ADVANCES IN MATHEMATICS 61, 1–48 (1986)

𝔊-Modules, Springer's Representations and Bivariant Chern Classes

V. GINSBURG

Vinokurova, 5/6-2-5, 117449 Moscow, USSR

0. INTRODUCTION

0.1. Let G be a connected simply connected complex semi-simple Lie group with Lie algebra $\mathfrak{G}$ and Borel subgroup B.

There are basically two different approaches to the study of simple G-modules. The first one is connected with the Flag manifold $X = G/B$ and G-modules are related to vector bundles and sheaves on X. The second approach, the so-called Kirillov–Kostant "orbit method," links representations with coadjoint orbits in the dual space $\mathfrak{G}^*$ of the Lie algebra $\mathfrak{G}$. In 1981 Borho and Brylinski and the author independently developed the idea of joining these two pictures together, thus describing $\mathfrak{G}$-modules and primitive ideals by their characteristic varieties in $\mathfrak{G}^* \times X$.

In a recent paper Kashiwara and Tanisaki [KT] related characteristic cycles of holonomic systems on the Flag manifold to Weyl group representations. Here we explain that relation by means of a new "Lagrangian" construction of Springer's representations. We also show that Springer's representations are in turn just a special case of the general bivariant theory of Chern classes for singular varieties.

With regard to $\mathfrak{G}$-modules we will give two proofs of the irreducibility of the associated variety of a primitive ideal in an enveloping algebra of a complex semi-simple Lie algebra. The first of them (see Section 1) appeared in [KT] and is heavily based on results of Joseph. The second (see Section 8), while independent of Joseph's results, uses a certain property of Kazhdan–Lusztig cells, verified by Barbasch and Vogan and Lusztig.

0001-8708/86 $7.50

0.2. Most of the recent advances in primitive ideal and representation theories were based on the observation that irreducible $\mathfrak{G}$-modules are related somehow to irreducible Weyl group representations. The simpliest way to explain this phenomenon is to refer to a finite field case. So let $G(\mathbb{F}_q)$ be a split reductive group over a finite field $\mathbb{F}_q$ and let B be a Borel subgroup of $G(\mathbb{F}_q)$ (defined over $\mathbb{F}_q$). A considerable amount of simple $G(\mathbb{F}_q)$-modules arises from decomposition of the regular $G(\mathbb{F}_q)$-representation on the space $\mathbb{C}(G(\mathbb{F}_q)/B)$ of complex-valued functions on $G(\mathbb{F}_q)/B$. This decomposition is governed by the Hecke algebra $H(q)$ consisting of B-bi-invariant functions on $G(\mathbb{F}_q)$ (with respect to convolution). $H(q)$ acts on $\mathbb{C}(G(\mathbb{F}_q)/B)$ on the right commuting with $G(\mathbb{F}_q)$-action on the left. So irreducible $G(\mathbb{F}_q)$-modules appearing in $\mathbb{C}(G(\mathbb{F}_q)/B)$ are in 1–1 correspondence with irreducible $H(q)$-modules. Further, the algebra $H(q)$ is known to be (unnaturally) isomorphic to the group-algebra $\mathbb{C}[W]$ of the Weyl group. Whence a desired correspondence: $G(\mathbb{F}_q)$-modules $\leftrightarrow$ W-modules.

What one wants to have for complex groups is a similar correspondence: $\mathfrak{G}$-modules $\leftrightarrow$ $H(q)$-modules (where q is regarded now as an indeterminate). This is beyond our reach. At present we are able only to establish some kind of that correspondence with $H(q)$ replaced by the group algebra $\mathbb{C}[W]$ that is by putting: $q=1$. Although $H(q)$ is isomorphic to $\mathbb{C}[W]$ this isomorphism is unnatural. Moreover, there is a strong evidence to expect that there is a *natural* geometric parametrization of irreducible $H(q)$-modules degenerating to Springer's theory [Spr 1] when $q=1$. It is essential however that this (conjectural) parametrization depends on q in a continuous fashion when $q \neq 1$ but has a discontinuity at $q=1$. We believe that $\mathfrak{G}$-modules should be related to the "assymptotical parametrization" as $q \underset{\neq}{\to} 1$ rather than to that given by Springer for $q=1$. The parametrization for $q \neq 1$ seems to be related to Kazhdan–Lusztig cells [KL 1] in the same way as Springer's theory is related to "geometric" cell's constructed in Section 5 of the present paper. The difference between these two pictures is perhaps responsible for various visible complications such as: non-irreducibility of Kazhdan–Lusztig cell's under W-action, non-irreducibility of characteristic varieties of simple $\mathfrak{G}$-modules, appearance of the "special" nilpotent orbits, etc....

Let us now indicate how $\mathfrak{G}$-modules are related to W-modules in this paper. The first more traditional way (see Sect. 1) is to consider a (formal) character of a $\mathfrak{G}$-module as function on a Cartan subalgebra h. Suppose for instance that V_λ is a finite dimensional irreducible module. Its character is given by the classical formula of Weyl: $\chi_\lambda = \sum a_w \exp w \cdot (\lambda+\rho) / \prod_{\alpha>0} (\exp(\alpha/2) - \exp(-\alpha/2))$, where $a_w = \varepsilon(w)$ is the irreducible "sign"-representation of the Weyl group. The same formula holds for an infinite dimensional $\mathfrak{G}$-module V, provided $\varepsilon(w)$'s are replaced by certain integers

a_w. If V is irreducible then a_w's generate a W-module which is almost irreducible in the sense that it contains a distinguished irreducible submodule and all other submodules are "smaller" (with respect to a certain ordering) than this main one. This irreducible W-submodule is attached to the original $\mathfrak{G}$-module.

There is another way to establish (essentially the same) relation between $\mathfrak{G}$-modules and W-modules. It consists of three steps. First, to a $\mathfrak{G}$-module M we associate, following [BeBe] and [BK], a $\mathscr{D}$-module $\mathscr{M} = \mathscr{D}_X \otimes_{U(\mathfrak{G})} M$. Then we consider its characteristic cycle gr $\mathscr{M}$ which is a Lagrangian cycle in the cotangent bundle T^*X. Finally the "Lagrangian" construction for Springer's representations developed in Section 5 is used to relate Lagrangian cycles to W-modules.

0.3. Let us fix some notations in order to state a few conjectures. Let $\mathbf{b} = \mathbf{h} + \mathbf{n}$ be the Borel subalgebra, corresponding to B; W be the Weyl group; Δ^- be the negative Weyl chamber in $\mathbf{h}^*$; and ρ be half the sum of positive roots. For $\chi \in \Delta^-$ denote by $M_{w \cdot \chi}$ the Verma module of the highest weight $w \cdot \chi - \rho$, by $L_{w \cdot \chi}$ its simple quotient, and by $I_{w \cdot \chi} = \operatorname{Ann} L_{w \cdot \chi}$ the corresponding primitive ideal in the enveloping algebra $U(\mathfrak{G})$.

On the Flag manifold consider the Schubert cell stratification: $X = \bigcup X_w$ and the twisted holonomic system $\mathscr{L}_{w \cdot \chi}$ corresponding to $L_{w \cdot \chi}$. Its characteristic variety $SS\mathscr{L}_{w \cdot \chi}$ is a union of certain conormal bundles $T^*_{X_w}X$. On the other hand consider a non-holonomic module, corresponding to $U(\mathfrak{G})/I_{w \cdot \chi}$, and set $S(I_{w\chi}) = SS(\mathscr{D}_\chi \otimes_U U(\mathfrak{G})/I_{w\chi})$ Note that according to Proposition 8.3:

$$S(I_{w \cdot \chi}) = G \cdot SS\mathscr{L}_{w \cdot \chi}.$$

Conjecture. For each $w \in W$ one can find $\chi \in \mathbf{h}^*$ such that $S(I_{w \cdot \chi}) = G \cdot \overline{T^*_{X_w}X}$.

This conjecture is supported by [BV, II, Theorem 4.5]. Of course, the weight χ may be non-integral.

0.4. Next suppose $\chi = -\rho$ is the integral weight. We set $I_w = I_{-w \cdot \rho}$, $\mathscr{L}_w = \mathscr{L}_{-w\rho, \ldots}$, etc.

THEOREM. *The associated variety* Var (I_w) *is the closure of a nilpotent orbit in* $\mathfrak{G}^*$.

This was proved by Joseph and Kashiwara–Tanisaki [KT] (see also Sections 1 and 8.6 of the present paper) and by Borho and Brylinski [BB]. The following strengthening of the theorem still remains a challenging problem.

Conjecture.[1] $S(I_w)$ is irreducible.

It is natural to suggest the following:

Conjecture. $S(I_w) \subset S(I_y)$ iff $I_y \subset I_w$.

0.5. Let $M = M_\rho$ be the Verma module of the zero highest weight. For any primitive ideal $I = I_w$ the quotient $M/I \cdot M$ is known to contain the unique simple submodule $L_y \subset M/I \cdot M$. The corresponding element $y = y(I) \in W$ is actually an involution.

Conjecture. $S(I) = G \cdot \overline{T^*_{X_y} X}$ for $y = y(I)$.

1. Irreducibility of the Associated Variety (after [Jo4; Jo5; KT])

1.1. We keep to the notations of 0.4. On $U(\mathfrak{G})$ consider the natural increasing filtration $U_0 \subset U_1 \subset U_2 \subset \cdots$. Then $\operatorname{gr} U(\mathfrak{G}) \simeq \oplus (U_i/U_{i-1}) \simeq \mathbb{C}[\mathfrak{G}^*]$. The associated variety $\operatorname{Var}(I_w)$ of the primitive ideal $I_w \subset U(\mathfrak{G})$ is by definition the zero variety of the associated graded ideal $\operatorname{gr} I_w = \oplus (I_w \cap U_i / I_w \cap U_{i-1}) \subset \mathbb{C}[\mathfrak{G}^*]$. We prove here, following [KT], that the variety $\operatorname{Var}(I_w)$ is irreducible.

THEOREM 1.1. [KT]. $\operatorname{Var}(I_w)$ *is the closure of a nilpotent orbit in* $\mathfrak{G}^*$.

In subsequent sections we will give proofs for some of the statements below.

1.2. Let $\mathbf{b} = \mathbf{h} + \mathbf{n}$ be a Borel subalgebra and T be the Cartan subgroup, with Lie algebra $\mathbf{h}$. Consider the polynomial algebra $\mathbb{C}[\mathbf{n}]$ and adjoint T-action on it. Suppose M is a $\mathbb{C}[\mathbf{n}]$-module of finite type with a T-action on M, compatible with that on $\mathbb{C}[\mathbf{n}]$, i.e., $g \cdot (a \cdot m) = (g \cdot a) \cdot (g \cdot m)$, $g \in T$, $a \in \mathbb{C}[\mathbf{n}]$, $m \in M$. Furthermore, we assume this action to be locally finite so that there is a root-space decomposition: $M = \oplus_{\lambda \in \mathbf{h}^*} M^\lambda$, where $M^\lambda = \{m \in M \mid h \cdot m = \lambda(h) \cdot m,\ h \in \mathbf{h}\}$. Introduce the formal character

$$\chi_M(h) = \sum_{\lambda \in \mathbf{h}^*} \dim M^\lambda \cdot e^{\lambda(h)}, \qquad h \in \mathbf{h}.$$

PROPOSITION 1.2. (i) χ_M *is a meromorphic function on* $\mathbf{h}$ *of the form*

$$\chi_M(h) = P(h) \Big/ \prod_{\alpha > 0} (e^{\alpha(h)/2} - e^{-\alpha(h)/2}),$$

[1] See also: Open problems suggested by W. Borho, J.-L. Brylinski, and R. MacPherson, *in* "Open Problems in Algebraic Groups," Conference held in Katata (Japan), August–September 1983.

where P is a holomorphic function on $\mathbf{h}$ and $\prod_{\alpha>0}$ denotes the product for all positive roots α.

(ii) *For regular h the meromorphic function $t \mapsto \chi_M(t \cdot h)$, $t \in \mathbb{C}$, has a pole at* 0 *of order equal to* dim supp M.

This is in fact nothing but the T-equivariant version of Hilbert–Serre's theorem on Poincaré series of a graded module. The proof is also usual (see, e.g., [AM])

In view of 1.2(i) we have the asymptotic expansion into a Laurent series

$$\chi_M(t \cdot h) = (t^k \cdot p_M^k(h) + t^{k+1} \cdot p_M^{k+1}(h) + \cdots) \Big/ \prod_{\alpha>0} \alpha(t \cdot h)$$

where p_M^j is a certain homogeneous polynomials on $\mathbf{h}$ of degree j. Note that according to (ii) the degree of the first of them equals $k = \dim \mathbf{n} - \dim \operatorname{supp} M$. Set $p_M = p_M^k$.

1.3. Let $\mathcal{O} \subset \mathfrak{G}^*$ be a nilpotent orbit.

LEMMA 1.3.1. [Gi1]; see also Section 4.3 of the present paper). *All irreducible components of $\mathcal{O} \cap \mathbf{n}$ are of the same dimension, equal to $\frac{1}{2} \cdot \dim \mathcal{O}$.*

Clearly each irreducible component of $\overline{\mathcal{O} \cap \mathbf{n}}$ is a homogeneous T-stable subvariety in $\mathbf{n}$. For such a component F consider the graded $\mathbb{C}[\mathbf{n}]$-module $\mathbb{C}[F]$. Let $p_F = p_{\mathbb{C}[F]}$ be the corresponding homogeneous polynomial on $\mathbf{h}$ of degrees $\dim \mathbf{n} - \frac{1}{2} \cdot \dim \mathcal{O}$. Consider the action of the Weyl group on $\mathbb{C}[\mathbf{h}]$.

PROPOSITION 1.3.2 ([H2]; see also the preprint version of [Gi3]). *The polynomials p_F (F runs through the irreducible component of $\overline{\mathcal{O} \cap \mathbf{n}}$) form a basis of an irreducible W-module. This module coincides with the irreducible W-module, associated with $\mathcal{O}$ via the Springer correspondence (see* [H1; BM; Spr1]).

Recall that according to Springer's theory there is a 1–1 correspondence between irreducible W-modules and pairs $(\mathcal{O}, \rho_{\mathcal{O}})$, where $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{G}^*$ and $\rho_{\mathcal{O}}$ is an irreducible representation of its Poincaré group $\pi_1(\mathcal{O})$ (see, e.g., [BM]). W-Modules emerging in the proposition are exactly those corresponding to pairs $(\mathcal{O}, \rho_{\mathcal{O}} = 1)$.

1.4. The proof of the following fact is standard:

LEMMA 1.4.1. *Let M be a graded $\mathbb{C}[\mathbf{n}]$-module with a T-action as in* 1.2. *Suppose that the variety $S = \operatorname{supp} M$ is irreducible and let m be the multiplicity of M at a generic point of S. Then*

$$p_M = m \cdot p_{\mathbb{C}[S]}.$$

COROLLARY 1.4.2. *Let M be a graded $\mathbb{C}[\mathfrak{n}]$-module with a T-action, $d = \dim \operatorname{supp} M$ and let $\{S_i\}$ be the collection of d-dimensional irreducible components of* supp M. *Then*

$$p_M = \sum m_i \cdot p_{\mathbb{C}[S_i]},$$

where m_i is the multiplicity of M at S_i.

1.5. Consider the simple $\mathfrak{G}$-module L_w (see the Introduction). Recall that its formal character looks like

$$\chi_{L_w} = \left(\sum_{y \in W} \chi_{w,y^{-1}} \cdot e^{y \cdot \rho} \right) \Big/ \prod_{\alpha > 0} (e^{\alpha/2} - e^{-\alpha/2}) \tag{1.5.1}$$

for certain integers $\chi_{w,y}$. As in 1.2, one has an asymptotic expansion $\chi_{L_w}(t \cdot h) = (t^k \cdot q_w^k(h) + t^{k+1} \cdot q_w^{k+1}(h) + \cdots)/\prod_{\alpha > 0} \alpha(t \cdot h)$. It clearly follows from (1.5.1) that $q_w^k = \sum \chi_{w,y} \cdot y^{-1} \cdot \rho^k$.

Choose a good filtration on L_w and consider the associated graded $\mathbb{C}[\mathfrak{G}^*]$-module gr L_w. Since the action of T on L_w is locally finite supp gr L_w is a homogeneous T-stable subvariety of $\mathfrak{n}$. Further note that L_w and gr L_w have the same formal character. Hence $p_{L_w} = p_{\operatorname{gr} L_w} = q_w^k$. Thus we get

PROPOSITION 1.5. (i) $p_w = \sum \chi_{w,y} \cdot y^{-1} \cdot \rho^{d(w)}$, $d(w) = \dim \operatorname{supp} \operatorname{gr} L_w$;
(ii) *if $k < d(w)$ then* $\sum \chi_{w,y} \cdot y^{-1} \cdot \rho^k = 0$.

It follows from (ii) that p_w is a harmonic polynomial on $\mathbf{h}$ with respect to W-action (see [Jo4]).

1.6. The following results of A. Joseph are crucial. Their proof involves a complicated analysis of Goldie-rank functions.

THEOREM 1.6.1 [Jo4]. *The polynomial p_w generates an irreducible W-module.*

THEOREM 1.6.2 [Jo4]. $I_w = I_{w'}$ *iff* $p_w = \text{const}\ p_{w'}$.

1.7. The following proposition was proved by Borho and Brylinski [BB] and the author (see Section 8.2).

PROPOSITION 1.7.1. $\operatorname{Var}(I_w) = G \cdot (\operatorname{supp} \operatorname{gr} L_w)$.

From this proposition it is possible to deduce (see Section 8):

PROPOSITION 1.7.2. [Jo6]. $\dim \operatorname{Var}(I_w) = 2 \cdot \dim \operatorname{supp} \operatorname{gr} L_w$.

One also has

THEOREM 1.7.3 [(Gabber and Kashiwara, see [Le] and also [Gi4])]. *All irreducible components of* supp gr L_w *are of the same dimension.*

1.8. *Proof of Theorem* 1.1. It is easy to see that $\mathrm{Var}(I_w)$ is contained in the nilpotent variety in $\mathfrak{G}^*$.

Since the number of nilpotent orbits is finite each irreducible component of $\mathrm{Var}(I_w)$ is the closure of a certain orbit $\mathcal{O}_i$ so that $\mathrm{Var}(I_w) = \bigcup \bar{\mathcal{O}}_i$. For trivial reasons supp $L_w \subset \mathrm{Var}(I_w)$ and $\mathrm{supp}\, L_w \subset \mathbf{n}$. Hence supp $L_w \subset U(\bar{\mathcal{O}}_i \cap \mathbf{n})$. Let S_i be the part of $\mathrm{supp}\, L_w$ contained in $\bar{\mathcal{O}}_i \cap \mathbf{n}$. Proposition 1.7.1 implies that each S_i is non-empty. According to Lemma 1.3.1 we have $\dim S_i \leqslant \frac{1}{2} \cdot \dim \mathcal{O}_i \leqslant \frac{1}{2} \cdot \dim \mathrm{Var}(I_w)$.

Theorem 1.7.3, combined with Proposition 1.7.2, shows that $\dim \mathcal{O}_i = \dim \mathrm{Var}(I_w)$ for all i, and that S_i is a union of some irreducible components of $\bar{\mathcal{O}}_i \cap \mathbf{n}$. Proposition 1.3.2 together with Corollary 1.4.2 implies $p_w = p_{\mathrm{gr}\, L_w} = \sum p_i$, where p_i generates an irreducible W-module, corresponding to $\mathcal{O}_i$ via 1.3. Since the module $\mathbb{C}[W] \cdot p_w$ is itself irreducible (Theorem 1.6.1) there is actually the only one orbit $\mathcal{O}_i$, so that $p_w = p_i$.

Q.E.D.

2. 𝔊-MODULES AND 𝒟-MODULES (AFTER [Bebe; BK]

2.1. We keep to the previous notations. In particular we assume that G is a complex semi-simple Lie group with Lie algebra $\mathfrak{G}$. Let $\mathfrak{G}^*$ be the space dual to $\mathfrak{G}$ (and identified with $\mathfrak{G}$ via the Killing form), and let X be the Flag manifold. It will be convenient for us to consider X as a set of all Borel subalgebras $\mathbf{b} \subset \mathfrak{G}$ on which G acts by conjugation. For a point $x \in X$ let $\mathbf{b}_x$ be the corresponding Borel subalgebra and let $\mathbf{n}_x$ be its nilpotent radical. There are natural identifications of tangent and cotangent spaces at x: $T_x X = \mathfrak{G}/\mathbf{b}_x$, $T_x^* X = (\mathfrak{G}/\mathbf{b}_x)^* = \mathbf{b}_x^\perp = \mathbf{n}_x$. Hence $T^*X = \{(x, n) | x \in X, n \in \mathbf{n}_x\} \subset X \times \mathfrak{G}^*$. The corresponding projections of T^*X to X and to $\mathfrak{G}^*$ will be denoted by π and μ, respectively. While $\pi : T^*X \to X$ is the usual projection the map μ: $T^*X \to \mathfrak{G}^*$ is called either the "Springer's resolution" or the moment map.

PROPOSITION 2.1 (see, e.g., [Spr2]). (i) *The image* $\mu(T^*X)$ *is the nilpotent variety* $N \subset \mathfrak{G}^*$.

