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Reducibility of the intersections of components of a Springer fiber ☆

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ABSTRACT

The description of the intersections of components of a Springer fiber is a very complex problem. Up to now only two cases have been described completely. The complete picture for the hook case has been obtained by N. Spaltenstein and J.A. Vargas, and for two-row case by F.Y.C. Fung. They have shown in particular that the intersection of a pair of components of a Springer fiber is either irreducible or empty. In both cases all the components are non-singular and the irreducibility of the intersections is strongly related to the non-singularity. As it has been shown in *J. Algebra* **298** (2006) 1–14, a bijection between orbital varieties and components of the corresponding Springer fiber in GL_n extends to a bijection between the irreducible components of the intersections of orbital varieties and the irreducible components of the intersections of components of Springer fiber preserving their codimensions. Here we use this bijection to compute the intersections of the irreducible components of Springer fibers for two-column case. In this case the components are in general singular. As we show the intersection of two components is non-empty. The main result of the paper is a necessary and sufficient condition for the intersection of two components of the Springer fiber to be irreducible in two-column case. The condition is purely combinatorial. As an application of this characterization, we give first examples of pairs of components with a reducible intersection having components of different dimensions.

Key words and phrases: Flag manifold, Springer fibers, Orbital varieties, Robinson–Schensted correspondence

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1.1. Let G denote the complex linear algebraic group GL_n with Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ on which G acts by the adjoint action. For $g \in G$ and $u \in \mathfrak{g}$ we denote this action by $g.u := gug^{-1}$.

We fix the standard triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ where $\mathfrak{n}$ is the subalgebra of strictly upper-triangular matrices, $\mathfrak{n}^-$ is the subalgebra of strictly lower triangular matrices and $\mathfrak{h}$ is the subalgebra of diagonal matrices of $\mathfrak{g}$. Let $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$ be the standard Borel subalgebra so that $\mathfrak{n}$ is its nilpotent radical. Let B be the (Borel) subgroup of invertible upper-triangular matrices in G so that $\mathfrak{b} = \mathrm{Lie}(B)$. The associated Weyl group $W = \langle s_i \rangle_{i=1}^{n-1}$ where s_i is a reflection w.r.t. a simple root α_i is identified with the symmetric group $\mathbf{S}_n$ by taking s_i to be an elementary permutation interchanging i and $i+1$.

Let $\mathcal{F} := G/B$ denote the flag manifold. Let $G \times^B \mathfrak{n}$ be the space obtained as the quotient of $G \times \mathfrak{n}$ by the right action of B given by $(g, x).b := (gb, b^{-1}.x)$ with $g \in G$, $x \in \mathfrak{n}$ and $b \in B$. By the Killing form we identify the space $G \times^B \mathfrak{n}$ with the cotangent bundle of the flag manifold $T^*(G/B)$. Let $g * x$ denote the class of (g, x). The map $G \times^B \mathfrak{n} \to \mathcal{F} \times \mathfrak{g}, g * x \mapsto (gB, g.x)$ is an embedding which identifies $G \times^B \mathfrak{n}$ with the following closed subvariety of $\mathcal{F} \times \mathfrak{g}$ (see [10, p. 19]):

$$\mathcal{Y} := \{(gB, x) \mid x \in g.\mathfrak{n}\}.$$

The map $f : G \times^B \mathfrak{n} \to \mathfrak{g}, g * x \mapsto g.x$ is called the *Springer resolution*. It embeds into the following commutative diagram:

$$\begin{tikzcd} G \times^B \mathfrak{n} \arrow[rr, "i"] \arrow[dr, "f"'] & & \mathcal{F} \times \mathfrak{g} \arrow[dl, "pr_2"] \\ & \mathfrak{g} & \end{tikzcd}$$

where $pr_2 : \mathcal{F} \times \mathfrak{g} \to \mathfrak{g}, (gB, x) \mapsto x$ is the natural projection. Since $\mathcal{F}$ is complete and i is closed embedding f is proper (because G/B is complete) and its image is exactly $G.\mathfrak{n} = \mathcal{N}$, the *nilpotent variety* of $\mathfrak{g}$ (cf. [13]).

Let x be a nilpotent element in $\mathfrak{n}$. By the diagram above we have:

$$(*) \qquad \mathcal{F}_x := f^{-1}(x) = \{gB \in \mathcal{F} \mid x \in g.\mathfrak{n}\} = \left\{gB \in \mathcal{F} \mid g^{-1}.x \in \mathfrak{n}\right\}.$$

The variety $\mathcal{F}_x$ is called the *Springer fiber* above x. It has been studied by many authors. Springer fibers arise as fibers of Springer's resolution of singularities of the nilpotent cone in [10,13]. In the course of these investigations, Springer defined $\mathcal{W}$-module structures on the rational homology groups $H_*(\mathcal{F}_x, \mathbb{Q})$ on which also the finite group $A(x) = Z_G(x)/Z_G^o(x)$ (where $Z_G(x)$ is a stabilizer of x and $Z_G^o(x)$ is its identity component) acts compatibly. Recall that $A(x)$ is trivial for $G = \mathrm{GL}_n$. For $d = \dim(\mathcal{F}_x)$ the $A(x)$-fixed subspace $H_{2d}(\mathcal{F}_x, \mathbb{Q})^{A(x)}$ of the top homology is known to be irreducible as a W-module [14].

In [4], D. Kazhdan and G. Lusztig tried to understand Springer's work connecting nilpotent classes and representations of Weyl groups. Among problems posed there, Conjecture 6.3 in [4] has stimulated the research of the relation between the Kazhdan–Lusztig basis and Springer fibers.

Let $x \in \mathfrak{n}$ be a nilpotent element and let $\mathcal{O}_x = G.x$ be its orbit. Consider $\mathcal{O}_x \cap \mathfrak{n}$. Its irreducible components are called *orbital varieties* associated to $\mathcal{O}_x$. By Spaltenstein's construction [12] $\mathcal{O}_x \cap \mathfrak{n}$ is a translation of $\mathcal{F}_x$ (see Section 2.1).

1.2. For $x \in \mathfrak{n}$ its Jordan form is completely defined by $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of n where λ_i is the length of ith Jordan block. Arrange the numbers in a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ in the decreasing order (that is $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$) and write $J(x) = \lambda$. Note that the nilpotent orbit $\mathcal{O}_x$ is completely defined by $J(x)$. We set $\mathcal{O}_{J(x)} := \mathcal{O}_x$ and $\mathrm{sh}(\mathcal{O}_x) := J(x)$.

In turn an ordered partition can be presented as a Young diagram D_λ – an array of k rows of boxes starting on the left with the ith row containing λ_i boxes. In such a way there is a bijection between Springer fibers (resp. nilpotent orbits) and Young diagrams.

Fill the boxes of Young diagram D_λ with n distinct positive integers. If the entries increase in rows from left to right and in columns from top to bottom we call such an array a Young tableau or simply a tableau of shape λ. Let $\mathbf{Tab}_\lambda$ be the set of all Young tableaux of shape λ. For $T \in \mathbf{Tab}_\lambda$ we put $\mathrm{sh}(T) := \lambda$.

By Spaltenstein [11] and Steinberg [16] for $x \in \mathfrak{n}$ such that $J(x) = \lambda$ there is a bijection between the set of irreducible components of $\mathcal{F}_x$ (resp. orbital varieties associated to $\mathcal{O}_\lambda$) and $\mathbf{Tab}_\lambda$ (cf. Section 2.2). For $T \in \mathbf{Tab}_\lambda$, set $\mathcal{F}_T$ to be the corresponding component of $\mathcal{F}_x$. Respectively set $\mathcal{V}_T$ to be the corresponding orbital variety associated to $\mathcal{O}_\lambda$. Moreover, as it has been established in [8] (cf. Section 2.1) for $T, T' \in \mathbf{Tab}_\lambda$ the number of irreducible components and their codimensions in $\mathcal{F}_T \cap \mathcal{F}_{T'}$ is equal to the number of irreducible components and their codimensions in $\mathcal{V}_T \cap \mathcal{V}_{T'}$. Thus, the study of intersections of irreducible components of $\mathcal{F}_x$ can be reduced to the study of the intersections of orbital varieties of $\mathcal{O}_x \cap \mathfrak{n}$.

The conjecture of Kazhdan and Lusztig mentioned above is equivalent to the irreducibility of certain characteristic varieties [1, Conjecture 4]. They have been shown to be reducible in general by Kashiwara and Saito [3]. Nevertheless, the description of pairwise intersections of the irreducible components of the Springer fibers is still open.

