to whether or not $\mathrm{ORD}(A) \in$ Standard Part of B is in fact unnecessary. Instead one may use the following mild strengthening of the Hard Core Theorem: If T is a theory Σ_1 over the countable admissible structure $\langle A, \epsilon, \ldots \rangle$ and x belongs to the standard part of every model B of T such that $\mathrm{ORD}(A) \notin B$, then $x \in A$.

(b) In case A is countable, resolvable and $A \models$ Every set can be mapped 1–1 into the ordinals, then the proof of Theorem 8 becomes much simpler. The reason is that in this case A is of the form $L_\alpha[P]$, $P \subseteq \alpha$, and the structure $\langle W, <, U \rangle$ can be taken to be a nonstandard version of $\langle \alpha, \epsilon, P \rangle$.

3. Applications to models of analysis

The moral of this section is the following: By infinitary model theory (as in the preceding sections) one can build structures $\mathcal{M}$ with specific definability-theoretic properties. By generically collapsing $\mathcal{M}$ to ω these definability properties are not damaged, thereby yielding structures built over ω (reals or sets of reals) with similar properties. This forcing can be viewed as a set-forcing over $\mathrm{HYP}(\mathcal{M})$ (the collection of forcing conditions forms an element of $\mathrm{HYP}(\mathcal{M})$) and thus is easily analyzed.

In a sense Steel forcing performs both of the above tasks simultaneously, thereby necessitating the use of class forcing. However it is more difficult to argue that admissibility is not destroyed when using class forcing.

We will do three applications. The first is a result of G. Sacks [ ] which states that if α is a countable admissible ordinal then $\alpha = \omega_1^T$ for some $T \subseteq \omega$. This example reveals the essence of our method. The remaining two applications are to results of Steel's thesis which appear in [9]. The first states that if $\lambda < \omega_1$ then there are $T \subseteq \omega$ and $\Pi_1^0(T)$-singletons f, g such that $f \notin L_\lambda(g, T)$, $g \notin L_\lambda(f, T)$. (f is a $\Pi_1^0(T)$-singleton if f is the unique solution to a predicate which is Π_1^0 in the parameter T.) This application was observed jointly by Leo Harrington and the author. Finally we prove the existence of an ω-model of Δ_1^1-CA which does not satisfy Σ_1^1-AC, the result which inspired the development of Steel forcing. An ω-model of Δ_1^1-CA is a set of reals $\mathcal{S}$ which is closed under pairing and satisfies:

$$\Delta_1^1\text{-CA:}\ \forall n\,(\exists X\,\varphi(n, X) \leftrightarrow \sim\exists Y\,\psi(n, Y)) \rightarrow \exists Z\,\forall n\,(n \in Z \leftrightarrow \exists X\,\varphi(n, X))$$

where φ and ψ are arithmetic formulas involving arbitrary parameters from $\mathcal{S}$. An ω-model of Σ_1^1-AC must also satisfy:

$$\Sigma_1^1\text{-AC:}\ \forall n\,\exists X\,\varphi(n, X) \rightarrow \exists Y\,\forall n\,\varphi(n, (Y)_n)$$

where φ is as before and $(Y)_n = \{m \mid 2^n 3^m \in Y\}$.

Theorem 11 (Sacks). *Every countable admissible ordinal $>\omega$ is of the form ω_1^T for some $T \subseteq \omega$.*

Proof. Recall that ω_1^T = first ordinal α such that $L_\alpha(T)$ is admissible. Fix α to be a countable admissible ordinal $>\omega$. We let $\mathcal{T}$ be the following tree (which is essentially the tree obtained by Steel forcing over L_α): $\mathcal{T}$ has a unique top node, tagged with ∞. If $s \in |\mathcal{T}|$ is tagged with ∞, then s has infinitely many immediate extensions tagged with ∞ and a unique immediate extension tagged with β, for each $\beta < \alpha$. If $s \in |\mathcal{T}|$ is tagged with β, then s has a unique immediate extension tagged with γ, for each $\gamma < \beta$. No tags appear other than those mentioned above.

