J. Math. Anal. Appl. 368 (2010) 382–390

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Dichotomies for L^p spaces

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A R T I C L E   I N F O

Article history:
Received 28 July 2009
Available online 10 February 2010
Submitted by T.D. Benavides

Keywords:
Measure space
L^p space
Porous sets
Porosity

A B S T R A C T

Assume that (X, Σ, μ) is a measure space and $p_1, \dots, p_n, r > 0$. We prove that $\{(f_1, \dots, f_n) \in L^{p_1} \times \dots \times L^{p_n}\colon f_1 \cdots f_n \in L^r\}$ is either $L^{p_1} \times \dots \times L^{p_n}$ or a σ-porous subset of $L^{p_1} \times \dots \times L^{p_n}$. This dichotomy depends on properties of μ and the sign of the number $\frac{1}{r} - \frac{1}{p_1} - \dots - \frac{1}{p_n}$.

1. Introduction

Among linear topological spaces there are spaces X consisting of sequences or functions such that a natural multiplication is defined on pairs $(x_1, x_2) \in X^2$, however, its result need not necessarily belong to X. It is an interesting question about the size of the set of such "bad" pairs in a various sense. Such a kind of studies was initiated in [1,4]. Balcerzak and Wachowicz proved in [1] that $\{(f, g) \in L^1[0,1] \times L^1[0,1]\colon f \cdot g \in L^1[0,1]\}$ is a meager subset of $L^1[0,1] \times L^1[0,1]$. They also proved that

$$\left\{(x, y) \in c_0 \times c_0\colon \left(\sum_{i=1}^{n} x(i)y(i)\right)_{n=1}^{\infty} \text{ is bounded}\right\}$$

is a meager subset of $c_0 \times c_0$. These meagerness results were generalized by Jachymski in the following extension of the classical Banach–Steinhaus theorem. Recall that a function $\varphi : X \to \mathbb{R}_+$ is L-subadditive for some $L \geqslant 1$, if $\varphi(x+y) \leqslant L(\varphi(x) + \varphi(y))$ for any $x, y \in X$.

Theorem 1. *(See Jachymski [4].) Given $k \in \mathbb{N}$, let $X_1, \dots, X_k$ be Banach spaces, $X = X_1$ if $k = 1$, and $X = X_1 \times \dots \times X_k$ if $k > 1$. Assume that $L \geqslant 1$, $F_n : X \to \mathbb{R}_+$ $(n \in \mathbb{N})$ are lower semicontinuous and such that all functions $x_i \mapsto F_n(x_1, \dots, x_k)$ $(i = 1, \dots, k)$ are L-subadditive and even. Let $E = \{x \in X\colon (F_n(x))_{n=1}^{\infty}$ is bounded$\}$. Then the following statements are equivalent:*

(i) *E is meager;*
(ii) *$E \neq X$;*
(iii) $\sup\{F_n(x)\colon n \in \mathbb{N},\ \|x\| \leqslant 1\} = \infty$.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.02.011

At first, we were interested in a further generalization of this theorem changing meagerness by σ-porosity. It turns out that this is not possible. To see it, consider the following set:

$$E = \left\{ x \in \mathbb{R}\colon \left(\sum_{k=1}^{n} \frac{|\sin(k!\pi x)|}{k} \right)_{n=1}^{\infty} \text{ is bounded} \right\}.$$

Using Theorem 1 for $F_n(x) = \sum_{k=1}^{n} |\sin(k!\pi x)|/k$ (clearly, each F_n is subadditive) we obtain that this set is meager ($E \neq \mathbb{R}$ since it is of measure zero) and is not σ-upper porous [5, p. 341]. Hence we could not generalize Jachymski's theorem in this manner.

Assume that (X, Σ, μ) is a measure space. In our paper we answer the question about a size of the set (in the following we will write L^p instead of $L^p(X, \Sigma, \mu)$):

$$\{(f_1, \dots, f_n) \in L^{p_1} \times \dots \times L^{p_n}\colon f_1 \cdots f_n \in L^r\}.$$

We do not restrict our attention only to Banach L^p spaces for $p \in [1, \infty]$, but we consider all linear metric L^p spaces for $p \in (0, \infty]$. It appears that this set is either $L^{p_1} \times \dots \times L^{p_n}$ or a σ–c-lower porous (for some $c > 0$) subset of $L^{p_1} \times \dots \times L^{p_n}$. So, it is either the whole space or a very small set. We determine this dichotomy for every type of a measure space (X, Σ, μ). Surprisingly it depends on the following parameters (in the sequel the symbol $\frac{1}{\infty}$ means 0):

- the sign of the number $\frac{1}{r} - \frac{1}{p_1} - \dots - \frac{1}{p_n}$;
- $\inf\{\mu(A)\colon \mu(A) > 0\}$ (it is important whether it is equal or greater than zero);
- $\sup\{\mu(A)\colon \mu(A) < \infty\}$ (it is important whether it is finite or infinite).

The dichotomy is stated in Proposition 2 and Theorems 9, 10.

