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The $\mathfrak{sl}_3$ Jones polynomial of the trefoil: A case study of q-holonomic sequences ☆

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ABSTRACT

The $\mathfrak{sl}_3$ colored Jones polynomial of the trefoil knot is a q-holonomic sequence of two variables with natural origin, namely quantum topology. The paper presents an explicit set of generators for the annihilator ideal of this q-holonomic sequence as a case study. On the one hand, our results are new and useful to quantum topology: this is the first example of a rank 2 Lie algebra computation concerning the colored Jones polynomial of a knot. On the other hand, this work illustrates the applicability and computational power of the employed computer algebra methods.

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1. The colored Jones polynomial: a case study of *q*-holonomic sequences

1.1. Introduction

The aim of this paper is to investigate the $\mathfrak{sl}_3$ colored Jones polynomial of the trefoil knot which is a q-holonomic sequence in two variables. Such sequences arise naturally in quantum topology when studying knots. The relevance of our results within this field become evident when taking into account that this is the first example of a rank 2 Lie algebra computation concerning the colored Jones polynomial of a knot, in accordance to the $\mathfrak{sl}_3$ AJ Conjecture of [6].

Using computer algebra, we compute an explicit (conjectured) set of generators for this q-holonomic sequence as a case study. We employ the Mathematica packages developed by M. Kauers (see [16,15]) and the second author (see [19,18,20]). These computations show the power of the underlying computer algebra methods and may serve well the theoretical and practical needs of quantum topology.

Since quantum topology and computer algebra are rather disjoint subjects, we need to briefly review some basic concepts before we present our results.

1.2. q-Holonomic sequences

A *q-holonomic sequence* $(f_n(q))$ for $n \in \mathbb{N}$ is a sequence (typically of rational functions $f_n(q) \in \mathbb{Q}(q)$ in one variable q) which satisfies a *linear recursion relation*:

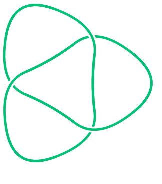
$$a_d\left(q^n, q\right) f_{n+d}(q) + \cdots + a_0\left(q^n, q\right) f_n(q) = 0$$

for all $n \in \mathbb{N}$, where $a_j(u, v) \in \mathbb{Q}[u, v]$ for all $j = 0, \ldots, d$. The algorithmic significance of such sequences was first recognized by Zeilberger in [30], where also a generalization to multivariate sequences $f_{\mathbf{n}}(q)$ for $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ was given. A down-to-earth introduction of (q-)holonomic sequences and closely related notions is given in [17]. It is well known that q-holonomic sequences enjoy several useful properties:

(a) Addition, multiplication, and specialization preserve q-holonomicity.
(b) Finite (multi-dimensional) sums of proper q-hypergeometric terms are q-holonomic in the remaining free variables. This is the *fundamental theorem* of WZ theory; see [29].
(c) The fundamental theorem is constructive, and computer-implemented; see [23,22,18,24].

1.3. The colored Jones polynomial is q-holonomic

Five years ago, a natural source of q-holonomic sequences was discovered: *Knot Theory and Quantum Topology*; see [8]. Let us review some basic concepts of knot theory and quantum topology here. For a detailed discussion, the reader may look into [1,8,12,13,27,28]. A *knot K* in 3-space is a smoothly embedded circle in the 3-sphere S^3 considered up to isotopy. The simplest non-trivial knot is the *trefoil*:



In his seminal paper [13], Jones introduced the *Jones polynomial* of a knot K. The Jones polynomial of a knot K is an element of $\mathbb{Z}[q^{\pm 1}] = \mathbb{Z}[q, q^{-1}]$ and can be generalized to a polynomial invariant $J_{K,V}(q) \in \mathbb{Z}[q^{\pm 1}]$ of a framed knot K colored by a representation V of a simple Lie algebra $\mathfrak{g}$. When the Lie algebra is $\mathfrak{sl}_2$ and $V = \mathbb{C}^2$ is the 2-dimensional irreducible representation, $J_{K,\mathbb{C}^2}(q)$ is the classical Jones polynomial. The definition of $J_{K,V}(q)$ uses the machinery of *quantum groups* and may be found in [12,27,28]. Without going into details, let us point out some features of this theory. Irreducible representations of a simple Lie algebra $\mathfrak{g}$ are parametrized by dominant weights, and once we choose a basis of fundamental weights, the set of dominant weights can be identified with $\mathbb{N}^r$, where r is the rank of the Lie algebra. Thus, the $\mathfrak{g}$ colored Jones polynomial of a knot is a function $\mathbf{n} \in \mathbb{N}^r \mapsto J_{K,\mathbf{n}}(q) \in \mathbb{Z}[q^{\pm 1}]$. In [8] the following result was shown.

Theorem 1.1. *(See [8, Theorem 1].) For every simple Lie algebra $\mathfrak{g}$ of rank r, and every knot K, the function $J_{K,\bullet}(q) : \mathbb{N}^r \to \mathbb{Z}[q^{\pm 1}]$ is q-holonomic.*

It will be useful to recall in brief the proof of the above theorem using principles of q-holonomic sequences and quantum group theory. The quantum group invariant $J_{K,V}(q)$ of a knot K is a local object, obtained by

- choosing a planar projection of the knot,
- assigning a tensor (the so-called R-matrix or its inverse, which depends on the representation V of the simple Lie algebra $\mathfrak{g}$) to each positive or negative crossing,
- summing over all contractions of indices, and adjust for the framing.