Now consider the theory KP($\mathcal{T}$) + Infinitary Diagram(L_α) + $\mathcal{T}$ is a tree of ordinals defined from ORD, ∞ in exactly the way $\mathcal{T}$ is defined from α, ∞ + $\forall\sigma$ ($\mathcal{T}(\sigma)$ is a set). This theory is Σ_1 over (L_α, ϵ) and $\mathcal{T}$ has an isomorphic copy in any model of this theory. As any theory Σ_1 over (L_α, ϵ) which extends KP has a model B with $\alpha \notin$ Standard Part(B), we see that HYP($\mathcal{T}$) $\cap$ ORD $\subseteq \alpha$. As $\mathcal{T}$ has a node of rank β, for each $\beta < \alpha$, we actually have HYP($\mathcal{T}$) $\cap$ ORD $= \alpha$. Thus HYP($\mathcal{T}$) $= L_\alpha(\mathcal{T})$.

Now collapse $|\mathcal{T}|$ to ω by doing Lévy forcing over HYP($\mathcal{T}$). Thus a condition is a finite function $p : n \xrightarrow{1-1} |\mathcal{T}|$, $n < \omega$, and $p \leq q$ iff p extends q. Forcing is defined à la Cohen. The set of conditions is an element of HYP($\mathcal{T}$) and from this it follows easily that the forcing relation is Σ_1 over HYP($\mathcal{T}$) when restricted to Σ_1 statements, and that if G is generic over HYP($\mathcal{T}$) for this forcing then $L_\alpha(\mathcal{T}, G)$ is admissible. Define a tree T_G on ω by: n extends m in T_G iff $G(n)$ extends $G(m)$ in $\mathcal{T}$. Then $T_G \in L_\alpha(\mathcal{T}, G)$ so $L_\alpha(T_G)$ is admissible. But as T_G is isomorphic to $\mathcal{T}$ and α = HYP($\mathcal{T}$) $\cap$ ORD, $L_\beta(T_G)$ cannot be admissible for any $\beta < \alpha$.

There are simpler proofs of Sacks' Theorem than that given above. However we included it as it is the prototype of our technique. Moreover a slight modification of it yields a simple proof of Steel's result that, if $\omega_1^R = \omega_1^S = \alpha$, then for some T $\omega_1^T = \omega_1^{(R,T)} = \omega_1^{(S,T)} = \alpha$. The only previously known proof of this strengthening used Steel's forcing.

Theorem 12 (Steel). *If λ is countable, then there are $T \subseteq \omega$ and $f, g \in \omega^\omega$ such that f, g are $\Pi_1^0(T)$-singletons and $f \notin L_\lambda(g, T)$, $g \notin L_\lambda(f, T)$.*

Proof (Jointly with Leo Harrington). Let λ be countable and admissible. Define terms $\tau \in L_\lambda$ so that for any f, T each element of $L_\lambda(f, T)$ is denoted by some term τ applied to f, T and such that the evaluation of terms is Σ_1 over $L_\lambda(f, T)$. Thus we wish to construct T and $\Pi_1^0(T)$-singletons f, g such that for any term $\tau \in L_\lambda$, $f \neq \tau(g, T)$ and $g \neq \tau(f, T)$.

Begin by defining the following tree $\mathcal{T}_\lambda$ with two distinguished paths $f_\lambda, g_\lambda : \mathcal{T}_\lambda$ has a unique top node (which equals $f_\lambda(0) = g_\lambda(0)$). Each node at level 1 except $f_\lambda(1)$, $g_\lambda(1)$ receives an ordinal tag $\gamma < \lambda$ and each tag $\gamma < \lambda$ is the tag of only 1 node at level 1. Also $f_\lambda(1) \neq g_\lambda(1)$. Any node tagged with γ has a unique immediate successor tagged with γ', for each $\gamma' < \gamma$. Any node $f_\lambda(n)$ has the immediate extension $f_\lambda(n+1)$ and an immediate extension tagged with γ, for each $\gamma < \lambda$. Similarly for $g_\lambda(n)$.