The complete picture of the intersections of the components have been described by J.A. Vargas for hook case in [18] and by F.Y.C. Fung for two-row case in [2]. Both in hook and two-row cases, all the components are non-singular, all the intersections are irreducible or empty.

In this paper we study the components of the intersection of a pair of components for two-column case (that is, $\lambda = (2, 2, \ldots)$). The two-column case and the hook case are two extreme cases in the following sense: For all nilpotent orbits of the given rank k the orbit $\lambda = (k, 1, 1 \ldots)$ is the most nondegenerate and the orbit $\lambda = (2, 2, \ldots)$ (with dual partition $\lambda^* = (n - k, k)$) is the most degenerate, in the following sense $\overline{\mathcal{O}}_{(k,1,\ldots)} \supset \overline{\mathcal{O}}_\mu \supset \mathcal{O}_{(2,\ldots,2,1,\ldots)}$ for any μ such that for $x \in \mathcal{O}_\mu$ one has $\operatorname{Rank} x = k$. However, it seems that the general picture must be more close to the two-column case than to the hook case, which is too simple and beautiful.

1.3. In general we have only Steinberg's construction for orbital varieties. Via this construction orbital varieties in $\mathcal{O}_x \cap \mathfrak{n}$ are as complex from geometric point of view as irreducible components of $\mathcal{F}_x$. There is, however a nice exception: the case of orbital varieties in $\mathfrak{gl}_n$ associated to two-column Young diagrams. In this case each orbital variety is a union of a finite number of B-orbits and we can apply [7] to get the full picture of intersections of orbital varieties. In [7] the special so-called rank matrix is attached to a B-orbit of $x \in \mathfrak{n}$. In the case of x of nilpotent order 2 it defines the corresponding B-orbit completely. Here we use the technique of these matrices to determine the intersection of two orbital varieties of nilpotent order two. In particular we show that the intersection of two orbital varieties associated to an orbit of nilpotent order 2 is not empty (see Proposition 3.14). We give the purely combinatorial and easy to compute necessary and sufficient condition for the irreducibility of the intersection of two orbital varieties of nilpotent order 2 and provide some examples showing that in general such intersections are reducible and not necessary equidimensional (see examples in Section 3.8).

In the subsequent paper (cf. [9]), we show that the intersections of codimension 1 in two-column case are irreducible. This together with computations in low rank cases permits us to the following conjecture.

Conjecture 1.1. Given $S, T \in \mathbf{Tab}_\lambda$. If $\operatorname{codim}_{\mathcal{F}_S} \mathcal{F}_T \cap \mathcal{F}_S = 1$ then $\mathcal{F}_T \cap \mathcal{F}_S$ is irreducible.

Let us now give a brief outline of the contents of the paper.

- To make the paper as self contained as possible we present in Section 2 Spaltenstein's and Steinberg's constructions and quote the connected results essential in further analysis.
- In Section 3 we provide the main result of this paper, namely, a purely combinatorial necessary and sufficient condition for the intersection of two components of the Springer fiber to be irreducible in two-column case; as an application of this characterization, we give the first examples for which the intersections of two components of the Springer fiber are reducible and are not of pure dimension. This is the most technical part of the paper.
- In Section 4 we give some other counter-examples concerning the possible simplification of the construction of orbital varieties and of their intersections in codimension one.

2. GENERAL CONSTRUCTION

2.1. Given $x \in \mathfrak{n}$ denote $G_x = \{g \in G \mid g^{-1}xg \in \mathfrak{n}\}$. Set $f_1 : G_x \to \mathcal{O}_x \cap \mathfrak{n}$ by $f_1(g) = g.x$ and $f_2 : G_x \to \mathcal{F}_x$ by $f_2(g) = gB$. Define $\pi : \mathcal{F}_x \to \mathcal{O}_x \cap \mathfrak{n}, gB \mapsto \pi(gB) := f_1(f_2^{-1}(gB))$. By Spaltenstein π induces a surjection $\hat{\pi}$ from the set of irreducible components of $\mathcal{F}_x$ onto the set of irreducible components of $\mathcal{O}_x \cap \mathfrak{n}$, moreover the fiber of this surjective map is exactly an orbit under the action of the component group $A(x) := Z_G(x)/Z_G^o(x)$ (cf. [12]). He showed also that $\mathcal{F}_x$ and $\mathcal{O}_x \cap \mathfrak{n}$ are equidimensional and got the following relations:

$$(2.1) \qquad (\mathcal{O}_x \cap \mathfrak{n}) + \dim(Z_G(x)) = \dim(\mathcal{F}_x) + \dim(B),$$

$$(2.2) \qquad \dim(\mathcal{O}_x \cap \mathfrak{n}) + \dim(\mathcal{F}_x) = \dim(\mathfrak{n}),$$

$$(2.3) \qquad \dim(\mathcal{O}_x \cap \mathfrak{n}) = \frac{1}{2}\dim(\mathcal{O}_x).$$

In our setting, for the case $G = \mathrm{GL}_n$, the component is always trivial, so $\hat{\pi}$ is actually a bijection. As an extension of his work, we established in [8] the following result.

Proposition 2.1. *Let $x \in \mathfrak{n}$ and let $\mathcal{F}_1, \mathcal{F}_2$ be two irreducible components of $\mathcal{F}_x$ and $\mathcal{V}_1 = \pi(\mathcal{F}_1), \mathcal{V}_2 = \pi(\mathcal{F}_2)$ the corresponding orbital varieties. Let $\{\mathcal{E}_l\}_{l=1}^t$ be the set of irreducible components of $\mathcal{F}_1 \cap \mathcal{F}_2$. Then $\{\pi(\mathcal{E}_l)\}_{l=1}^t$ is exactly the set of irreducible components of $\mathcal{V}_1 \cap \mathcal{V}_2$ and* $\operatorname{codim}_{\mathcal{F}_1}(\mathcal{E}_l) = \operatorname{codim}_{\mathcal{V}_1}(\pi(\mathcal{E}_l))$.

This simple proposition shows that in the case of GL_n, orbital varieties associated to $\mathcal{O}_x$ are equivalent to the irreducible components of $\mathcal{F}_x$.

2.2. The parametrization of the irreducible components of $\mathcal{F}_x$ in GL_n by standard Young tableaux is as follows.

In this case $\mathcal{F}$ is identified with the set of complete flags $\xi = (V_1 \subset \cdots \subset V_n = \mathbb{C}^n)$ and $\mathcal{F}_x \cong \{\xi = (V_i) \in \mathcal{F} \mid x(V_i) \subset V_{i-1}\}$.

Given $x \in \mathfrak{n}$ let $J(x) = \lambda$. By a slight abuse of notation we will not distinguish between the partition λ and its Young diagram. By R. Steinberg [17] and N. Spaltenstein [11] we have a parametrization of the irreducible components of $\mathcal{F}_x$ by the set $\mathbf{Tab}_\lambda$: Let $\xi = (V_i) \in \mathcal{F}_x$, then we get a satured chain in the poset of Young diagrams

$$\mathrm{St}(\xi) := \big(J(x) \supset J(x_{|V_{n-1}}) \supset \cdots \supset J(x_{|V_2}) \supset J(x_{|V_1})\big),$$

where $x_{|V_i}$ is the nilpotent endomorphism induced by x by restriction to the subspace V_i and $J(x_{|V_{i+1}})$ differs from $J(x_{|V_i})$ by one corner box. It is easy to see that the data of such a satured chain is equivalent to give a standard Young tableau. So we get a map $\mathrm{St} : \mathcal{F}_x \to \mathbf{Tab}_\lambda$. Then the collection $\{\mathrm{St}^{-1}(T)\}_{T \in \mathbf{Tab}_\lambda}$ is a partition of $\mathcal{F}_x$ into smooth irreducible subvarieties of the same dimension and $\{\overline{\mathrm{St}^{-1}(T)}\}_{T \in \mathbf{Tab}_\lambda}$ are the set of the irreducible components of $\mathcal{F}_x$ which will be denoted by $\mathcal{F}_T := \overline{\mathrm{St}^{-1}(T)}$ where $T \in \mathbf{Tab}_\lambda$.