It follows that $J_{K,V}(q)$ is a finite multi-dimensional sum where the summand is a product of entries of the R-matrix with certain framing factors. In [8] it was shown that if we choose a basis of the quantum group suitably, the entries of the R-matrix are q-proper hypergeometric multisums. In addition, the framing factor is a monomial in q raised to a quadratic form in the summation variables. It follows from property (b) of Section 1.2 that $J_{K,V}(q)$ is q-holonomic. This summarizes the proof of Theorem 1.1.

The proof of Theorem 1.1 is algorithmic. When the Lie algebra is $\mathfrak{sl}_2$ and K is a knot with a planar projection with d crossings, there is an explicit proper q-hypergeometric term in $d+1$ variables $(k_1, \dots, k_d, n)$ such that summation in all $(k_1, \dots, k_d) \in \mathbb{N}^d$ (this is a finite sum) gives $J_{K,n}(q)$. The algorithm is discussed in detail in [8, Section 3] and has been computer-implemented in the `KnotAtlas` via the command `ColouredJones`; see [1]. The above presentation shows the limitation of the WZ method. Although $J_{K,n}(q)$ is a q-holonomic sequence presented by an explicit q-hypergeometric multisum, when K is a knot with several (e.g. 8) crossings, the computation of a recursion relation for $J_{K,n}(q)$ via WZ theory of [29] becomes impractical.

1.4. The $\mathfrak{sl}_3$-colored Jones polynomial of the trefoil

Concrete formulas for the colored Jones polynomial $J_{K,V}(q)$ are hard to find in the case of higher rank Lie algebras, and for good reasons. For *torus knots* T, Jones and Rosso gave a formula for $J_{T,V}(q)$ which involves a *plethysm map* of V, unknown in general; see [25]. The plethysm map was computed explicitly by [9] for $\mathfrak{sl}_3$ (and in forthcoming work of [10] for all simple Lie algebras). This gives an explicit formula for the trefoil, and more generally for the $T(2,b)$ torus knots, where b is an odd natural number. Let us recall this formula, which is the starting point of our paper, using the notation of [9]. Let

$$f_{b,n_1,n_2}(q) = J_{T(2,b),n_1,n_2}(q) \tag{1}$$

denote the $\mathfrak{sl}_3$ quantum group invariant of the torus knot $T(2,b)$ colored with the irreducible representation V_{n_1,n_2} of $\mathfrak{sl}_3$ of highest weight $\lambda = n_1\omega_1 + n_2\omega_2$ (0-framed and normalized to be 1 at the unknot), where n_1, n_2 are non-negative natural numbers and ω_1, ω_2 are the fundamental weights of $\mathfrak{sl}_3$ [5,11].

Theorem 1.2. *(See [9].) For all odd natural numbers b we have*

$$\begin{aligned} f_{b,n_1,n_2}(q) = \frac{\theta_{n_1,n_2}^{-2b}}{d_{n_1,n_2}} \Bigg(& \sum_{l=0}^{\min\{n_1,n_2\}} \sum_{k=0}^{n_1-l} (-1)^k d_{2n_1-2k-2l,2n_2+k-2l} \theta_{2n_1-2k-2l,2n_2+k-2l}^{\frac{b}{2}} \\ & + \sum_{l=0}^{\min\{n_1,n_2\}} \sum_{k=0}^{n_2-l} (-1)^k d_{2n_1+k-2l,2n_2-2k-2l} \theta_{2n_1+k-2l,2n_2-2k-2l}^{\frac{b}{2}} \\ & - \sum_{l=0}^{\min\{n_1,n_2\}} d_{2n_1-2l,2n_2-2l} \theta_{2n_1-2l,2n_2-2l}^{\frac{b}{2}} \Bigg) \end{aligned}$$

where the quantum integer $[n]$, *the* quantum dimension d_{n_1,n_2} *and the* twist parameter θ_{n_1,n_2} *of* V_{n_1,n_2} *are defined by*

$$[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \tag{2}$$

$$d_{n_1,n_2} = \frac{[n_1+1][n_2+1][n_1+n_2+2]}{[2]}, \tag{3}$$

$$\theta_{n_1,n_2} = q^{\frac{1}{3}(n_1^2+n_1n_2+n_2^2)+n_1+n_2}. \tag{4}$$

1.5. The q-Weyl algebra

Our results below are phrased in terms of the q-Weyl algebra, and its corresponding module theory and geometry. Let us recall the appropriate q-Weyl algebra for bivariate q-holonomic sequences.