Thus f_λ, g_λ are the unique paths through $\mathcal{T}_\lambda$. We claim that, if τ is a term in L_λ and $s \in |\mathcal{T}_\lambda|^{<\omega}$, then $f_\lambda \neq \tau(g_\lambda, \mathcal{T}_\lambda, s)$ as elements of $L_\lambda(f_\lambda, g_\lambda, \mathcal{T}_\lambda)$. Otherwise consider the theory KP$(\mathcal{T}, \mathbf{f}, \mathbf{g})$ + Infinitary Diagram(L_λ) + $(\mathcal{T}, \mathbf{f}, \mathbf{g})$ is built from $\boldsymbol{\lambda}$ exactly the way $(\mathcal{T}_\lambda, f_\lambda, g_\lambda)$ was built from λ + $\mathbf{s} \in |\mathcal{T}|^{<\omega}$ + $\mathbf{f} = \tau(\mathbf{g}, \mathcal{T}, \mathbf{s})$. This theory has a model B whose standard part has height λ. But there is an automorphism φ of $\mathcal{T}^B$ fixing g^B, s^B and moving f^B, obtained by choosing a descending sequence $\lambda_0 > \lambda_1 > \cdots$ through ORD(B) and (a) switching $f^B(n+1)$ with some immediate extension of $\varphi(f^B(n))$ tagged with λ_n, whenever $f^B(m) \notin s^B$ for all $m > n$, (b) corresponding the immediate extensions of $f^B(n)$ with nonstandard tags with the remaining immediate extensions of $\varphi(f^B(n))$ which do not have standard tags. Thus we get $f^B \neq \varphi(f^B) = \tau(\varphi(g^B), \varphi(\mathcal{T}^B), \varphi(s^B)) = \tau(g^B, \mathcal{T}^B, s^B) = f^B$. Contradiction.

Now Lévy collapse $|\mathcal{T}_\lambda|$ to ω using finite conditions, viewing $L_\lambda(\mathcal{T}_\lambda, f_\lambda, g_\lambda)$ as the ground model. Note that for any $\gamma < \lambda$, the forcing relation is an element of $L_\lambda(\mathcal{T}_\lambda, g_\lambda)$ when restricted to statements of rank $\leq \gamma$ not mentioning f_λ and so can be named by a term $\sigma(\mathcal{T}_\lambda, g_\lambda)$, $\sigma \in L_\lambda$.

Let $G : \omega \to |\mathcal{T}_\lambda|$ be generic over $L_\lambda(\mathcal{T}_\lambda, f_\lambda, g_\lambda)$ for this forcing and define T_G by: n extends m in T_G iff $G(n)$ extends $G(m)$ in $\mathcal{T}_\lambda$. Also let $f_G = G^{-1}(f_\lambda)$, $g_G = G^{-1}(g_\lambda)$. Then f_G, g_G are $\Pi_1^0(T_G)$-singletons as f_G = unique path through T_G extending $f_G(1)$, g_G = unique path through T_G extending $g_G(1)$.

We claim that for any term $\tau \in L_\lambda$, $f_G \neq \tau(T_G, g_G)$. Otherwise choose a condition

p such that $p \Vdash f_G = \tau(T_G, g_G)$. Then $s \in f_\lambda \leftrightarrow$

$$\exists q \leq p\, q \Vdash \boldsymbol{G}^{-1}(s) \in f_G \leftrightarrow \exists q \leq p\, q \Vdash \boldsymbol{G}^{-1}(s) \in \tau(T_G, g_G)$$

and this last condition can be described by a term $\sigma(\mathcal{T}_\lambda, g_\lambda, p)$, $\sigma \in L_\lambda$. This contradicts our earlier claim. By symmetry $g_G \neq \tau(T_G, f_G)$ and we are done.

We note that Steel's result (included in [9]) on the relativized McLaughlin conjecture can also be obtained by techniques similar to those used in Theorem 12. We now proceed to our most elaborate application.

Theorem 13 (Steel). *There is an ω-model of $\boldsymbol{\Delta}_1^1$-CA which does not satisfy $\boldsymbol{\Sigma}_1^1$-AC.*

Proof. Begin by defining the following tree $\mathcal{T}$ and collection $\mathcal{P}$ of paths through $\mathcal{T}$: $\mathcal{T}$ has a unique top node which is untagged. Each untagged node of $\mathcal{T}$ has infinitely many untagged immediate extensions, infinitely many immediate extensions tagged with ∞ and a unique immediate extension tagged with α, for each $\alpha < \omega_1^{\text{CK}}$. Each node tagged with ∞ has infinitely many immediate extensions tagged with ∞ and a unique immediate extension tagged with α, for each $\alpha < \omega_1^{\text{CK}}$. Each node tagged with $\alpha < \omega_1^{\text{CK}}$ has a unique immediate extension tagged with β, for each $\beta < \alpha$. $\mathcal{P}$ is obtained by choosing for each untagged node τ a path through $\mathcal{T}$ which passes through τ and only untagged nodes.