(ii) μ: $T^*X \to N$ *is a resolution of* N (*i.e.,* μ *is birational and proper*).

(iii) *The defining ideal of N is generated by the set I_+ of G-invariant homogeneous polynomials on $\mathfrak{G}^*$ of positive degree.*

(iv) *If $\mathcal{O}_{T^*X}$ is the sheaf of regular functions on T^*X, then $\mu^*: \mathbb{C}[N] \xrightarrow{\sim} \Gamma(T^*X, \mathcal{O}_{T^*X})$ is the isomorphism.*

Here (i) and (ii) are easy; (iii) is due to Kostant [Kost] and (iv) follows from the fact that N is normal [Kost].

2.2. The natural G-action on X gives rise to a homomorphism of the enveloping algebra $U(\mathfrak{G})$ into the ring $\Gamma(X, \mathcal{D}_X)$ of global differential operators on X. Let $Z(\mathfrak{G})$ be the center of $U(\mathfrak{G})$ and $Z_+ = Z(\mathfrak{G}) \cap \mathfrak{G} \cdot U(\mathfrak{G})$ be the augmentation ideal in $Z(\mathfrak{G})$.

PROPOSITION 2.2 [BeBe]. *There are exact sequences:*

(i) $0 \to \mathbb{C}[\mathfrak{G}^*] \cdot I_+ \to \mathbb{C}[\mathfrak{G}^*] \to \Gamma(T^*X, \mathcal{O}_{T^*X}) \to 0$;

(ii) $0 \to U(\mathfrak{G}) \cdot Z_+ \to U(\mathfrak{G}) \to \Gamma(X, \mathcal{D}_X) \to 0$.

Proof. Part (i) follows from Proposition 2.1(iii), (iv).

The fact that the composite of both arrows in (ii) equals 0 is a property of the Harish-Chandra homomorphism. Further, consider the natural filtrations on $U(\mathfrak{G})$ and D_X. It is easy to see that after applying the functor "gr" (ii) turns into (i).

Set $U = U(\mathfrak{G})/U(\mathfrak{G}) \cdot Z_+$. We identify $\Gamma(X, \mathcal{D}_X)$ with U by means of 2.2(ii).

THEOREM 2.3 [BeBe]. (i) *Any quasi-coherent $\mathcal{D}_X$-module $\mathcal{M}$ is generated by its global sections (i.e., $\mathcal{D}_X \otimes_U \Gamma(X, \mathcal{M}) = \mathcal{M}$) and $H^i(X, \mathcal{M}) = 0$ for $i > 0$.*

(ii) *The categories of coherent $\mathcal{D}_X$-modules and of finitely generated U-modules are equivalent. The equivalence is given by the mutually inverse functors $\mathcal{D}_X \otimes_U (-)$ and $\Gamma(X, -)$.*

2.3. Recall that there are natural filtrations on $U(\mathfrak{G})$, U, and $\mathcal{D}_X$ compatible with 2.2. Note that $\operatorname{gr} U(\mathfrak{G}) = \mathbb{C}[\mathfrak{G}^*]$, $\operatorname{gr} U = \mathbb{C}[N]$ and $\operatorname{gr} \mathcal{D}_X = \pi.\, \mathcal{O}_{T^*X}$. Let M be a finitely generated U-module and let $\mathcal{M} = \mathcal{D}_X \otimes_U M$ be the corresponding $\mathcal{D}_X$-module. On $\mathcal{M}$ (resp. M) choose a good filtration so that $\operatorname{gr} \mathcal{M}$ (resp. $\operatorname{gr} M$) is a coherent $\operatorname{gr} \mathcal{D}_X$ (resp. U)-module. Let $SS\mathcal{M}$ (resp. SSM) be its support. This is a homogeneous subvariety in T^*X (resp. N).

Suppose in particular that $I \subset U(\mathfrak{G})$ is a primitive ideal such that $I \cap Z(\mathfrak{G}) = Z_+$. Then $U(\mathfrak{G})/I$ is a U-module. We set $S(I) = SS(\mathcal{D}_X \otimes_U U(\mathfrak{G})/I)$. This variety takes part in the conjectures of the Introduction. Note that the image $\mu(S(I)) \subset \mathfrak{G}^*$ is the associated variety of the ideal I, studied in Section 1 (see [BB2] or Section 8).

2.4. Choose a Borel subalgebra $\mathbf{b}=\mathbf{h}+\mathbf{n}$ with the Cartan subalgebra $\mathbf{h}$. Call the roots of $\mathbf{h}$ on $\mathbf{n}$ positive. Let $\rho\in\mathbf{h}^*$ be half the sum of positive roots. For $\chi\in\mathbf{h}^*$ denote by M_χ the Verma module of the highest weight $\chi-\rho$.

Suppose that $\chi\in\mathbf{h}^*$ is a dominant integral and regular weight. Identify it with a character of $Z(\mathfrak{G})$ by means of the Harish-Chandra homomorphism. Let Z_χ be the kernel of this character. In other words Z_χ is the annihilator in $Z(\mathfrak{G})$ of an irreducible finite-dimensional $\mathfrak{G}$-module E_χ with the highest weight $(\chi-\rho)$.

Set $U_\chi=U(\mathfrak{G})/U(\mathfrak{G})\cdot Z_\chi$ and let $\mathscr{D}_\chi$ be the sheaf of differential operators on the line bundle on X corresponding to E_χ via the Borel–Weyl theorem. As in 2.2, one can show that $U_\chi=\Gamma(X,\mathscr{D}_\chi)$.

Let $\tilde{\mathcal{O}}_\chi$ be the category of finitely generated U_χ-modules M such that dim $U(\mathbf{b})\cdot m<\infty$ for any $m\in M$. If W is the Weyl group and $w\in W$ then $M_{w\cdot\chi}\in\tilde{\mathcal{O}}_\chi$.

Let $X=\bigcup X_w$, $X_w=B\cdot w\cdot B/B$ be the Schubert-cell decomposition of the Flag manifold X. The conormal bundle on a cell X_w is denoted by $T^*_{X_w}X$. In addition to Theorem 2.3 there is

THEOREM 2.4 [BK, BeBe]. (i) *The category of regular holonomic $\mathscr{D}_\chi$-modules whose characteristic variety is contained in $\bigcup T^*_{X_w}X$ is equivalent to $\tilde{\mathcal{O}}_\chi$, the equivalency being provided by $\mathscr{D}_\chi\otimes_{U_\chi}(-)$ and $\Gamma(X,-)$.*

(ii) *The natural duality $M\rightsquigarrow M^*$ for $U(\mathfrak{G})$-modules (see, e.g., [BGG1]) corresponds to the Verdier duality for $\mathscr{D}$-modules.*

Remark. If $\chi=\rho$ then $U_\chi=U$ and $\mathscr{D}_\chi=\mathscr{D}_X$ is the ring of usual differential operators. We set $\tilde{\mathcal{O}}=\tilde{\mathcal{O}}_\rho$.

2.5. In the future the reader can either work directly with U_χ- and $\mathscr{D}_\chi$-modules or restrict considerations to the special case $\chi=\rho$. There is actually no loss of generality in such a restriction because of the following.

THEOREM 2.5 (Translation principle [*BJ*; *Z*]). *For integral regular weights χ and χ' the categories $\tilde{\mathcal{O}}_\chi$ and $\tilde{\mathcal{O}}_{\chi'}$ are equivalent.*

This result is in fact an easy consequence of Theorem 2.4 since the categories of $\mathscr{D}_\chi$- and $\mathscr{D}_{\chi'}$-modules are equivalent. The equivalence is defined as $\mathscr{M}\rightsquigarrow\mathscr{L}\otimes_{\mathcal{O}_X}\mathscr{M}$, where $\mathscr{L}$ is an appropriate line bundle (invertible sheaf) on X.

2.6. From now on we suppose that $\chi=\rho$ so that $\mathscr{D}_\chi=\mathscr{D}_X$. For $w\in W$ set $M_w=M_{-w\cdot\rho}$ and let L_w be the simple quotient of M_w. Let $j_w\colon X_w\hookrightarrow X$ be the inclusion of the Schubert cell and let $(j_w)_*$, $(j_w)_!$ be the corresponding "direct image" functors on $\mathscr{D}_X$-modules.

PROPOSITION 2.6 [BK; BeBe]. (i) $\mathscr{D}_X \otimes_U M_w = (j_w)_! \mathcal{O}_{X_w}$.

(ii) $\mathscr{D}_X \otimes_U M_w^* = (j_w)_* \mathcal{O}_{X_w}$.

(iii) $\mathscr{D}_X \otimes_U L_w$ is equal to the image of natural map

$$(j_w)_! \mathcal{O}_{X_w} \to (j_w)_* \mathcal{O}_{X_w}.$$

On the U-module level the natural map, mentioned in (iii), coincides with the morphism $M_w \to M_w^*$, $m \mapsto \langle m, - \rangle$, where $\langle -, - \rangle$ is Shapovalov's form on M_w. Its image clearly equals L_w.

Let δ_w be the unique $\mathbf{n}$-invariant "δ-function" supported at X_w.

COROLLARY 2.6.1. (i) *δ_w is an eigenvector relative to the action of h and the corresponding weight equals* $-w \cdot \rho - \rho$.

(ii) *The $U(\mathfrak{G})$-module, generated by δ_w, is irreducible and hence isomorphic to L_w.*

This corollary is, of course, nothing but another form of Proposition 2.6(iii). Let us however give its elementary proof, which may throw some light on the role of Schubert cells. Verification of (i) is left for the reader. In order to show that $L = U(\mathfrak{G}) \cdot \delta_w$ is irreducible it is enough to prove that δ_w is the only extreme vector (i.e., $\mathbf{b}$-eigenvector) in L. Suppose $u \in L$ is another one of the weight $\lambda \in \mathbf{h}^*$. Consider u as a distribution on X of the order k (by definition $u = P \cdot \delta_w$ for some differential operator on X). Let $\sigma_k(u)$ be its principal symbol. After an appropriate trivialization of $T^*_{X_w} X$ by means of δ_w this symbol may be regarded as a function on $T^*_{X_w} X$ of the weight $(\lambda + w \cdot \rho + \rho)$. We identify the fibre of $T^*_{X_w} X$ at $w \in X_w$ with ${}^w\mathbf{n} \cap \mathbf{n}$ (${}^w\mathbf{n} := w \cdot \mathbf{n} \cdot w^{-1}$). When restricted to this fibre the symbol $\sigma_k(u)$ becomes a polynomial function on ${}^w\mathbf{n} \cap \mathbf{n}$ still of the weight $\lambda + w \cdot \rho + \rho$. As a weight of a polynomial function it must be equal to $-\Sigma k_i \cdot \alpha_i$ for certain non-negative integers k_i and roots α_i in ${}^w\mathbf{n} \cap \mathbf{n}$. Whence $\lambda + \rho = -w \cdot \rho - \Sigma k_i \cdot \alpha_i$. Since α_i is a root of ${}^w\mathbf{n}$, the root $\beta_i = w^{-1} \cdot \alpha_i$ is positive and we get $\lambda + \rho = -w \cdot (\rho + \Sigma k_i \cdot \beta_i)$.

On the other hand $\lambda + \rho \in -W \cdot \rho$ since all extreme weights on L belong to the same W-orbit in $\mathbf{h}^*$ corresponding to the central character. It remains to note that $\rho + \Sigma k_i \cdot \beta_i \in W \cdot \rho$ only if $k_i = 0$ for all i. That implies that $\sigma_k(u) = 0$. The only remaining possibility is for u to be the distribution of the zero-order. Thus $u = \text{const}\, \delta_w$. Q.E.D.

3. HARISH-CHANDRA MODULES

3.1. Consider the Lie group $G \times G$ with Lie algebra $\mathfrak{G} \times \mathfrak{G}$ and the subgroup $G_\Delta = \{(g, g), g \in G\} \subset G \times G$ with Lie algebra $\mathfrak{G}_\Delta \subset \mathfrak{G} \times \mathfrak{G}$. A finitely generated $U(\mathfrak{G}\times\mathfrak{G})$-module V is called a Harish-Chandra module if dim $U(\mathfrak{G}_\Delta)\cdot v<\infty$ for any $v\in V$. For the motivation of this definition consider the complex group G as a real Lie group $G_{\mathbb{R}}$ and pick up a maximal compact subgroup $K\subset G_{\mathbb{R}}$. Then the pair $((G_{\mathbb{R}})_{\mathbb{C}},\ K_{\mathbb{C}})$ is isomorphic to the pair $(G\times G,\ G_\Delta)$ so that our definition of Harish-Chandra modules agrees with the usual one for $(G_{\mathbb{R}},\ K)$-modules.

Identify $U(\mathfrak{G}\times\mathfrak{G})$ with $U(\mathfrak{G})\otimes U(\mathfrak{G})$. Then $Z(\mathfrak{G}\times\mathfrak{G})\simeq Z(\mathfrak{G})\otimes Z(\mathfrak{G})$ so that $U(\mathfrak{G}\times\mathfrak{G})/U(\mathfrak{G}\times\mathfrak{G})\cdot(Z_+\otimes Z_+)\simeq U\otimes U$. Following [CD] we introduce a category H of Harish-Chandra modules, annihilated by $Z_+\otimes Z_+$; that is a subcategory of $U\otimes U$-modules.

The map $x\mapsto -x$, $x\in\mathfrak{G}$, can be extended to the involution $u\mapsto\check{u}$ of $U(\mathfrak{G})$ such that $(u_1\cdot u_2)\check{}=\check{u}_2\cdot\check{u}_1$. Since $\check{Z}_+=Z_+$ the corresponding involution is well defined on U. Any $(U\otimes U)$-module V can be equivalently regarded as a left–right U-bimodule via the action $u_1\cdot v\cdot u_2=(u_1\otimes\check{u}_2)\cdot v$. For U-bimodules V_1 and V_2 it is clear that $V_1\otimes_U V_2$ is also a U-bimodule. If $V_1,\ V_2\in H$ then $V_1\otimes_U V_2\in H$. Similarly, for $V\in H$ and a U-module $M\in\tilde{\mathcal{O}}$ there is a U-module $V\otimes_U M\in\tilde{\mathcal{O}}$.

3.2. Let $M_\rho\in\tilde{\mathcal{O}}$ be the Verma module with the zero highest weight (in the notations of Section 0, $M_\rho=M_{w_0}$, where $w_0\in W$ is the element of maximal length).

THEOREM 3.2 [BG]; see also [Jo3]. *The categories $\tilde{\mathcal{O}}$ and H are equivalent; the equivalency is provided by mutually inverse functors $Q: H\rightsquigarrow\tilde{\mathcal{O}}$ and $P:\tilde{\mathcal{O}}\rightsquigarrow H$*

$$Q: V\rightsquigarrow V\otimes_U M_\rho;$$

$$P: M\rightsquigarrow U(\mathfrak{G}_\Delta)\text{-finite part of } \mathrm{Hom}_{\mathbb{C}}(M_\rho, M).$$

For example, the Harish-Chandra modules $P_w=P(M_w)$ form the so-called "principal series" representations.

3.3. It is known that there is a 1–1 correspondence between G_Δ-orbits in $X\times X$ and elements of W. Let C_w be the orbit, corresponding to $w\in W$ and let $T^*_{C_w}(X\times X)$ be its conormal bundle. As in Section 2.4 one can prove

PROPOSITION 3.3 [BeBe]. The category H is equivalent to the category $\mathscr{H}$ of regular holonomic $D_{X\times X}$-modules, whose characteristic variety is contained in $\bigcup T^*_{C_w}(X\times X)$.

Theorem 3.2 is an immediate consequence of Theorem 2.4 and Proposition 3.3: the respective categories of $\mathscr{D}_X$-modules and $\mathscr{D}_{X\times X}$-

modules are equivalent since the manifolds X and $X \times X$ have the same orbit structure. If $B \subset G$ is a Borel subgroup and x_0 is a corresponding point in X then the $\mathscr{D}$-module counterpart of the functor $Q: H \rightsquigarrow \tilde{\mathcal{O}}$ is just the restriction to the submanifold: $X \times \{x_0\} \subset X \times X$. Note that $C_{w'} \cap (X \times \{x_0\}) = X_{w'} \times \{x_0\}$.

COROLLARY 3.3.1. *Any irreducible Harish-Chandra module in H is isomorphic to $P(L_w)$ for some $w \in W$.*

THEOREM 3.4. [Du; BG]. *For a two-sided ideal $I \subset U$ the following conditions are equivalent:*

(i) *I is prime;*

(ii) *I is primitive;*

(iii) *I is the annihilator of a certain $L_{w'}$, $w \in W$.*

COROLLARY [Du; Jo3]. *Any primitive ideal $I \subset U(\mathfrak{G})$ such that $I \cap Z(\mathfrak{G}) = Z_+$ is the annihilator of a certain $L_{w'}$.*

Proof of the theorem (after [BG]). Implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. Let us show that (i) $\Rightarrow$ (iii). Consider U/I as a U-bimodule. Clearly $U/I \in H$. Let $N = Q(U/I)$ $(= M_\rho/I \cdot M_\rho)$ be the element of $\tilde{\mathcal{O}}$ corresponding to U/I via Theorem 3.2. If $0 = N_0 \subset N_1 \subset \cdots \subset N_k = N$ is a composition series for N then each simple quotient N_i/N_{i-1} is isomorphic to certain $L_{w'}$. Set $I_i = \mathrm{Ann}(N_i/N_{i-1})$. Therefore it is enough to prove that (a) Ann $N = I$, and (b) Ann $N = I_i$ for some i. If (a) is verified, then (b) is easy: on the one hand Ann $N \subset I_i$ for all i. On the other hand $I_k \cdot N \subset N_{k-1}$, $I_{k-1} \cdot I_k \cdot N \subset N_{k-2}, \ldots$, $(I_1 \cdot I_2 \cdot \cdots \cdot I_k) \cdot N = 0$. Hence $I_1 \cdot I_2 \cdot \cdots \cdot I_k \subset \mathrm{Ann}\, N = I$. Since I is a prime ideal the above inclusions imply that Ann $N = I_i$ for some i.

It remains to prove that Ann $N = I$. For any bimodule $V \in H$ let L Ann V be its left annihilator in U. The explicit form of functors P and Q, combined with the equality $P \circ Q = \mathrm{id}_H$, shows that Ann $Q(V) = L$ Ann V for $V \in H$. In particular $I = L\ \mathrm{Ann}(U/I) = \mathrm{Ann}\ Q(U/I) = \mathrm{Ann}\ N$. Q.E.D.

3.5. For a Harish-Chandra module V and a finite-dimensional $\mathfrak{G}$-module E define $\mathfrak{G} \times \mathfrak{G}$-action on $V \otimes_{\mathbb{C}} E$ as follows:

$$x \cdot (v \otimes e) = x \cdot v \otimes e \qquad \text{if } x \in \mathfrak{G} \times \{0\} \subset \mathfrak{G} \times \mathfrak{G},\ v \in V,\ e \in E;$$

$$x \cdot (v \otimes e) = x \cdot v \otimes e + v \otimes x \cdot e \qquad \text{if } x \in \{0\} \times \mathfrak{G} \subset \mathfrak{G} \times \mathfrak{G}.$$

As in the proof of Theorem 3.4 denote by L Ann V the left annihilator of V and consider the $U(\mathfrak{G} \times \mathfrak{G})$-bimodule $U(\mathfrak{G})/L$ Ann V.

LEMMA [V]. *For a Harish-Chandra module V there are finite-dimensional modules E_1 and E_2 such that*:

(a) *V is a subquotient of* $(U(\mathfrak{G})/L \text{ Ann } V) \otimes E_1$ *and*

(b) $U(\mathfrak{G})/L$ Ann V *is a subquotient of* $V \otimes E_2$.

Proof of (a) is trivial: since V is finitely generated as a left $U(\mathfrak{G})$-module it is a quotient of $(U(\mathfrak{G})/L \text{ Ann } V) \otimes \mathbb{C}^n$. In order to prove (b) choose a $U(\mathfrak{G}_\Delta)$-stable finite-dimensional subspace $E \subset V$ such that $V = U(\mathfrak{G}) \cdot E$. Let E' be the $\mathfrak{G}_\Delta$-module dual to E. Identify $E \otimes E'$ with the subspace of $V \otimes E'$. Consider the distinguished $\mathfrak{G}_\Delta$-invariant element $a \in V \otimes E'$, corresponding to $1 \in \text{Hom}(E, E) = E \otimes E'$. One can easily verify that the map $U(\mathfrak{G})/L \text{ Ann } V \to V \otimes E'$, $u \mapsto u \cdot a$ is injective. So we may take E' as E_2. Q.E.D.

PROPOSITION 3.5 [V]. *If V_1 and V_2 are irreducible Harish-Chandra modules then conditions*

(i) L Ann $V_1 \subset L$ Ann V_2,

(ii) *V_2 is a subquotient of $V_1 \otimes E$ for some (finite-dimensional) E, are equivalent.*

Clearly (ii) $\Rightarrow$ (i). Conversely, if L Ann $V_1 \subset L$ Ann V_2 then according to the lemma, $V_2 = \text{sbq}((U(\mathfrak{G})/L\text{Ann } V_2) \otimes E_1) = \text{sbq}((U(\mathfrak{G})/L\text{Ann } V_1) \otimes E_1) = \text{sbq}((V_1 \otimes E_2) \otimes E_1) = \text{sbq}(V_1 \otimes E)$, where $E = E_2 \otimes E_1$ and "sbq" means "subquotient of". Q.E.D.