On the level of orbital varieties the construction is as follows. For $1 \leqslant i < j \leqslant n$ consider the canonical projections $\pi_{i,j} : \mathfrak{n}_n \to \mathfrak{n}_{j-i+1}$ acting on a matrix by deleting the first $i-1$ columns and rows and the last $n-j$ columns and rows. For any $u \in \mathcal{O}_\lambda \cap \mathfrak{n}$ set $J_n(u) := J(u) = \lambda$ and $J_{n-i}(u) := J(\pi_{1,n-i}(u))$ for any i: $1 \leqslant i \leqslant n-1$. Exactly as in the previous construction we get a standard Young tableau corresponding to the satured chain $(J_n(u) \supset \cdots \supset J_1(u))$, therefore we get a map $\mathrm{St}_1 : \mathcal{O}_\lambda \cap \mathfrak{n} \to \mathbf{Tab}_\lambda$. Again the collection $\{\mathrm{St}_1^{-1}(T)\}_{T \in \mathbf{Tab}_\lambda}$ is a partition of $\mathcal{O}_\lambda \cap \mathfrak{n}$ into smooth irreducible subvarieties of the same dimensions and $\{\overline{\mathrm{St}_1^{-1}(T)} \cap \mathcal{O}_\lambda\}_{T \in \mathbf{Tab}_\lambda}$ are orbital varieties associated to $\mathcal{O}_\lambda$. Put $\mathcal{V}_T := \overline{\mathrm{St}_1^{-1}(T)} \cap \mathcal{O}_\lambda$ where $T \in \mathbf{Tab}_\lambda$; in particular, $\coprod_{\lambda \vdash n} \mathbf{Tab}_\lambda$ parameterizes the set of orbital varieties contained in $\mathfrak{n}$.

2.3. A general construction for orbital varieties by R. Steinberg (cf. [16]) is as follows. For $w \in \mathbf{S}_n$ consider the subspace

$$\mathfrak{n} \cap {}^w\mathfrak{n} := \bigoplus_{\alpha \in \mathcal{R}^+ \cap {}^w\mathcal{R}^+} \mathfrak{g}_\alpha$$

contained in $\mathfrak{n}$. Then $G.(\mathfrak{n} \cap {}^w\mathfrak{n})$ is an irreducible locally closed subvariety of the nilpotent variety $\mathcal{N}$. Since $\mathcal{N}$ is a finite union of nilpotent orbits, it follows that there is a unique nilpotent orbit $\mathcal{O}$ such that $\overline{G.(\mathfrak{n} \cap {}^w\mathfrak{n})} = \overline{\mathcal{O}}$. Moreover, $\overline{B.(\mathfrak{n} \cap {}^w\mathfrak{n})} \cap \mathcal{O}$ is an orbital variety associated to $\mathcal{O}$ and the fundamental result in Steinberg's work is that every orbital variety can be obtained in this way [16]; in particular there is a surjective map $\varphi : \mathbf{S}_n \to \coprod_{\lambda \vdash n} \mathbf{Tab}_\lambda$. The preimages of this map $\mathcal{C}_T := \varphi^{-1}(T)$ are called the *geometric* (or *left*) *cells* of $\mathbf{S}_n$. The geometric cells are given by Robinson–Schensted correspondence, namely for $T \in \mathbf{Tab}_\lambda$, one has $\mathcal{C}_T = \{\mathrm{RS}(T, S)\colon\ S \in \mathbf{Tab}_\lambda\}$, where RS represents the Robinson–Schensted correspondence.

3. TWO-COLUMN CASE

3.1. In this section we use intensively the results of [7] and we adopt its notation.

Set $\mathcal{X}_2 := \{x \in \mathfrak{n} \mid x^2 = 0\}$ to be the variety of nilpotent upper-triangular matrices of nilpotent order 2. Denote $\mathbf{S}_n^2 := \{\sigma \in \mathbf{S}_n \mid \sigma^2 = id\}$ the set of involutions of $\mathbf{S}_n$. For every $\sigma \in \mathbf{S}_n^2$, set N_σ to be the "strictly upper-triangular part" of its corresponding permutation matrix, that is

$$(N_\sigma)_{i,j} := \begin{cases} 1 & \text{if } i < j \text{ and } \sigma(i) = j; \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Let $\mathbf{Tab}_n^2$ be the set of all Young tableaux of size n with two columns. For $T \in \mathbf{Tab}_n^2$, write it as $T = (T_1, T_2)$, where $T_1 = \begin{pmatrix} t_{1,1} \\ \vdots \\ t_{n-k,1} \end{pmatrix}$ is the first column of T and $T_2 = \begin{pmatrix} t_{1,2} \\ \vdots \\ t_{k,2} \end{pmatrix}$ is the second column of T. And define the following involution

$$\sigma_T := (i_1, j_1) \cdots (i_k, j_k), \tag{3.2}$$

where $j_s := t_{s,2}$; $i_1 := t_{1,2} - 1$, and $i_s := \max\{d \in T_1 - \{i_1, \ldots, i_{s-1}\} \mid d < j_s\}$ for any $s > 1$. For example, take

$$T = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 7 \\ \hline 6 & 8 \\ \hline \end{array}\,.$$

Then $\sigma_T = (3,4)(2,5)(6,7)(1,8)$.

Remark 3.1. To define $T \in \mathbf{Tab}_n$ it is enough to know columns T_i as sets (we denote them by $\langle T_i \rangle$), or equivalently the different column positions $c_T(i)$ of integers i: $1 \leqslant i \leqslant n$ since the entries increase from up to down in the columns. Thus given σ_T we can reconstruct T. Indeed, $\langle T_2 \rangle = \{j_1, \ldots, j_k\}$ and $\langle T_1 \rangle = \{i\}_{i=1}^n \setminus \langle T_2 \rangle$.

One has the following theorem.

Theorem 3.2 ([6, 2.2], [5, 4.13]).

(i) *The variety $\mathcal{X}_2$ is a finite union of B-orbits, namely*

$$\mathcal{X}_2 = \coprod_{\sigma \in \mathbf{S}_n^2} B.N_\sigma.$$

(ii) *For any $T \in \mathbf{Tab}_n^2$, one has $\bar{\mathcal{V}}_T = \overline{B.N_{\sigma_T}}$.*

The finiteness property is particular for $\mathcal{X}_2$. The fact that each orbital variety has a dense B-orbit is also particular for very few types of nilpotent orbits including orbits of nilpotent order 2 (cf. [5]). The first property permits us to compute the intersections of any two B-orbit closures in $\mathcal{X}_2$. The second one permits us to apply the results to the intersections of orbital varieties of nilpotent order 2.

We begin with the general theory of the intersections of $\overline{B.N_\sigma}$ for $\sigma \in \mathbf{S}_n^2$.

3.2. In this section we prefer to use the dual partition λ^* instead of λ since it will be more convenient to write it down for nilpotent orbits of nilpotent order 2. Indeed, for $x \in \mathcal{X}_2$ one has $J^*(x) = (n-k, k)$ where k is number of Jordan blocks of length two in $J(x)$.

Remark 3.3. For every element $x \in \mathcal{X}_2$, the integer $\mathrm{rk}(x)$ is exactly the number of blocks of length 2 in $J(x)$, so it defines the GL_n-orbit of x.

Any element $\sigma \in \mathbf{S}_n^2$ can be written as a product of disjoint cycles of length 2. Order elements in increasing order inside the cycle and order cycles in increasing order according to the first entries. In that way we get a unique writing of every involution. Thus, $\sigma = (i_1, j_1)(i_2, j_2)\dots(i_k, j_k)$ where $i_s < j_s$ for any $1 \leqslant s \leqslant k$ and $i_s < i_{s+1}$ for any $1 \leqslant s < k$. Set $L(\sigma) := k$ (do not confuse this notation with the length function), and denote by $\mathcal{O}_\sigma$ the GL_n-orbit of N_σ. By definition we have $L(\sigma) = \mathrm{rk}(N_\sigma)$.

Let us define the following number

$$r_s(\sigma) := \mathrm{card}\{i_p < i_s \mid j_p < j_s\} + \mathrm{card}\{j_p \mid j_p < i_s\}. \tag{3.3}$$

Note that the definition of $r_s(\sigma)$ is independent of ordering cycles in increasing order according to the first entries. However if it is ordered then $r_1(\sigma) = 0$ and to compute $r_s(\sigma)$ it is enough to check only the pairs (i_p, j_p) where $p < s$. For example, take $\sigma = (1,6)(3,4)(5,7)$. Then $L(\sigma) = 3$ and $r_1(\sigma) = 0, r_2(\sigma) = 0, r_3(\sigma) = 2 + 1 = 3$.