We consider the theory $T = \text{KP} + \text{Infinitary Diagram}(L_{\omega_1^{\text{CK}}}) + \boldsymbol{a}$ is an ordinal $+ \{\boldsymbol{a} > \boldsymbol{\alpha} \mid \alpha < \omega_1^{\text{CK}}\} + \boldsymbol{\mathcal{T}}$ is a tree, $\boldsymbol{\mathcal{P}}$ a set of paths through $\boldsymbol{\mathcal{T}}$ defined from $\boldsymbol{a}$, ∞ in the same way that $(\mathcal{T}, \mathcal{P})$ was defined from ω_1^{CK}, ∞. If B is a countable model of T such that $\omega_1^{\text{CK}} \notin \text{Standard Part}(B)$, then $(\mathcal{T}, \mathcal{P})$ is isomorphic to $(\mathcal{T}^B, \mathcal{P}^B)$. So $\text{HYP}(\mathcal{T}, \mathcal{P}) \cap \text{ORD} = \omega_1^{\text{CK}}$.

Claim. *If $X \subseteq \omega$ is Σ_1 over* $\text{HYP}(\mathcal{T}, \mathcal{P})$ *in parameters $\mathcal{T}, f_1, \ldots, f_n$, $\mathcal{P}$ (where $f_1, \ldots, f_n \in \mathcal{P}$), then X is Σ_1 over* $\text{HYP}(\mathcal{T}, f_1, \ldots, f_n)$.

Proof. Consider the theory T' over $\text{HYP}(\mathcal{T}, f_1, \ldots, f_n)$ whose axioms are $\text{KPU} + \text{Infinitary Diagram}(\text{HYP}(\mathcal{T}, f_1, \ldots, f_n)) + \boldsymbol{\mathcal{P}}$ is a set of paths through $\mathcal{T} + f_1, \ldots, f_n \in \boldsymbol{\mathcal{P}} +$ Any node of $\mathcal{T}$ extendible to a member of $\boldsymbol{\mathcal{P}}$ has infinitely many immediate extensions extendible to members of $\boldsymbol{\mathcal{P}}$ + Every set belongs to $\text{HYP}(\mathcal{T}, \boldsymbol{\mathcal{P}})$. If B' is a countable model of T' such that $\omega_1^{\text{CK}} \notin \text{Standard Part}(B')$, then $(\mathcal{T}, \mathcal{P}^{B'})$ is isomorphic to $(\mathcal{T}, \mathcal{P})$. (This is easy to see once it is realized that any node not in the well-founded part of $\mathcal{T}$ must have infinitely many immediate extensions which neither are in the well-founded part of $\mathcal{T}$ nor can be extended to a path in $\mathcal{P}^{B'}$. This follows as otherwise $\omega_1^{\text{CK}} \in \text{Standard Part}(B')$.) Moreover this isomorphism can be chosen to fix $\mathcal{T}, f_1, \ldots, f_n$.

Now suppose that $X \subseteq \omega$ is Σ_1 over $\text{HYP}(\mathcal{T}, \mathcal{P})$ in $\mathcal{T}, f_1, \ldots, f_n$, $\mathcal{P}$; thus $m \in X \leftrightarrow \text{HYP}(\mathcal{T}, \mathcal{P}) \models \varphi(m, \mathcal{P})$ where φ is Σ_1 (and we suppress the parameters $\mathcal{T}, f_1, \ldots, f_n$). We assert that $m \in X \leftrightarrow T' \vdash \varphi(m, \boldsymbol{\mathcal{P}})$, and then we will be done.

Clearly $T' \vdash \varphi(m, \mathscr{P})$ implies $m \in X$ so suppose $T' + \sim\varphi(m, \mathscr{P})$ is consistent. Then $T' + \sim\varphi(m, \mathscr{P})$ has a countable model B' such that $\omega_1^{CK} \notin$ Standard Part(B'), as this theory is Σ_1 over HYP$(\mathscr{T}, f_1, \ldots, f_n)$ and HYP$(\mathscr{T}, f_1, \ldots, f_n) \cap \mathrm{ORD} = \omega_1^{CK}$. But B' is isomorphic to HYP$(\mathscr{T}, \mathscr{P})$ with isomorphism fixing $\mathscr{T}, f_1, \ldots, f_n$ so HYP$(\mathscr{T}, \mathscr{P}) \vDash \sim\varphi(m, \mathscr{P})$. So $m \notin X$. The claim is proved.

Now Lévy collapse $|\mathscr{T}|$ to ω, generically over HYP$(\mathscr{T}, \mathscr{P})$. Choose a generic $G : \omega \to |\mathscr{T}|$ for this forcing and for H a finite subset of $\mathscr{P}$ let $M(H) =$ HYP$(G, \mathscr{T}, H)$. The model we are looking for is $M \cap \omega^\omega$ where $M = \bigcup \{M(H) \mid H$ a finite subset of $\mathscr{P}\}$.