Let X be a metric space. $B(x, R)$ stands for the ball with a radius R centered at a point x. Let $c \in (0, 1]$. We say that $M \subset X$ is *c-lower porous* [6], if

$$\forall x \in M, \quad \liminf_{R \to 0^+} \frac{\gamma(x, M, R)}{R} \geqslant \frac{c}{2},$$

where

$$\gamma(x, M, R) = \sup\{r \geqslant 0\colon \exists z \in X,\ B(z, r) \subset B(x, R) \setminus M\}.$$

Clearly, M is c-lower porous iff

$$\forall x \in M, \quad \forall \beta \in (0, c/2), \quad \exists R_0 > 0, \quad \forall R \in (0, R_0), \quad \exists z \in X, \quad B(z, \beta R) \subset B(x, R) \setminus M.$$

The set is σ–c-lower porous if it is a countable union of c-lower porous sets. Note that a σ–c-lower porous set is meager, and the notion of σ-porosity is essentially stronger than that of meagerness.

Note that the sets investigated in this paper will be c-porous in some stronger sense, namely,

$$\forall x \in X, \quad \forall \beta \in (0, c/2), \quad \forall R > 0, \quad \exists z \in X, \quad B(z, \beta R) \subset B(x, R) \setminus M.$$

However, we do not want to define any new notion of porosity, so in the formulations of theorems we will deal only with c-lower porosity.

2. Algebraic product of functions from $L^{p_1} \times \dots \times L^{p_n}$

Throughout the paper, (X, Σ, μ) is a measure space. If $p \in (0, 1)$, then we consider L^p as a metric linear space with the metric

$$d(f, g) = \int_X |f - g|^p \, d\mu.$$

Additionally we put

$$\|f\|_p = d(f, 0) = \int_X |f|^p \, d\mu.$$

If $p \in [1, \infty)$, then we consider L^p as a normed linear space with the norm

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

Finally, if $p=\infty$, then we consider L^p as a normed linear space with the norm $\|f\|_\infty = \operatorname{supess}|f|$. Note that in all cases L^p is a complete space.

For every $n \in \mathbb{N}$ and any $p_1, \dots, p_n, r \in (0,\infty]$, we define the set (we allow n to be 1):

$$E_r^{(p_1,\dots,p_n)} = \left\{(f_1,\dots,f_n) \in L^{p_1} \times \dots \times L^{p_n}\colon\ f_1 \cdots f_n \in L^r\right\}.$$

In this paper we consider $L^{p_1} \times \dots \times L^{p_n}$ as a space with the metric defined as the maximum of distances on all coordinates in $L^{p_1}, \dots, L^{p_n}$.

Using the general Hölder inequality [3, p. 10] we obtain that:

Proposition 2. *Let $p_1, \dots, p_n, r \in (0,\infty]$ be such that*

$$\frac{1}{r} = \frac{1}{p_1} + \dots + \frac{1}{p_n}.$$

Then $E_r^{(p_1,\dots,p_n)} = L^{p_1} \times \dots \times L^{p_n}$.

Now we will give some helpful lemmas.

Lemma 3. *Let $h \geqslant 0$, $h \in L^1$, $\varepsilon > 0$. Then*

(i) *if $\inf\{\mu(A)\colon A \in \Sigma,\ \mu(A) > 0\} = 0$, there is $A \in \Sigma$ with $0 < \mu(A) \leqslant \varepsilon$ and $\int_A h\,d\mu \leqslant \varepsilon$;*
(ii) *if $\sup\{\mu(A)\colon A \in \Sigma,\ \mu(A) < \infty\} = \infty$, there is $A \in \Sigma$ with $1/\varepsilon \leqslant \mu(A) < \infty$ and $\int_A h\,d\mu \leqslant \varepsilon$.*

Proof. (i) Follows immediately from the absolute continuity of the function $B \mapsto \int_B h\,d\mu$ $(B \in \Sigma)$ with respect to μ.

(ii) Let, for any $n \in \mathbb{N}$, A_n be such that $n < \mu(A_n) < \infty$. Set $F_n = \bigcup_{k=1}^n A_k$. Then (F_n) is increasing, $\mu(F_n) < \infty$ and $\mu(F_n) \to \infty$. Put $F = \bigcup_{n=1}^\infty F_n$. We have

$$\lim_{n\to\infty} \int_{F_n} h\,d\mu = \int_F h\,d\mu < \infty.$$

Then there is $n_0 \in \mathbb{N}$ with

$$\int_{F_{n_0}} h\,d\mu > \int_F h\,d\mu - \varepsilon.$$

Hence

$$\int_{F\setminus F_{n_0}} h\,d\mu < \varepsilon.$$

On the other hand, $\lim_{n\to\infty} \mu(F_n \setminus F_{n_0}) = \infty$, so there is $N \in \mathbb{N}$ such that $\mu(F_N \setminus F_{n_0}) > 1/\varepsilon$. Put $A = F_N \setminus F_{n_0}$. □

Lemma 4. *Let $p_1, \dots, p_n, r \in (0,\infty)$, $(f_1,\dots,f_n) \in L^{p_1} \times \dots \times L^{p_n}$ and let A be a measurable subset of X. Suppose that for some numbers $a_1, \dots, a_n$ and for each $i = 1, \dots, n$, the following holds*

$$\int_A |f_i - 1|^{p_i}\,d\mu \leqslant a_i.$$

Then for any numbers $c_1, \dots, c_n \in (0,1)$, we have

$$\int_A |f_1 \cdots f_n|^r\,d\mu \geqslant c_1^r \cdots c_n^r \left(\mu(A) - \frac{a_1}{(1-c_1)^{p_1}} - \dots - \frac{a_n}{(1-c_n)^{p_n}}\right).$$