3.6. Let $l(w)$ be the length of $w \in W$. The following statement is well known:

LEMMA 3.6.1. *Let A be a $\mathbb{Z}$-algebra with basis a_w, $w \in W$, subject to the relations*:

(i) $a_s \cdot a_s = 1$ *if $s \in W$ is a simple reflection*;

(ii) $a_{w_1} \cdot a_{w_2} = a_{w_1 \cdot w_2}$ *if* $l(w_1) + l(w_2) = l(w_1 \cdot w_2)$.

Then A is isomorphic to the group-algebra $\mathbb{Z}[W]$.

Now we will introduce three algebras which are in fact isomorphic to $\mathbb{Z}[W]$.

Let $K[H]$ be the Grothendieck group generated by the elements of H. On $K[H]$ define a multiplication:

$$V_1 \circ V_2 = \sum (-1)^i \cdot \text{Tor}^i_U(V_1, V_2).$$

(Higher derived functors are necessary here since $V_1, V_2 \rightsquigarrow V_1 \otimes_U V_2$ is not an exact functor.) One can show that $K[H]$ becomes an associative $\mathbb{Z}$-algebra.

Next consider the category $\mathscr{H}$ of $\mathscr{D}_{X\times X}$-modules, described in the Proposition 3.3, and the corresponding Grothendieck group $K[\mathscr{H}]$.

Finally let $F(X\times X)$ be the group of constructible functions on $X\times X$, constant on each G_Δ-orbit C_w. Multiplication on $K[\mathscr{H}]$ and on $F(X\times X)$ are convolution-like operations defined as follows. Let $P_{ij}: X\times X\times X \to X\times X$ be natural projections on three possible pairs of factors. For constructible functions $\varphi, \psi \in F(X\times X)$ set $\varphi \circ \psi = (P_{13})_* [(P_{12}^*\varphi)\cdot(P_{23}^*\psi)]$. Here P_{ij}^* (resp. $(P_{ij})_*$) stands for a pull-back to $X\times X\times X$ (resp. push-forward to $X\times X$).

Similarly for $\mathscr{D}_{X\times X}$-modules $\mathscr{M}, \mathscr{N} \in \mathscr{H}$ the element $\mathscr{M}\circ\mathscr{N} \in K[\mathscr{H}]$ is defined as

$$\mathscr{M}\circ\mathscr{N} = \sum (-1)^i \cdot \int_{P_{13}}^i (P_{12}^*\mathscr{M} \otimes_{\mathcal{O}_{X\times X\times X}} P_{23}^*\mathscr{N}),$$

where direct images $\int^i$ were introduced by Kashiwara.

PROPOSITION 3.6.2. (a) *All three algebras $K[H]$, $K[\mathscr{H}]$ and $F(X\times X)$ are isomorphic to the group-algebra $\mathbb{Z}[W]$.*

(b) *If $w \in W$ and $j_w: C_w \hookrightarrow X\times X$ is the inclusion of a cell then the element $1\cdot w \in \mathbb{Z}[W]$ corresponds by these isomorphisms to $P(M_w)$ in $K[H]$, $(j_w)_! \mathcal{O}_{C_w}$ in $K[\mathscr{H}]$ and the characteristic function $\mathbb{1}_{C_w}$ in $F(X\times X)$ respectively.*

This proposition is well known (see, e.g., [LV; Spr2]). The isomorphism $K(H) \simeq K(\mathscr{H})$ is provided by Proposition 3.3. An arrow $K(\mathscr{H}) \to F(X\times X)$ attaches $\chi(y, \mathscr{M}) := \sum (-1)^i \cdot \dim H^i DR\, \mathscr{M}_{|y}$, $y \in X\times X$ to a $\mathscr{D}_{X\times X}$-module $\mathscr{M}$.

Here DR $\mathscr{M}$ is the De Rham complex, associated with $\mathscr{M}$ [Br]. Finally, in order to prove that $K(\mathscr{H})$ and $F(X\times X)$ are isomorphic to $\mathbb{Z}[W]$ it is enough to verify that elements $(j_w)_!\, \mathcal{O}_{C_w}$ and $\mathbb{1}_{C_w}$ satisfy conditions of Lemma 3.6.1. The first condition follows from a direct checkup for SL_2. The second is an easy consequence of the fact that for $l(w_1)+l(w_2) = l(w_1\cdot w_2)$ the cell $C_{w_1\cdot w_2}$ is isomorphic to the fibre-product $C_{w_1}\times_X C_{w_2}$.

3.7. The Grothendieck group $K(\tilde{\mathcal{O}})$ is, of course, isomorphic to $K(H)$ as a group. However, there is no way to equip it with multiplicative structure. What is possible is to consider $K(\tilde{\mathcal{O}})$ as a $K(H)$-module via the action

$$V\circ M = \sum (-1)^i \cdot \mathrm{Tor}_U^i (V, M).$$

This clearly gives rise to the regular representation of W if $K(H)$ is identified with $\mathbb{Z}[W]$.

Suppose $M \in K[\tilde{\mathcal{O}}]$ corresponds to the element $\sum \chi_w(M) \cdot w \in \mathbb{Z}[W]$. That just means that $[M] = \sum \chi_w(M) \cdot [M_w]$ in $K[\tilde{\mathcal{O}}]$. Consequently the formal character of M is equal to (cf. (1.5.1):

$$\left(\sum \chi_w(M) \cdot e^{w\rho}\right) \Big/ \prod_{\alpha > 0} (e^{\alpha/2} - e^{-\alpha/2}).$$

Note that for M, $M' \in K[\tilde{\mathcal{O}}]$ convolution $P(M) \circ P(M')$ in $K(H)$ corresponds to the usual convolution of functions $\chi_w(M)$ and $\chi_w(M')$ on the Weyl group.

4. Geometry of the Moment Map (after [Gi1; St])

4.1. Let A be a complex algebraic Lie group with Lie algebra $\mathbf{a}$ and the dual space $\mathbf{a}^*$. Suppose we are given a hamiltonian action of A on a symplectic (complex algebraic) manifold M. That means, in particular, the existence of Lie algebra homomorphism $\mathbf{a} \to \mathcal{O}_M$, $a \mapsto H_a$. Therefore one can define the moment map $\mu: M \to \mathbf{a}^*$ (see; e.g., [BB]) by the formula $\mu(m): a \mapsto H_a(m)$, $m \in M$, $a \in \mathbf{a}$.

Recall that a subvariety of a symplectic manifold is called coisotropic (= involutive) if tangent spaces to its regular points contain their orthogonal complements relative to the symplectic form. There is also another definition: a subvariety in M is coisotropic if its defining ideal is a Lie subalgebra in $\mathcal{O}_M$.

Theorem 4.1 [Gi1]. *Suppose* $\mathbf{a}$ *is a solvable Lie algebra. Then for any coadjoint orbit* $\Omega \subset \mathbf{a}^*$ *the inverse image* $\mu^{-1}(\Omega)$ *is either empty or a coisotropic subvariety of* M.

The proof of this theorem is given in Appendix B.

Remark. It is impossible to drop the assumption that $\mathbf{a}$ is solvable. For example, consider the natural $SL_2(\mathbb{C})$-action on $M = \mathbb{C}^2$. Then the subvariety $\mu^{-1}\{0\} = \{0\}$ is not coisotropic.

4.2. Let G be a connected complex Lie group with Lie algebra $\mathfrak{G}$ and $\lambda \in \mathfrak{G}^*$. Suppose $\mathbf{b}$ is a solvable subalgebra in $\mathfrak{G}$ and $\mathbf{b}^\perp$ its annihilator in $\mathfrak{G}^*$.

Proposition 4.2 [Gi1]. *If* $\lambda/[\mathbf{b}, \mathbf{b}] = 0$ *then* $G \cdot \lambda \cap (\lambda + \mathbf{b}^\perp)$ *is a coisotropic subvariety in the orbit* $G \cdot \lambda$.

Proof. One should apply Theorem 4.1 to $M = G \cdot \lambda$, $\mathbf{a} = \mathbf{b}$ and $\Omega = \{\lambda|_{\mathbf{b}}\} \subset \mathbf{b}^*$.

4.3. From this moment we keep to the notations of Sections 1–3. In particular we fix a Borel subalgebra $\mathbf{b} = \mathbf{h} + \mathbf{n}$ in $\mathfrak{G}$ and identify $\mathbf{n}$ with $\mathbf{b}^\perp \subset \mathfrak{G}^*$ via the Killing form.

PROPOSITION 4.3 [Gil]. *For any nilpotent orbit $\mathcal{O} \subset \mathfrak{G}^*$ the intersection $\mathcal{O} \cap \mathbf{n}$ is a Lagrangian subvariety in $\mathcal{O}$.*

The proof will be indicated in Section 4.4.

COROLLARY. *All irreducible components of $\mathcal{O} \cap \mathbf{n}$ are of the same dimension, equal to $\frac{1}{2} \cdot \dim \mathcal{O}$.*

4.4. Consider the flag manifold X, the moment map μ: $T^*X \to \mathfrak{G}^*$ and the conormal bundles $T^*_{X_w} X$ to Schubert cells X_w. The following fact is almost trivial.

LEMMA 4.4. $\mu^{-1}(\mathbf{n}) = \bigcup_{w \in W} T^*_{X_w} X$.

Proof of Proposition 4.3. $\mathcal{O} \cap \mathbf{n}$ is a coisotropic subvariety in $\mathcal{O}$ according to Proposition 4.2. It is also isotropic since $\mathcal{O} \cap \mathbf{n} \subset \mathbf{n} = \mu(\bigcup T^*_{X_w} X)$ (Lemma 4.4) and $T^*_{X_w} X$ is an isotropic subvariety in T^*X, while the moment map μ is compatible with symplectic structures on T^*X and $\mathcal{O}$.

4.5. Suppose that $n \in \mathbf{n}$ and $\mathcal{O} = G \cdot n$. Let Y be an irreducible component of $\mu^{-1}(n)$ and let $G \cdot Y$ be an irreducible component of $\mu^{-1}(\mathcal{O})$.

LEMMA [St; Gil]. $\dim Y + \dim \mathcal{O} = \dim G \cdot Y = \dim(\mathcal{O} \cap \mathbf{n}) + \dim X$.

The first equality is clear. To obtain the second consider the projection $G \cdot Y \hookrightarrow T^*X \to X$ to X. Its fibres can be identified with components of $\mathcal{O} \cap \mathbf{n}$ Q.E.D.

From this lemma and the crucial Proposition 4.3. it is not hard to deduce the following

THEOREM 4.5 [Gil]. *If $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{G}^*$ then:*

(i) *$\mu^{-1}(\mathcal{O})$ is a coisotropic subvariety in T^*X and the fibres of the map $\mu: \mu^{-1}(\mathcal{O}) \to \mathcal{O}$ coincide with the leaves of the natural null-foliation on the coisotropic subvariety;*

(ii) *the closure of each irreducible component of $\mu^{-1}(\mathcal{O} \cap \mathbf{n})$ equals the closure of a certain conormal bundle $T^*_{X_w} X$.*

COROLLARY 4.5.1. *All irreducible components of* $\mu^{-1}(\mathcal{O})$ *have the same dimension, equal to* $\dim X + \frac{1}{2} \cdot \dim \mathcal{O}$.

COROLLARY 4.5.2. [Spa; Gi1]. *For any nilpotent element* $n \in \mathfrak{G}^*$ *all components of* $\mu^{-1}(n)$ *have the same dimension, equal to* $\dim \mathbf{n} - \frac{1}{2} \cdot \dim G \cdot n$.

COROLLARY 4.5.3. *For any irreducible component* F *of* $\mathcal{O} \cap \mathbf{n}$ *there is* $w \in W$ *such that* $\bar{F} = B \cdot (w\, \mathbf{n}\, w^{-1} \cap \mathbf{n})$ *(here* B *is the Borel subgroup).*

4.6. Later we will make use of similar results for $T^*(X \times X)$ instead of T^*X and G_Δ-orbits $C_w \subset X \times X$ instead of B-orbits $X_w \subset X$. Recall that $G_\Delta = \{(g, g) | g \in G\} \subset G \times G$ and $\mathfrak{G}_\Delta \subset \mathfrak{G} \times \mathfrak{G}$ is its Lie algebra. Let N_Δ be the nilpotent variety in $\mathfrak{G}_\Delta^* \subset \mathfrak{G}^* \times \mathfrak{G}^*$ and let $\mu_\Delta = \mu \times (-\mu)$: $T^*X \times T^*X \to \mathfrak{G}^* \times \mathfrak{G}^*$ the moment map (note the sign on the second factor!). The $\mathfrak{G} \times \mathfrak{G}$-counterpart of Lemma 4.4 is

LEMMA 4.6. $\mu_\Delta^{-1}(N_\Delta) = \bigcup_w T^*_{C_w}(X \times X)$.

4.7. The following result is a formal consequence of Theorem 4.5:

THEOREM 4.7 [St; Gi 1]. *Let* $\mathcal{O}$ *be a nilpotent orbit in* $\mathfrak{G}_\Delta^*$.

(a) *The closure of an irreducible component of* $\mu_\Delta^{-1}(\mathcal{O})$ *is equal to the closure of a certain conormal bundle* $T^*_{C_w}(X \times X)$;

(b) *all* $T^*_{C_w}(X \times X)$ *are obtained in this way.*

Consider the projections p_i: $T^*X \times T^*X \to T^*X$ $(i = 1, 2)$ on both factors.

COROLLARY. *For any* $w \in W$ *there is a nilpotent orbit* $\mathcal{O} \subset \mathfrak{G}^*$ *such that* $p_i(T^*_{C_w}(X \times X))$ *equals the closure of an irreducible component of* $\mu^{-1}(\mathcal{O})$.

Remark. While the orbit $\mathcal{O}$ is the same for p_1 and p_2 irreducible component of $\mu^{-1}(\mathcal{O})$, corresponding to these projections, may be different!

5. CONSTRUCTION OF SPRINGER REPRESENTATIONS

5.1. Suppose N is a complex manifold and Λ a homogeneous Lagrangian subvariety of T^*N. Consider the Grothendieck group, generated by coherent $\mathcal{O}_{T^*N}$ sheaves, supported at Λ. Let $L(\Lambda)$ be its quotient modulo the subgroup generated by all sheaves F such that $\dim \operatorname{supp} F < \dim \Lambda$. If $\{\Lambda_\alpha\}$ are irreducible components of Λ then $L(\Lambda)$ is clearly a free abelian group with the basis $\mathcal{O}_{\Lambda_\alpha}$.

Let N_i, $i = 1, 2, 3$, be complex manifolds and P_{ij}: $T^*(N_1 \times N_2 \times N_3) \to$

$T^*(N_i \times N_j)$ be the natural projections. Suppose further that $\Lambda \subset T^*(N_1 \times N_2)$, $\Lambda' \subset T^*(N_2 \times N_3)$ are homogeneous Lagrangian subvarieties and that the map $P_{13}: P_{12}^{-1}(\Lambda) \cap P_{23}^{-1}(\Lambda') \to T^*(N_1 \times N_3)$ is proper. Then its image is a closed isotropic homogeneous subvariety of $T^*(N_1 \times N_2)$, denoted $\Lambda \circ \Lambda'$ We will define a multiplication $L(\Lambda) \otimes L(\Lambda') \to L(\Lambda \circ \Lambda')$ as follows (if $\dim(\Lambda \circ \Lambda') < \dim(X_1 \times X_3)$ then $L(\Lambda \circ \Lambda') = 0$ by definition). For sheaves F and F' on Λ and Λ', respectively, consider the complex (in the derived category) $F \circ F' = (RP_{13})_* (P_{12}^* F \otimes_{\mathcal{O}_{T^*(N_1 \times N_2 \times N_3)}} P_{23}^* F')$. The cohomology sheaves $\mathscr{H}^i(F \circ F')$ are clearly supported by $\Lambda \circ \Lambda'$. One can show that the class $[F \circ F'] = \sum (-1)^i \cdot \mathscr{H}^i(F \circ F')$ in $L(\Lambda \circ \Lambda')$ is determined by classes of F and F' in $L(\Lambda)$ and $L(\Lambda')$, respectively. Thus the map $[F] \times [F'] \mapsto [F \circ F']$ is well defined, giving rise to the multiplication $L(\Lambda) \otimes L(\Lambda') \to L(\Lambda \circ \Lambda')$.

Remark. Let us give the geometric interpretation of $L(\Lambda)$. Suppose $\Lambda \subset T^*N$ and $F \in L(\Lambda)$. Consider all purely ($\dim \Lambda$)-dimensional components of supp F (counted with their multiplicities) as an analytic cycle in T^*N. In this way, we get an isomorphism of $L(\Lambda)$ onto the group of analytic cycles of maximal dimension in Λ, where both are regarded as abelian groups. Next suppose that $\Lambda_1 \subset T^*(N_1 \times N_2)$, $\Lambda_2 \subset T^*(N_2 \times N_3)$ and consider the multiplication $L(\Lambda_1) \otimes L(\Lambda_2) \to L(\Lambda_1 \circ \Lambda_2)$. In terms of analytic cycles it is, roughly speaking, defined by the formula

$$\Lambda \circ \Lambda' = (P_{13})_* (P_{12}^* \Lambda \cap P_{23}^* \Lambda'), \tag{5.1.1}$$

where Λ and Λ' are components of Λ_1 and Λ_2 respectively and inverse and direct images of cycles are defined in the usual way. However, the geometric intersection $P_{12}^* \Lambda \cap P_{23}^* \Lambda'$ may have the "wrong" dimension. Therefore the right-hand side of (5.1.1) should be understood as follows. Choose a flat family of analytic cycles $\Lambda_t \subset T^*(N_1 \times N_2)$, $t \in \mathbb{C} \setminus \{0\}$ which specializes to Λ when $t \to 0$ and such that all the intersections $P_{12}^* \Lambda_t \cap P_{23}^* \Lambda'$ are proper (i.e., of the "right" dimension which is equal to ($\dim N_1 + \dim N_3$)). Then *define* the right-hand side of (5.1.1) as a specialization of the family $(P_{13})_* (P_{12}^* \Lambda_t \cap P_{23}^* \Lambda')$ at $t = 0$. At first glance it may depend on a choice of the family Λ_t. Coincidence with the previous sheaf-theoretic definition shows that it actually does not.

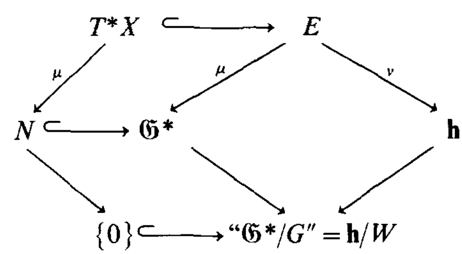
5.2. Let us consider a special case of the above construction. Suppose X is a complex manifold and let $\Lambda \subset T^*(X \times X)$ be a homogeneous Lagrangian subvariety such that $\Lambda \circ \Lambda \subset \Lambda$. On $L(\Lambda)$ one can define according to Section 5.1 a structure of $\mathbb{Z}$-algebra with multiplication $L(\Lambda) \otimes L(\Lambda) \to L(\Lambda \circ \Lambda) \subsetneq L(\Lambda)$, This algebra is associative but not necessarily commutative.

5.3. We apply Section 5.2 to the flag manifold X and $\Lambda = \bigcup_w T^*_{C_w}(X \times X) \subset T^*(X \times X)$. We have (see 4.6) $\Lambda \circ \Lambda = \mu_\Delta^{-1}(N_\Delta) \circ \mu_\Delta^{-1}(N_\Delta) \subset \mu_\Delta^{-1}(N_\Delta \circ N_\Delta) \subset \mu_\Delta^{-1}(N_\Delta) = \Lambda$. We can now state the crucial

PROPOSITION 5.3. *There is an algebra isomorphism* lim: $\mathbb{Z}[W] \xrightarrow{\sim} L(\Lambda)$

Proof. Recall that X is the set of Borel subalgebras of $\mathfrak{G}$ (see Section 2.1). Let E be a vector bundle on X with the fibre $\mathfrak{b}_x$ at $x \in X$. Identify T^*X with the subbundle of E with fibres $T^*_x X \simeq \mathfrak{n}_x \subset \mathfrak{b}_x$. The moment map $\mu: T^*X \to \mathfrak{G}^*$ extends to E via the imbedding of Borel subalgebras into $\mathfrak{G} \simeq \mathfrak{G}^*$. This map $E \to \mathfrak{G}^*$ is surjective and will be also denoted by μ.