Consider the operators M_i, L_i for $i = 1, 2$ which act on a sequence $f_{n_1,n_2}(q) \in \mathbb{Q}(q)$ by

$$\begin{aligned} (L_1 f)_{n_1,n_2}(q) &= f_{n_1+1,n_2}(q), & (M_1 f)_{n_1,n_2}(q) &= q^{n_1} f_{n_1,n_2}(q), \\ (L_2 f)_{n_1,n_2}(q) &= f_{n_1,n_2+1}(q), & (M_2 f)_{n_1,n_2}(q) &= q^{n_2} f_{n_1,n_2}(q). \end{aligned}$$

It is easy to see that the operators q, M_1, L_1, M_2, L_2 commute except in the following cases:

$$L_1 M_1 = q M_1 L_1, \qquad L_2 M_2 = q M_2 L_2.$$

It follows that if $R(q, M_1, M_2) \in \mathbb{Q}(q, M_1, M_2)$ is a rational function, then we have the commutation relations

$$I = \big\{ L_1 R(q, M_1, M_2) - R(q, qM_1, M_2) L_1, L_2 R(q, M_1, M_2) - R(q, M_1, qM_2) L_2 \big\}.$$

Consider the (localized) q-*Weyl algebra*

$$W_{q,\mathrm{loc}} = \mathbb{Q}(q, M_1, M_2)\langle L_1, L_2 \rangle / (I) \tag{5}$$

generated by the operators L_1, L_2 over the field $\mathbb{Q}(q, M_1, M_2)$, modulo the 2-sided ideal I above. $W_{q,\mathrm{loc}}$ is an example of an *Ore extension* of the field $\mathbb{Q}(q, M_1, M_2)$; see [21]. Given a bivariate sequence $f_{n_1,n_2}(q) \in \mathbb{Q}(q)$, consider the annihilating ideal

$$\mathrm{Ann}_f = \{ P \in W_q \mid Pf = 0 \}.$$

Ann_f is always a left ideal in W_q, which gives rise to the cyclic $W_{q,\mathrm{loc}}$-module $M_f = W_{q,\mathrm{loc}}/\mathrm{Ann}_f$. Finitely generated $W_{q,\mathrm{loc}}$-modules M have a well-defined theory of *dimension* (and *Hilbert polynomial*), and satisfy the key *Bernstein inequality*; see [26, Theorem 2.1.1]. Thus, one can define q-holonomic modules. When $M = W_{q,\mathrm{loc}}/J$ for a left ideal J, the Hilbert polynomial of the left ideal J can be computed by *Gröbner bases* [2] and their *Gröbner fans* which are well defined for the case of the Ore algebra $W_{q,\mathrm{loc}}$ under consideration; see [26,17]. For a computer implementation of Gröbner bases in the Ore algebra $W_{q,\mathrm{loc}}$, see [3,17].

1.6. Our results

One may try to compute generators for the ideal of recursion relations using techniques which are well known in computer algebra, but which are recalled in Section 2 for sake of self-containedness. Given elements P_i for $i \in S$ of $W_{q,\mathrm{loc}}$, let $\langle P_i \mid i \in S\rangle$ denote the left ideal of $W_{q,\mathrm{loc}}$ that they generate. Furthermore, let τ denote the symmetry map that exchanges n_1 and n_2, i.e., for $P \in W_{q,\mathrm{loc}}$ we have $\tau(P(M_1, M_2, L_1, L_2)) = P(M_2, M_1, L_2, L_1)$.

Theorem 1.3.

(a) *Let P_1, P_2, P_3 be the operators of Appendix* A *and*

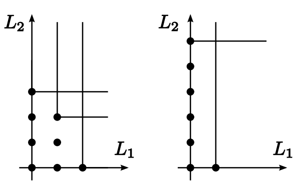
$$J = \langle P_1, P_2, P_3\rangle. \tag{6}$$

Then $\{P_1, P_2, P_3\}$ is a Gröbner basis for J with respect to total degree lexicographic order $(L_1 > L_2)$, *and J is a zero-dimensional ideal of rank* 5.

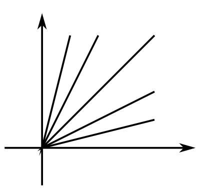
(b) *We have*

$$J = \langle Q_1, Q_2\rangle$$

where $\{Q_1, Q_2\}$ is a Gröbner basis with respect to the lexicographic order $(L_1 > L_2)$. *The five monomials below the staircases of $\langle P_1, P_2, P_3\rangle$ and $\langle Q_1, Q_2\rangle$ are given by*



(c) *The Gröbner fan* $\mathrm{GF}(J)$ *of J is a fan in $\mathbb{R}^2_+$ with rays generated by the vectors* $(4,1)$, $(2,1)$, $(1,1)$, $(1,2)$, $(1,4)$ *on* (L_1, L_2) *coordinates.*



(d) *The left ideal J is invariant under the symmetry map τ. The operator P_1 is itself symmetric* (*modulo sign change*).

Proof. The computer proof of statements (a)–(d) is given in the electronic supplementary material [20]. □

Conjecture 1.1. *Let J be the left ideal defined in* (6)*. Then*

$$\mathrm{Ann}_{f_{3,n_1,n_2}} = J.$$

Remark 1.2. The colored Jones polynomial satisfies the symmetry $J_{K,V^*}(q) = J_{K,V}(q)$ where V^* is the dual representation of the simple Lie algebra. Since $V^*_{n_1,n_2} = V_{n_2,n_1}$ for the representations of $\mathfrak{sl}_3$, it follows that $f_{b,n_1,n_2}(q) = f_{b,n_2,n_1}(q)$. Thus, the fact that J is invariant under τ gives additional evidence for the above conjecture.