Proof. Observe that the above assumptions imply that $\mu(A) < \infty$. Let $A_i = \{x \in A\colon\ f_i(x) < c_i\}$ for $i = 1, \dots, n$. Then for any i, we have

$$a_i \geqslant \int_A |f_i - 1|^{p_i}\,d\mu \geqslant \int_{A_i} |f_i - 1|^{p_i}\,d\mu \geqslant \int_{A_i} |1 - c_i|^{p_i}\,d\mu = (1-c_i)^{p_i}\mu(A_i).$$

Hence

$$\int_A |f_1 \cdots f_n|^r \, d\mu \geqslant \int_{A \setminus \bigcup_{i=1}^n A_i} |f_1 \cdots f_n|^r \, d\mu \geqslant \int_{A \setminus \bigcup_{i=1}^n A_i} c_1^r \cdots c_n^r \, d\mu$$
$$\geqslant c_1^r \cdots c_n^r \left(\mu(A) - \mu\left(\bigcup_{i=1}^n A_i \right) \right)$$
$$\geqslant c_1^r \cdots c_n^r \left(\mu(A) - \frac{a_1}{(1-c_1)^{p_1}} - \cdots - \frac{a_n}{(1-c_n)^{p_n}} \right). \quad \square$$

Lemma 5. *Let $A, A_1, \dots, A_n$ be measurable with $A_i \subset A$ and $\mu(A_i) > (1 - \frac{1}{n})\mu(A)$ for any $i = 1, \dots, n$. Then*

$$\mu\left(\bigcap_{i=1}^n A_i \right) > 0.$$

Proof. Using the induction principle, it is easy to show that

$$\mu\left(\bigcap_{i=1}^k A_i \right) > (1 - k/n)\mu(A) \quad \text{for any } k = 1, \dots, n.$$

In particular, for $k = n$, we get that $\mu(\bigcap_{i=1}^n A_i) > 0$. $\square$

The next theorem is a main result of the paper. It is rather technical, but it shows when $E_r^{(p_1,\dots,p_n)}$ can be σ-porous and how good are porosity estimations in each of the considered cases. For any $n \in \mathbb{N}$ and any $p_1, \dots, p_n$, put $c(p_1, \dots, p_n) = 2/(1+m)$ if there is at least one finite p_i, where m is the number of finite p_i's, and put $c(p_1, \dots, p_n) = 1$ if $p_i = \infty$ for every $i = 1, \dots, n$.

Theorem 6. *Let $n \in \mathbb{N}$ and let $p_1, \dots, p_n, r \in (0, \infty]$. Assume that one of the following conditions holds*:

(i) $\frac{1}{p_1} + \cdots + \frac{1}{p_n} > \frac{1}{r}$ *and* $\inf\{\mu(A)\colon \mu(A) > 0\} = 0$;
(ii) $\frac{1}{p_1} + \cdots + \frac{1}{p_n} < \frac{1}{r}$ *and* $\sup\{\mu(A)\colon \mu(A) < \infty\} = \infty$.

Then for any $u > 0$, the set

$$E_u = \{(f_1, \dots, f_n) \in L^{p_1} \times \cdots \times L^{p_n}\colon \|f_1 \cdots f_n\|_r \leqslant u\}$$

is c-lower porous, where $c = c(p_1, \dots, p_n)$. In particular, the set $E_r^{(p_1,\dots,p_n)}$ is σ–c-lower porous.

Proof. We will consider two cases.

Case 1. $p_1 = \cdots = p_n = \infty$.

Then our assumptions imply that $r < \infty$ and $\sup\{\mu(A)\colon \mu(A) < \infty\} = \infty$. Let $(f_1, \dots, f_n) \in L^\infty \times \cdots \times L^\infty$, $R > 0$, $\alpha \in (0, \frac{1}{2})$ (note that in this case $c(p_1, \dots, p_n) = 1$). Fix a measurable set A of finite measure such that

$$\mu(A) > \frac{u^r}{((\frac{1}{2} - \alpha)R)^{rn}}.$$

For any $i = 1, \dots, n$, we define

$$\tilde{f}_i(x) = \begin{cases} f_i(x) + \frac{1}{2}R, & f_i(x) \geqslant 0; \\ f_i(x) - \frac{1}{2}R, & f_i(x) < 0. \end{cases}$$

Clearly, for any $i = 1, \dots, n$, $\|\tilde{f}_i - f_i\|_\infty = R/2$ and $B((\tilde{f}_1, \dots, \tilde{f}_n), \alpha R) \subset B((f_1, \dots, f_n), R)$. Now if $(h_1, \dots, h_n) \in B((\tilde{f}_1, \dots, \tilde{f}_n), \alpha R)$, then for any $i = 1, \dots, n$ and for μ-almost every $x \in A$, we have

$$|h_i(x)| \geqslant \left(\frac{1}{2} - \alpha \right) R.$$

Hence

$$\int_A |h_1 \cdots h_n|^r \geqslant \left(\left(\frac{1}{2} - \alpha\right) R\right)^{rn} \cdot \mu(A) > u^r,$$

and

$$\|h_1 \cdots h_n\|_r > u.$$

This ends the proof in Case 1.