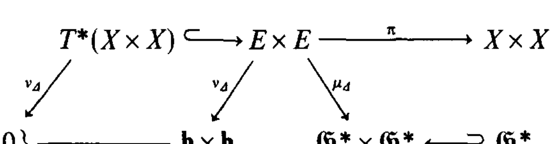
Note that if $\mathfrak{b}_x = \mathfrak{h}_x + \mathfrak{n}_x$ then $\mathfrak{h}_x \cong \mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x]$. If $\mathfrak{b}_{x'} = g \cdot \mathfrak{b}_x \cdot g^{-1}$ then the corresponding conjugation-map $\mathfrak{h}_x = \mathfrak{b}_{x'}/[\mathfrak{b}_{x'}, \mathfrak{b}_{x'}] \to \mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x] = \mathfrak{h}_x$ does not depend on a choice of $g \in G$. Therefore there are canonical isomorphisms $\mathfrak{h}_{x'} \simeq \mathfrak{h}_x$ so that all Cartan subalgebras can be identified with the fixed one $\mathfrak{h}$. Thus there is a well-defined morphism ν: $E \to \mathfrak{h}$ mapping $\mathfrak{b}_x$ into $\mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x] \simeq \mathfrak{h}$. We have the following commutative diagram (see, e.g., [Spr 1; Spr2]):



Here "$\mathfrak{G}^*/G$" denotes the spectrum of the ring of G-invariant polynomials on $\mathfrak{G}^*$.

Recall that for $h \in \mathfrak{h}$ the inverse image $\nu^{-1}(h)$ is a smooth submanifold of E with the natural symplectic structure of "twisted cotangent bundle" on X (see [Gi5]). Also note that over a set of semi-simple regular elements of $\mathfrak{G}^*$ the map μ is a covering with the free W-action on its fibres.

Return to the proof of the Proposition. Set: $\mu_\Delta = \mu \times (-\mu)$, $\nu_\Delta = \nu \times (-\nu)$ and consider the diagram



Choose a regular element $h \in \mathfrak{h}$ and for all $s \in \mathbb{C}$ and $w \in W$ set $\Lambda_w(s) = \nu_\Delta^{-1}(s \cdot h \times s \cdot w \cdot h) \cap \mu_\Delta^{-1}(\mathfrak{G}^*_\Delta)$.

LEMMA. *If $s \neq 0$ then*:

(a) μ_Δ *maps* $\Lambda_w(s)$ *isomorphically onto a regular* G_Δ*-orbit in* $\mathfrak{G}_\Delta^*$;

(b) $\pi(\Lambda_w(s)) = C_w$.

We also have

LEMMA. *If $s \neq 0$ then*:

(a) $\Lambda_w(s)$ *is a smooth connected Lagrangian submanifold in the symplectic manifold* $v^{-1}(s \cdot h) \times v^{-1}(-s \cdot w \cdot h)$;

(b) *in notations of Section* 5.1, $\Lambda_{w_1}(s) \circ \Lambda_{w_2}(s) \subset \Lambda_{w_1 \cdot w_2}(s)$ *and* (*for the multiplication* $L(\Lambda_{w_1}(s)) \otimes L(\Lambda_{w_2}(s)) \to L(\Lambda_{w_1 w_2}(s))$) *we have* $\mathcal{O}_{\Lambda_{w_1}(s)} \circ \mathcal{O}_{\Lambda_{w_2}(s)} = \mathcal{O}_{\Lambda_{w_1 w_2}}(s)$.

Part(a) is trivial; (b) is also easy since all the intersections involved are transversal. Set

$$\Lambda(s) = \bigcup_w \Lambda_w(s).$$

COROLLARY. *If $s \neq 0$ then* $L(\Lambda(s)) \simeq \mathbb{Z}[W]$.

Now we can finish the proof of Proposition 5.3. First of all note that $\Lambda(0) = \bigcup T^*_{C_w}(X \times X) = \Lambda$ according to Lemma 4.6. Next for each $w \in W$ let us vary s and consider a flat family of symplectic manifolds $v^{-1}(s \cdot h \times s \cdot w \cdot h)$. On it we have for $s \neq 0$ a family of sheaves $\mathcal{O}_{\Lambda_w(s)}$. Let $\mathcal{O}_w = \lim_{s \to 0} \mathcal{O}_{\Lambda_w(s)}$ be its specialization at $v_\Delta^{-1}(0) = T^*(X \times X)$. Clearly $\operatorname{supp} \mathcal{O}_w \subset \Lambda(0) = \Lambda$ so that $\mathcal{O}_w \in L(\Lambda)$. Since the specialization commutes with the multiplication $\circ$ the equality $\mathcal{O}_{w_1} \circ \mathcal{O}_{w_2} = \mathcal{O}_{w_1 w_2}$ still holds in $L(\Lambda)$. Hence, in order to finish the proof it remains to show that the elements $\mathcal{O}_w$ form a basis of $L(\Lambda)$.

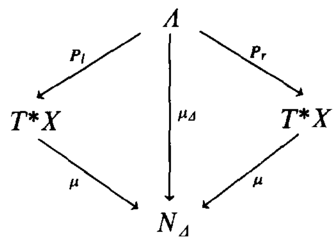
Since $\Lambda = \bigcup T^*_{C_w}(X \times X)$ the group $L(\Lambda)$ has the natural basis $T^*_{C_w} = \mathcal{O}_{\overline{T^*_{C_w}(X \times X)}}$. We can therefore write

$$\mathcal{O}_w = \sum_{y \in W} b_{w,y} \cdot T^*_{C_y}$$

for certain integers $b_{w,y} \in \mathbb{Z}$. It is easy to prove that $b_{w,y} = 0$ unless $C_y \subset \bar{C}_w$ and $b_{w,w} = 1$. Hence, the matrix $b_{w,y}$ is strictly upper triangular relative to the Bruhat ordering. So it is invertible and $\mathcal{O}_w$ is a basis. Q.E.D.

Remark. Integers $b_{w,y}$ are equal to those introduced by Kazhdan and Lusztig in [KL2] (see also [KT] and Section 6 of the present paper).

5.4. Consider the diagram (∗) below.



(∗)

Here P_l (resp. P_r) are projections of $T^*X \times T^*X$ to the left-hand (resp. right-hand) factors and N_Δ is the nilpotent variety in $\mathfrak{G}^*_\Delta \subset \mathfrak{G}^* \times \mathfrak{G}^*$. It is clear that $\mu_\Delta(\operatorname{supp}(F_1 \circ F_2) \subset \mu_\Delta(\operatorname{supp} F_1) \cap \mu_\Delta(\operatorname{supp} F_2)$, $P_l(\operatorname{supp} F_1 \circ F_2) \subset P_l(\operatorname{supp} F_1)$ (and similarly for P_r), where $F_1, F_2 \in L(\Lambda)$. Therefore one can determine two-sided (resp. left or right) ideals in $L(\Lambda)$ by fixing images of supports (of the sheaves in question) relative to μ_Δ, P_r, or P_l, respectively. Accordingly, for a nilpotent orbit $\mathcal{O} \subset N_\Delta$ we define two-sided ideals $L_{\bar{\mathcal{O}}}$ and $L_{\partial\mathcal{O}}$ in $L(\Lambda)$ (here $\partial\mathcal{O} = \bar{\mathcal{O}} \setminus \mathcal{O}$) as

$$L_{\bar{\mathcal{O}}} = \{F \in L(\Lambda) \mid \mu_\Delta(\operatorname{supp} F) \subset \bar{\mathcal{O}}\}$$

and for $L_{\partial\mathcal{O}}$ similarly. Consider the algebra $L_{\mathcal{O}} := L_{\bar{\mathcal{O}}}/L_{\partial\mathcal{O}}$. Since $L(\Lambda) \simeq \mathbb{Z}(W)$ this is also a W-bimodule with distinguished basis $\{T^*_{c_w} \mid \mu(T^*_{c_w}(X \times X)) = \bar{\mathcal{O}}\}$. In view of complete reducibility of $\mathbb{C}[W]$ we have

$$\mathbb{C}[W] \simeq \oplus_{\mathcal{O} \subset N_\Delta} \mathbb{C} \otimes L_{\mathcal{O}}. \tag{5.4.1}$$

5.5. Let us now reinterpret these results in terms of T^*X instead of $T^*(X \times X)$. Choose a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ and set $\Lambda_\mathfrak{b} = \mu^{-1}(\mathfrak{n}) = \bigcup T^*_{X_w}X$. One may regard $\Lambda_\mathfrak{b}$ as a Lagrangian correspondence in $T^*(X \times \text{point})$. It is then clear that $\Lambda \circ \Lambda_\mathfrak{b} \subset \Lambda_\mathfrak{b}$. So the convolution $L(\Lambda) \otimes L(\Lambda_\mathfrak{b}) \to L(\Lambda_\mathfrak{b})$ defines a $L(\Lambda) \simeq \mathbb{Z}[W]$-module structure on $L(\Lambda_\mathfrak{b})$.

On the other hand the components of $\Lambda_\mathfrak{b}$ are just the irreducible components of $\mu^{-1}(\mathcal{O} \cap \mathfrak{n})$ for various nilpotent orbits $\mathcal{O} \subset \mathfrak{G}^*$. For each $\mathcal{O}$ let us pick up a component F of $\mathcal{O} \cap \mathfrak{n}$ with the boundary $\partial F = \bar{F} \setminus F$. The same argument as in Section 5.4. shows that sheaves in $L(\Lambda_\mathfrak{b})$ whose support projects into $\bar{F}$ (resp. ∂F) form $L(\Lambda)$-stable subspace $V_{\bar{F}}$ and $V_{\partial F}$ in $L(\Lambda_\mathfrak{b})$. Set $V_{\mathcal{O}} = V_{\bar{\mathcal{O}}}/V_{\partial\mathcal{O}}$.

In order to investigate each $V_{\mathcal{O}}$ more closely identify $\mathfrak{G}^*_\Delta$ with $\mathfrak{G}^*$, choose an element $n \in F$ and consider its centralizer $G(n) \subset G$. There is natural $G(n)$-action on the fibre $\mu^{-1}(n) \subset X$.

Let $C(n)$ be the group of connected components of $G(n)$. The group $C(n)$ acts on the set of components of $\mu^{-1}(n)$ by permutations (induced by

$G(n)$-action on $\mu^{-1}(n)$). Note that components of $\mu^{-1}(F)$ are in 1–1 correspondence with those of $\mu^{-1}(n) \subset \mu^{-1}(F)$ (if $n \in F \subset \mathcal{O}$). Hence one can define the action of $C(n)$ by permutation of components of $\mu^{-1}(F)$.

One can show that the $C(n)$-action on $V_{\mathcal{O}}$, so defined, commutes with the action of W. Hence we can write (see also [KL2; Spr1],...): $\mathbb{C} \otimes_{\mathbb{Z}} V_{\mathcal{O}} \simeq \oplus (E_x \otimes V_{\mathcal{O},x})$, where E_x are irreducible representations of $C(n)$ and $V_{\mathcal{O},x}$-representations of W.

Following [KL2] and [St] we note that

$$\mu_{\Lambda}^{-1}(\mathcal{O}) \simeq G \times_{G(n)} (\mu^{-1}(n) \times \mu^{-1}(n)), \tag{5.5.1}$$

(here $G(n)$ acts on both factors of $\mu^{-1}(n) \times \mu^{-1}(n)$ simultaneously). Equation (5.5.1) yields $L_{\mathcal{O}} \simeq (V_{\mathcal{O}} \otimes V_{\mathcal{O}})^{C(n)}$. Thus we obtain the decomposition

$$\mathbb{C} \otimes L_{\mathcal{O}} \simeq \oplus_{x \in C(n)} (V_{\mathcal{O},x} \otimes V_{\mathcal{O},x})^{C(n)}. \tag{5.5.2}$$

Finally (5.4.1) implies the following (cf. [KL2; BM; Spr2]):

PROPOSITION 5.5. *$\mathbb{C}[W] \simeq \oplus_{\mathcal{O},x}(V_{\mathcal{O},x} \otimes V_{\mathcal{O},x})$, where $V_{\mathcal{O},x}$ are exactly all irreducible W-modules (without repetition).*

Irreducibility of $V_{\mathcal{O},x}$ follows from the equality $\sum (\dim V_{\mathcal{O},x})^2 = \# W$, which can be verified as in [St].

5.6. In the previous paragraphs we defined a W-module structure on the spaces $V_{\mathcal{O}}$ with the distinguished base, indexed by irreducible components of $X_n = \mu^{-1}(n)$, $n \in \mathcal{O}$. These components form as they are a base of the top dimensional homology space $H_d(X_n)$, $d = \dim X_n$. Let us now indicate how a W-action on each homology space $H_i(X_n)$, $i \leqslant d$ can be defined. That was originally done by Springer [Spr 1] (see also [BM]).

For a nilpotent element $n \in N$ let U be its small open neighborhood in the nilpotent cone N and let $\tilde{U} = \mu^{-1}(U)$ be the corresponding "tubular" neighborhood of $X_n := \mu^{-1}(n)$ in $T^*X = \mu^{-1}(N)$. We assume $\tilde{U}$ to be contractible to X_n. Let $p: \tilde{U} \to X_n$ be such a contraction. Then we have the isomorphisms:

$$H^*(X_n) \xrightarrow[\approx]{p^*} H^*(\tilde{U}) \cong H_*(\tilde{U}) \ (= \text{Borel–Moore homology}) \tag{5.6.1}$$

where the second isomorphism is due to Poincaré duality for a *smooth* variety $\tilde{U}$ (note that $\tilde{U}$ is smooth as an open part of a smooth variety T^*X).

Consider the Lagrangian correspondence $\Lambda = T^*X \times T^*X$ (see n. 5.3) and set: $\Lambda_U = \Lambda \cap (\tilde{U} \times \tilde{U}) = \tilde{U} \times_N \tilde{U}$. Thus Λ_U is a Lagrangian correspondence in $\tilde{U} \times \tilde{U}$ and $\Lambda_U \circ \Lambda_U = \Lambda_U$. So the group $L(\Lambda_U)$ acquires a ring structure and the restriction map: $L(\Lambda) \to L(\Lambda_U)$ is a ring homomorphism. Hence there is a homomorphism $\mathbb{Z}[W] \to L(\Lambda_U)$ due to Proposition 5.3. Our job will be therefore completed if we define an $L(\Lambda_U)$-action on $H_*(X_n)$. To do that set $r = \dim \Lambda$ $(= \dim X)$ and identify $L(\Lambda_U)$ with Borel–Moore homology group $H_r(\Lambda_U)$. We shall define a convolution-like bilinear pairing:

$$H_r(\Lambda_U) \times H_i(X_n) \to H_i(X_n) \tag{5.6.2}$$

as follows. Consider the contraction $p \times id$: $\tilde{U} \times \tilde{U} \to X_n \times \tilde{U}$. Its restriction to Λ_U is a proper map p^Λ, giving rise to the Borel–Moore homology morphism:

$$p_*^\Lambda \colon H_r(\Lambda_U) \to H_r(X_n \times \tilde{U}) = \sum_i H_i(X_n) \otimes H_{r-i}(\tilde{U})$$

(by (5.6.1))

$$\cong H_i(X_n) \otimes H^i(X_n) \cong \mathrm{Hom}(H_i(X_n), H_i(X_n)).$$

The definiton of (5.6.2) is now clear.

6. Characteristic Cycles and the Weyl Group

6.1. Suppose N is a complex manifold, $\mathcal{M}$ a holonomic module on N and $\Lambda \subset T^*N$ a Lagrangian subvariety such that $SS\mathcal{M} \subset \Lambda$. For a good filtration on $\mathcal{M}$ the associated graded $\mathcal{O}_{T^*N}$-sheaf is supported at Λ. One can show (see, e.g., [La; Gi4, Sect. 9]) that its class in the group $L(\Lambda)$ (see Section 5.1) does not depend on a choice of good filtration. We denote this class by gr $\mathcal{M}$.

Let $\{\Lambda_\alpha\}$ be the collection of irreducible components of Λ. Following Kashiwara define the multiplicity $m_\alpha(\mathcal{M})$ as the multiplicity of the sheaf gr $\mathcal{M}$ at a generic point of Λ_α. In the group $L(\Lambda)$ we can write an equality

$$\mathrm{gr}\, \mathcal{M} = \sum m_\alpha(\mathcal{M}) \cdot \mathcal{O}_{\Lambda_\alpha}.$$

For that reason the formal linear combination of irreducible components of the characteristic variety SSM counted with their multiplicities is called the characteristic cycle of $\mathcal{M}$ (see, e.g., [Gi4; KT],...).

6.2. Returning to the group-theoretic situation, we assume that X is the Flag manifold, $N = X \times X$, $\Lambda = \bigcup T^*_{C_w}(X \times X)$ and $T^*_{C_w} := \mathcal{O}_{T^*_{C_w}(X \times X)}$ is the basis of $L(\Lambda)$. For $\mathcal{M} \in \mathcal{H}$ we can write: gr $\mathcal{M} = \sum m_w(\mathcal{M}) \cdot T^*_{C_w}$. Further it is clear that the map $\mathcal{M} \rightsquigarrow \mathrm{gr}\, \mathcal{M}$ gives rise to an additive homomorphism gr: $K(\mathscr{H}) \to L(\Lambda)$. We have shown respectively in Sections 3 and 5 that both $K(\mathscr{H})$ and $L(\Lambda)$ have natural structures of $\mathbb{Z}$-algebras and that these algebras are isomorphic to $\mathbb{Z}[W]$. Keeping this in mind we can state

THEOREM 6.2. *The diagram*

$$\begin{array}{ccc} K(\mathscr{H}) & \xrightarrow{\text{gr}} & L(\Lambda) \\ & \searrow{\scriptstyle \chi} \quad \swarrow{\scriptstyle \lim} & \\ & \mathbb{Z}[W] & \end{array}$$

is commutative (the arrow χ assigns the element $\sum \chi(w, \mathscr{M}) \cdot w$ (see Proposition 3.6.2) *to* $\mathscr{M} \in K(\mathscr{H})$).

COROLLARY. *The map* gr *is an isomorphism of algebras.*

Remark. In the next section we will explain directly why "gr" is compatible with multiplicative structures.

Suppose that the image $\chi(V) \in \mathbb{Z}[W]$ of a Harish-Chandra module $V \in H$ equals $\sum \chi_w(V) \cdot w$. That means the equality in the Grothendieck group: $[V] = \sum \chi_w(V) \cdot P(M_w)$. So the function $\chi_w(V)$ may be interpreted as the formal character of V. Theorem 6.3 is equivalent to the following fundamental relation between characters and characteristic varieties. In notations of the proof of Proposition 5.3 (cf. [KT, Theorem 6]):

$$\text{gr}(\mathscr{D}_{X \times X} \otimes_{U \otimes U} V) = \lim_{s \to 0} \Big(\sum_w \chi_w(V) \cdot \Lambda_w(s) \Big). \tag{6.2.1}$$

In particular we have

$$\text{gr}\, P(M_w) = \lim_{s \to 0} \Lambda_w(s). \tag{6.2.2}$$

There is a similar formula for characteristic cycles of $\mathscr{D}_X$-modules, corresponding to elements of $\tilde{\mathcal{O}}$. If $\mathbf{b} = \mathbf{h} + \mathbf{n}$ is a fixed Borel subalgebra, $\lambda \in \mathbf{h}^*$ is a dominant regular weight and $\mathscr{M}_w = \mathscr{D}_X \otimes_U M_w$, then in notations used in Section 5.3:

$$\text{gr}\, \mathscr{M}_w = \lim_{s \to 0} [\mu^{-1}(s \cdot w \cdot \lambda + \mathbf{n}) \cap \nu^{-1}(s \cdot \lambda)]. \tag{6.2.3}$$

6.3. In order to prove Theorem 6.2 recall one general result from [Gi4].

Suppose Z is a one-codimensional subvariety in a complex manifold N.

Set $U = N \setminus Z$, $j\colon U \hookrightarrow N$. Let $\mathcal{M}$ be a regular holonomic module on U and $j_*\mathcal{M}$ its "meromorphic" direct image to N. We will express gr $(j_*\mathcal{M})$ in terms of gr $\mathcal{M}$.

Consider Z as a divisor in N. Let $p\colon Y \to N$ be the principal $\mathbb{C}^*$-bundle on N, associated with that divisor. There is a hamiltonian $\mathbb{C}^*$-action on T^*Y induced by the natural $\mathbb{C}^*$-action on Y. It gives rise to a moment map $\mu\colon T^*Y \to \mathbb{C}$. Fibres of μ are stable under action of $\mathbb{C}^*$ and for each $s \in \mathbb{C}$ the quotient-space $\mu^{-1}(s)/\mathbb{C}^*$ is a complex manifold with natural symplectic structure. It is known as "twisted cotangent bundle" T^sN (see [Gi5]). In particular $T^0N = T^*N$.

Suppose $\Lambda \subset T^*U$ is a homogeneous Lagrangian cycle. Let $p^*\Lambda \subset p^*(T^*N) \subset T^*Y$ be its pull-back to Y. This is a Lagrangian cycle in $T^*(p^{-1}(U))$. Further, there is a function f on Y such that $f^{-1}(0) = p^{-1}(Z)$ and $f(t \cdot y) = t \cdot f(y)$ for $y \in Y$, $t \in \mathbb{C}^*$. It can be shown that for $s \neq 0$ the intersection $(p^*\Lambda + \mathbb{C}^* \cdot df) \cap \mu^{-1}(s)$ is transverse and closed in $\mu^{-1}(s)$, giving rise to a Lagrangian cycle Λ^s in $T^sN = \mu^{-1}(s)/\mathbb{C}^*$. Finally denote by $\lim_{s\to 0} \Lambda^s$ the specialization of the family of analytic cycles Λ^s, $s \in \mathbb{C} \setminus \{0\}$ at $s = 0$. By definition, $\lim_{s \to 0} \Lambda^s$ is a homogeneous Lagrangian cycle in T^*N (of course, it is not equal to Λ since $\Lambda \subset T^*U$ is *not* closed in T^*N).