By [6, 3.1] one has

Theorem 3.4. *For $\sigma = (i_1, j_1)(i_2, j_2)\cdots(i_k, j_k) \in \mathbf{S}_n^2$ one has*

$$\dim(B.N_\sigma) = kn - \sum_{s=1}^{k}(j_s - i_s) - \sum_{s=2}^{k} r_s(\sigma).$$

Remark 3.5. By Theorem 3.2(ii), the orbits $B.N_{\sigma_T}$ (where $(\mathrm{sh}(T))^* = (n-k, k)$) are the only B-orbits of maximal dimension inside the variety $\mathcal{O}_{(n-k,k)^*} \cap \mathfrak{n}$ and $\dim(B.N_{\sigma_T}) = k(n-k)$: Indeed any orbit $B.N_\sigma$ is irreducible and therefore lies inside an orbital variety $\mathcal{V}_T$, in particular it lies in $\overline{\mathcal{V}}_T$, so if $\dim B.N_\sigma = \dim \mathcal{V}_T$ we get that $\overline{B.N_\sigma} = \overline{\mathcal{V}}_T$ thus by Theorem 3.2(ii) $\overline{B.N_\sigma} = \overline{B.N_{\sigma_T}}$ which provides $\sigma = \sigma_T$.

In particular if $\sigma = (i_1, j_1)\cdots(i_k, j_k)$ is such that $\dim(B.N_\sigma) = k(n-k)$, then $\sigma = \sigma_T$ where T is the tableau obtained by

$$c_T(s) = \begin{cases} 2 & \text{if } s = j_p \text{ for some } p\text{: } 1 \leqslant p \leqslant k, \\ 1 & \text{otherwise.} \end{cases}$$

3.3. In [7] the combinatorial description of $\overline{B.N_\sigma}$ (with respect to Zariski topology) for $\sigma \in \mathbf{S}_n^2$ is provided. Let us formulate this result.

Recall from Section 2.2 the notion $\pi_{i,j} : \mathfrak{n}_n \to \mathfrak{n}_{j-i+1}$ and define the rank matrix R_x of $x \in \mathfrak{n}$ to be

$$(R_x)_{i,j} := \begin{cases} 0 & \text{if } i \geqslant j, \\ \mathrm{rk}(\pi_{i,j}(x)) & \text{otherwise.} \end{cases} \tag{3.4}$$

Note that for any element $b \in B$, $\pi_{i,j}(b)$ is an invertible upper-triangular matrix in GL_{j-i+1}. Therefore we can define an action of B on $\mathfrak{n}_{j-i+1}$ by: $b.y := \pi_{i,j}(b).y$ for $y \in \mathfrak{n}_{j-i+1}$ and $b \in B$.

Let us first establish the following result:

Lemma 3.6.

(i) *If $x, y \in \mathfrak{n}$ are in the same B-orbit, then they have the same rank matrix.*
(ii) *The morphism $\pi_{i,j}$ is B-invariant.*

Proof. Note that for any two upper-triangular matrices a, b and for any i, j: $1 \leqslant i < j \leqslant n$ one has $\pi_{i,j}(ab) = \pi_{i,j}(a)\pi_{i,j}(b)$. In particular, if $a \in B$ then $\pi_{i,j}(a^{-1}) = (\pi_{i,j}(a))^{-1}$. Applying this to $x \in \mathfrak{n}$ and y in its B orbit (that is, $y = b.x$ for some $b \in B$) we get $\pi_{i,j}(y) = \pi_{i,j}(b).\pi_{i,j}(x)$ so that the morphism $\pi_{i,j}$ is B-invariant and in particular $\mathrm{rk}(\pi_{i,j}(y)) = \mathrm{rk}(\pi_{i,j}(x))$. Hence $R_x = R_y$. $\square$

By this lemma we can define $R_\sigma := R_{N_\sigma}$ as the rank matrix associated to orbit $B.N_\sigma$.

Remark 3.7. Note that computation of $(R_{N_\sigma})_{i,j}$ is trivial – this is exactly the number of non-zero entries in submatrix of $1, \ldots, j$ columns and $i, \ldots, n$ rows of N_σ or in other words the number of ones in N_σ to the left-below of position (i, j) (including position (i, j)).

Let $\mathbb{Z}^+$ be the set of non-negative integers. Put $\mathbf{R}_n^2 := \{R_\sigma \mid \sigma \in \mathbf{S}_n^2\}$. By [7, 3.1, 3.3], one has the following proposition.

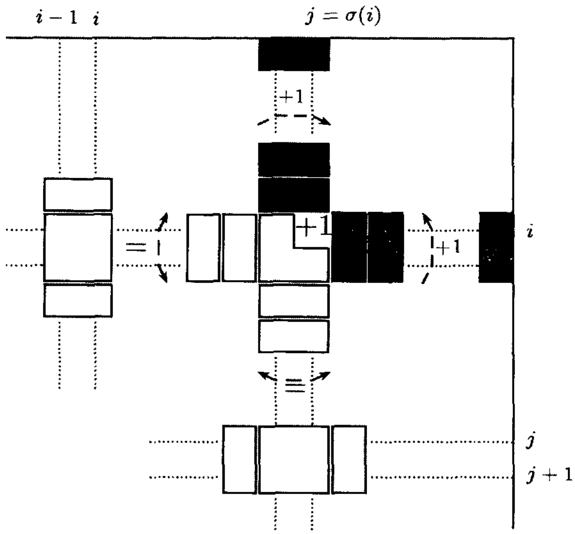
Proposition 3.8. *$R = (R_{i,j}) \in M_{n\times n}(\mathbb{Z}^+)$ belongs to $\mathbf{R}_n^2$ if and only if it satisfies*

(i) $R_{i,j} = 0$ *if* $i \geqslant j$;
(ii) *For $i < j$ one has $R_{i+1,j} \leqslant R_{i,j} \leqslant R_{i+1,j} + 1$ and $R_{i,j-1} \leqslant R_{i,j} \leqslant R_{i,j-1} + 1$;*
(iii) *If $R_{i,j} = R_{i+1,j} + 1 = R_{i,j-1} + 1 = R_{i+1,j-1} + 1$ then*
  (a) *$R_{i,k} = R_{i+1,k}$ for any $k < j$ and $R_{i,k} = R_{i+1,k} + 1$ for any $k \geqslant j$;*
  (b) *$R_{k,j} = R_{k,j-1}$ for any $k > i$ and $R_{k,j} = R_{k,j-1} + 1$ for any $k \leqslant i$;*
  (c) *$R_{j,k} = R_{j+1,k}$ and $R_{k,i} = R_{k,i-1}$ for any k: $1 \leqslant k \leqslant n$.*

Fix $\sigma \in \mathbf{R}_n^2$, then the conditions (i) and (ii) are obvious from Remark 3.7, and the conditions (iii) appears exactly for the coordinates (i, j) in the matrix when $j = \sigma(i)$, with $i < j$; we draw the following picture to help the reader to visualize the constraints (a), (b), (c) of (iii), with the following rule: the integers which are inside a same white polygon, are equal, and the integers in a same gray rectangle differ by one.

The first part of (c) can be explained in the following: since the integer j appears already in the second entry of the cycle (i, j), so it cannot appear again in any other cycle; therefore in the matrix N_σ, the integers of the jth row are all 0, and that explains why we should have $(R_\sigma)_{j,k} = (R_\sigma)_{j+1,k}$ for $1 \leqslant k \leqslant n$; the same explanation can also be done for the second part of (c).

When the constrain (iii) appears, let us call the couple (i, j) a *position* of constrain (iii).



Remark 3.9. If two horizontal (resp. vertical) consecutive boxes of a matrix in $\mathbf{R}_n^2$ differ by one, then it is also the same for any consecutive horizontal (resp. vertical) boxes above (resp. on the right).

As an immediate corollary of Proposition 3.8 we get the following lemma.