Case 2. For some $i = 1, \ldots, n$, $p_i < \infty$.

Without loss of generality, we assume that $p_i \in (0,1)$ for $i = 1, \ldots, m$, $1 \leqslant p_i < \infty$ for $i = m+1, \ldots, m+k$ and $p_i = \infty$ for $i = m+k+1, \ldots, m+k+j$, where j is such that $m+k+j = n$ (clearly, m, k or j can be equal to zero, but $m+k \neq 0$). Additionally define $q_i = p_{m+i}$ for $i = 1, \ldots, k$. Then the product space $L^{p_1} \times \cdots \times L^{p_n}$ can be written in the following way:

$$L^{p_1} \times \cdots \times L^{p_m} \times L^{q_1} \times \cdots \times L^{q_k} \times L^\infty \times \cdots \times L^\infty.$$

Let $(f_1, \ldots, f_m, g_1, \ldots, g_k, l_1, \ldots, l_j)$ be a member of that space, and let $R > 0$, $\delta \in (0, \frac{1}{m+k+1})$ (note that in this case $c(p_1, \ldots, p_n) = 2/(m+k+1)$). Then, clearly, $1 - \delta > (m+k)\delta$ and hence we can take $\eta \in ((m+k)\delta, 1-\delta)$. Since $\delta/\eta < 1/(m+k)$ and hence $(\delta/\eta)^{q_i} < 1/(m+k)$ for $i = 1, \ldots, k$, there exist $c \in (0,1)$ and $\varepsilon > 0$ such that

$$\frac{\delta}{\eta} \leqslant \frac{(1-c)^{p_i}}{m+k+\varepsilon} \quad \text{for every } i = 1, \ldots, m \tag{1}$$

and

$$\left(\frac{\delta}{\eta}\right)^{q_i} \leqslant \frac{(1-c)^{q_i}}{m+k+\varepsilon} \quad \text{for every } i = 1, \ldots, k. \tag{2}$$

Now we will define a positive number β. To define β consider three cases.

If $r < \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \cdots + \frac{1}{q_k} > \frac{1}{r}$, then $r(\frac{1}{r} - \frac{1}{p_1} - \cdots - \frac{1}{p_m} - \frac{1}{q_1} - \cdots - \frac{1}{q_k}) < 0$, so we can find $\beta > 0$ be such that for any $\beta' \in (0, \beta]$, we have

$$u^r \left(\left(R(1-2\delta)\right)^{rj} (\eta R)^{kr + r(\frac{1}{p_1} + \cdots + \frac{1}{p_m})} \cdot c^{(m+k)r} \frac{\varepsilon}{m+k+\varepsilon} \right)^{-1} < \left(\beta'\right)^{r(\frac{1}{r} - \frac{1}{p_1} - \cdots - \frac{1}{p_m} - \frac{1}{q_1} - \cdots - \frac{1}{q_k})} < \infty. \tag{3}$$

If $r < \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \cdots + \frac{1}{q_k} < \frac{1}{r}$, then $r(\frac{1}{r} - \frac{1}{p_1} - \cdots - \frac{1}{p_m} - \frac{1}{q_1} - \cdots - \frac{1}{q_k}) > 0$, so we can find $\beta > 0$ be such that for any $\beta' \in (0, \beta]$, we have

$$u^r \left(\left(R(1-2\delta)\right)^{rj} (\eta R)^{kr + r(\frac{1}{p_1} + \cdots + \frac{1}{p_m})} \cdot c^{(m+k)r} \frac{\varepsilon}{m+k+\varepsilon} \right)^{-1} < \left(\frac{1}{\beta'}\right)^{r(\frac{1}{r} - \frac{1}{p_1} - \cdots - \frac{1}{p_m} - \frac{1}{q_1} - \cdots - \frac{1}{q_k})} < \infty. \tag{4}$$

If $r = \infty$, then our assumptions imply $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \cdots + \frac{1}{q_k} > 0 = \frac{1}{r}$, so we can find $\beta > 0$ such that for any $\beta' \in (0, \beta]$, we have

$$u \left(\left(R(1-2\delta)\right)^j c^{m+k} \cdot (\eta R)^{k + \frac{1}{p_1} + \cdots + \frac{1}{p_m}} \right)^{-1} < \left(\beta'\right)^{-(\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \cdots + \frac{1}{q_k})} < \infty. \tag{5}$$

Using Lemma 3 with $h = \max\{|f_1|^{p_1}, \ldots, |f_m|^{p_m}, |g_1|^{q_1}, \ldots, |g_k|^{q_k}\}$ (note that $h \in L^1$) and

$$\varepsilon = \min\{\beta, (1-\delta-\eta)R, \left((1-\delta-\eta)R\right)^{q_1}, \ldots, \left((1-\delta-\eta)R\right)^{q_k}\},$$

we infer that there is $A \in \Sigma$ with $0 < \mu(A) \leqslant \varepsilon$ if $\inf\{\mu(A)\colon \mu(A) > 0\} = 0$, or with $1/\varepsilon \leqslant \mu(A) < \infty$ if $\sup\{\mu(A)\colon \mu(A) < \infty\} = \infty$, such that the following conditions hold

$$\int_A |f_i|^{p_i} \, d\mu \leqslant (1-\delta-\eta)R \quad \text{for every } i = 1, \ldots, m; \tag{6}$$

$$\left(\int_A |g_i|^{q_i} \, d\mu \right)^{1/q_i} \leqslant (1-\delta-\eta)R \quad \text{for every } i = 1, \ldots, k. \tag{7}$$