Now we can state the following twisted version of [Gi4, Theorem 3.2]:

THEOREM 6.3. $\operatorname{gr}(j_*\mathcal{M}) = \lim_{s \to 0} (\operatorname{gr} \mathcal{M})^s$.

Proof. The result being local we assume that there is a regular function g on N such that $Z = g^{-1}(0)$. Then $Y = N \times \mathbb{C}^*$, $T^*Y = T^*N \times \mathbb{C}^* \times \mathbb{C}$ and $\mu(\xi, t, \tau) = t \cdot \tau$ (here t and τ are coordinate functions on $\mathbb{C}^*$ and $\mathbb{C}$, respectively, and $\xi \in T^*N$). We also have $f(n, t) = t \cdot g(n)$, $n \in N$. Consequently $(p^*(\operatorname{gr} \mathcal{M}) + \mathbb{C}^* \cdot df) \cap \mu^{-1}(s) = \{(\xi + s \cdot g^{-1} \cdot dg,\ t,\ s \cdot t^{-1}) \in T^*N \times \mathbb{C}^* \times \mathbb{C} \mid \xi \in \operatorname{gr} \mathcal{M},\ s,\ t \in \mathbb{C}^*\}$. Whence $\lim_{s \to 0} \Lambda^s = \lim_{s \to 0} (\operatorname{gr} \mathcal{M} + s \cdot d \log g)$. It remains to apply [Gi4, Theorem 3.2.].

6.4. *Proof of Theorem* 6.2. It is clear that the statement is equivalent to (6.2.1). Since elements $P(M_w)$ form a basis in $K(H)$ its enough to verify (6.2.2.). The proof of (6.2.2.) and (6.2.3) being absolutely identical (in fact these formulas are equivalent), we will prove only (6.2.3) in order to avoid complicated notations. Let $j_w\colon X_w \hookrightarrow X$ be the imbedding of a Schubert cell. Then $\mathcal{M}_w = (j_w)_! \mathcal{O}_{X_w}$ (see Section 2.6). Also note that $\operatorname{gr}(j_w)_! \mathcal{O}_{X_w} = \operatorname{gr}(j_w)_* \mathcal{O}_{X_w}$.

Denote by B, T, U the groups corresponding to **b**, **h** and **n**, so that $B = T \cdot U$. Let λ be the highest weight of a finite-dimensional irreducible $\mathfrak{G}$-module E and $e \in E$ the highest-weight vector in E. Consider the subgroup $T_0 = \ker(\exp \lambda) \subset T$, the manifold $Y = G/(T_0 \cdot U)$ and the principal $\mathbb{C}^*$-bundle $p\colon Y \to G/B = X$. We regard E as a space of holomorphic sections of the line bundle on X. These sections can be also identified with functions on Y. Let f be the function, corresponding to the extreme vector $w \cdot e \in E$. It was shown in [BGG2] that $p(f^{-1}(0)) \cap \bar{X}_w = \bar{X}_w \backslash X_w = \partial X_w$.

Now we are in a position to apply Theorem 6.2. According to that theorem:

$$\operatorname{gr}(j_w)_* \mathcal{O}_{X_w} = \lim_{s \to 0} [(p^* T^*_{X_w} X + \mathbb{C}^* \cdot df) \cap \mu^{-1}(s)].$$

It can be verified that the twisted cotangent bundle $T^s X$, introduced in Section 6.3., identifies with the symplectic manifold $\nu^{-1}(s \cdot \lambda)$. Further, let e^* be the lowest-weight vector of the dual $\mathfrak{G}$-module E'. Then $f(g \cdot T_0 \cdot U) = \langle e^*, w^{-1} \cdot g \cdot e \rangle$ and computation shows that for $x \in \mathfrak{G}$, $u \in U$ and $y = u \cdot w \in G/(T_0 \cdot U) = Y$ we have $df_y(x) = (u \cdot w \cdot \lambda)(x)$. It is now easy to identify $p^* T^*_{X_w} X + \mathbb{C}^* \cdot df$ with $\nu^{-1}(\mathbb{C}^* \cdot \lambda) \cap \mu^{-1}(\mathbb{C}^* \cdot w \cdot \lambda)$. Q.E.D.

7. Bivariant Chern Classes

The subject of this section is not directly related to the problem of classification of primitive ideals. The uninterested reader can omit it without any trouble.

7.1. First of all we generalize the concept of bivariant theory, introduced in [FM]. By a correspondence between algebraic varieties X_1 and X_2 we mean any subvariety $Z \subset X_1 \times X_2$. Suppose X_i, $i = 1, 2, 3$, are three algebraic varieties. Let $P_{ij}: X_1 \times X_2 \times X_3 \to X_i \times X_j$ be natural projections. For $Z \subset X_1 \times X_2$ and $Z' \subset X_2 \times X_3$ consider the map $P_{13}: P_{12}^{-1}(Z) \cap P_{23}^{-1}(Z') \to X_1 \times X_3$. If it is proper denote its image by $Z \circ Z'$ and say that $Z \circ Z'$ "is defined."

We say that a bivariant theory T is given if for any correspondence $Z \subset X_1 \times X_2$ there is a group $T(Z)$ with a convolution-operation: $T(Z) \otimes T(Z') \to T(Z \circ Z')$, provided $Z \circ Z'$ is defined. This operation is assumed to be bi-additive and associative.

For example, bivariant homology theory H, introduced in [FM] associates to $Z \subset X_1 \times X_2$ the group $H(Z) = H^*(N \times X_2, (N \times X_2) \backslash Z)$, where N is a smooth manifold containing X_1 and H^* denotes the usual cohomology group. In particular for $Z = X_1 \times X_2$ denote this group by $H(X_1, X_2)$. Clearly $H(X_1, X_2) \simeq H_*(X_1) \otimes H^*(X_2)$.

7.2. Suppose that X_1 and X_2 are complex manifolds. For $x \in X_2$ consider the inclusion $i_x: X_1 \times \{x\} \hookrightarrow X_1 \times X_2$ and the projection $p: X_1 \times X_2 \to X_2$. For a function f let ψ_f and ϕ_f be the vanishing cycles functors, introduced by Deligne [De]. Further let $R\psi_x$ be the functor of Verdier-specialization [Ver] at the normal bundle $T_{X_1\times\{x\}}(X_1\times X_2)$.

Let $\mathcal{M}$ be a holonomic system on $X_1\times X_2$ with regular singularities. Consider the constructible complex $A^{\cdot}=\mathrm{DR}\mathcal{M}$, the lagrangian variety $\Lambda=SS\mathcal{M}\subset T^*(X_1\times X_2)$ and the coherent $\mathcal{O}_{T^*(X_1\times X_2)}$-sheaf $F=\mathrm{gr}\,\mathcal{M}$ (see Section 6.1). We will also make use of the projection $\bar{P}\colon T^*X_1\times T^*X_2\to(T^*X_1)\times X_2$.

PROPOSITION 7.2. *The following conditions are equivalent*:

(T1) $\phi_{p^*f}(A^{\cdot})=0$ *for any* $x\in X_2$ *and any function* f *on* X *such that* $f(x)=0$ *and* $df(x)\neq 0$;

(T2) *for any* $x\in X_2$ *the morphism* (*in the derived category*): $i_x^*\mathcal{M}\to R\psi_x\mathcal{M}$ *is an isomorphism*;

(T3) $\mathcal{M}$ *is a coherent* $P^{\cdot}(\mathcal{D}_{X_1}\otimes\mathcal{O}_{X_2})$-*module*;

(T4) F *is a coherent* $\bar{P}^{\cdot}\mathcal{O}_{(T^*X_1)\times X_2}$-*module*;

(T5) *the map* $\bar{P}\colon\Lambda\to(T^*X_1)\times X_2$ *is finite.*

Let us indicate the proof. Consider the subbundle $P^*(T^*X_2)$ of the cotangent bundle $T^*(X_1\times X_2)$. According to [Br, théorème 4.2.8] the condition T1 is equivalent to the following one:

T6. *$SS\mathcal{M}\cap P^*(T^*X_2)$ is contained at the zero section of $T^*(X_1\times X_2)$.*

If (T6) holds then the map $\Lambda\hookrightarrow T^*(X_1\times X_2)\setminus p^*(T^*X_2)\to^{\bar{P}}(T^*X_1)\times X_2$ is proper and hence finite. Therefore (T6)$\Rightarrow$(T5). It is easy to see that (T5)$\Rightarrow$(T4)$\Leftrightarrow$(T3)$\Rightarrow$(T6). In order to prove that (T2)$\Leftrightarrow$(T6) one shows that the Fourier transform of $R\psi_x\mathcal{M}$ is supported by the zero-section of the conormal bundle to $X_1\times\{x\}$. This can be done by means of [Gi4, Theorem 7.1].

7.3. Keeping the notations of Section 7.2. suppose $Z\subset X_1\times X_2$ is an arbitrary subvariety. We will now introduce a number of bivariant groups.

—$\mathrm{Hol}_{\mathrm{T}}(Z)$ is the Grothendieck group generated by regular holonomic modules $\mathcal{M}$ on $X_1\times X_2$ satisfying (T2) or (T3) and supported at Z.

—$D_{\mathrm{Eu}}(Z)$ is the Grothendieck group generated by constructible complexes $A^{\cdot}$ on Z such that the following local Euler condition (see [FM]) holds:

(Eu) for any $x\in X_2$ and any function f on X_2 such that $f(x)=0$ and $df(x)\neq 0$ the sum $\sum(-1)^i\cdot\mathcal{H}^i(\phi_{p^*f}A^{\cdot})$ (of cohomology sheaves) vanishes in the Grothendieck group of constructible sheaves.

As was pointed out to me by C. Sabbah, condition (T1) is much stronger than (Eu) since it deals with the complex $\phi_{p^*f}(A^\cdot)$ while (Eu) imposes restriction only on a combination of its cohomologies sheaves. It is therefore natural to introduce in addition to $\mathrm{Hol}_{\mathrm{T}}(Z)$ the larger group $\mathrm{Hol}_{\mathrm{Eu}}(Z)$ generated by holonomic regular complexes $\mathcal{M}^\cdot$ such that $\mathrm{DR}\,\mathcal{M}^\cdot \in D_{\mathrm{Eu}}(Z)$.

Next consider (see [FM]) the group $F(Z)$ of constructible functions φ on Z (extended by 0 to $X_1 \times X_2$) subjected to similar Euler condition: $i_x^*\varphi = \psi_x\varphi$ for any $x \in X_2$. Here $i_x^*\varphi$ denotes the restriction to $X_1 \times \{x\}$ and $\psi_x\varphi$ is the vanishing cycles counterpart for constructible functions. Note that if $A^\cdot \in D_{\mathrm{Eu}}(Z)$ then the constructible function.

$$\chi(-, A^\cdot) = \sum (-1)^i \cdot \dim \mathscr{H}^i(A^\cdot)$$

belongs to $F(Z)$.

Finally let $L_{\mathrm{T}}(Z)$ be the group of homogeneous Lagrangian cycles $\Lambda \subset T^*(X_1 \times X_2)_{|Z}$ such that the map $\bar{p}\colon \Lambda \to (T^*X_1) \times X_2$ is finite. For $\Lambda = \overline{T^*_Y(X_1 \times X_2)}$ this means in particular that $Y \subset Z$ and that $\overline{p(Y)} = X_2$. The finiteness of $\bar{p}$ implies however somewhat more, imposing certain extraconditions on the behaviour of the map $p\colon \bar{Y} \to X_2$ at its singular locus.

Following n. 5.1 we may regard $L_{\mathrm{T}}(Z)$ as a Grothendieck group of coherent $\mathcal{O}_{T^*(X_1 \times X_2)}$-sheaves supported by Lagrangian subvarieties (modulo the sheaves supported by subvarieties of lower dimension). To a sheaf F one attaches the cycle **supp** F. It is clear that $L_{\mathrm{T}}(Z)$ is generated by exactly those sheaves F that (T4) holds.

It is desirable to have a group $L_{\mathrm{Eu}}(Z)$, similar to $L_{\mathrm{T}}(Z)$ but with the condition imposed on cycles in $L_{\mathrm{T}}(Z)$ being relaxed. I do not know a good definition of such a group (cf. [Sab 2]). If $\dim X_2 = 1$ the group $L_{\mathrm{Eu}}(Z)$ should be generated by all $\Lambda = T^*_Y(X_1 \times X_2)$ such that $Y \subset Z$ and $\overline{p(Y)} = X_2$. If $\dim X_2 > 1$ one may try to define it as follows (cf. [Sab 2]). Consider the projection: $(T^*X_1) \times X_2 \to X_2$. For $\Lambda \subset T^*(X_1 \times X_2)$ let $\Lambda_x \subset T^*X_1$ be the fibre of $\bar{p}(\Lambda)$ over $x \in X_2$. If $\Lambda = T^*_Y(X_1 \times X_2)$ then Λ_x is the relative conormal bundle: $\Lambda_x = T^*_{Y_x}X_1$, where $Y_x := p^{-1}(x)$ $(\subset X_1 \times \{x\})$.

Fix a point $x \in X_2$. Given a Lagrangian cycle $\Lambda \subset T^*(X_1 \times X_2)$ and a germ of an algebraic curve $\gamma\colon \mathbb{C} \to X_2$, $\gamma(0) = x$ consider a family $\{\Lambda_{\gamma(t)}, t \neq 0\}$ of the above defined Lagrangian cycles in T^*X_1. Let $\Lambda_{\gamma,x}$ be its specialization at $t = 0$.

Define $L_{\mathrm{Eu}}(Z)$ as a group of Lagrangian cycles $\Lambda \subset T^*(X_1 \times X_2)_{|Z}$ satisfying the following condition: if $x \in X_2$ and γ, $\gamma'\colon \mathbb{C} \to X_2$ are two *generic* curves such that $\gamma(0) = x = \gamma'(0)$ then: $\Lambda_{\gamma,x} = \Lambda_{\gamma',x}$.

7.4. Recall that the Grothendieck transformation from a bivariant theory T_1 to a bivariant theory T_2 is a collection of group homomorphisms

$f_Z: T_1(Z) \to T_2(Z)$ (one for each Z) compatible with convolution in the sense that all the following rectangles are commutative

$$\begin{array}{lccc} f_Z \times f_{Z'}: & T_1(Z) \otimes T_1(Z') & \longrightarrow & T_2(Z) \otimes T_2(Z') \\ & \downarrow & & \downarrow \\ f_{Z \circ Z'}: & T_1(Z \circ Z') & \longrightarrow & T_2(Z \circ Z') \end{array}$$

With that understood we can state a

Conjecture. There is a commutative diagram of Grothendieck transformations:

$$\begin{array}{ccccc} \mathrm{Hol}_{\mathrm{Eu}}(Z) & \xrightarrow{\mathrm{gr}} & L_{\mathrm{Eu}}(Z) & \searrow^{\mathrm{Chern}} & \\ \mathrm{DR}\downarrow \wr & & \uparrow m & & H(Z) \\ D_{\mathrm{Eu}}(Z) & \xrightarrow{\chi} & F(Z) & \nearrow_{\mathrm{McP}} & \end{array}$$

Here DR assigns the De Rham complex DR $\mathscr{M}$ to a $\mathscr{D}$-module $\mathscr{M}$. The map "gr" was defined in n. 6.1 and the map "χ" in n. 7.3. It was shown by Kashiwara [K] (see also [BDK]] and [Gi 4]) how the characteristic cycle of a holonomic system $\mathscr{M}$ can be explicitly expressed in terms of the constructible function $\chi(\cdot, \mathrm{DR}\,\mathscr{M})$. The class of gr $\mathscr{M}$ is therefore completely determined by $\chi \circ \mathrm{DR}\,\mathscr{M}$ giving rise to the punctured arrow "m" (provided we know that gr $\mathscr{M} \in L_{\mathrm{Eu}}(Z)$). The map "Chern" will be defined later.

At present we are able to prove the following weaker form of the conjecture sufficient nevertheless for many interesting applications (see, e.g., 7.4.1, 7.4.2, and n. 7.5)

THEOREM 7.4. *The following arrows are Grothendieck transformations*:

$$F(Z) \xleftarrow{\chi} D_{\mathrm{Eu}}(Z) \xleftarrow{\mathrm{DR}} \mathrm{Hol}_{\mathrm{T}}(Z) \xrightarrow{\mathrm{gr}} L_{\mathrm{T}}(Z) \xrightarrow{\mathrm{Chern}} H(Z).$$

By taking here the group $\mathrm{Hol}_{\mathrm{T}}$ instead of the larger group L_{Eu} we have destroyed the surjectivity of the arrow "DR". That makes it impossible to define the arrow "m" of the conjecture. Suppose, however, that $X_2 = \text{point}$. Then both the local Euler condition and conditions (T1)–(T5) reduce to nothing. Therefore the map DR becomes surjective and we can derive from our theorem that the composite $\mathrm{McP} = \mathrm{Chern} \circ m$ is a Grothendieck transformation. Further $F(Z)$ in this case is actually the group of all constructible functions on Z. So we obtain

COROLLARY 7.4.1 [M]. *There is a natural transformation from constructible functions to homology, commuting with proper direct images.*

MacPherson's theorem was in fact the starting point for our analysis. Its proof in the spirit of Theorem 7.4 (but in purely geometric terms) was given by Sabbah [Sab].

Still assuming that $X_2 = \text{point}$ we also obtain a simple explanation of the following fact:

COROLLARY 7.4.2 [BDK]. *Suppose $F: X \to Y$ is a projective morphism of complex algebraic manifolds and $\mathcal{M}$ is a holonomic algebraic $\mathcal{D}_X$-module. Then*

$$\text{Chern}\left(\text{gr} \int_F \mathcal{M}\right) = F_* \,\text{Chern}(\text{gr}\, \mathcal{M}).$$

(here $\text{gr} \int_F \mathcal{M}$ *means* $\sum (-1)^i \cdot \text{gr} \int_F^i \mathcal{M}$*).*

7.5. Let us now assume that $X_1 = X_2 = X$ is a Flag manifold and let $\Lambda = \bigcup T^*_{C_w}(X \times X) \subset T^*(X \times X)$. Note that the map $\bar{p}: \Lambda \to (T^*X) \times X$ is injective. Hence the condition (T5) holds. Proposition 7.2 therefore shows that $K(\mathcal{H}) \subset \text{Hol}_T(X, X)$. Theorem 7.4 then gives the promised direct proof of the following

COROLLARY 7.5. *The map* $\text{gr}: K(\mathcal{H}) \to L(\Lambda)$ *is an algebra homomorphism.*

Remark. The isomorphism $F(X \times X) \simeq \mathbb{Z}[W]$ being already known, we get therefore another proof of Proposition 5.3.

Furthermore consider the bivariant homology group $H(X, X) \simeq H_*(X) \otimes H^*(X)$. According to Theorem 7.4 the map Chern: $L(\Lambda) \to H(X, X)$ is an algebra homomorphism. Identify $H(X, X)$ with $H_*(X \times X)$ by means of Poincaré duality and recall that $L(\Lambda) \simeq \mathbb{Z}[W]$. Thus we get a two-sided W-action on the image of Chern in $H_*(X \times X)$ and this action gives rise to the regular representation of W. For $T^*_{C_w}(X \times X)$ the total class Chern $\overline{T^*_{C_w}(X \times X)}$ can be decomposed into its components $\text{Chern}_i\, \overline{T^*_{C_w}(X \times X)} \in H_i(X \times X)$. One can easily verify that these components are zero unless $\dim X \leqslant i \leqslant \dim C_w$ (and that $\text{Chern}_{\dim C_w} \overline{T^*_{C_w}(X \times X)} = [\bar{C}_w] \in H_{\dim C_w}(X \times X)$). Hence the map into the lowest nontrivial component $\text{Chern}_{\dim X}: L(\Lambda) \to H_{\dim X}(X \times X)$ is still an algebra homomorphism. That gives rise to the regular $W \times W$-representation in a subspace of $H_{\dim X}(X \times X)$ with the distinguished basis $\text{Chern}_{\dim X}(\overline{T^*_{C_w}(X \times X)})$. It is not hard to show that the basis so defined coincides with that of Kazhdan and Lusztig [KL2].