Lemma 3.10. *Let σ, σ_1 and σ_2 be involutions such that $\sigma = \sigma_1.\sigma_2$ and $L(\sigma) = L(\sigma_1) + L(\sigma_2)$, then $R_\sigma = R_{\sigma_1} + R_{\sigma_2}$; in particular, we have $\sigma_1, \sigma_2 \preceq \sigma$.*

Proof. The hypothesis $L(\sigma) = L(\sigma_1) + L(\sigma_2)$ means exactly that any integer appearing a cycle of σ_1 does not appear in any cycle of σ_2 and conversely (note that it is also equivalent to say $\sigma_1.\sigma_2 = \sigma_2.\sigma_1 = \sigma$); this means in particular that when the coefficient 1 appears in the matrix R_{σ_1} for the coordinate $(i, \sigma_1(i))$, then it cannot appear in the ith line and in the $\sigma_1(i)$th column of R_{σ_2} and conversely; therefore we get $N_\sigma = N_{\sigma_1} + N_{\sigma_2}$ and the result follows. $\square$

3.4. Define the following partial order on $M_{n\times n}(\mathbb{Z}^+)$. For $A, B \in M_{n\times n}(\mathbb{Z}^+)$ put $A \preceq B$ if for any i, j: $1 \leqslant i, j \leqslant n$ one has $A_{i,j} \leqslant B_{i,j}$.

The restriction of this order to $\mathbf{R}_n^2$ induces a partial order on $\mathbf{S}_n^2$ by setting $\sigma' \preceq \sigma$ if $R_{\sigma'} \preceq R_\sigma$ for $\sigma, \sigma' \in \mathbf{S}_n^2$. By [7, 3.5] this partial order describes the closures of $B.N_\sigma$ for $\sigma \in \mathbf{S}_n^2$. Combining [7, 3.5] with Remark 3.5 we get the following theorem.

Theorem 3.11. *For any $\sigma \in \mathbf{S}_n^2$, one has*

$$\overline{B.N_\sigma} = \coprod_{\sigma' \preceq \sigma} B.N_{\sigma'}.$$

In particular, for $T \in \mathbf{Tab}_n^2$,

$$\mathcal{V}_T = \coprod_{\substack{\sigma' \preceq \sigma_T \\ L(\sigma') = L(\sigma_T)}} B.N_{\sigma'}.$$

3.5. Let $\pi_{i,j} : \mathfrak{n}_n \to \mathfrak{n}_{j-i+1}$. If we denote by $\hat{\pi}_{s,t} : \mathfrak{n}_{j-i+1} \to \mathfrak{n}_{t-s+1}$ the same projection, but with the starting-space $\mathfrak{n}_{j-i+1}$, then we can easily check the following relation:

$$(3.5) \qquad \hat{\pi}_{s,t} \circ \pi_{i,j} = \pi_{s+i-1,t+i-1}.$$

Now if $R \in \mathbf{R}_n^2$, it is obvious by Remark 3.7 that $\pi_{i,j}(R)$ fulfills the constraints (i), (ii) and (iii) of Proposition 3.8. Thus, we get the following lemma.

Lemma 3.12. *If $R \in \mathbf{R}_n^2$, then $\pi_{i,j}(R) \in \mathbf{R}_{j-i+1}^2$ for $1 \leqslant i \leqslant j \leqslant n$.*

Obviously, the converse is not true, as one can check for the matrix $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

By this lemma, for any $R_\sigma \in \mathbf{R}_n^2$, we have $\pi_{i,j}(R_\sigma) \in \mathbf{R}_{j-i+1}^2$; therefore $\pi_{i,j}$ induces a natural map from $\mathbf{S}_n^2$ onto $\mathbf{S}_{\langle i,j \rangle}^2 \cong \mathbf{S}_{j-i+1}^2$, symmetric group of the set $\{i, \ldots, j\}$. This projection will be also denoted by $\pi_{i,j}$. Moreover, by (3.5) and Remark 3.7 one gets immediately:

$$(3.6) \qquad \pi_{i,j}(N_\sigma) = N_{\pi_{i,j}(\sigma)} \quad \text{and} \quad \pi_{i,j}(R_\sigma) = R_{\pi_{i,j}(\sigma)}.$$

Note that the resulting element $\pi_{i,j}(\sigma)$ is obtained from σ by deleting all the cycles in which at least one entry does not belong to $\{i, \ldots, j\}$. For every $\delta \in \mathbf{S}_{\langle i,j \rangle}^2$, any element $\sigma \in \pi_{i,j}^{-1}(\delta)$ will be called a *lifting* of δ. In the same way we will call the matrix R_σ a lifting of R_δ.

Remark 3.13.

(i) We will consider sometimes $\sigma \in \mathbf{S}_{\langle i,j \rangle}^2$ as an element of $\mathbf{S}_n^2$ (cf. proofs of Proposition 3.14, Lemma 3.16 and Theorem 3.15); in particular, with the description above we have $\sigma = \pi_{i,j}(\sigma)$.

(ii) By note (i) and Lemma 3.10 for any $\delta \in \mathbf{S}_{\langle i,j \rangle}^2$ and any σ its lifting in $\mathbf{S}_n^2$ one has $\delta \preceq \sigma$.

(iii) By the relations (3.6), the projection $\pi_{i,j}$ respect the order $\preceq$: If $\sigma_1 \preceq \sigma_2$, then $\pi_{i,j}(\sigma_1) \preceq \pi_{i,j}(\sigma_2)$.

3.6. Put $\mathbf{S}_n^2(k) := \{\sigma \in \mathbf{S}_n^2 \mid L(\sigma) = k\}$ and respectively $\mathbf{Tab}_n^2(k) := \{T \in \mathbf{Tab}_n^2 \mid \mathrm{sh}(T) = (n-k, k)^*\}$. As a corollary of partial order $\preceq$ on $\mathbf{S}_n^2$ we get the following proposition.

Proposition 3.14. *$\sigma_o(k) := (1, n-k+1)(2, n-k+2)\cdots(k, n)$ is the unique minimal involution in $\mathbf{S}_n^2(k)$ and for any $\sigma \in \mathbf{S}_n^2(k)$ one has $\sigma_o(k) \preceq \sigma$. In particular, for any $S, T \in \mathbf{Tab}_n^2(k)$ one has $\mathcal{V}_T \cap \mathcal{V}_S \neq \emptyset$.*

Proof. Note that $N_{\sigma_o(k)}$ and respectively $R_{\sigma_o(k)}$ are

$$
N_{\sigma_o(k)} = \begin{array}{ccccccc}
\overbrace{0 \cdots 0}^{n-k-1} & 1 & 0 & \cdots & 0 & & \\
\vdots & & 0 & 1 & 0 & & \\
\vdots & & & \ddots & 1 & \{k, & \\
\vdots & & & & 0 & & \\
0 & \cdots & \cdots & \cdots & 0 & &
\end{array}
\qquad
R_{\sigma_o}(k) = \begin{array}{cccccc}
0 \cdots 0 & 1 & 2 & \cdots & k \\
\vdots & & 0 & 1 & 2 \\
\vdots & & & \ddots & 1 \\
\vdots & & & & 0 \\
0 & \cdots & \cdots & \cdots & 0
\end{array}
\tag{3.7}
$$

so that

$$
(R_{\sigma_o(k)})_{i,j} = \begin{cases} j - i + 1 - (n-k) & \text{if } j - i > n - k - 1, \\ 0 & \text{otherwise.} \end{cases}
$$

On the other hand, by Proposition 3.8(ii) for any $\sigma \in \mathbf{S}_n^2$, one has $(R_\sigma)_{i,j} \geqslant (R_\sigma)_{i-1,j} - 1 \geqslant (R_\sigma)_{i-2,j} - 2 \geqslant \cdots \geqslant (R_\sigma)_{1,j} - (i-1)$. In turn, $(R_\sigma)_{1,j} \geqslant (R_\sigma)_{1,j+1} - 1 \geqslant \cdots \geqslant (R_\sigma)_{1,n} - (n-j)$ so that $(R_\sigma)_{i,j} \geqslant (R_\sigma)_{1,n} - (n-j+i-1)$. Thus, for any $\sigma \in \mathbf{S}_n^2(k)$, one has $(R_\sigma)_{i,j} \geqslant j - i + 1 - (n-k)$. As well one has $(R_\sigma)_{i,j} \geqslant 0$ so that $(R_\sigma)_{i,j} \geqslant \max\{0, j-i+1-(n-k)\} = (R_{\sigma_o(k)})_{i,j}$. Thus, $\sigma \succeq \sigma_o(k)$.

The second part is now a corollary of this result and Theorem 3.11. □

3.7. Given $\sigma, \sigma' \in \mathbf{S}_n^2$ we define $R_{\sigma,\sigma'}$ by

$$
(R_{\sigma,\sigma'})_{i,j} := \min\{(R_\sigma)_{i,j}, (R_{\sigma'})_{i,j}\}. \tag{3.8}
$$

One has the following theorem.