Next, let $M_1, \ldots, M_m$, $N_1, \ldots, N_k$ be such that

$$M_i^{p_i} \mu(A) = \eta R \quad \text{for every } i = 1, \ldots, m; \tag{8}$$

$$N_i \left(\mu(A)\right)^{1/q_i} = \eta R \quad \text{for every } i = 1, \ldots, k. \tag{9}$$

Now, let us define $\tilde{f}_1,\dots,\tilde{f}_m$, $\tilde{g}_1,\dots,\tilde{g}_k$, $\tilde{l}_1,\dots,\tilde{l}_j$ by formulas:

$$\tilde{f}_i(x)=\begin{cases} M_i, & x\in A;\\ f_i(x), & x\notin A,\end{cases}\qquad \tilde{g}_i(x)=\begin{cases} N_i, & x\in A;\\ g_i(x), & x\notin A,\end{cases}$$
$$\tilde{l}_i(x)=\begin{cases} l_i(x)+(1-\delta)R, & \text{if } l_i(x)\geqslant 0;\\ l_i(x)-(1-\delta)R, & \text{if } l_i(x)<0.\end{cases}$$

Using (6)–(9) we obtain

$$\begin{aligned} d(\tilde{f}_i,f_i)&=\int_A |M_i-f_i|^{p_i}\,d\mu\leqslant\int_A M_i^{p_i}\,d\mu+\int_A |f_i|^{p_i}\,d\mu\\ &\leqslant \eta R+(1-\delta-\eta)R=R-\delta R,\\ \|\tilde{g}_i-g_i\|_{q_i}&=\left(\int_A |N_i-g_i|^{q_i}\,d\mu\right)^{1/q_i}\leqslant\left(\int_A N_i^{q_i}\,d\mu\right)^{1/q_i}+\left(\int_A |g_i|^{q_i}\,d\mu\right)^{1/q_i}\\ &\leqslant \eta R+(1-\delta-\eta)R=R-\delta R,\end{aligned}$$

and

$$\|\tilde{l}_i-l_i\|_\infty=(1-\delta)R.$$

Hence $B((\tilde{f}_1,\dots,\tilde{f}_m,\tilde{g}_1,\dots,\tilde{g}_k,\tilde{l}_1,\dots,\tilde{l}_j),\delta R)\subset B((f_1,\dots,f_m,g_1,\dots,g_k,l_1,\dots,l_j),R)$. It is enough to show that $B((\tilde{f}_1,\dots,\tilde{f}_m,\tilde{g}_1,\dots,\tilde{g}_k,\tilde{l}_1,\dots,\tilde{l}_j),\delta R)\cap E_u=\emptyset$. Let

$$(h_1,\dots,h_m,s_1,\dots,s_k,w_1,\dots,w_j)\in B\big((\tilde{f}_1,\dots,\tilde{f}_m,\tilde{g}_1,\dots,\tilde{g}_k,\tilde{l}_1,\dots,\tilde{l}_j),\delta R\big).$$

Clearly, since $\|\tilde{l}_i\|_\infty\geqslant(1-\delta)R$, for μ-almost every $x\in A$, we have

$$\big|w_i(x)\big|\geqslant R(1-2\delta). \tag{10}$$

Assume now that $r<\infty$. For any $i=1,\dots,m$, we have

$$\delta R\geqslant\int_A |h_i-\tilde{f}_i|^{p_i}\,d\mu=\int_A |h_i-M_i|^{p_i}\,d\mu=M_i^{p_i}\int_A\left|\frac{h_i}{M_i}-1\right|^{p_i}d\mu.$$

Using (1) and (8) we obtain

$$\int_A\left|\frac{h_i}{M_i}-1\right|^{p_i}d\mu\leqslant\frac{\delta R}{M_i^{p_i}}=\frac{\delta}{\eta}\mu(A)\leqslant\frac{1}{m+k+\varepsilon}\mu(A)(1-c)^{p_i}.$$

Similarly for any $i=1,\dots,k$,

$$(\delta R)^{q_i}\geqslant\int_A |s_i-\tilde{g}_i|^{q_i}\,d\mu=N_i^{q_i}\int_A\left|\frac{s_i}{N_i}-1\right|^{q_i}d\mu,$$

and using (2) and (9) we have

$$\int_A\left|\frac{s_i}{N_i}-1\right|^{q_i}d\mu\leqslant\left(\delta\frac{R}{N_i}\right)^{q_i}=\left(\frac{\delta}{\eta}\right)^{q_i}\mu(A)\leqslant\frac{1}{m+k+\varepsilon}\mu(A)(1-c)^{q_i}.$$