1.7. The $\mathfrak{sl}_3$ AJ Conjecture for the trefoil

One of the best-known geometric invariants of holonomic D-modules are their characteristic varieties [14]. In the case of holonomic D-modules over the complex torus $(\mathbb{C}^*)^r$, the characteristic variety is a Lagrangian subvariety of the cotangent bundle $T^*((\mathbb{C}^*)^r)$, namely a conormal bundle. q-Holonomic D-modules have a characteristic variety, too. In the case of the q-holonomic D-modules that come from knot theory, namely the $\mathfrak{g}$ colored Jones polynomial of a knot K, their characteristic variety is a subvariety of $\mathbb{C}^{2r}$, where r is the rank of the simple Lie algebra $\mathfrak{g}$. In [6] the first author formulated the AJ Conjecture which identifies the characteristic variety of the colored Jones polynomial of a knot with its deformation variety. The latter is the variety of G-representations of the boundary torus of the knot complement that extend to G-representations of the knot complement, where G is the simple, simply connected Lie group with Lie algebra $\mathfrak{g}$.

In down-to-earth terms, the $\mathfrak{sl}_3$ AJ Conjecture involves the image of the $\mathfrak{sl}_3$ recursion ideal of a knot under the partially defined map:

$$\epsilon : W_{q,\mathrm{loc}} \to \mathbb{Q}(M_1, M_2)[L_1, L_2], \qquad e(P) = P|_{q=1}. \tag{7}$$

For the ideal J of Proposition 1.3, we have

$$\epsilon(J) = \langle p_1, p_2, p_3 \rangle \tag{8}$$

where p_1, p_2, p_3 are given in Appendix B. In general, it is hard to compute the variety of $\mathrm{SL}_3(\mathbb{C})$ representations of a knot complement and to compare it with the above image. For the trefoil knot, however such a computation is possible (see [7]). It turns out that $\epsilon(J)$ coincides with the $\mathrm{SL}_3(\mathbb{C})$-deformation variety of the trefoil, thus confirming the AJ Conjecture.

2. The computation

2.1. Guessing generators for the annihilator ideal

The method of guessing is nothing else but an ansatz with undetermined coefficients. Assume that a multivariate sequence $(f_{\mathbf{n}}(q))$, $\mathbf{n} \in \mathbb{N}^r$, of rational functions in q, i.e., $f_{\mathbf{n}}(q) \in \mathbb{Q}(q)$, is given by some expression that allows to compute the values of $(f_{\mathbf{n}}(q))$ for small $\mathbf{n}$, e.g., for all $\mathbf{n}$ inside the hypercube $[0, n_0]^r$. For finding linear recurrences with polynomial coefficients (they are guaranteed to exist if $(f_{\mathbf{n}}(q))$ is known to be q-holonomic), an ansatz of the following form is used

$$\Bigg(\sum_{(\mathbf{a},\mathbf{b}) \in S \subseteq \mathbb{N}^{2r}} c_{\mathbf{a},\mathbf{b}} \mathbf{M}^{\mathbf{a}} \mathbf{L}^{\mathbf{b}}\Bigg) f_{\mathbf{n}}(q) = 0 \tag{9}$$

where the *structure set* S is finite and the unknowns $c_{\mathbf{a},\mathbf{b}}$ are to be determined in $\mathbb{Q}(q)$. The multi-index notation $\mathbf{M}^{\mathbf{a}} = M_1^{a_1} \cdots M_r^{a_r}$ is employed here. Linear equations for the $c_{\mathbf{a},\mathbf{b}}$ are obtained by substituting concrete integer tuples for $\mathbf{n}$ into (9) and by plugging in the known values of $(f_{\mathbf{n}}(q))$. If sufficiently many equations are generated this way, i.e., if the resulting system of equations is overdetermined, then it is to be expected with high probability that the solutions of this system are indeed valid recurrences for the sequence in question.

In the case of a univariate sequence $(f_n(q))$, usually the recursion relation with minimal order is desired. The analogue in the multivariate case is a set of recurrence operators that generate a left ideal in $W_{q,\mathrm{loc}}$ of minimal rank. For their many nice properties it is natural to aim at a *Gröbner basis* [2] of this annihilating ideal. There are two strategies how to obtain such a basis of recurrences. The first strategy consists in finding a bunch of recurrences by guessing and then apply Buchberger's algorithm to obtain a Gröbner basis of the ideal they generate. The more recurrences are used as input, the higher is the probability of ending up with the full annihilating ideal (the one with minimal rank). Since this latter computation can be quite tedious, we follow a different strategy that doesn't require Buchberger's algorithm: the structure set S in the ansatz (9) is adjusted by trial and error until the linear system for the $c_{\mathbf{a},\mathbf{b}}$ admits a single solution, which (supposedly) is an element of the desired Gröbner basis. The latter is achieved by searching systematically (in the same fashion as in the FGLM algorithm) through the monomials $\mathbf{L}^{\mathbf{b}}$ that appear in the ansatz. This second strategy will finally deliver a set of operators that are very likely to be elements of the annihilator of $(f_{\mathbf{n}}(q))$ (as it is the case for the first strategy), and that are very likely to form a Gröbner basis (which is not the case for the first strategy).