7.6. Let us give some indications for the proof of Theorem 7.4. For the arrow χ the statement is trivial. For DR it is well known and is due to Deligne, Kashiwara and Kawai, and Mebkhout. In order to prove the theorem for "gr" recall that multiplication in $\mathrm{Hol_T}$ and L_{T} are defined in terms of direct and inverse images so that these cases can be treated separately. For proper direct images the statement follows from [La] or [Gi4, Sect. 9]. In the case of inverse images consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hol_T} & \xrightarrow{h} & \mathscr{E} \\ & {\scriptstyle \mathrm{gr}}\searrow \quad \swarrow{\scriptstyle \widetilde{\mathrm{gr}}} & \\ & L_{\mathrm{T}} & \end{array}$$

where $\mathscr{E}$ is an appropriate Grothendieck group of holonomic systems of micro-differential operators, h-means micro-localization and $\widetilde{\mathrm{gr}}$ is a functor, defined in the same way as gr. Formal homological algebra shows that $\widetilde{\mathrm{gr}}$ is always a Grothendieck transformation. Suppose that $\mathscr{M} \in \mathrm{Hol_T}(X_1, X_2)$, $\mathscr{N} \in \mathrm{Hol_T}(X_2, X_3)$. Consider the diagonal $\Delta \subset X_2 \times X_2$, the inclusion $i\colon Y = X_1 \times \Delta \times X_3 \hookrightarrow X_1 \times X_2 \times X_2 \times X_3 = X$ and the $\mathscr{D}_X$ module $\mathscr{M} \times \mathscr{N}$. We must prove that $h(i^*(\mathscr{M} \times \mathscr{N})) = i^*h(\mathscr{M} \times \mathscr{N})$, i.e., that the micro-localization of $\mathscr{M} \times \mathscr{N}$ commutes with its restriction to Y. This will be done if we show that $\mathscr{M} \times \mathscr{N}$ is non-characteristic to Y, i.e., that $SS(\mathscr{M} \times \mathscr{N}) \cap T^*_Y X \subset T^*_X X$. Suppose $(\xi_1, \xi_2) \in SS\mathscr{M}$, $(\xi_2', \xi_3) \in SS\mathscr{N}$. If $(\xi_1, \xi_2, \xi_2', \xi_3) \in T^*_Y X$ then $\xi_1 = 0 = \xi_3$ and $\xi_2 + \xi_2' = 0$. But since $\mathscr{M} \in \mathrm{Hol_T}(X_1, X_2)$ its characteristic variety SSM satisfies (T6). In our notations that means that $\xi_2 = 0$. Hence $\xi_2' = 0$.

Finally, the statement of the theorem for "Chern" follows from homological computations carried out in [Sab, Appendix].

7.7. We shall define here the map Chern: $L_{\mathrm{T}}(Z) \to H(Z)$. We begin with a general construction interesting for its own.

For a complex manifold X consider the natural $\mathbb{C}^*$-action on T^*X by multiplication. Let $K_{\mathbb{C}^*}(T^*X)$ (resp. $K_{\mathbb{C}^*}(X)$) be the Grothendieck group of $\mathbb{C}^*$-equivariant coherent $\mathcal{O}_{T^*X}$-sheaves (resp. $\mathcal{O}_X$-sheaves). Here $\mathbb{C}^*$ acts trivially on X so that: $K_{\mathbb{C}^*}(X) = \mathbb{Z}[q, q^{-1}] \otimes K(X)$, where $K(X)$ is the usual K-group of coherent sheaves on X and $\mathbb{Z}[q, q^{-1}] = K_{\mathbb{C}^*}(pt)$ stands for the representation ring of the group $\mathbb{C}^*$. Note that $K_{\mathbb{C}^*}(T^*X)$ is also a $\mathbb{Z}[q, q^{-1}]$-module.

Let $\pi_X\colon T^*X \to X$ be the projection and let $i_X\colon X \hookrightarrow T^*X$ be the zero-section inclusion. These maps induce the Thom isomorphisms:

$$\pi_X^*\colon K_{\mathbb{C}^*}(X) \xrightarrow{\sim} K_{\mathbb{C}^*}(T^*X) \quad \text{and} \quad i_X^*\colon K_{\mathbb{C}^*}(T^*X) \xrightarrow{\sim} K_{\mathbb{C}^*}(X) \qquad (7.7.1)$$

inverse to each other.

Consider the graded algebra $\pi.\ \mathcal{O}_{T^*X}$. The category of $\mathbb{C}^*$-equivariant $\mathcal{O}_{T^*X}$-sheaves is equivalent to the category of graded $\pi\cdot\mathcal{O}_{T^*X}$-modules. So $K_{\mathbb{C}^*}(T^*X)$ may be regarded as a Grothendieck group of graded $\pi\cdot\mathcal{O}_{T^*X}$-modules. Multiplication by q corresponds then to the shift of gradation.

Given two smooth varieties X_1, X_2 and a coherent sheaf $F\in K_{\mathbb{C}^*}(T^*(X_1\times X_2))$ define a homomorphism: $K_{\mathbb{C}^*}(T^*X_2)\to K_{\mathbb{C}^*}(T^*X_1)$. It arises from the "convolution"-operation: $K_{\mathbb{C}^*}(T^*(X_1\times X_2))\otimes K_{\mathbb{C}^*}(T^*X_2)\to K_{\mathbb{C}^*}(T^*X_1)$ and takes a sheaf E on T^*X_2 to the alternating sum of cohomology sheaves of the complex $Rf_*(F\overset{L}{\otimes}_{\mathcal{O}_{T^*(X_1\times X_2)}}f^*E)$, where $f\colon T^*(X_1\times X_2)\to T^*X_2$ is the projection. Of course one needs a certain properness condition in order to get a coherent complex at the end. We assume it to be satisfied. Next we use the Thom isomorphisms (7.7.1) to define a homomorphism $\hat{F}\colon K_{\mathbb{C}^*}(X_2)\to K_{\mathbb{C}^*}(X_1)$ as a composition:

$$K_{\mathbb{C}^*}(X_2)\xrightarrow{\pi^*_{X_2}}K_{\mathbb{C}^*}(T^*X_2)\xrightarrow{F\circ}K_{\mathbb{C}^*}(T^*X_1)\xrightarrow{i^*_{X_1}}K_{\mathbb{C}^*}(X_1)$$

We would like to express $\hat{F}$ in terms involving no cotangent bundles. To do that consider a commutative diagram:

$$\begin{array}{ccccc} T^*(X_1\times X_2) & \xrightarrow{\bar{p}} & (T^*X_1)\times X_2 & \overset{\bar{i}}{\hookleftarrow} & X_1\times X_2 \\ & \searrow^{\bar{p}_1} & \swarrow^{p_1} & & \swarrow^{p_1} \\ & & T^*X_1 & \underset{i_{X_1}}{\hookleftarrow} & X_1 \end{array}$$

Using the equality: $i^*_{X_1}\cdot(p_1)_*=(p_1)_*\cdot\bar{i}^*\cdot p_*$ we get (for $E\in K_{\mathbb{C}^*}(X_2)$):

$$\begin{aligned}\hat{F}(E)&=i^*_{X_1}(\bar{p}_1)_*(F\otimes\pi^*_{X_2}E)=(p_1)_*\bar{i}^*\bar{p}_*(F\otimes\pi^*_{X_2}E)\\&=(p_1)_*\,\bar{i}^*(\bar{p}_*F\otimes E)=(p_1)_*(\bar{i}^*\bar{p}_*F\otimes p_1^*E).\end{aligned}$$

Thus we see that the operator $\hat{F}$ is induced by the "convolution": $K_{\mathbb{C}^*}(X_1\times X_2)\otimes K_{\mathbb{C}^*}(X_2)\to K_{\mathbb{C}^*}(X_1)$ with the class $\bar{i}^*\cdot\bar{p}_*F\in K_{\mathbb{C}^*}(X_1\times X_2)$. We set $T(F):=\bar{i}^*\bar{p}_*F$.

Let X_3 be a third variety and let F' be a sheaf on $T^*(X_2\times X_3)$. It gives rise to a morphism $\hat{F}'\colon K_{\mathbb{C}^*}(X_3)\to K_{\mathbb{C}^*}(X_2)$ and it is obvious from our construction that: $\hat{F}\circ\hat{F}'=\widehat{F\circ F'}$. Since $\hat{F}$, $\hat{F}'$ are represented by the classes $T(F)$, $T(F')$ this equality suggests that one should have: $T(F\circ F')=T(F)\circ T(F')$. To make a precise statement let us fix notations. For a subvariety $Z\subset X_1\times X_2$ let $L_{\mathbb{C}^*}(Z)$ be the Grothendieck group of $\mathbb{C}^*$-equivariant coherent $\mathcal{O}_{T^*(X_1\times X_2)}$-sheaves F such that: (i) $\operatorname{supp}F$ is an isotropic subvariety; (ii) $\pi_{X_1\times X_2}(\operatorname{supp}F)\subset Z$; (iii) the map $\bar{p}\colon\operatorname{supp}F\to(T^*X_1)\times X_2$ is finite. The homomorphism $T=\bar{i}^*\bar{p}_*\colon L_{\mathbb{C}^*}(Z)\to K_{\mathbb{C}^*}(Z)$ is then well defined and we have

LEMMA 7.7.2. *T is a Grothendieck transformation.*

The proof is a formal exercise based on the identity: $\pi_X^* \circ i_X^* = id$ holding in $K_{\mathbb{C}^*}(T^*X)$ (cf. (7.7.1)).

Remark. The "convolution": $K(Z) \otimes K(Z') \to K(Z \circ Z')$ is defined by a formula similar to that for $\mathscr{D}$-modules (see n. 3.6). It essentially depends, as it stands, on ambient manifolds X_1, X_2, X_3. Suppose however that the group $L_T(Z)$ is non-trivial. We have seen in n. 7.3 that this is possible only if $p(Z)$ is a Zariski-open part of X_2. It is not hard to verify that the convolution: $K(Z) \otimes K(Z') \to K(Z \circ Z')$ is intrinsically defined and agrees with that of [FM], provided $p(Z)$ is open in X_2 and $p(Z')$ in X_3. This will be tacitly assumed in the future.

7.8. Following [FM] one can define the Chern character ch: $K(Z) \to H(Z)$ as follows. For $E \in K(Z)$ choose a finite locally free $\mathcal{O}_{X_1 \times X_2}$-resolution and take the alternating sum of the corresponding chern characters, considered as an element of $H_Z^*(X_1 \times X_2)$. Finally identify $H_Z^*(X_1 \times X_2)$ with $H(Z)$. This can be extended to a map Ch: $K_{\mathbb{C}^*}(Z) \to H(Z)[[t]]$ defined by the rule:

$$E = \sum E_k q^k \mapsto \sum \text{ch } E_k \cdot \exp(k \cdot t).$$

Let $\text{Ch}^j E$ be the component of Ch E contained in $\sum_i H_Z^{2i}(X_1 \times X_2) \cdot t^{j-i}$.

LEMMA 7.8.1. *Suppose that* $\dim(X_1 \times X_2) = n$ *and consider a sheaf* $F \in L_{\mathbb{C}^*}(Z)$. *Then*

(i) $\text{Ch}^j\, T(F) = 0$ *for* $j < n$;

(ii) *the class* $\text{Ch}^n\, T(F) \in H(Z)$ *is completely determined by the component of the cycle* supp F *of pure dimension n.*

The n-dimensional component of supp F is a homogeneous Lagrangian cycle in $T^*(X_1 \times X_2)$, i.e., an element of $L_T(Z)$. According to the statement (ii) there is a well defined homomorphism **Chern**: $L_T(Z) \to H(Z)$ assigning $\text{Ch}^n\, T(F)$ to the n-dimensional component of supp F. For $\Lambda \in L_T(Z)$ the element $\mathbf{Chern}(\Lambda)$ will be called the Chern class of Λ. The terminology is motivated by the following fact: let $X_2 =$ point, let Z be a singular subvariety of $X := X_1$ and let T_Y^*X be the closure of $T_{Z^{\text{reg}}}^*X$. Then $\mathbf{Chern}(T_Z^*X) \in H_Z^*(X) \simeq H_*(Z)$ is (up to a sign) the Chern–Mather class of Y (cf. [Sab]).

Proof of Lemma 7.8.1. We will temporarily write X instead of X_1. Consider the projective bundle $\mathbb{P}^*X = \mathbb{P}(\mathbb{C} + T^*X)$ on X with $\mathbb{P}^d$, $d = \dim X$ as a fibre. It is well known that:

$$H^*(\mathbb{P}^*X) \cong H^*(X)[t]/(t^{d+1}). \tag{7.8.2}$$

Let: $T^*X \to \mathbb{P}^*X$ be the open embedding. A $\mathbb{C}^*$-equivariant $\mathcal{O}_{T^*X}$-sheaf F can be uniquely extended to a coherent $\mathcal{O}_{\mathbb{P}^*X}$-sheaf $\bar{F}$. In this way we get a homomorphism ε: $K_{\mathbb{C}^*}(T^*X) \to K(\mathbb{P}^*X)$. Consider the diagram:

$$\begin{array}{ccccc} & K(\mathbb{P}^*X) & \xrightarrow{\text{ch}} & H^*(\mathbb{P}^*X) & \\ {}^{\varepsilon}\nearrow & & & & \searrow^{\alpha} \\ K_{\mathbb{C}^*}(T^*X) & & & & H^*(X)[t]/(t^{d+1}) \\ {}_{i_X^*}\searrow & & & & \nearrow_{\beta} \\ & K_{\mathbb{C}^*}(X) & \xrightarrow{\text{Ch}} & H^*(X)[[t]] & \end{array} \tag{7.8.3}$$

Here α is the isomorphism (7.8.2) and the arrow β is induced by the quotient map: $\mathbb{Q}[[t]] \to \mathbb{Q}[[t]]/(t^{d+1})$. We claim that the diagram (7.8.3) is commutative. To prove it consider a sheaf $G \in K(X)$ and set: $F = \pi_X^*(q^k \cdot G) \in K_{\mathbb{C}^*}(X)$. If $\bar{\pi}$: $\mathbb{P}^*X \to X$ is the projection, then clearly: $\varepsilon(F) = \bar{\pi}^*G \otimes \mathcal{O}_{\mathbb{P}^*X}(k)$.

Therefore,

$$\text{ch}(\varepsilon(F)) = (\bar{\pi}^* \text{ ch } G) \cdot \text{ch } \mathcal{O}_{\mathbb{P}^*X}(k) = (\bar{\pi}^* \text{ ch } G) \cdot \exp(k \cdot t).$$

Hence (7.8.3) is commutative for $F = \pi_X^*(q^k \cdot G)$. But the sheaves of that type generate $K_{\mathbb{C}^*}(T^*X)$, since π_X^*: $K_{\mathbb{C}^*}(X) \to K_{\mathbb{C}^*}(T^*X)$ is an isomorphism. So our claim is proved.

Now we are ready to prove the Lemma. Let $F \in L_{\mathbb{C}^*}(Z)$. The map $\bar{p}$: $\text{supp } F \to (T^*X_1) \times X_2$ being finite, it is clear that $\bar{p}_* F$ is a $\mathbb{C}^*$-equivariant coherent sheaf on $(T^*X_1) \times X_2$ and that $\dim \text{supp}(p_* F) \leqslant n$. Nothing will change in (7.8.3) if a parameter-space X_2 is added so that T^*X is replaced there by $(T^*X_1) \times X_2$ and $\mathbb{P}^*X$ by $(\mathbb{P}^*X_1) \times X_2$. Applying the mappings of the lower way of this new diagram to $\bar{p}_* F$ we obtain exactly $\sum_{i \leqslant n} \text{Ch}^i\, T(F)$. Let us now go along the upper way. Consider the sheaf $\varepsilon(\bar{p}_* F)$ on $(\mathbb{P}^*X_1) \times X_2$. We apply to it the following general result which can be easily derived from Grothendieck's version of Riemann–Roch: let E be a coherent $\mathcal{O}_Y$-sheaf on a certain manifold Y. If $\dim(\text{supp } E) = d$ then:

(i′) $\text{ch}^j E = 0$ for $j < \dim Y - d$

(ii′) $(\text{ch}^d E) \cap [Y] =$ the class of supp E in $H_d(Y)$. The statement follows. ∎

7.9. We are now in a position to complete the proof of Theorem 7.4 by showing that "Chern" is a Grothendieck transformation.

Let τ be the Todd class of X_1. Following [FM] define the Riemann–Roch map RR: $K_{\mathbb{C}*}(Z) \to H(Z)[[t]]$ as: $F \mapsto \tau \cdot \mathrm{Ch}\, F \in H_Z^*(X_1 \times X_2)[[t]] \cong H(Z)[[t]]$. This is a Grothendieck transformation by the Riemann–Roch theorem. Taking 7.7.2 into account we see that $L_{\mathbb{C}*}(Z) \xrightarrow{T} K_C{}^*(Z) \xrightarrow{RR} H(Z)[[t]]$ is a composite of Grothendieck transformations. We also have: $\tau = 1 + \tau_2 + \tau_4 + \cdots$, $\tau_i \in H_Z^{2i}(X_1 \times X_2)$ and: $\mathrm{Ch}\, T(F) = \mathrm{Ch}^n\, T(F) + \mathrm{Ch}^{n+2}\, T(F) + \cdots$ (by Lemma 7.8.1). Hence: $\mathrm{RR} \cdot T(F) =$ **Chern** (supp F) + **terms of higher dimension.** Thus the mapping Chern has to be a Grothendieck transformation also. Q.E.D.

8. Characteristic Varieties and Primitive Ideals

The results of 8.1–8.4 were also obtained by Borho and Brylinski [BB2]. Our proofs appear to be the same.

8.1. Let M be a filtered $U(\mathfrak{G})$-module. We will frequently use the following observation:

LEMMA 8.1.1. *A filtration on M is good iff* gr M *is a finitely generated* $\mathbb{C}[\mathfrak{G}^*]$*-module.*

For a finitely generated U-module M let $\mathcal{M} = \mathcal{D}_X \otimes_U M$ be the corresponding $\mathcal{D}_X$-module and $SSM \subset \mathfrak{G}^*$, resp. $SS\mathcal{M} \subset T^*X$ their characteristic varieties.

PROPOSITION 8.1 (cf. [BB, Sect. 4; KT]). $SSM = \mu(SS\mathcal{M})$.

Proof. Let us show that $SS\mathcal{M} \subset \mu^{-1}(SSM)$. Suppose first that $M = U/J$ for some left ideal $J \subset U$. Then $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X \cdot J$. Therefore the zero variety of $\mathrm{gr}(\mathcal{D}_X \cdot J)$ is contained in the inverse image of the zero variety of gr J and we are done. In general M is the finite sum of submodules $M_i = U/J_i$ so that $SSM = \bigcup_i SSM_i$ and $SS\mathcal{M} \subset \bigcup SS(D_X \otimes M)$. Thus $SS\mathcal{M} \subset \mu^{-1}(SSM)$.

The proof of the opposite inclusion $SSM \subset \mu(SS\mathcal{M})$ copies that of [BB1, Theorem 4.6]. Choose a good filtration $\{\mathcal{M}_i\}$ on $\mathcal{M}$ and let $M_i = M \cap \mathcal{M}_i$ be the induced filtration on M. Clearly $\mathrm{gr}\, M \subsetneq \Gamma(X, \mathrm{gr}\, \mathcal{M}) \simeq \Gamma(\mathfrak{G}^*, \mu_\cdot(\mathrm{gr}\, \mathcal{M}))$. Since μ is a proper morphism the sheaf $\mu_\cdot(\mathrm{gr}\, \mathcal{M})$ is coherent. Therefore $\Gamma(\mathfrak{G}^*, \mu_\cdot(\mathrm{gr}\, \mathcal{M}))$ and, hence, gr M are finitely generated $\mathbb{C}[\mathfrak{G}^*]$-modules. So, according to Lemma 8.1.1, $\mathrm{supp}\, \mathrm{gr}\, M \subset \mathrm{supp}\, \mu_\cdot(\mathrm{gr}\, \mathcal{M}) \subset \mu\, (\mathrm{supp}\, \mathrm{gr}\, \mathcal{M}) = \mu(SS\mathcal{M})$. Q.E.D.

8.2. Here we compare characteristic varieties of U-modules and those of their annihilators.

Consider the projections $p: X\times X\to X$ and $\bar{p}: T^*X\times T^*X\to T^*X$ on first factors. The following is clear:

LEMMA 8.2.1. $\bar{p}(T^*_{C_w}(X\times X))=G\cdot T^*_{X_w}X$.

Recall that this set is an irreducible component of $\overline{\mu^{-1}(\mathcal{O})}$ for some nilpotent orbit $\mathcal{O}\subset\mathfrak{G}^*$.

Next suppose that $M\in\tilde{\mathcal{O}}$ is an U-module, Ann $M\subset U$ is its annihilator and $\mathcal{M}$ (resp. $\mathcal{D}_X/\mathcal{D}_X$. Ann M) are the corresponding $\mathcal{D}_X$-modules.

PROPOSITION 8.2. $SS(\mathcal{D}_X/\mathcal{D}_X\cdot \text{Ann } M)=G\cdot SS\mathcal{M}$.