By (3), (4), (8)–(10) and Lemma 4 used for $c_i=c$, we obtain the following

$$\begin{aligned}&\int_X |h_1\cdots h_m\cdot s_1\cdots s_k\cdot w_1\cdots w_j|^r\,d\mu\\ &\quad\geqslant\big(R(1-2\delta)\big)^{rj}\int_A |h_1\cdots h_m\cdot s_1\cdots s_k|^r\,d\mu\\ &\quad=\big(R(1-2\delta)\big)^{rj}M_1^r\cdots M_m^r\cdot N_1^r\cdots N_k^r\int_A\left|\frac{h_1}{M_1}\cdots\frac{h_m}{M_m}\cdot\frac{s_1}{N_1}\cdots\frac{s_k}{N_k}\right|^r d\mu\end{aligned}$$

$$
\begin{aligned}
&\geqslant \big(R(1-2\delta)\big)^{rj} M_1^r\cdots M_m^r\cdot N_1^r\cdots N_k^r\cdot c^{(m+k)r}\left(\mu(A)-(m+k)\frac{1}{m+k+\varepsilon}\mu(A)\right)\\
&= \big(R(1-2\delta)\big)^{rj} M_1^r\cdots M_m^r\cdot N_1^r\cdots N_k^r\cdot c^{(m+k)r}\frac{\varepsilon}{m+k+\varepsilon}\mu(A)\\
&= \big(R(1-2\delta)\big)^{rj}\big[M_1^{p_1}\mu(A)\big]^{\frac{r}{p_1}}\cdots\big[M_m^{p_m}\mu(A)\big]^{\frac{r}{p_m}}\cdot\big[N_1\mu(A)^{\frac{1}{q_1}}\big]^r\cdots\big[N_k\mu(A)^{\frac{1}{q_k}}\big]^r\cdot c^{(m+k)r}\\
&\quad\cdot\big(\mu(A)\big)^{r(\frac{1}{r}-\frac{1}{p_1}-\cdots-\frac{1}{p_m}-\frac{1}{q_1}-\cdots-\frac{1}{q_k})}\cdot\frac{\varepsilon}{m+k+\varepsilon}\\
&= \big(R(1-2\delta)\big)^{rj}(\eta R)^{\frac{r}{p_1}}\cdots(\eta R)^{\frac{r}{p_m}}\cdot(\eta R)^r\cdots(\eta R)^r\cdot c^{(m+k)r}\\
&\quad\cdot\big(\mu(A)\big)^{r(\frac{1}{r}-\frac{1}{p_1}-\cdots-\frac{1}{p_m}-\frac{1}{q_1}-\cdots-\frac{1}{q_k})}\cdot\frac{\varepsilon}{m+k+\varepsilon}\\
&= \big(R(1-2\delta)\big)^{rj}(\eta R)^{kr+r(\frac{1}{p_1}+\cdots+\frac{1}{p_m})}\cdot c^{(m+k)r}\cdot\big(\mu(A)\big)^{r(\frac{1}{r}-\frac{1}{p_1}-\cdots-\frac{1}{p_m}-\frac{1}{q_1}-\cdots-\frac{1}{q_k})}\cdot\frac{\varepsilon}{m+k+\varepsilon}>u^r.
\end{aligned}
$$

For the last inequality, observe that if $\frac{1}{p_1}+\cdots+\frac{1}{p_m}+\frac{1}{q_1}+\cdots+\frac{1}{q_k}>\frac{1}{r}$, then by hypothesis, we infer that $\mu(A)\leqslant\varepsilon\leqslant\beta$, so we may use (3) with $\beta'=\mu(A)$. If $\frac{1}{p_1}+\cdots+\frac{1}{p_m}+\frac{1}{q_1}+\cdots+\frac{1}{q_k}<\frac{1}{r}$, then $\frac{1}{\mu(A)}\leqslant\varepsilon\leqslant\beta$, and we may use (4) with $\beta'=\frac{1}{\mu(A)}$. Hence

$$\|h_1\cdots h_m\cdot s_1\cdots s_k\cdot w_1\cdots w_j\|_r>u.$$

Assume now that $r=\infty$. As was mentioned, this case is possible only if $\inf\{\mu(A)\colon \mu(A)>0\}=0$. For any $i=1,\dots,m$, we define

$$A_i^1=\big\{x\in A\colon h_i(x)\geqslant cM_i\big\},\qquad A_i^2=A\setminus A_i^1,$$

and for any $i=1,\dots,k$, we define

$$B_i^1=\big\{x\in A\colon s_i(x)\geqslant cN_i\big\}\quad\text{and}\quad B_i^2=A\setminus B_i^1.$$

Then

$$\delta R>\int_A|h_i-M_i|^{p_i}\,d\mu\geqslant\int_{A_i^2}|h_i-M_i|^{p_i}\,d\mu\geqslant M_i^{p_i}(1-c)^{p_i}\mu\big(A_i^2\big).$$

Hence by (1) and (8), we have

$$\mu\big(A_i^2\big)<\frac{\delta R}{M_i^{p_i}(1-c)^{p_i}}=\frac{\delta}{\eta}\frac{1}{(1-c)^{p_i}}\mu(A)\leqslant\frac{1}{m+k}\mu(A).$$

Then $\mu(A_i^1)>(1-\frac{1}{m+k})\mu(A)$ for each $i=1,\dots,m$. The same estimations (by (2) and (9)) hold for s_i:

$$(\delta R)^{q_i}>\int_A|s_i-N_i|^{q_i}\,d\mu\geqslant\int_{B_i^2}|s_i-N_i|^{q_i}\,d\mu\geqslant N_i^{q_i}(1-c)^{q_i}\mu\big(B_i^2\big).$$