The above described method involves a lot of trials until the structure set for each element of the Gröbner basis has been figured out exactly. In order to make this reasonably fast, modular techniques are used. First, the input data $(f_{\mathbf{n}}(q))$ for $\mathbf{n} \in [0, n_0]^r$ is computed only for a specific choice of q, say $q = 64$, and modulo some prime number, say $p = 2\,147\,483\,647$. While in other examples of this type it doesn't really matter which value is substituted for q (there may be a small finite set of non-eligible integers only), the situation here is slightly different and the choice $q = 64$ not at random: note the rational numbers $\frac{1}{2}$ and $\frac{1}{3}$ that appear in the exponents of Theorem 1.2. Although in the end all fractional powers cancel, yielding a Laurent polynomial, the do appear in intermediate expressions. Thus setting $q = m^6$ for some $m \in \mathbb{N}$ simplifies our job significantly. All following computations (constructing and solving the linear system) are now done with the homomorphic images in $\mathbb{Z}_p$ instead of $\mathbb{Q}(q)$. The computational complexity is reduced drastically since no expression swell can happen. Once the exact shape of (9) is found this way, the final computation over $\mathbb{Q}(q)$ can be started with a refined ansatz by omitting all unknowns $c_{\mathbf{a},\mathbf{b}}$ that become zero in the modular computation. This methodology of guessing as described above is implemented in the Mathematica package `Guess.m` (see [16,15]).

In our case study of the $\mathfrak{sl}_3$ colored Jones polynomial of the trefoil knot we have $r = 2$ and Theorem 1.2 allows to compute $f_{3,n_1,n_2}(q)$ for sufficiently many $(n_1, n_2) \in \mathbb{N}^2$. Trial and error gave evidence that the (L_1, L_2)-supports of the Gröbner basis elements are

$$\begin{aligned}
\mathrm{supp}_{\mathbf{L}}(P_1) &= \{L_1^2, L_1L_2, L_2^2, L_1, L_2, 1\},\\
\mathrm{supp}_{\mathbf{L}}(P_2) &= \{L_2^3, L_1L_2, L_2^2, L_1, L_2, 1\},\\
\mathrm{supp}_{\mathbf{L}}(P_3) &= \{L_1L_2^2, L_1L_2, L_2^2, L_1, L_2, 1\}.
\end{aligned}$$

As stated in Proposition 1.3, the ideal generated by these three elements has rank 5 and the monomials that cannot be reduced by P_1, P_2, P_3 are the following

$$U = \{L_1L_2, L_2^2, L_1, L_2, 1\}.$$

We tried to guess an operator with support U and total degree 60 with respect to M_1 and M_2 in its coefficients, but did not succeed. This indicates that we found the ideal of minimal rank, taking into account that P_1, P_2, P_3 have (M_1, M_2)-total degrees 23, 28, and 27, respectively. The modular computations involved solving linear systems with several thousand unknowns, which is not a big deal with today's hardware. The details of this section are given in the Mathematica notebook `GuessIdeal.CJ.2.3.nb` of [20].

2.2. Gröbner bases in the q-Weyl algebra

Once a Gröbner basis with respect to some specified term order is found, the FGLM algorithm (see [4]) can be employed to obtain a Gröbner basis of the same ideal with respect to any other term order. Using the FGLM implementation for noncommutative operator algebras in the Mathematica package `HolonomicFunctions.m` (see [19,18]), the second basis given in Proposition 1.3 was computed. Fixing a certain term order destroys the symmetry with respect to n_1 and n_2 that is inherent in the problem of the $\mathfrak{sl}_3$-colored Jones polynomials, because either $L_1 > L_2$ or vice versa. Therefore the generators P_1, P_2, P_3 are not expected to be symmetric themselves (P_1 however is). The symmetry of the problem is revealed when the Gröbner fan is computed. The resulting universal Gröbner basis is symmetric with respect to n_1 and n_2. Moreover, specific symmetric recurrences in the ideal can be found by calling the command `FindRelation` [19,18]. For example, by specifying the support $\{1, L_1L_2, L_1^2L_2^2, L_1^3L_2^3, L_1^4L_2^4\}$ we found an operator with this support and coefficients that are symmetric with respect to M_1 and M_2. This operator gives rise to a recurrence for the diagonal sequence $f_{3,n,n}(q)$. The details of this section are given in the Mathematica notebook `GroebnerBasis.CJ.2.3.nb` of [20].