Theorem 3.15 (Main theorem). *For any $\sigma, \sigma' \in \mathbf{S}_n^2$ one has*

$$
\overline{B.N_\sigma} \cap \overline{B.N_{\sigma'}} = \coprod_{R_\varsigma \preceq R_{\sigma,\sigma'}} B.N_\varsigma.
$$

This intersection is irreducible if and only if $R_{\sigma,\sigma'} \in \mathbf{R}_n^2$.

Proof. To establish this equivalence we need only to prove the "only if" part and to do this we need some preliminary result.

Lemma 3.16. *Suppose that $\overline{B.N_\sigma} \cap \overline{B.N_{\sigma'}}$ is irreducible. Denote B' the Borel subgroup in GL_{j-i+1}. Then $\overline{B'.N_{\pi_{i,j}(\sigma)}} \cap \overline{B'.N_{\pi_{i,j}(\sigma')}}$ is irreducible.*

Proof. Let α, β be two maximal involutions in $\mathbf{S}^2_{\langle i,j\rangle}$ such $\alpha, \beta \preceq \pi_{i,j}(\sigma), \pi_{i,j}(\sigma')$. By Remark 3.13(ii), we have also $\alpha, \beta \preceq \sigma, \sigma'$. By hypothesis we have $\overline{B.N_\sigma} \cap \overline{B.N_{\sigma'}} = \overline{B.N_\delta}$ for an element $\delta \in \mathbf{S}^2_n$. In particular we get $\alpha, \beta \preceq \delta$. By Remark 3.13(i) and (iii) we get $\alpha = \pi_{i,j}(\alpha), \beta = \pi_{i,j}(\beta) \preceq \pi_{i,j}(\delta) \preceq \pi_{i,j}(\sigma), \pi_{i,j}(\sigma')$. Since α and β are maximal, we get $\alpha = \beta = \pi_{i,j}(\delta)$. $\square$

We prove the theorem by induction on n. For $n = 3$ all the intersections are irreducible so that the claim is trivially true.

Let now n be minimal such that $\overline{B.N_\sigma} \cap \overline{B.N_{\sigma'}}$ is irreducible and $R_{\sigma,\sigma'} \notin \mathbf{R}^2_n$. Note that constrains (i) and (ii) of Proposition 3.8 are satisfied by any $R_{\sigma,\sigma'}$. If $R_{\sigma,\sigma'} \notin \mathbf{R}^2_n$ then at least one of the conditions (a), (b) and (c) of the constrain (iii) of Proposition 3.8 is not fulfilled. By symmetry around the anti diagonal it is enough to check only condition (a) and the first part of condition (c).

As for the first relation in (3.6), we can easily check that

$$R_{\pi_{i,j}(\sigma),\pi_{i,j}(\sigma')} = \pi_{i,j}(R_{\sigma,\sigma'}). \tag{3.9}$$

Let B' be the Borel subgroup of GL_{n-1}. By Lemma 3.16 and relation (3.9), we get that the varieties $\overline{B'.N_{\pi_{1,n-1}(\sigma)}} \cap \overline{B'.N_{\pi_{1,n-1}(\sigma')}}$, $\overline{B'.N_{\pi_{2,n}(\sigma)}} \cap \overline{B'.N_{\pi_{2,n}(\sigma')}}$ are irreducible. Thus by induction hypothesis

$$\pi_{1,n-1}(R_{\sigma,\sigma'}), \pi_{2,n}(R_{\sigma,\sigma'}) \in \mathbf{R}^2_{n-1}. \tag{3.10}$$

Put $\zeta \in \mathbf{S}^2_{n-1}$ to be such that $R_\zeta = \pi_{1,n-1}(R_{\sigma,\sigma'})$ and $\eta \in \mathbf{S}^2_{\langle 2,n\rangle}$ be such that $R_\eta = \pi_{2,n}(R_{\sigma,\sigma'})$.

Suppose that $R_{\sigma,\sigma'} \notin \mathbf{R}^2_n$, denote (i_o, j_o) the position of a constrain (iii) $\begin{array}{|c|c|}\hline k & k+1 \\ \hline k & k \\ \hline\end{array}$ which is not satisfied by the matrix $R_{\sigma,\sigma'}$.

Condition (a). If the first part of Condition (a) is not satisfied, it means that we can find two horizontal consecutive boxes below of the two boxes $\begin{array}{|c|c|}\hline k & k \\ \hline\end{array}$ which differ by one; but these two boxes and $\begin{array}{|c|c|}\hline k & k \\ \hline\end{array}$ will lies in $\pi_{2,n}(R_{\sigma,\sigma'}) \in \mathbf{R}^2_{n-1}$, which is impossible by Remark 3.9.

Now if the second part of condition (a) is not satisfied, it means that we can find two equal vertical consecutive boxes $\begin{array}{|c|}\hline m \\ \hline m \\ \hline\end{array}$ on the right of the boxes $\begin{array}{|c|}\hline k+1 \\ \hline k \\ \hline\end{array}$. By relation (3.10), these four last boxes cannot lie inside $\pi_{1,n-1}(R_{\sigma,\sigma'}), \pi_{2,n}(R_{\sigma,\sigma'})$; we deduce in particular that $i_o = 1$ and that the boxes $\begin{array}{|c|}\hline m \\ \hline m \\ \hline\end{array}$ belong to the last column. Since $R_{\sigma,\sigma'}$ satisfies condition (ii) of Proposition 3.8, the "North-East" corner of $R_{\sigma,\sigma'}$ must be $\begin{array}{|c|c|}\hline m & m \\ \hline m-1 & m \\ \hline\end{array}$. Now if we look at ζ (resp. η) as its own lifting in $\mathbf{S}^2_n$, then its configuration in the "North-East" corner will be of the following $\begin{array}{|c|c|}\hline m & m \\ \hline m-1 & m-1 \\ \hline\end{array}$ (resp. $\begin{array}{|c|c|}\hline m-1 & m \\ \hline m-1 & m \\ \hline\end{array}$). Since the intersection is irreducible, we should find

$\delta \in \mathbf{S}_n^2$ such that $\delta \succeq \zeta, \eta$ and $R_\delta \preceq R_{\sigma,\sigma'}$. Since $(R_\zeta)_{2,n-1} = (R_{\sigma,\sigma'})_{2,n-1} = m-1$ we get that also $(R_\delta)_{2,n-1} = m-1$. Since $(R_\zeta)_{1,n-1} = (R_{\sigma,\sigma'})_{1,n-1} = m$ we get that also $(R_\delta)_{1,n-1} = m$. Since $(R_\eta)_{2,n} = (R_{\sigma,\sigma'})_{2,n} = m$ we get that also $(R_\delta)_{2,n} = m$. But then by Remark 3.9 the "North-East" corner of R_δ should be of the following configuration $\begin{array}{|c|c|}\hline m & m+1 \\ \hline m-1 & m \\ \hline\end{array}$, this is impossible since $(R_\delta)_{1,n} \leqslant (R_{\sigma,\sigma'})_{1,n} = m$.

Condition (c). Suppose that the first part of condition (c) is not satisfied, it means that we can find two vertical consecutive boxes $\begin{array}{|c|}\hline m+1 \\ \hline m \\ \hline\end{array}$ lying in the j_oth and (j_o+1)th lines. As above this problem cannot appear inside the matrices $\pi_{1,n-1}(R_{\sigma,\sigma'})$ and $\pi_{2,n}(R_{\sigma,\sigma'})$; we deduce in particular that $i_o = 1$ and that the boxes $\begin{array}{|c|}\hline m+1 \\ \hline m \\ \hline\end{array}$ lie on the last column. Since $R_{\sigma,\sigma'}$ satisfies condition (ii) of Proposition 3.8, on the right side of the j_oth and (j_o+1)th lines of $R_{\sigma,\sigma'}$ we should find $\begin{array}{|c|c|}\hline m & m+1 \\ \hline m & m \\ \hline\end{array}$. Let us draw its configuration

$$R_{\sigma,\sigma'} = \begin{array}{cccc} \boxed{k} & \boxed{k+1} & & \\ \boxed{k} & \boxed{k} & & \\ \vdots & \vdots & & \\ \cdots & \cdots & \boxed{m} & \boxed{m+1} \\ \cdots & & \boxed{m} & \boxed{m} \end{array} \tag{3.11}$$