Suppose $X \subseteq \omega$ is Σ_1 over M in parameters $G, \mathscr{T}, f_1, \ldots, f_n$ where $f_1, \ldots, f_n \in \mathscr{P}$. Thus for some $\Sigma_1 \varphi$ we have $m \in X \leftrightarrow \exists p \in G\, p \Vdash (M \vDash \varphi(m, \boldsymbol{G}, \mathscr{T}, f_1, \ldots, f_n)) \leftrightarrow \exists p \in G\, p \Vdash \exists$ finite $H \subseteq \mathscr{P}$, $(f_1, \ldots, f_n \in H$ and $M(H) \vDash \varphi(m, \boldsymbol{G}, \mathscr{T}, f_1, \ldots, f_n))$. This is in the form $\exists p \in G\, \psi(m, p)$ where ψ is Σ_1 over HYP$(\mathscr{T}, \mathscr{P})$ in parameters $\mathscr{T}, f_1, \ldots, f_n, \mathscr{T}$, as the forcing relation is Σ_1 over HYP$(\mathscr{T}, \mathscr{P})$ when restricted to Σ_1 statements. By our earlier claim we see that X is Σ_1 over HYP$(G, \mathscr{T}, f_1, \ldots, f_n)$. Thus if X is both Σ_1 and Π_1 over M in parameters $G, \mathscr{T}, f_1, \ldots, f_n$ then X is Δ_1 over HYP$(G, \mathscr{T}, f_1, \ldots, f_n)$ and hence $X \in M$. So $M \cap \omega^\omega \vDash \boldsymbol{\Delta}_1^1$-CA.

It remains to show that $M \cap \omega^\omega \nvDash \boldsymbol{\Sigma}_1^1$-AC and for this it suffices to see that for any finite $H \subseteq \mathscr{P}$, the members of H are the only paths through $\mathscr{T}$ in $M(H)$. (For then $M \cap \omega^\omega \vDash \forall n\, \exists F : n \xrightarrow{1-1}$ Paths through T_G but $\sim\exists F : \omega \xrightarrow{1-1}$ Paths through T_G, where T_G is defined to make G an isomorphism from T_G onto $\mathscr{T}$.) Suppose $s \in \mathscr{T}$ and s does not lie on any path in H. We show that for any condition $p : n \to |\mathscr{T}|$, $p \nVdash s$ can be extended to a path in HYP$(\boldsymbol{G}, \mathscr{T}, H)$. For, choose a model B of the theory T such that $\omega_1^{CK} \notin$ Standard Part(B) and let φ be an isomorphism of $(\mathscr{T}, \mathscr{P})$ onto $(\mathscr{T}^B, \mathscr{P}^B)$. Choose a nonstandard ordinal $a \in \mathrm{ORD}(B) - \omega_1^{CK}$ and consider $\hat{\mathscr{T}} \in B$ defined by $\hat{\mathscr{T}} = \{s \in \mathscr{T}^B \mid \mathrm{rank}(s) < a$ or s lies on a path in $\varphi(H)\}$. Then $(\mathscr{T}, H)$ is isomorphic to $(\hat{\mathscr{T}}, \varphi(H))$; let ψ be such an isomorphism. Now if $p \Vdash s$ can be extended to a path in HYP$(\boldsymbol{G}, \mathscr{T}, H)$, then $\psi(p) \Vdash \psi(s)$ can be extended to a path in HYP$(\boldsymbol{G}, \hat{\mathscr{T}}, \varphi(H))$. The point is that $B \vDash \psi(s)$ has an ordinal rank in $\hat{\mathscr{T}}$. Therefore for no condition q can we have $q \Vdash \psi(s)$ can be extended to a path in HYP$(\boldsymbol{G}, \hat{\mathscr{T}}, \varphi(H))$ as otherwise $\{b \in \mathrm{ORD}(B) \mid \exists r \leq q\, \exists t \in \hat{\mathscr{T}}$ of rank b $(r \Vdash t$ can be extended to a path in HYP$(\boldsymbol{G}, \hat{\mathscr{T}}, \varphi(H)))\}$ is a definable class of ordinals of B with no least element, contradicting $B \vDash$ KP. Thus we have shown that for no condition p can we have $p \Vdash s$ can be extended to a path in HYP$(\boldsymbol{G}, \mathscr{T}, H)$ and thus $M(H) \vDash$ Any path through $\mathscr{T}$ belongs to H. This completes the proof of Theorem 13.

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