2.3. Changing the normalization of the colored Jones polynomial

In quantum topology, one often uses other normalizations of the colored Jones function of a knot. Theorem 1.2 uses the normalization where the 0-framed unknot has colored Jones polynomial 1. If we use the TQFT (i.e., topological quantum field theory) normalization, where the colored Jones of the unknot has value d_{n_1,n_2} and an arbitrary fixed framing $c \in \mathbb{Z}$, the corresponding colored Jones function is given by

$$F_{b,c,n_1,n_2}(q) = d_{n_1,n_2}\theta^c_{n_1,n_2} f_{b,n_1,n_2}(q)$$

where d_{n_1,n_2} and θ_{n_1,n_2} are given by Eqs. (3) and (4) respectively. The result is still a bivariate sequence of Laurent polynomials in q. Since d_{n_1,n_2} and θ_{n_1,n_2} are proper q-hypergeometric (in the language of [29]), it follows that the generators for the annihilator ideal of $f_{b,n_1,n_2}(q)$ can easily be translated to generators of the annihilator ideal of $F_{b,n_1,n_2}(q)$ and vice versa. This is explained in detail in the Mathematica notebook `GroebnerBasis.CJ.2.3.nb` of [20].

Acknowledgments

The paper came into maturity following requests for explicit formulas for the recursion of the colored Jones polynomial of a knot, during visits of the first author in the Max-Planck-Institut für Mathematik in 2009–2010, and during an Oberwolfach workshop 1033/2010 in August 2010. The authors met during the conference in honor of Doron Zeilberger's 60th birthday, held at Rutgers University in May 2010, and wish to thank the organizers for their hospitality. The authors especially wish to thank Don Zagier and Doron Zeilberger for their interest, their encouragement and for stimulating conversations.

Appendix A. Generators for the recursion ideal of $f_{3,n_1,n_2}(q)$

$$\begin{aligned}
P_1 = &-q^6M_1^3M_2\big(q^3M_1-1\big)(qM_2-1)\big(q^4M_1M_2-1\big)\big(q^5M_1M_2^2-1\big)F_1L_1^2\\
&-q^4M_1M_2\big(q^2M_1-1\big)\big(q^2M_2-1\big)(M_1-M_2)\big(q^4M_1M_2-1\big)F_6L_1L_2\\
&+q^6M_1M_2^3(qM_1-1)\big(q^3M_2-1\big)\big(q^4M_1M_2-1\big)\big(q^5M_1^2M_2-1\big)\tau(F_1)L_2^2
\end{aligned}$$

$$
\begin{aligned}
&+ M_2\big(q^2M_1-1\big)(qM_2-1)\big(q^3M_1M_2-1\big)\big(q^5M_1^2M_2-1\big)F_7L_1\\
&+ (-M_1)(qM_1-1)\big(q^2M_2-1\big)\big(q^3M_1M_2-1\big)\big(q^5M_1M_2^2-1\big)\tau(F_7)L_2\\
&+ q^3(qM_1-1)(qM_2-1)(M_1-M_2)\big(q^2M_1M_2-1\big)F_2,\\
P_2 = &-q^{21}M_1^3M_2^7\big(q^4M_2-1\big)\big(q^5M_1M_2-1\big)\big(q^5M_1M_2^2-1\big)F_1L_2^3\\
&+ q^7M_1^2M_2^3\big(q^2M_1-1\big)\big(q^2M_2-1\big)\big(q^4M_1M_2-1\big)\big(q^6M_1M_2^2-1\big)F_8L_1L_2\\
&+ q^6M_2^3\big(q^3M_2-1\big)\big(q^4M_1M_2-1\big)F_{16}L_2^2\\
&+ q^8M_1^2M_2^2\big(q^2M_1-1\big)(qM_2-1)\big(q^3M_1M_2-1\big)\big(q^6M_1M_2^2-1\big)F_3L_1\\
&+ \big(q^2M_2-1\big)\big(q^3M_1M_2-1\big)\big(q^5M_1M_2^2-1\big)F_{14}L_2 - q^2(qM_2-1)\big(q^2M_1M_2-1\big)F_9,\\
P_3 = &-q^{17}M_1^3M_2^5\big(q^2M_1-1\big)\big(q^3M_2-1\big)\big(q^5M_1M_2-1\big)\big(q^5M_1M_2^2-1\big)F_1L_1L_2^2\\
&+ q^5M_1M_2^2\big(q^2M_1-1\big)\big(q^2M_2-1\big)\big(q^4M_1M_2-1\big)\big(q^6M_1M_2^2-1\big)F_{12}L_1L_2\\
&+ q^6M_2^3(qM_1-1)\big(q^3M_2-1\big)\big(q^4M_1M_2-1\big)F_{13}L_2^2\\
&+ q^7M_1M_2^2\big(q^2M_1-1\big)(qM_2-1)\big(q^3M_1M_2-1\big)F_{11}L_1\\
&- (qM_1-1)\big(q^2M_2-1\big)\big(q^3M_1M_2-1\big)\big(q^5M_1M_2^2-1\big)F_{15}L_2\\
&+ q^2(qM_1-1)(qM_2-1)\big(q^2M_1M_2-1\big)\big(q^6M_1M_2^2-1\big)F_4,\\
Q_1 = &\, q^{47}M_1^5M_2^{10}\big(q^6M_2-1\big)\big(q^7M_1M_2-1\big)\big(q^6M_1M_2^2-1\big)\big(q^7M_1M_2^2-1\big)F_5L_2^5\\
&- q^{24}M_1^2M_2^6\big(q^5M_2-1\big)\big(q^6M_1M_2-1\big)\big(q^6M_1M_2^2-1\big)F_{20}L_2^4\\
&+ q^9M_2^3\big(q^4M_2-1\big)\big(q^5M_1M_2-1\big)\big(q^9M_1M_2^2-1\big)F_{23}L_2^3\\
&+ \big(q^3M_2-1\big)\big(q^4M_1M_2-1\big)\big(q^7M_1M_2^2-1\big)F_{24}L_2^2\\
&+ q^2\big(q^2M_2-1\big)\big(q^3M_1M_2-1\big)\big(q^{10}M_1M_2^2-1\big)F_{18}L_2\\
&- q^6(qM_2-1)\big(q^2M_1M_2-1\big)\big(q^9M_1M_2^2-1\big)\big(q^{10}M_1M_2^2-1\big)F_{10},\\
Q_2 = &\, q^8M_1^3M_2^2(q-1)\big(q^2M_1-1\big)\big(q^2M_1^2+qM_1+1\big)(qM_2-1)\big(q^3M_1M_2-1\big)\\
&\times \big(q^6M_1M_2^2-1\big)\big(q^7M_1M_2^2-1\big)\big(q^8M_1M_2^2-1\big)F_5L_1\\
&- q^{37}M_1^5M_2^{10}\big(q^5M_2-1\big)\big(q^6M_1M_2-1\big)\big(q^6M_1M_2^2-1\big)F_8L_2^4\\
&+ q^{18}M_1^2M_2^6\big(q^4M_2-1\big)\big(q^5M_1M_2-1\big)F_{19}L_2^3\\
&- q^6M_2^3\big(q^3M_2-1\big)\big(q^4M_1M_2-1\big)\big(q^7M_1M_2^2-1\big)F_{21}L_2^2 - \big(q^2M_2-1\big)\big(q^3M_1M_2-1\big)F_{22}L_2\\
&+ q^2(qM_2-1)\big(q^2M_1M_2-1\big)\big(q^8M_1M_2^2-1\big)F_{17}
\end{aligned}
$$