Note that $SS(U/\text{Ann } M)$ is just Var(Ann M) (notations of Sect. 1). So by 8.1 and 8.2: $\text{Var}(\text{Ann } M)=\mu(SS(\mathcal{D}_X/\mathcal{D}_X\cdot\text{Ann } M))$. So we get

COROLLARY 8.2.1. If $M\in\tilde{\mathcal{O}}$, *then*

$$\text{Var}(\text{Ann } M)=G\cdot SS\dot{M}.$$

In view of Corollary 4.3 we also have (see Section 1.7):

COROLLARY 8.2.2. *If* $M\in\tilde{\mathcal{O}}$, *then*

$$\dim \text{Var}(\text{Ann } M)=2\cdot\dim(SSM).$$

Proof of the proposition. Consider the Harish-Chandra module $P(M)$, corresponding to M. We have already mentioned that $L\,\text{Ann } P(M)=\text{Ann } M$. It follows from Lemma 3.5 that there is a finite-dimensional $\mathfrak{G}$-module E such that $U/\text{Ann } M$ is a subquotient of $P(M)\otimes_{\mathbb{C}} E$. Extending the left-hand module structure to $\mathcal{D}_X$ we see that $\mathcal{D}_X/\mathcal{D}_X\cdot\text{Ann}(M)$ is a subquotient of $\mathcal{D}_X\otimes_U P(M)\otimes_{\mathbb{C}} E$. Consequently $SS(\mathcal{D}_X/\mathcal{D}_X\cdot\text{Ann } M)\subset SS(\mathcal{D}_X\otimes_U P(M)\otimes_{\mathbb{C}} E)\subset SS(\mathcal{D}_X\otimes_U P(M))$, where the second inclusion is trivial since $\dim E<\infty$.

Next consider the $\mathcal{D}_{X\times X}$-module $P(\mathcal{M}):=\mathcal{D}_{X\times X}\otimes_{U\otimes U} P(M)$ corresponding to $P(M)$ (note that the notation $P(\mathcal{M})$ is consistent with the previous notation $\mathcal{M}=\mathcal{D}_X\otimes_U M$).

Let $p_{.}P(\mathcal{M})$ be its sheaf-theoretic direct image to X. It easily follows from Theorem 2.3 that $\mathcal{D}_X\otimes_U P(M)=p_{.}P(\mathcal{M})$. Thus we have shown that

$$SS(\mathcal{D}_X/\mathcal{D}_X\cdot\text{Ann } M)\subset SSp_{.}P(\mathcal{M}). \tag{8.2.3}$$

Let us now interrupt the proof in order to state

LEMMA 8.2.4. *Suppose* $\mathcal{N}\in\mathcal{H}$ *is a* $\mathcal{D}_{X\times X}$*-module and* $Q(\mathcal{N})$ *the corresponding* $\mathcal{D}_X$*-module* (*cf. Theorem* 3.2). *Then*

$$\bar{p}(SS\mathcal{N})=G\cdot SSQ(\mathcal{N}).$$

This follows from Lemma 8.2.1 and the relation between characteristic cycles: if $\operatorname{gr} \mathcal{N} = \sum m_w \cdot \overline{T^*_{C_w}(X \times X)}$ then $\operatorname{gr} Q(\mathcal{N}) = \sum m_w \cdot \overline{T^*_{X_w} X}$. Such a relation is, for example, a consequence of the fact that $Q(\mathcal{N})$ is the restriction of $\mathcal{N}$ to a noncharacteristic submanifold $X \times \{x_0\} \subset X \times X$.

Let us resume the proof of Proposition 8.2 and set $\Lambda = \bigcup T^*_{C_w}(X \times X)$. In order to estimate the right-hand side of (8.2.3) note that the map $\bar{p}: \Lambda \to T^*X$ is proper and that $SSP(\mathcal{M}) \subset \Lambda$. Exactly the same argument as in the proof of Proposition 8.1 shows that $SSp_* P(\mathcal{M}) = \bar{p}(SSP(\mathcal{M}))$. Further, according to Lemma 8.2.4, $\bar{p}(SSP(\mathcal{M})) = G \cdot SS\mathcal{M}$. Thus, in view of (8.2.3),

$$SS(\mathcal{D}_X / \mathcal{D}_X \cdot \operatorname{Ann} M) \subset G \cdot SS\mathcal{M}.$$

The opposite inclusion is easy: the two-sided ideal Ann M is stable under the adjoint G-action. Therefore $SS(\mathcal{D}_X / \mathcal{D}_X \cdot \operatorname{Ann} M)$ is a G-stable subvariety. For trivial reasons it contains $SS\mathcal{M}$. Consequently, $G \cdot SS\mathcal{M} \subset SS(\mathcal{D}_X / \mathcal{D}_X \cdot \operatorname{Ann} M)$. Q.E.D.

8.3. Let $\mathbf{b}$ be a fixed Borel subalgebra and $x_0 \in X$ the corresponding point. Denote by L_w the simple quotient of the Verma module M_w and by $\mathcal{L}_w$ the corresponding $\mathcal{D}_X$-module. According to Proposition 2.6, $\operatorname{supp} \mathcal{L}_w = \bar{X}_w$.

LEMMA 8.3.1. *If $\mathcal{N} \in \mathcal{H}$ is a simple $\mathcal{D}_{X \times X}$-module with* supp $\mathcal{N} = \bar{C}_w$ *then $\mathcal{N}|_{X \times \{x_0\}} = \mathcal{L}_w$ and $\mathcal{N}|_{\{x_0\} \times X} = \mathcal{L}_{w^{-1}}$.*

Proof. It is clear that $\bar{C}_w \cap (X \times \{x_0\}) = \bar{X}_w$ and $\bar{C}_w \cap (\{x_0\} \times X) = \bar{X}_{w^{-1}}$. On the other hand both $\mathcal{N}|_{X \times \{x_0\}}$ and $\mathcal{N}|_{\{x_0\} \times X}$ are simple $\mathcal{D}_X$-modules since these restrictions are equivalences of categories (see Section 3.3). Hence there is no possibility other than $\mathcal{N}|_{X \times \{x_0\}} = \mathcal{L}_w$ and $\mathcal{N}|_{\{x_0\} \times X} = \mathcal{L}_{w^{-1}}$. Q.E.D.

For an Harish-Chandra $U \otimes U$-module N denote by L Ann N and R Ann N its annihilators in $U \otimes 1$ (resp. $1 \otimes U$). Recall that $I_w = \operatorname{Ann} L_w$.

PROPOSITION 8.3 [Jo3; V]. *L Ann $P(L_w) = I_w$ and R Ann $P(L_w) = I_{w^{-1}}$.*

The first equality is a consequence of the general fact $L \operatorname{Ann} P(M) = \operatorname{Ann} M$. It was established during the proof of Theorem 3.4. Of course the same remains true for L Ann replaced by R Ann if in the relation between $\tilde{\mathcal{O}}$ and H (see Theorem 3.2) the role of two U-actions on Harish-Chandra modules is interchanged. In $\mathcal{D}$-module language that means the change of factors of $X \times X$. So the statement follows from Lemma 8.3.1.

8.4. We keep to the notations of Sections 8.2 and 8.3 Identify T^*X with $G\times_B \mathbf{n}$ and consider the subvariety $SSL_w \subset \mathbf{n}$. Also set: $S(I_w)=SS(\mathcal{D}_X/\mathcal{D}_X\cdot I_w)$.

PROPOSITION 8.4. $S(I_w)=G\times_B SS(L_{w^{-1}})$.

Proof. Let $\mathcal{N}=P(\mathcal{L}_w)$ be the simple $D_{X\times X}$-module, corresponding to $\mathcal{L}_w$. Proposition 8.2 and Lemma 8.2.4 show that $S(I_w)=\tilde{p}(SS\mathcal{N})$. On the other hand, according to Lemma 8.3.1, $\mathcal{L}_{w^{-1}}=\mathcal{N}|_{\{x_0\}\times X}$. Since this restriction is non-characteristic, $SS(\mathcal{L}_{w^{-1}})$ equals the projection of $SS\mathcal{N}\cap (T^*_{x_0}X\times T^*X)$ to T^*X (the second factor). If q is this projection and μ is the moment map then according to Proposition 8.1, $SS(L_{w^{-1}})=\mu\circ q(SS\mathcal{N}\cap(T^*_{x_0}X\times T^*X))$. It follows from the commutative diagram

$$\begin{array}{ccc} \Lambda & =\bigcup T^*_{C_w}(X\times X) \\ \downarrow{\scriptstyle \mu_\Delta} & & \downarrow{\scriptstyle q} \\ \mathfrak{G}^*_\Delta & \xleftarrow{\ \mu\ } & T^*X \end{array}$$

that $\mu\circ q$ may be replaced by μ_Δ. So the statement reduces to the following purely geometric equality

$$\tilde{P}(T^*_{C_w}(X\times X))=G\times_B \mu_\Delta[T^*_{C_w}(X\times X)\cap(T^*_{x_0}X\times T^*X)]. \qquad (8.4.1)$$

Here is its direct check-up for the G_Δ-orbit $C_w=G_\Delta\cdot z$, where $z=(w\cdot x_0, x_0)\in X\times X$. Set ${}^w\mathbf{n}=w\cdot\mathbf{n}\cdot w^{-1}$. Clearly $T^*_{C_w}(X\times X)=G\times_{{}^wB\cap B}({}^w\mathbf{n}\cap\mathbf{n})$. Hence the left-hand side of (8.4.1) equals

$$G\times_B B\cdot(\mathbf{n}\cap{}^{w^{-1}}\mathbf{n}).$$

One can also verify that

$$\mu_\Delta[T^*_{C_w}(X\times X)\cap(T^*_{x_0}X\times T^*X)]=B\cdot(\mathbf{n}\cap{}^{w^{-1}}\mathbf{n}). \qquad \text{Q.E.D.}$$

8.5. LEMMA 8.5. *Suppose $I\subset J$ are two-sided ideals in U. If I is prime then $S(J)\subset\partial S(I)$ (that is, an intersection of $S(J)$ with each irreducible component of $S(I)$ is strictly contained in that component).*

Proof. If I is prime J/I is the essential ideal in U/I. Therefore there is a non-zero divisor $a\in J/I$.

Let $\mathcal{E}_X$ be the sheaf on micro-differential operators on T^*X and let S be an irreducible component of $\mathcal{E}_X\otimes_U(U/I)$. It suffices to show that the module $\mathcal{E}_X\otimes_U(U/J)$ has the zero multiplicity at a generic point of S. Consider an exact sequence: $0 \to U/I \xrightarrow{\cdot a} U/I \to \operatorname{coker}(a) \to 0$ arising from the right multiplication by a. By tensoring it by $\mathscr{E}_X$ we get:

$$\mathbf{mult}_S(\mathscr{E}_X \otimes \operatorname{coker} a) = \mathbf{mult}_S(\mathscr{E}_X \otimes (U/I)) - \mathbf{mult}_S(\mathscr{E}_X \otimes (U/I)) = 0$$

(by additivity of $\mathbf{mult}_S$ and the exactness of $\mathscr{E}_X \otimes (\cdot)$). Further, U/J is a quotient of $\operatorname{coker}(a)$. Hence, $\operatorname{supp}(\mathscr{E}_X \otimes (U/J)) \subset \operatorname{supp}(\mathscr{E}_X \otimes \operatorname{coker}(a)) \subset \partial S(I)$. Q.E.D.

COROLLARY 8.5.1. *Under the same assumptions*, $\operatorname{Var}(J) \subset \partial \operatorname{Var}(I)$.

Remark. This strengthens the earlier result of Borho: $\dim \operatorname{Var}(J) < \dim \operatorname{Var}(I)$.

In order to prove the corollary note that each irreducible component of either $S(I)$ or $S(J)$ is of the form $\overline{G \cdot T^*_{X_w} X}$. Any such variety is a component of the inverse image $\overline{\mu^{-1}(\mathcal{O})}$ for some nilpotent orbit $\mathcal{O} \subset \mathfrak{G}^*$. It is therefore clear that the inclusion $S(J) \subset \partial S(I)$ implies $\mu(S(J)) \subset \partial \mu(S(I))$. The result now follows from Proposition 8.1 and Lemma 8.5.

Remark. One can prove Corollary 8.5.1 using the arguments of the proof of Lemma 8.5 directly by means of the formal micro-localization of $U(\mathfrak{G})/I$ (see [Gi4, Sect. 1]).

8.6. We identify $K(\tilde{\mathcal{O}})$ and $K(H)$ with $\mathbb{Z}[W]$. Following [Jo2; KL1] on W define a preorder $\leqslant_L$ and an equivalence relation $\sim_L$ by

$$w_1 \underset{L}{\leqslant} w_2 \text{ iff } I_{w_1} \subset I_{w_2}; \qquad w_1 \underset{L}{\sim} w_2 \text{ iff } I_{w_1} = I_{w_2}.$$

For $w \in W$ the subspaces in $\mathbb{C}[W] = K(\tilde{\mathcal{O}})$

$$\bar{V}^L_w = \bigoplus_{y \underset{L}{\leqslant} w} \mathbb{C} \cdot L_y$$

$$K^L_w = \bigoplus_{y \underset{L}{\leqslant} w,\, y \underset{L}{\not\sim} w} \mathbb{C} \cdot L_y$$

are known to be stable under the left W-action and the quotient $V^L_w = \bar{V}^L_w / K^L_w$ is called the left cell representation. It has a natural basis consisting of L_y, $y \sim_L w$.

Let us identify $K(\tilde{\mathcal{O}})$ with the group $L(\Lambda_{\mathfrak{b}})$ of Lagrangian cycles in $\Lambda_{\mathfrak{b}} = \bigcup_{y \in W} T^*_{X_y} X$ via the map $L_y \mapsto \operatorname{gr} L_y$. Set $\dim \operatorname{Var}(I_w) = d$ and recall notations of Section 5.4. According to Corollary 8.5.1 we get a non-trivial map:

$$V^L_w = \bar{V}^L_w / K^L_w \to \bigoplus_{\dim \mathcal{O} = 2d} \mathbb{C} \otimes L_{\mathcal{O}}. \tag{8.6.1}$$

Theorem 6.2 shows that it is compatible with W-action.

Suppose that $\mathfrak{G} = \mathrm{sl}_n(\mathbb{C})$. It was then proved in [KL1] that the representation of W in V_w^L is irreducible. Hence the map (8.6.1) is injective. In view of Proposition 5.4 its image is contained in the unique term $\mathbb{C} \otimes L_{\mathcal{O}}$ in (8.6.1). It follows from results of Sections 8.1 and 8.2 that $\mathrm{Var}(I_w) = \bar{\mathcal{O}}$. Thus we have obtained the following.

PROPOSITION [BB]. *If* $\mathfrak{G} = \mathrm{sl}_n(\mathbb{C})$ *then* $\mathrm{Var}(I_w)$ *is irreducible.*

In the case of an arbitrary semi-simple Lie algebra $\mathfrak{G}$ representations V_w^L are generally not irreducible. However, one can derive from explicit computations by Barbasch Vogan and Lusztig [Lu] that the following fact is still true (see also [KT, Proposition 12]):

PROPOSITION. *For each left cell representation* V_w^L *there is a special irreducible W-module* $V_{\mathcal{O},1}$ *contained in* V_w^L *with multiplicity one such that for any other irreducible component* $V_{\mathcal{O}',\rho}$ *of* V_w^L *one has* $\mathcal{O}' \subset \bar{\mathcal{O}}$.

This proposition together with (8.6.1) and Propositions 8.1 and 8.2 gives another proof of Theorem 1.1:

THEOREM 8.6. *For any primitive ideal* I_w *the associated variety* $\mathrm{Var}(I_w)$ *is irreducible.*

APPENDIX: SYMBOLIC QUANTIZATION

This section is inspired by ideas of Guillemin, Sternberg and Weinstein [GS1]. The next few phrases are borrowed from [GS2].

A.1. The Heisenberg uncertainty principle says that it is impossible to determine simultaneously the position and momentum of a quantum-mechanical particle. More generally the smallest subsets of classical phase space in which the presence of a quantum-mechanical particle can be detected are its Lagrangian submanifolds. (For instance one can determine exactly the position of a particle at the expense of remaining in total ignorance about its momentum.) For this reason it makes sense to regard Lagrangian subvarieties of phase space M as being its "quantum points." Thus "quantum phase space" is the set of Lagrangian subvarieties of the classical phase space M.

It is therefore more natural to realize elements of the quantum-mechanical Hilbert space as functions (or vector bundle sections) on the variety of Lagrangian subsets in M rather than functions on M itself as is usually done. Since the variety of all Lagrangian subsets is infinite-dimensional, in practice one should pick up a certain finite-dimensional family

$\{\Lambda_x,\ x \in X\}$ of Lagrangian varieties $\Lambda_x \subset M$. Here X is assumed to be a finite-dimensional manifold. For example, if $M = T^*X$ and $\{\Lambda_x = T^*_x X,\ x \in X\}$ is the family of fibres of cotangent bundle we arrive at the simplest situation where position and momentum are separated. Our scheme works, however, in considerably more complicated cases in which various Λ_x intersect each other in an arbitrary way. We will be able, in particular, to construct representations of Lie algebras associated with any non-polarized coadjoint orbits.

A.2. Returning to mathematics consider the category with symplectic manifolds as objects and Lagrangian correspondences as morphisms. For symplectic manifolds N_1, N_2 and a Lagrangian subvariety $\Lambda \subset N_1 \times N_2$ consider the projections $p_i\colon N_1 \times N_2 \to N_i$. Then $\Sigma_i := p_i(\Lambda)$ are coisotropic subvarieties. On Σ_1 and Σ_2 there are natural null-foliations.

LEMMA A.2. *If Q is a generic leaf of the null-foliation on Σ_1 then $p_2(p_1^{-1}(Q) \cap \Lambda)$ is a finite union of leaves of the null-foliation on Σ_2.*

Proof. Let $x \in \Lambda$ be a generic point. By Sard's lemma both maps $T_x\Lambda \to T_{p_1(x)}\Sigma_1$ and $T_x\Lambda \to T_{p_2(x)}\Sigma_l$ are surjective. So it remains to show that the inverse image of $T_{p_1(x)}Q$ in $T_x\Lambda$ maps onto the tangent space of the null-foliation on Σ_2. That is easy. Q.E.D.

A.3. Recall that the map $\mu\colon \Sigma \to M$ is called the reduction of the coisotropic variety Σ if generic fibres of μ are the leaves of the null-foliation on Σ and M is a symplectic variety with its symplectic structure induced from Σ. We allow M to be singular: the symplectic structure then means a Poisson bracket on $\mathcal{O}_M$ such that the Lie algebra $(\mathcal{O}_M, \{-,-\})$ has no center.

Conjecture. Suppose Σ is a coisotropic subvariety of a projective (algebraic) complex symplectic manifold N. Then one can find a proper reduction $\mu\colon \Sigma \to M$.

In the situation of Section A.2 assume that there are reductions $\mu_i\colon \Sigma_i \to M_i$. Lemma A.2 just means that:

COROLLARY A.3. *There is a finite correspondence between M_1 and M_2.*

Note that if this correspondence is bijective, i.e., $M_1 = M_2 = M$, we have $\Lambda = \Sigma_1 \times_M \Sigma_2$. In the general case the correspondence gives rise to some equivalence relation between points of M_1 and M_2. Set $M = \{(m_1, m_2) \in M_1 \times M_2 \mid m_1 \sim m_2\}$. Natural maps: $M \to M_i$ are clearly finite. One can introduce $\tilde{\Sigma}_i = M \times_{M_i} \Sigma_i$ and regard them as "immersed" coisotropic varieties. Then $\tilde{\Lambda} := \tilde{\Sigma}_1 \times_M \tilde{\Sigma}_2$ is the "immersed" Lagrangian variety $\tilde{\Lambda} \twoheadrightarrow \Lambda \subsetneq N_1 \times N_2$, where the map $\tilde{\Lambda} \to \Lambda$ is also finite.

A.4. Let us move in the opposite direction. Given a symplectic manifold M call its symplectic resolution any diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{i} & N \\ {\scriptstyle\mu}\downarrow & & \\ M & & \end{array}$$

where Σ is an immersed coisotropic subvariety in a symplectic manifold N and the map $\mu: \Sigma \to M$ is a reduction. The following statement is trivial:

LEMMA A.4.1. *If $\mu_i: \Sigma_i \to M$, $\Sigma_i \to N_i$ are two symplectic resolutions of M then $\Sigma_1 \times_M \Sigma_2$ is an immersed Lagrangian subvariety of $N_1 \times N_2$.*

Combining all previous remarks we get (modulo existence of reductions; see Section A.3):

PROPOSITION A.4.2. *Immersed Lagrangian correspondences are* (*up to finite lifting*) *fibre-products $\Sigma_1 \times_M \Sigma_2$ of symplectic resolutions.*

A.5. For a correspondence $\Lambda \subset N_1 \times N_2$ let $\Lambda' = \{(n_2, n_1) \in N_2 \times N_1 \mid (n_1, n_2) \in \Lambda\}$ be the adjoint correspondence. We call a Lagrangian correspondence $\Lambda \subset N \times N$ such that $\Lambda' = \Lambda$ and $\Lambda \circ \Lambda = \Lambda$ a self-adjoint idempotent ($\Lambda \circ \Lambda$ means the composite of the correspondence; see Section 7.1).

COROLLARY A.5. *If $\mu: \Sigma \to M$, $\Sigma \subset N$, is a symplectic resolution then $\Lambda = \Sigma \times_M \Sigma \subset N \times N$ is a self-adjoint idempotent and any self-adjoint idempotent is obtained in this way.*

A.6. As in Section A.1 suppose that $\{\Lambda_x,\ x \in X\}$ is a family of Lagrangian subvarieties of a symplectic manifold M. Set $\Sigma = \{(m, x) \in M \times X \mid x \in X,\ m \in \Lambda_x\}$. Clearly Σ determines the family $\{\Lambda_x\}$ and vice versa. We assume X to be a complex manifold.