Then

$$\mu\big(B_i^2\big)<\left(\frac{\delta R}{N_i(1-c)}\right)^{q_i}\leqslant\left(\frac{\delta}{\eta(1-c)}\right)^{q_i}\mu(A)\leqslant\frac{1}{m+k}\mu(A).$$

Hence $\mu(B_i^1)>(1-\frac{1}{m+k})\mu(A)$ for each $i=1,\dots,k$. Now by Lemma 5 we obtain that $\mu(A_1^1\cap\cdots\cap A_m^1\cap B_1^1\cap\cdots\cap B_k^1)>0$. Also, for μ-almost every $x\in A_1^1\cap\cdots\cap A_m^1\cap B_1^1\cap\cdots\cap B_k^1$, using (8)–(10) and (5) we have

$$
\begin{aligned}
&\big|h_1(x)\cdots h_m(x)\cdot s_1(x)\cdots s_k(x)\cdot w_1(x)\cdots w_j(x)\big|\\
&\quad\geqslant\big(R(1-2\delta)\big)^j c^{m+k}M_1\cdots M_m\cdot N_1\cdots N_k\\
&\quad=\big(R(1-2\delta)\big)^j c^{m+k}(\eta R)^{\frac{1}{p_1}+\cdots+\frac{1}{p_m}}(\eta R)^k\big(\mu(A)\big)^{-(\frac{1}{p_1}+\cdots+\frac{1}{p_m}+\frac{1}{q_1}+\cdots+\frac{1}{q_k})}>u,
\end{aligned}
$$

and hence

$$\|h_1\cdots h_m\cdot s_1\cdots s_k\cdot w_1\cdots w_j\|_r>u.$$

This ends the proof. □

Lemma 7. *Assume that*

$$\inf\{\mu(A)\colon \mu(A) > 0\} > 0.$$

Then:

(i) *for every* $r \in (1, \infty)$, $L^1 \subset L^r$;
(ii) *for every* $p > 0$, $L^p \subset L^\infty$.

The proof of Lemma 7 is known (see, e.g. [2, 224X(e)]).

Proposition 8. *Let* $p_1, \dots, p_n, r \in (0, \infty]$. *If one of the following conditions holds*:

(i) $\sup\{\mu(A)\colon \mu(A) < \infty\} < \infty$ *and* $0 < \frac{1}{p_1} + \dots + \frac{1}{p_n} < \frac{1}{r}$;
(ii) $\inf\{\mu(A)\colon \mu(A) > 0\} > 0$ *and* $\frac{1}{p_1} + \dots + \frac{1}{p_n} > \frac{1}{r}$,

then $E_r^{(p_1,\dots,p_n)} = L^{p_1} \times \dots \times L^{p_n}$.

Proof. Assume (i). Then r is finite and at least one $p_i < \infty$.

Let $M = \sup\{\mu(A)\colon \mu(A) < \infty\}$. For any $k \in \mathbb{N}$, let D_k be a measurable set with $M - 1/k \leqslant \mu(D_k) \leqslant M$. Set $D = \bigcup_{k=1}^\infty D_k$. Since $\mu(\bigcup_{s=1}^k D_s) \leqslant M$ for any k, then $\mu(D) = M$ and for a measurable $F \subset X \setminus D$ we have $\mu(F) = 0$ or $\mu(F) = \infty$. Hence if $p < \infty$ and $f \in L^p$, then $\mu(\{x \in X \setminus D\colon f(x) \neq 0\}) = 0$.

Assume that for some $1 \leqslant m \leqslant n$, we have $p_1, \dots, p_m < \infty$ and $p_{m+1}, \dots, p_n$ are equal to ∞. Let $M > 0$ be such that $|f_i| \leqslant M$ μ-a.e. on X for $i = m+1, \dots, n$, and set $h = \max\{|f_1|^{p_1}, \dots, |f_m|^{p_m}\}$. Then $h \in L^1$. Since $f_1 \in L^{p_1}$ and $p_1 < \infty$, we have that

$$\mu\bigl(\bigl\{x \in X \setminus D\colon f_1(x)\cdots f_n(x) \neq 0\bigr\}\bigr) = 0.$$

Hence

$$\int_X |f_1 \cdots f_n|^r \, d\mu = \int_D |f_1 \cdots f_n|^r \, d\mu \leqslant M^{n-m} \int_D |f_1 \cdots f_m|^r \, d\mu \leqslant M^{n-m} \int_D h^{r(\frac{1}{p_1} + \dots + \frac{1}{p_m})} \, d\mu.$$

We only have to observe that $\int_D h^{r(\frac{1}{p_1} + \dots + \frac{1}{p_m})} \, d\mu < \infty$, but this follows from the fact that $\mu(D) < \infty$ and

$$r\left(\frac{1}{p_1} + \dots + \frac{1}{p_m}\right) < 1.$$

Now assume (ii). We have to consider two cases:

Case 1. $r < \infty$. Then at least one of $p_1, \dots, p_n$ is finite. Assume again, that for some $1 \leqslant m \leqslant n$, we have $p_1, \dots, p_m < \infty$ and $p_{m+1} = \dots = p_n = \infty$. Let $(f_1, \dots, f_n) \in L^{p_1} \times \dots \times L^{p_n}$. Set $h = \max\{|f_1|^{p_1}, \dots, |f_m|^{p_m}\}$. Then $h \in L^1$. Let $M > 0$ be such that $|f_i| \leqslant M$ μ-a.e. on X for all $i = m+1, \dots, n$. Then by Lemma 7, we obtain

$$\int_X |f_1 \cdots f_n|^r \, d\mu \leqslant M^{n-m} \int_X h^{r(\frac{1}{p_1} + \dots + \frac{1}{p_n})} \, d\mu < \infty,$$

since $r(\frac{1}{p_1} + \dots + \frac{1}{p_n}) > 1$.