where $F_1,\dots,F_{24}$ are irreducible polynomials in $\mathbb{Q}[q,M_1,M_2]$, too large to print all of them here:

$$
\begin{aligned}
F_1 = &\, q^{18}M_1^6M_2^6 - q^{15}M_1^6M_2^4 - q^{15}M_1^4M_2^6 + q^{14}M_1^5M_2^4 - q^{14}M_1^4M_2^5 + q^{13}M_1^5M_2^4 - q^{12}M_1^5M_2^4\\
&- q^{11}M_1^3M_2^4 + q^{11}M_1^2M_2^5 - q^{10}M_1^5M_2^2 + q^{10}M_1^4M_2^3 + q^9M_1^4M_2^2 - q^9M_1^3M_2^3 + q^8M_1^4M_2^2\\
&+ 2q^8M_1^3M_2^3 + q^8M_1^2M_2^4 - q^7M_1^4M_2^2 - q^7M_1^3M_2^3 - q^6M_1^2M_2^2 - q^4M_1^2M_2 - q^4M_2^3\\
&+ q^3M_1^3 + q^3M_1M_2^2 + qM_2 - M_1,
\end{aligned}
$$

$$\begin{aligned}
F_2 &= q^{19}M_1^6M_2^6 - q^{16}M_1^6M_2^4 - q^{16}M_1^4M_2^6 - q^{13}M_1^5M_2^4 - q^{13}M_1^4M_2^5 + q^{11}M_1^4M_2^3 + q^{11}M_1^3M_2^4 \\
&\quad + q^{10}M_1^5M_2^2 + q^{10}M_1^2M_2^5 + q^9M_1^4M_2^2 + q^9M_1^3M_2^3 + q^9M_1^2M_2^4 - 2q^8M_1^3M_2^3 + q^7M_1^3M_2^3 \\
&\quad - q^5M_1^2M_2^2 - q^4M_1^2M_2 - q^4M_1M_2^2 - q^3M_1^3 - q^3M_1^2M_2 - q^3M_1M_2^2 - q^3M_2^3 \\
&\quad + q^2M_1^2M_2 + q^2M_1M_2^2 + M_1 + M_2, \\
&\ \ \vdots \\
F_6 &= q^{22}M_1^7M_2^7 - q^{20}M_1^7M_2^7 - q^{18}M_1^7M_2^6 - q^{18}M_1^6M_2^7 + q^{15}M_1^7M_2^4 + q^{15}M_1^6M_2^5 + q^{15}M_1^5M_2^6 \\
&\quad - 2q^{15}M_1^5M_2^5 + q^{15}M_1^4M_2^7 + q^{14}M_1^6M_2^4 + q^{14}M_1^4M_2^6 + q^{13}M_1^5M_2^5 + q^{12}M_1^5M_2^5 - q^{12}M_1^4M_2^4 \\
&\quad - q^{11}M_1^6M_2^2 - q^{11}M_1^2M_2^6 - q^{10}M_1^4M_2^4 + q^9M_1^4M_2^3 + q^9M_1^3M_2^4 - q^8M_1^5M_2^2 - 3q^8M_1^4M_2^3 \\
&\quad - 3q^8M_1^3M_2^4 - q^8M_1^2M_2^5 + q^7M_1^4M_2^3 + q^7M_1^3M_2^4 + q^6M_1^3M_2^2 + q^6M_1^2M_2^3 + q^4M_1^4 + q^4M_1^3M_2 \\
&\quad + 2q^4M_1^2M_2^2 + q^4M_1M_2^3 + q^4M_2^4 - q^3M_1^3M_2 - q^3M_1M_2^3 - qM_1^2 - qM_1M_2 - qM_2^2 + M_1M_2, \\
F_7 &= q^{24}M_1^8M_2^7 - q^{21}M_1^8M_2^5 - q^{21}M_1^6M_2^7 + q^{19}M_1^6M_2^6 - q^{18}M_1^7M_2^5 - q^{18}M_1^6M_2^6 + q^{16}M_1^5M_2^5 \\
&\quad - q^{16}M_1^4M_2^6 + q^{15}M_1^7M_2^3 + q^{15}M_1^4M_2^6 + q^{14}M_1^6M_2^3 + q^{14}M_1^5M_2^4 + 2q^{14}M_1^4M_2^5 - 3q^{13}M_1^5M_2^4 \\