THEOREM A.6. *Suppose that*: (a) *the projection $\mu: \Sigma \to M$ is generically submersive and* (b) *the projection*: *$\Sigma \to X$ is a locally trivial fibration with simply-connected fibres. Then there is a twisted cotangent bundle $T^{\#}X$* (*see* [Gi5]) *and a morphism i: $\Sigma \to T^{*}X$ over X such that the following holds*:

(1) *i is an immersion with a coisotropic image*;

(2) *the diagram*: $\begin{array}{ccc}\Sigma & \xrightarrow{i} & T^{\#}X \\ {\scriptstyle\mu}\downarrow & & \\ M & & \end{array}$ *is a symplectic resolution of* M;

(3) $\Lambda_x = \mu \cdot i^{-1}(T_x^{\#} X)$ *for all* $x \in X$.

This theorem shows that any Lagrangian family can be turned after an appropriate symplectic resolution into a standard one: $[T_x^{\#} X, x \in X\}$.

Most important for us is the special case of homogeneous symplectic manifolds (i.e., manifolds with vector field ξ such that $L_\xi \omega = \omega$, where ω is the symplectic 2-form). Note that nilpotent orbits in $\mathfrak{G}^*$ ($\mathfrak{G}$-semi-simple Lie algebra) are homogeneous symplectic manifolds. This example should be constantly kept in mind.

COROLLARY A.6.1. *If* $\{\Lambda_x\}$ *is a family of homogeneous Lagrangian subvarieties in a homogeneous symplectic manifold M then in Theorem* A.6 Σ *is a homogeneous immersed subvariety of the usual cotangent bundle* T^*X.

Let us indicate the proof of Theorem A.6. Assume for simplicity that the symplectic 2-form ω on M is exact: $\omega = d\alpha$ and that $\alpha\,|_{\Lambda_x} = 0$ for all x (this is always so in the homogeneous case). Let $\tilde{\alpha}$ be the pull-back of α to Ξ. Consider the projection $\pi: \Xi \to X$. The assumption $\alpha\,|_{\Lambda_x} = 0$ implies that $\tilde{\alpha}$ vanishes on fibres of π. Hence $\tilde{\alpha}$ may be regarded as a section of the subbundle $\pi^*(T^*X) \subset T^*\Sigma$. That gives the map i: $\Sigma \to T^*X$. It is not hard to show that $i(\Sigma)$ is a coisotropic subvariety in T^*X and that the map $i: \Sigma \to i(\Sigma)$ is "etale." It remains to verify that fibres of the projection $\Sigma \to M$ maps into leaves of the null-foliation on $i(\Sigma)$.

A.7. Suppose we are given a symplectic manifold M and a Lagrangian family $\{\Lambda_x, x \in X\}$ on it. Let us explain at a physical level how the programm outlined in Section A.1 can be carried out. For simplicity we assume M to be homogeneous (as in Corollary A.6.1) and the map $i: \Sigma \to T^*X$ to be injective (the first assumption is not essential). Then according to Section A.5, $\Lambda = \Sigma \times_M \Sigma$ is a homogeneous Lagrangian idempotent in $T^*(X \times X)$. In the C^∞ case it is possible to consider the holonomic system $\mathscr{C}^\Lambda$ of all microfunctions supported at Λ. Elements of $\mathscr{C}^\Lambda$ are also called Fourier integral operators associated with Λ. Since $\Lambda \circ \Lambda = \Lambda$ and $\Lambda^t = \Lambda$, $\mathscr{C}^\Lambda$ is actually an algebra with involution (cf. [GS1]). This algebra should be regarded as a quantization of the classical system (M, $\{\Lambda_x\}$).

In order to understand $\mathscr{C}^\Lambda$ better, consider irreducible components of Λ. Any such component is a homogeneous Lagrangian variety in $T^*(X \times X)$. Hence it is equal to $T^*_{C_i}(X \times X)$ for some $C_i \subset X \times X$. Thus $\Lambda = \bigcup T^*_{C_i}(X \times X)$. Further for each i consider projections $X \xleftarrow{p_i} C_i \xrightarrow{q_i} X$

on both factors. As a bimodule over the ring of pseudo-differential operators on X integral operators in $\mathscr{C}^{\Lambda}$ are generated by "generalized Radon transformations" $R_i\colon \varphi \mapsto (q_i)_* p_i^* \varphi$, $\varphi \in C_0^\infty(X)$. It is therefore interesting to describe fibres of maps p_i and q_i. Note that for $x \in X$ the set $\mu^{-1}(\Lambda_x) = \mu^{-1} \circ \mu(\Sigma \cap T_x^* X)$ is a Lagrangian subvariety in T^*X. One can prove:

LEMMA A.7. *$\mu^{-1}(\Lambda_x) = \bigcup_i T^*_{p_i^{-1}(x)} X$ and similarly for q_i. In particular $\{p_i^{-1}(x)\}$ differs from the collection $\{q_i^{-1}(x)\}$ only by permutation of indices.*

Remark. One undoubtedly noticed similarities between the geometry of Section 4 and the present one. Clearly X corresponds to a Flag manifold, subvarieties $C_i \subset X \times X$ to G_Δ-orbits C_w, the subvarieties $p_i^{-1}(x)$ (for fixed $x \in X$) to Schubert cells, etc.... That will be elaborated in Remark (ii) below.

In contrast with the C^∞ case in the complex (possibly singular) situation there is no module C^Λ naturally associated with Λ. All that remains is the algebra $L(\Lambda)$ (see Section 5.1). It can be verified that Λ satisfies condition (T5) of Proposition 7.2, provided $\Sigma \to X$ is a fibration. Then, according to Theorem 6.2, $L(\Lambda)$ is isomorphic to the algebra generated by $\mathscr{D}_{X \times X}$-modules (such that $SS\mathscr{M} \subset \Lambda$), that is, by genuine quantum objects.

Conclusion

Any coisotropic subvariety $\Sigma \subset T^*X$ gives rise to the involutive $\mathbb{Z}$-algebra $L(\Sigma \times_M \Sigma)$.

Remarks. (i) If X is a flag manifold and $\Lambda = T^*_{C_w}(X \times X)$ then $L(\Lambda) \simeq \mathbb{Z}[W]$ according to Proposition 5.3. Note that the involution on $L(\Lambda)$ is induced by the usual one: $w \mapsto w^{-1}$ on the Weyl group.

(ii) As another example consider the nilpotent orbit $\mathcal{O} \subset \mathfrak{G}^*$. This is a symplectic manifold with the distinguished family of Lagrangian subvarieties (see Proposition 4.3) $\Lambda_x = \mathcal{O} \cap \mathbf{n}_x$, $x \in G/B$. The coisotropic subvariety Σ associated with that family via Corollary A.6.1 coincides with the inverse image $\mu^{-1}(\mathcal{O}) \subset T^*X$. The corresponding algebra $L_{\mathcal{O}} = L(\Sigma \times_{\mathcal{O}} \Sigma)$ was studied in Section 5.4.

This can be extended to an arbitrary complex Lie algebra $\mathfrak{G}$ and an *arbitrary* coadjoint orbit $\mathcal{O} \subset \mathfrak{G}^*$. For $\lambda \in \mathcal{O}$ one can find an appropriate subalgebra $\mathbf{p} \subset \mathfrak{G}$ such that $\Lambda_x = \mathcal{O} \cap x \cdot (\lambda + \mathbf{p}^\perp) \cdot x^{-1}$, $x \in G/P$, is the Lagrangian family in $\mathcal{O}$ (P is a group, corresponding to $\mathbf{p}$). Then it is possible to set $X = G/P$, $\Sigma = \mu^{-1}(\mathcal{O})$, $\Lambda = \Sigma \times_{\mathcal{O}} \Sigma$. Further, there are P-orbits in G/P similar to Schubert cells (see Lemma A.7) and $\mathfrak{G}$-modules connected with these P-orbits as in Corollary 2.6.1. It can be shown that *all* primitive ideals in $U(\mathfrak{G})$ may be obtained as annihilators of such modules. For details the reader is referred to [Gi1; Gi2].

APPENDIX B: A COISOTROPICNESS THEOREM

We shall give here a proof of Theorem 4.1. It proceeds by induction on $\dim A$. We may clearly assume A to be connected. Let A_1 be a connected codimension 1 normal subgroup of A and let $\mathfrak{a}_1$ be the corresponding ideal in the Lie algebra $\mathfrak{a}$ of A. Consider the projection $p\colon \mathfrak{a}^* \to \mathfrak{a}_1^*$ induced by inclusion: $\mathfrak{a}_1 \hookrightarrow \mathfrak{a}$. It fits into a commutative triangle involving the moment maps with respect to A and A_1:

$$\begin{array}{ccc} & M & \\ {}^{\mu}\swarrow & & \searrow^{\mu_1} \\ \mathfrak{a}^* & \xrightarrow[p]{} & \mathfrak{a}_1^* \end{array} \tag{B1}$$

Let $\Omega \subset \mathfrak{a}^*$ be an A-orbit. Set $N := \mu_1^{-1} \cdot p(\Omega) = \mu^{-1}(p^{-1} \cdot p(\Omega))$. Since $p(\Omega)$ is a union of A_1-orbits and the statement of the theorem holds for A_1 by induction hypothesis, $\mu_1^{-1} \cdot p(\Omega)$ is a union of coisotropic subvarieties. So N is itself a coisotropic A-stable subvariety containing $\mu^{-1}(\Omega)$.

For an orbit Ω there are two alternatives:

(i) $\dim p(\Omega) = \dim \Omega - 1$;

(ii) $\dim p(\Omega) = \dim \Omega$.

In the case (i) we clearly have: $\dim p^{-1} \cdot p(\Omega) = 1 + \dim p(\Omega) = \dim \Omega$ so that Ω is an open subset of $p^{-1} \cdot p(\Omega)$. Hence $\mu^{-1}(\Omega)$ is coisotropic as an open part of the coisotropic subvariety $N = \mu^{-1}(p^{-1} \cdot p(\Omega))$.

In the second case we have: $\dim p^{-1} \cdot p(\Omega) = 1 + \dim p(\Omega) = 1 + \dim \Omega$. So Ω is a locally closed 1-codimensional subvariety of $p^{-1} \cdot p(\Omega)$. We can choose a polynomial function on $\mathfrak{a}^*$ vanishing on Ω but not identically zero on $p^{-1} \cdot p(\Omega)$. Let P be its pull-back to M via the moment map μ.

We shall show that for any $c \in \mathbb{C}$ the intersection $N_c := N \cap P^{-1}(c)$ is either empty or a coisotropic subvariety of M. The statement being local we may assume that N is irreducible and also that $P_{|N} \neq \text{const}$. Let c be a generic value of P, let x be a generic point of N_c and let V be the kernel of restriction to T_xN of the symplectic 2-form on M. It follows from our assumptions that dP does not vanish on T_xN so that $T_xN_c = T_xN \cap \ker dP$. To prove that N_c is a coisotropic subvariety it suffices to show that $V \subset \ker dP$ (due to the coisotropicness of N). Consider the hamiltonian vector field ξ_P on the symplectic manifold M associated to the 1-form $d\bar{P}$. This vector field is tangent to N since N is an A-stable subvariety and the function P came from a function on $\mathfrak{a}^*$. So for $v \in V$ we have: $dP(v) = \langle \xi_P, v \rangle = 0$, by the definition of V. That proves the coisotropicness of N_c for generic $c \in \mathbb{C}$.

To handle the general case we will make use of the following general result which seems to be known to algebraic geometers. In any case its proof can be derived easily by means of Hironaka's resolution of singularities theorem.

LEMMA. *Let P be a regular function on a complex algebraic variety N and let $\{x_i\}$ be a sequence of generic points of N such that $x_i \to x_0 \in N$. Set $N_i := P^{-1}(P(x_i))$ and $N_0 := P^{-1}(P(x_0))$. Suppose that x_0 is a regular ($=$ smooth) point of N_0. Then for the sequence of tangent spaces (viewed as points of an appropriate Grassmann bundle) we have: $T_x N_i \to T_{x_0} N_0$.*

We are now ready to complete the proof of case (ii) of the theorem by showing that $N_0 = N \cap P^{-1}(0)$ is a coisotropic subvariety of M. Let x_0 be a regular point of N_0 and let $\{x_i\}$ be a sequence of generic points of N such that $x_i \to x_0$. For generic values $c_i := P(x_i)$ it has been already proved that $N_i := P^{-1}(c_i) \cap N$ is a coisotropic subvariety of M. Hence we can choose a sequence of Lagrangian subspaces $\Lambda_i \subset T_{x_i} N_i$. By taking a subsequence of this sequence and using the lemma we conclude that there exists a subspace $\Lambda \subset T_{x_0} N_0$ such that $\Lambda_i \to \Lambda$. Hence Λ is also a Lagrangian subspace. Therefore $T_{x_0} N_0$ is a coisotropic subspace. Q.E.D.

REFERENCES

[AM] M. ATIYAH AND I. MACDONALD, "Introduction to Commutative Algebra," Addison–Wesley, Reading, Mass., 1969.

[BV] D. BARBASCH AND D. VOGAN, Primitive ideals and orbital integrals in complex groups, I, II, *Math. Ann.* **259** (1982), 153–199; *J. Algebra* **80** (1983), 350–382.

[BeBe] A. BEILINSON AND J. BERNSTEIN, Localisation de 𝔊-modules, *C. R. Acad. Sci. Paris* **292** (1981), 15–18.

[BGG1] J. N. BERNSTEIN, I. M. GEL'FAND, AND S. I. GEL'FAND, Category of 𝔊-modules, *Functional Anal. Appl.* **10** (1976), 1–18.

[BGG2] J. N. BERNSTEIN, I. M. GEL'FAND, AND S. I. GEL'FAND, Schubert cells and cohomologies of Flag manifolds, *Russian Math. Surveys*, (1973).

[BG] J. BERNSTEIN AND S. GEL'FAND Tensor products of finite and infinite dimensional representations, *Compositio Math.* **41** (1980), 245–285.

[BB] W. BORHO AND J.-L. BRYLINSKI, Differential operators on homogeneous spaces, I, II, *Invent. Math.* **69** (1982), 437–476.

[BBM] W. BORHO, J.-L. BRYLINSKI, AND R. MACPHERSON, A note on primitive ideals and characteristic classes, preprint, 1985.

[BJ] W. BORHO AND J.-C. JANTZEN, Uber primitive Ideale in der Einhullenden einer halbeinfachen Lie-Algebra, *Invent. Math.* **39** (1977), 1–53.

[BM] W. BORHO AND R. MACPHERSON, Représentations des groupes de Weyl et homologie d'intersection pour les variétés nilpotentes, *C. R. Acad. Sci. Paris* **292** (1981), 707–710.

[Bras] J.-P. BRASSELET, Existence des classes de Chern en théorie bivariante, *Astérisque* **101-102** (1983), 7–23.

[Br] J.-L. BRYLINSKI, Homologie d'intersection et faisceaux pervers. *Séminaire Bourbaki*, exp. 585 (1982); *Astérisque* **92–93** (1982).

[BDK] J.-L. BRYLINSKI, A. DUBSON, AND M. KASHIWARA, Formule de l'indice pour les modules holonômes et obstruction d'Euler locale, *C. R. Acad. Sci. Paris* (1981).
[BK] J.-L. BRYLINSKI, AND M. KASHIWARA, Kazhdan–Lusztig conjecture and holonomic systems, *Invent. Math.* **64** (1981), 387–410.
[CD] N. CONZE-BERLINE, AND M. DUFLO, Sur les representations induites des groupes semi-simples complexes, *Compositio Math.* **34** (1977), 307–336.
[De] P. DELIGNE, SGA7, exposé XIV, *in* "Lecture Notes in Mathematics Vol. 340," Springer-Verlag, Berlin/New York/Heidelberg.
[Du] M. DUFLO, Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semisimple, *Ann. Math.* **105** (1977), 107–120.
[FM] W. FULTON, AND R. MACPHERSON, Categorical framework for the study of singular spaces, *Mem. Amer. Math. Soc.* **31** (1981).
[Gi1] V. GINSBURG, Symplectic geometry and representations, preprint, 1981.
[Gi2] V. GINSBURG, Symplectic geometry and representations, *Functional Anal, Appl.* **17**, No. 3 (1983), 75–76 [In Russian]
[Gi3] V. GINSBURG, Intégrales sur les orbites nilpotentes et représentations des groupes de Weyl, *C. R. Acad. Sci. Paris* **296** (1983), 249–252.
[Gi4] V. GINSBURG, Characteristic varieties and vanishing cycles, preprint, 1983 (*Invent. Math.*, in press).
[Gi5] V. GINSBURG, Twisted cotangent bundles and twisted differential operators, preprint, 1983.
[GS1] V. GUILLEMIN, AND S. STERNBERG, Some problems of integral geometry and micro-local calculus, *Amer. J. Math.* **101** (1979), 915–955.
[GS2] V. GUILLEMIN AND S. STERNBERG, Geometric quantization and multiplicities of group representations, *Invent. Math.* **67** (1982), 515–538.
[H1] R. HOTTA, On Springer's representations, *Math. Proc. Kyoto Univ.* (1981), 683–876.
[H2] R. HOTTA, On Joseph's construction of Weyl group representations, *Tohoku Math. J.* **36** (1984), 49–74.
[HK1] R. HOTTA AND M. KASHIWARA, The invariant holonomic system on a semisimple Lie algebra, *Invent. Math.* **75** (1984), 327–358.
[HK2] R. HOTTA AND M. KASHIWARA, Quotients of Harish–Chandra system by primitive ideals, preprint, 1985.
[Jo1] A. JOSEPH, Towards the Jantzen conjecture, I, II, III, *Compositio Math.* **40** (1980), 35–78; **41** (1981), 23–30.
[Jo2] A. JOSEPH, *W*-module structure in the primitive spectrum of the enveloping algebra of a semisimple Lie algebra, *in* "Lecture Notes in Mathematics Vol. 728" pp. 116–135, Springer-Verlag, Berlin/New York/Heidelberg, 1978.
[Jo3] A. JOSEPH, Dixmier's problem for Verma modules and principal series submodules, *J. London Math. Soc.* **20** (1979), 193–204.
[Jo4] A. JOSEPH, Goldie rank in enveloping algebra of a semisimple Lie algebra, I, II, *J. Algebra* **65** (1981), 269–306.
[Jo5] A. JOSEPH, On the variety of a highest weight module, *J. Algebra.*
[Jo6] A. JOSEPH, On the annihilators of the simple subquotients of the principal series, *Ann. Sci. Ecole Norm. Sup.* **10** (1977), 419–440.
[K] M. KASHIWARA, *Proc. Japan Acad.* **49** (1973), 803–804.
[KK] M. KASHIWARA, AND T. KAWAI, On holonomic systems of linear differential equations III (systems with regular singularities). *Publ. Res. Inst. Math. Sci.* **17** (1981), 813–979.
[KT] M. KASHIWARA, AND T. TANISAKI, The characteristic cycles of holonomic systems on a flag manifold, preprint.
[KL1] D. KAZHDAN, AND G. LUSZTIG, Representation of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.

[KL2] D. KAZHDAN AND G. LUSZTIG, A topological approach to Springer's representations, *Adv. in Math.* **38** (1980), 222–228.

[Kost] B. KOSTANT, Lie group representations on polynomial rings, *Amer. J. Math.* **81** (1959), 937–1032.

[La] G. LAUMON, Sur la catégorie derivée des *D*-modules filtrés, preprint, 1983.

[Lu] G. LUSZTIG, A class of irreducible representations of a Weyl group, I, II, *Proc. Kon. Nederl. Acad.* **82** (1979), 323–335; **85** (1982), 219–226.

[Le] T. LEVASSEUR (after O. Gabber), Equidimensionalité de la variété caractéristique, Paris, 1982.

[LV] G. LUSZTIG, AND D. VOGAN, Singularities of closures of *K*-orbits on flag manifolds, *Invent. Math.* **71** (1983), 365–379.

[M] R. MACPHERSON Chern classes for singular varieties, *Ann. of Math.* **100** (1974), 423–432.

[Sab1] C. SABBAH, Quelques remarques sur la géometrie des espaces conormaux, *Astérisque* **130** (1985).

[Sab2] C. SABBAH, Espaces conormaux bivariantes, prépublication, 1984.

[Spa] N. SPALTENSTEIN, The fixed point set of a unipotent element on a flag manifold, *Topology* **16** (1977), 203–204.

[Spr1] T. SPRINGER, Trigonometric sums, Green functions and representations of Weyl groups, *Invent. Math.* **36** (1976), 173–207.

[Spr2] T. SPRINGER, Quelques applications de la cohomologie d'intersection, Séminaire Bourbaki, exp. 589 (1981), *Astérisque* **92-93** (1982), 249–273.

[St] R. STEINBERG, On the desingularisation of the unipotent variety, *Invent. Math.* **36** (1976), 209–224.

[Ver] J.-L. VERDIER, Specialisation de faisceaux et monodromie moderée, *Astérisque* **101–102** (1983), 332–364.

[V] D. VOGAN, Ordering of the primitive spectrum of a semisimple Lie algebra, *Math. Ann.* **248** (1980), 195–203.

[Z] C. ZUCKERMAN, Tensor products of finite and infinite deminsional representations of semisimple Lie groups, *Ann. of Math.* **106** (1977), 295–308.