In the same way if we look at R_ζ and R_η as elements of $\mathbf{R}_n^2$, then their configurations will be of the following

$$R_\zeta = \begin{array}{cccc} \boxed{k} & \boxed{k+1} & & \\ \boxed{k} & \boxed{k} & & \\ \vdots & \vdots & & \\ \cdots & \cdots & \boxed{m} & \boxed{m} \\ \cdots & & \boxed{m} & \boxed{m} \end{array} \tag{3.12}$$

and

$$(3.13) \qquad R_\eta = \begin{array}{l} \begin{array}{|c|c|}\hline k & k \\ \hline k & k \\ \hline \end{array} \\ \quad\vdots \qquad \vdots \\ \qquad\qquad\qquad \begin{array}{|c|c|}\hline m & m+1 \\ \hline m & m \\ \hline \end{array} \end{array}$$

Since $\delta \succeq \zeta, \eta$ and $R_\delta \preceq R_{\sigma,\sigma'}$ combining the pictures in (3.11), (3.12) and (3.13), we get

$$(3.14) \qquad R_\delta = \begin{array}{l} \begin{array}{|c|c|}\hline k & k+1 \\ \hline k & k \\ \hline \end{array} \\ \quad\vdots \qquad \vdots \\ \qquad\qquad\qquad \begin{array}{|c|c|}\hline m & m+1 \\ \hline m & m \\ \hline \end{array} \end{array}$$

which is impossible, because it does not satisfy condition (iii)(c). $\square$

3.8. Let us apply the previous subsection to the elements of the form σ_T to show that in general the intersection $\mathcal{V}_T \cap \mathcal{V}_{T'}$ is reducible and not equidimensional.

Example 3.17.

(i) For $n \leqslant 4$ all the intersections of B-orbit closures of nilpotent order 2 are irreducible. The first examples of reducible intersections of B orbit closures occur in $n = 5$. In particular there is the unique example of the reducible intersection of orbital varieties and it is

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}, \qquad R_{\sigma_T} = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$T' = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}, \qquad R_{\sigma_{T'}} = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so that

$$R_{\sigma_T,\sigma_{T'}} = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $(R_{\sigma_T,\sigma_{T'}})_{1,3} = (R_{\sigma_T,\sigma_{T'}})_{1,2} + 1 = (R_{\sigma_T,\sigma_{T'}})_{2,2} + 1 = (R_{\sigma_T,\sigma_{T'}})_{2,3} + 1$ and $(R_{\sigma_T,\sigma_{T'}})_{3,5} = (R_{\sigma_T,\sigma_{T'}})_{4,5} + 1$ we get that $R_{\sigma_T,\sigma_{T'}}$ does not satisfy condition (iii)(c) of Proposition 3.8, therefore $R_{\sigma_T,\sigma_{T'}} \notin \mathbf{R}_5^2$. As well $(R_{\sigma_T,\sigma_{T'}})_{1,4}$, $(R_{\sigma_T,\sigma_{T'}})_{2,5}$ do not satisfy Remark 3.9. Accordingly we find three maximal elements $R, R', R'' \in \mathbf{R}_5^2$ for which $R, R', R'' \prec R_{\sigma_T,\sigma_{T'}}$

$$R = R_{(1,3)(2,5)} = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R' = R_{(1,4)(3,5)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R'' = R_{(1,5)(2,4)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $\dim(B.N_{(1,3)(2,5)}) = \dim(B.N_{(1,4)(3,5)}) = \dim(B.N_{(1,5)(2,4)}) = 4$ so that $\mathcal{V}_T \cap \mathcal{V}_{T'}$ contains three components of codimension 2.

(ii) The first example of non-equidimensional intersection of orbital varieties occurs in $n = 6$ and it is

$$T = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}, \qquad R_{\sigma_T} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$T' = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}, \qquad R_{\sigma_{T'}} = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so that

$$R_{\sigma_T,\sigma_{T'}} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $(R_{\sigma_T,\sigma_{T'}})_{1,3} = (R_{\sigma_T,\sigma_{T'}})_{1,2} + 1 = (R_{\sigma_T,\sigma_{T'}})_{2,2} + 1 = (R_{\sigma_T,\sigma_{T'}})_{2,3} + 1$ and $(R_{\sigma_T,\sigma_{T'}})_{1,5} = (R_{\sigma_T,\sigma_{T'}})_{2,5}$ we get that $R_{\sigma_T,\sigma_{T'}}$ does not satisfy condition (iii)(a) of Proposition 3.8 and $(R_{\sigma_T,\sigma_{T'}})_{1,5}, (R_{\sigma_T,\sigma_{T'}})_{2,6}$ do not satisfy Remark 3.9 so that $R_{\sigma_T,\sigma_{T'}} \notin \mathbf{R}_6^2$ and the maximal elements $R, R' \in \mathbf{R}_6^2$ for which $R, R' \prec R_{\sigma_T,\sigma_{T'}}$ are

$$R = R_{(1,3)(4,6)} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$R' = R_{(1,6)(2,5)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $\dim(B.N_{(1,3)(4,6)}) = 6$ and $\dim(B.N_{(1,6)(2,5)}) = 4$ so that $\mathcal{V}_T \cap \mathcal{V}_{T'}$ contains one component of codimension 2 and another component of codimension 4.

4.1. Cell graphs

Let $T \in \mathbf{Tab}_\lambda$ be a standard tableau and $\mathcal{C}_T$ its corresponding left cell (cf. Section 2.3). Steinberg's construction provides the way to construct $\mathcal{V}_T$ with the help of elements of $\mathcal{C}_T$. In [8], we got another geometric interpretation of $\mathcal{C}_T$:

Theorem 4.1 ([8]). *Let $T \in \mathbf{Tab}_\lambda$ and let $w = \mathrm{RS}(T, T') \in \mathcal{C}_T$. Then for a $x \in \mathcal{V}_T \cap B.(\mathfrak{n} \cap^w \mathfrak{n})$ in general position, the unique Schubert cell whose intersection with the irreducible component $\mathcal{F}_{T'}$ of the Springer fiber is open and dense in $\mathcal{F}_{T'}$ is indexed by w.*

The cell $\mathcal{C}_T$ in $\mathbf{S}_n$ can be visualized as a *cell graph* Γ_T where the vertices are labeled by $\mathbf{Tab}_\lambda$, and two vertices T' and T'' are joined by an edge labeled by k if $s_k\,\mathrm{RS}(T, T') = \mathrm{RS}(T, T'')$. One can easily see (cf. [8], for example) that if T' and T'' are joined in Γ_T, then $\mathrm{codim}_{\mathcal{F}_{T'}} \mathcal{F}_{T'} \cap \mathcal{F}_{T''} = 1$.

Note that T' and T'' can be joined by an edge in Γ_T and not joined by an edge in Γ_S for some $S, T \in \mathbf{Tab}_\lambda$. Is it true that $\mathrm{codim}_{\mathcal{F}_{T'}} \mathcal{F}_{T'} \cap \mathcal{F}_{T''} = 1$ if and only if there exists $T \in \mathbf{Tab}_\lambda$ such that T' and T'' are joined by an edge in Γ_T?

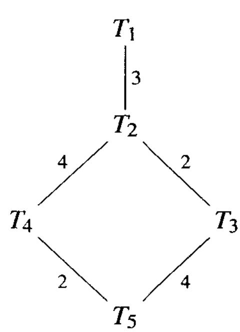
The answer is negative as we show by the example below.