Case 2. $r = \infty$. By Case 1, we obtain that for $r' < \infty$ with

$$\frac{1}{r'} < \frac{1}{p_1} + \dots + \frac{1}{p_n},$$

if $(f_1, \dots, f_n) \in L^{p_1} \times \dots \times L^{p_n}$, then $f_1 \cdots f_n \in L^{r'}$. Hence by Lemma 7, we have $\|f_1 \cdots f_n\|_\infty < \infty$. $\square$

Note that Proposition 8 is not valid if each p_i is infinite. Indeed, if we consider the following measure

$$\mu(A) = 0 \quad \text{if } A = \emptyset \quad \text{and} \quad \mu(A) = \infty \quad \text{if } A \neq \emptyset,$$

and we set $f = g = 1$, then $(f, g) \in L^\infty \times L^\infty$, but $(f, g) \notin E_r^{(\infty,\infty)}$.

Now we can summarize our results in the two following theorems. We write c instead of $c(p_1, \dots, p_n)$, where $c(p_1, \dots, p_n)$ was defined before the statement of Theorem 6.

Theorem 9. *Let (X, Σ, μ) be a measure space. The following conditions are equivalent*:

(i) *for any $n \in \mathbb{N}$ and $p_1, \dots, p_n, r > 0$ such that $\frac{1}{p_1} + \dots + \frac{1}{p_n} > \frac{1}{r}$, the set $E_r^{(p_1,\dots,p_n)}$ is σ–c-lower porous*;
(ii) *for any $n \in \mathbb{N}$ and $p_1, \dots, p_n, r > 0$ such that $\frac{1}{p_1} + \dots + \frac{1}{p_n} > \frac{1}{r}$, the set $E_r^{(p_1,\dots,p_n)}$ is not equal to $L^{p_1} \times \dots \times L^{p_n}$*;
(iii) *there are $n \in \mathbb{N}$ and $p_1, \dots, p_n, r > 0$ such that $\frac{1}{p_1} + \dots + \frac{1}{p_n} > \frac{1}{r}$ and the set $E_r^{(p_1,\dots,p_n)}$ is σ–c-lower porous*;
(iv) *there are $n \in \mathbb{N}$ and $p_1, \dots, p_n, r > 0$ such that $\frac{1}{p_1} + \dots + \frac{1}{p_n} > \frac{1}{r}$ and the set $E_r^{(p_1,\dots,p_n)}$ is not equal to $L^{p_1} \times \dots \times L^{p_n}$*;
(v) $\inf\{\mu(A)\colon \mu(A) > 0\} = 0$.

Proof. The following implications are trivial: (i) ⇒ (ii), (i) ⇒ (iii), (ii) ⇒ (iv) and (iii) ⇒ (iv). Implication (iv) ⇒ (v) follows from Proposition 8. Finally, (v) ⇒ (i) follows from Theorem 6. □

Theorem 10. *Let (X, Σ, μ) be a measure space. The following conditions are equivalent*:

(i) *for any $n \in \mathbb{N}$ and $p_1, \dots, p_n, r > 0$ such that $0 < \frac{1}{p_1} + \dots + \frac{1}{p_n} < \frac{1}{r}$, the set $E_r^{(p_1,\dots,p_n)}$ is σ–c-lower porous*;
(ii) *for any $n \in \mathbb{N}$ and $p_1, \dots, p_n, r > 0$ such that $0 < \frac{1}{p_1} + \dots + \frac{1}{p_n} < \frac{1}{r}$, the set $E_r^{(p_1,\dots,p_n)}$ is not equal to $L^{p_1} \times \dots \times L^{p_n}$*;
(iii) *there are $n \in \mathbb{N}$ and $p_1, \dots, p_n, r > 0$ such that $0 < \frac{1}{p_1} + \dots + \frac{1}{p_n} < \frac{1}{r}$ and the set $E_r^{(p_1,\dots,p_n)}$ is σ–c-lower porous*;
(iv) *there are $n \in \mathbb{N}$ and $p_1, \dots, p_n, r > 0$ such that $0 < \frac{1}{p_1} + \dots + \frac{1}{p_n} < \frac{1}{r}$ and the set $E_r^{(p_1,\dots,p_n)}$ is not equal to $L^{p_1} \times \dots \times L^{p_n}$*;
(v) $\sup\{\mu(A)\colon \mu(A) < \infty\} = \infty$.

Proof. The following implications are trivial: (i) ⇒ (ii), (i) ⇒ (iii), (ii) ⇒ (iv) and (iii) ⇒ (iv). Implication (iv) ⇒ (v) follows from Proposition 8. Finally, (v) ⇒ (i) follows from Theorem 6. □

Acknowledgments

We are grateful to an anonymous referee for a very careful reading of the text which has led to several improvements. In particular, thanks to this we clarified our argument in the proof of Theorem 6, we modified the formulation of Proposition 8, and we added the example after Proposition 8 which the referee had suggested.

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