&\quad + q^{12}M_1^5M_2^4 - q^{12}M_1^4M_2^3 + q^{11}M_1^3M_2^4 - q^{11}M_1^2M_2^5 + q^{10}M_1^5M_2^2 - q^{10}M_1^4M_2^3 - 2q^{10}M_1^4M_2^2 \\
&\quad + q^9M_1^3M_2^3 + q^9M_1^2M_2^4 - q^8M_1^5M_2 + q^8M_1^4M_2^2 - 3q^8M_1^3M_2^3 - q^8M_1^2M_2^4 + q^7M_1^4M_2^2 + q^7M_1^4 \\
&\quad + q^7M_1^3M_2^3 - q^5M_1^4 + q^5M_1^3M_2 + q^4M_1^2M_2 + q^4M_2^3 - q^3M_1^3 - q^3M_1M_2^2 - qM_2 + M_1 \\
&\ \ \vdots
\end{aligned}$$

The complete list is available at [20].

Appendix B. The $q = 1$ limit for the recursion ideal of $f_{3,n_1,n_2}(q)$

Let $p_i = \epsilon(P_i)$ denote the image of P_i when $q = 1$. Then, up to polynomial factors of M_1, M_2, we have

$$\begin{aligned}
p_1 &= M_1^2 - L_2M_1^2 - L_1M_1M_2 + L_2M_1M_2 - L_1^2M_1^4M_2 + L_1L_2M_1^4M_2 - M_2^2 + L_1M_2^2 - M_1^3M_2^2 \\
&\quad + L_1^2M_1^3M_2^2 + L_2M_1^3M_2^2 - L_1L_2M_1^3M_2^2 + M_1^2M_2^3 - L_1M_1^2M_2^3 + L_1L_2M_1^2M_2^3 - L_2^2M_1^2M_2^3 \\
&\quad + L_1M_1^5M_2^3 - L_1L_2M_1^5M_2^3 - L_1L_2M_1M_2^4 + L_2^2M_1M_2^4 + L_1M_1^4M_2^4 - L_1^2M_1^4M_2^4 - L_2M_1^4M_2^4 \\
&\quad + L_2^2M_1^4M_2^4 - L_2M_1^3M_2^5 + L_1L_2M_1^3M_2^5 - L_1M_1^6M_2^5 + L_1^2M_1^6M_2^5 + L_2M_1^5M_2^6 - L_2^2M_1^5M_2^6, \\
p_2 &= (-1 + L_2)\big(M_1^2 - M_1M_2 + M_2^2 - L_1M_1^2M_2^3 + L_2M_1^2M_2^3 + L_1M_1^5M_2^3 - L_2M_1M_2^4 - L_2M_1^4M_2^4 \\
&\quad + L_2M_2^5 + L_2M_1^3M_2^5 - L_2M_1^2M_2^6 - L_2^2M_1^4M_2^7 + L_2^2M_1^3M_2^8 - L_2^2M_1^5M_2^9\big), \\
p_3 &= (-1 + L_2)\big(-1 + M_1^2M_2 - M_1M_2^2 + L_1M_1M_2^2 - L_2M_2^3 - L_1M_1^3M_2^3 + L_2M_1^2M_2^4 \\
&\quad + L_1M_1^5M_2^4 - L_2M_1^4M_2^5 + L_1L_2M_1^4M_2^5 - L_1L_2M_1^3M_2^6 + L_1L_2M_1^5M_2^7\big), \\
q_1 &= (-1 + L_2)^2\big(1 + L_2M_2^3\big)\big(1 + L_2M_1^3M_2^3\big)\big(-1 + L_2M_1^2M_2^4\big), \\
q_2 &= -(-1 + L_2)\big(1 + L_2M_2^3\big)\big(1 + L_2M_1^3M_2^3\big)\big(-1 + L_2M_1^2M_2^4\big).
\end{aligned}$$

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