As we show in [9] if $k \leqslant 2$ then $\mathrm{codim}_{\mathcal{V}_T}(\mathcal{V}_T \cap \mathcal{V}_S) = 1$ if and only if there exists $P \in \mathbf{Tab}_{(n-k,k)^*}$ such that T and S are joined by an edge in Γ_P so that the first example occurs in $n = 6$ for $\mathbf{Tab}_{(3,3)^*}$. In that case $(3,3)^* = (2,2,2)$ and the corresponding orbital varieties are 9-dimensional. Let us put

$$T_1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline\end{array}, \qquad T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline\end{array}, \qquad T_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline\end{array},$$

$$T_4 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline\end{array}, \qquad T_5 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline\end{array}.$$

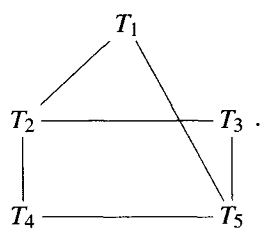
One can check that all the cell graphs are the same this graph is



On the other hand, one has

$$R_{\sigma_{T_1},\sigma_{T_5}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R_{(1,5)(2,6)(3,4)}$$

and $\dim(B.\mathcal{N}_{(1,5)(2,6)(3,4)}) = 8$, so that $\operatorname{codim}_{\mathcal{V}_{T_1}}(\mathcal{V}_{T_1} \cap \mathcal{V}_{T_5}) = 1$. As well the straight computations show that $\dim(\mathcal{V}_{T_1} \cap \mathcal{V}_{T_4}) = \dim(\mathcal{V}_{T_1} \cap \mathcal{V}_{T_3}) = \dim(\mathcal{V}_{T_2} \cap \mathcal{V}_{T_5}) = \dim(\mathcal{V}_{T_3} \cap \mathcal{V}_{T_4}) = 7$ so that all these intersections are of codimension 2. Further, $\mathcal{V}_{T_1} \cap \mathcal{V}_{T_4}$, $\mathcal{V}_{T_1} \cap \mathcal{V}_{T_3}$ and $\mathcal{V}_{T_3} \cap \mathcal{V}_{T_4}$ are irreducible. $\mathcal{V}_{T_2} \cap \mathcal{V}_{T_5}$ has three components with the following dense B-orbits: $B.\mathcal{N}_{(1,3)(2,5)(4,6)}$, $B.\mathcal{N}_{(1,5)(2,4)(3,6)}$, and $B.\mathcal{N}_{(1,4)(2,6)(3,5)}$. Below we draw the graph where two vertices are joined if the corresponding intersection is of codimension 1.



4.2. Orbital variety's construction

Let us go back to Steinberg's construction of an orbital variety (see Section 2.3). Given $T \in \mathbf{Tab}_\lambda$ one has $\mathcal{V}_T = \overline{B.(\mathfrak{n} \cap {}^w\mathfrak{n})} \cap \mathcal{O}_\lambda$ for any $w \in \mathcal{C}_T$. Obviously,

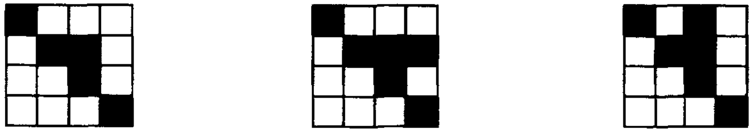
$$\dim\big(\overline{B.(\mathfrak{n} \cap {}^w\mathfrak{n})} \cap \mathcal{O}_\lambda\big) = \dim\big(B.(\mathfrak{n} \cap {}^w\mathfrak{n}) \cap \mathcal{O}_\lambda\big),$$

so that $\dim(B.(\mathfrak{n} \cap {}^w\mathfrak{n}) \cap \mathcal{O}_\lambda) = \dim(\mathcal{O}_\lambda \cap \mathfrak{n})$, therefore $B.(\mathfrak{n} \cap {}^w\mathfrak{n}) \cap \mathcal{O}_\lambda$ is also irreducible in $\mathcal{O}_\lambda \cap \mathfrak{n}$; in particular $B.(\mathfrak{n} \cap {}^w\mathfrak{n}) \cap \mathcal{O}_\lambda$ is an orbital variety if and only if $B.(\mathfrak{n} \cap {}^w\mathfrak{n}) \cap \mathcal{O}_\lambda$ is closed in $\mathcal{O}_\lambda \cap \mathfrak{n}$. The natural questions connected to the construction are the following ones.

Q1. May be one can always find $w \in \mathcal{C}_T$ such that $\mathcal{V}_T = B.(\mathfrak{n} \cap {}^w\mathfrak{n}) \cap \mathcal{O}_\lambda$?

Q2. Or may be $\mathcal{V}_T = \bigcup_{y \in \mathcal{C}_T} B.(\mathfrak{n} \cap {}^y\mathfrak{n}) \cap \mathcal{O}_\lambda$?

The answers to both these questions are negative as we show by the following counter-example.



(1) (2) (3)

Figure 1. Counter-example.

Example 4.2. Let $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$. The corresponding left cell is given by

$$\mathcal{C}_T = \left\{ w_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}, w_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, w_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \right\}.$$

We draw here in grey (see Fig. 1) the corresponding space $\mathfrak{n} \cap {}^{w}\mathfrak{n}$:

On the other hand, by Theorem 3.11, $\mathcal{V}_T = B.N_{(2,3)} \cup B.N_{(2,4)} \cup B.N_{(1,3)} \cup B.N_{(1,4)}$. As one can see from the picture $N_{(1,4)} \notin B.(\mathfrak{n} \cap {}^{w}\mathfrak{n})$ for $w \in \{w_1, w_2, w_3\}$.

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INDEX OF NOTATION

Symbols appearing frequently are given below in order of appearance.

1.1. $\mathfrak{n}, \mathfrak{g}_\alpha, \alpha_i, \prod, \alpha_{i,j}, B, \mathbf{S}_n, s_i, g.u, \mathcal{F}_x, \mathcal{O}_x$;
1.2. $J(x), \mathcal{O}_\lambda, \mathrm{sh}(\mathcal{O}), \mathrm{sh}(T), \mathbf{Tab}_\lambda, \mathcal{F}_T, \mathcal{V}_T$;
2.2. $\pi_{i,j} : \mathfrak{n}_n \to \mathfrak{n}_{j+1-i}$;
3.1. $\mathcal{X}_2, \mathbf{S}_n^2, N_\sigma, \mathbf{Tab}_n^2, \sigma_T$;
3.2. $L(\sigma), \mathcal{O}_\sigma$;
3.3. $R_\sigma, \mathbf{R}_n^2$;
3.5. $\mathbf{S}^2_{\langle i,j\rangle}, \pi_{i,j} : \mathbf{S}_n^2 \to \mathbf{S}^2_{\langle i,j\rangle}$;
3.6. $\mathbf{S}_n^2(k), \mathbf{Tab}_n^2(k)$.

REFERENCES

[1] Borho W., Brylinski J.L. – Differential operators on homogeneous spaces III, Invent. Math. **80** (1985) 1–68.

[2] Fung F.Y.C. – On the topology of components of some Springer fibers and their relation to Kazhdan–Lusztig theory, Advances in Math. **178** (2003) 244–276.

[3] Kashiwara M., Saito Y. – Geometric construction of Crystal bases, Duke Math. J. **89** (1) (1997) 9–36.

[4] Kazhdan D., Lusztig G. – Representations of Coxeter groups and Hecke algebras, Invent. Math. **53** (1979) 165–184.
[5] Melnikov A. – Orbital varieties in $\mathfrak{sl}_n$ and the Smith conjecture, J. Algebra **200** (1998) 1–31.
[6] Melnikov A. – B-orbits in solutions to the equation $X^2 = 2$ in triangular matrices, J. Algebra **223** (2000) 101–108.
[7] Melnikov A. – Description of B-orbit closures of order 2 in upper-triangular matrices, Transform. Groups **11** (2) (2006) 217–247.
[8] Melnikov A., Pagnon N.G.J. – On the intersections of orbital varieties and components of Springer fiber, J. Algebra **298** (2006) 1–14.
[9] Melnikov A., Pagnon N.G.J. – Intersections of components of a Springer fiber of codimension one for the two-column case, arXiv:math/0701178.
[10] Slodowy P. – Four lectures on simple groups and singularities, Commun. Math. Inst., Rijksuniv. Utr. **11** (1980).
[11] Spaltenstein N. – The fixed point set of a unipotent transformation on the flag manifold, Indag. Mathem. **38** (1976) 452–456.
[12] Spaltenstein N. – On the fixed point set of a unipotent element on the variety of Borel subgroups, Topology **16** (1977) 203–204.
[13] Springer T.A. – The unipotent variety of a semisimple group, in: S. Abhyankar (Ed.), Proc. of the Bombay Colloqu. in Algebraic Geometry, Oxford Univ. Press, London, 1969, pp. 373–391.
[14] Springer T.A. – A construction of representations of Weyl groups, Invent. Math. **44** (1978) 279–293.
[15] Springer T.A. – Linear Algebraic Groups, 2nd ed., Progress in Mathematics, Birkhäuser, Boston, 1998.
[16] Steinberg R. – On the desingularisation of the unipotent variety, Invent. Math. **36** (1976) 209–224.
[17] Steinberg R. – An occurrence of the Robinson–Schensted correspondence, J. Algebra **113** (2) (1988) 523–528.
[18] Vargas J.A. – Fixed points under the action of unipotent elements of SL_n in the flag variety, Bol. Soc. Mat. Mexicana **24** (1979) 1–14.

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