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Finite Groups and Even Lattices

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I would like to record a consequence of what appears to be a rare occurrence.

THEOREM. *Suppose G is a finite group and M is a finitely generated torsion free ZG-module such that for each prime p, M/pM is irreducible. Then, either $M = Z$ or there is a G-admissible positive definite integral inner product on M that is unimodular and even.*

Proof. The hypotheses guarantee that $\{nM \mid n = 0, 1,\ldots\}$ is the set of all submodules of M.

Since G is finite, there are G-admissible positive definite integral inner products on M. Take one and call it $(\ ,\ )_0$. Let $M^* = \{m \in QM \mid (m, M)_0 \subseteq Z\}$ be the dual lattice and let k be the smallest positive integer such that $kM^* \subseteq M$. Since M^* admits G, we get $kM^* = lM$, for some positive integer l, whence, $M^* = (l/k)M$. Since $M^* \supseteq M$, we get $l \mid k$, and minimality of k gives $l = 1$. Define $(\ ,\ )$ by $(m_1, m_2) = (1/k)(m_1, m_2)_0$. Then, $(\ ,\ )$ is positive definite, integral, G-admissible, and if $m \in QM$ satisfies $(m, M) \subseteq Z$, then $m \in M$; that is, M is self dual, or equivalently, M is unimodular.

Let $M_0 = \{m \in M \mid (m, m) \in 2Z\}$ be the even sublattice of M. Then, $M_0 \supseteq 2M$, so $M_0 = 2M$ or M. If $M_0 = 2M$, then $M/2M$ inherits an inner product with values in $Z/2Z$ with the property that 0 is the only isotropic vector, whence, $M/2M$ is of order 2 and $M = Z$. If $M_0 = M$, M is even (by definition) and we are done.

Suppose now that M is a unimodular even lattice and that $(\mathrm{Aut}\, M, M)$ satisfies the hypotheses of the theorem. Let M be the orthogonal sum of indecomposable sublattices $M_1, \ldots, M_r$. The M_i are obviously pairwise isomorphic lattices and $(\mathrm{Aut}\, M_1, M_1)$ also satisfies the hypotheses of the theorem, while $\mathrm{Aut}\, M = (\mathrm{Aut}\, M_1) \sim \Sigma_r$. Conversely, if $(\mathrm{Aut}\, M_1, M_1)$ satisfies the hypotheses of the theorem, then so does $(\mathrm{Aut}(M_1{}^r), M_1{}^r)$ for all $r = 1, 2, \ldots$.

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It is straightforward to verify that if M is an even indecomposable unimodular lattice and we set $G = \text{Aut } M$, and if (G, M) satisfies the hypotheses of the theorem, then the largest solvable normal subgroup of G has order 2. At present, the only available M are the Leech lattice and the lattice E_8, whose groups are, respectively, Conway's and the Weyl group of E_8.