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Nowhere-zero 3-flows and Z_3-connectivity in bipartite graphs

Liangchen Li, Xiangwen Li *

Department of Mathematics, Huazhong Normal University, Wuhan 430079, China

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ABSTRACT

Tutte conjectured that every 4-edge-connected graph admits a nowhere-zero 3-flow. Let $\mathcal{F}_{12}$ be a family of graphs such that $G \in \mathcal{F}_{12}$ if and only if G is a simple bipartite graph on 12 vertices and $\delta(G) = 4$. Let G be a simple bipartite graph on n vertices. It is proved in this paper that if $\delta(G) \geq \lceil \frac{n}{4} \rceil + 1$, then G admits a nowhere-zero 3-flow with only one exceptional graph. Moreover, if $G \notin \mathcal{F}_{12}$ with the minimum degree at least $\lceil \frac{n}{4} \rceil + 1$ is Z_3-connected. The bound is best possible in the sense that the lower bound for the minimum degree cannot be decreased.

1. Introduction

Graphs in this paper are finite, loopless, and may have multiple edges. Terminology and notation not defined here are from [2].

Let G be a graph and let D be an orientation of an undirected graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let $\text{tail}(e) = u$ and $\text{head}(e) = v$. For every vertex $v \in V(G)$, $E^+(v)$ is the set of all edges with tails at v and $E^-(v)$ is the set of all edges with heads at v. For two subsets $A, B \subseteq V(G)$ and $A \cap B = \emptyset$, let $e_G(A, B)$ (or simply $e(A, B)$) denote the number of edges with one endpoint in A and the other endpoint in B. For simplicity, if H_1 and H_2 are two disjoint subgraphs of G, we write $e(H_1, H_2)$ instead of $e(V(H_1), V(H_2))$. Throughout this paper, we use δ to denote the minimum degree of G rather than $\delta(G)$.

The theory of k-flows was introduced by Tutte as a generalization of face k-coloring of planar graphs. A graph admits a *nowhere-zero k-flow* if its edges can be oriented and assigned numbers $\pm(k-1), \pm(k-2), \ldots, \pm 1$ so that for every vertex, the sum of the values on incoming edges equals the sum of the outgoing edges. It is well-known that graphs with bridges have no nowhere-zero k-flow for $k \geq 2$ and that if a graph admits a nowhere-zero k-flow, then it admits a nowhere-zero $(k+1)$-flow.

The group connectivity was introduced by Jaeger et al. [6] as a generalization of nowhere-zero flows. Let A denote an (additive) abelian group with identity 0, and let A^* denote the set of nonzero elements in A. Define $F(G, A) = \{f | f : E(G) \to A\}$ and $F^*(G, A) = \{f | f : E(G) \to A^*\}$. For each $f \in F(G, A)$, the *boundary* of f is a function $\partial f : V(G) \to A$ given by

$$\partial f(v) = \sum_{e \in E^+(v)} f(v) - \sum_{e \in E^-(v)} f(v),$$

where "$\sum$" refers to the addition in A.

A function $b : V(G) \to A$ is called an *A-valued zero-sum function* on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A-valued zero-sum functions on G is denoted by $\mathbb{Z}(G, A)$. For a given $b \in \mathbb{Z}(G, A)$, if G has an orientation D and a function $f \in F^*(G, A)$ such that $\partial f = b$, then f is an (A, b)-nowhere-zero flow. A *nowhere-zero A-flow* is an $(A, 0)$-nowhere-zero flow. More specifically,

* Corresponding author.
E-mail addresses: xwli68@yahoo.cn, xwli2808@yahoo.com (X. Li).

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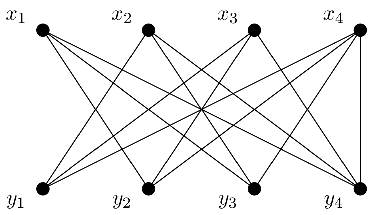

Fig. 1. The graph G_1.

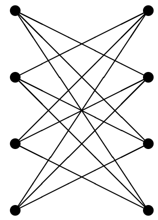

Fig. 2. The graph G_2.

a nowhere-zero k-flow is a nowhere-zero Z_k-flow, where Z_k is the cyclic group of order k. Tutte [14] proved that G admits a nowhere-zero A-flow with $|A| = k$ if and only if G admits a nowhere-zero k-flow.

A graph G is *A-connected* if G has an orientation D such that for any $b \in \mathbb{Z}(G, A)$, there is a function $f \in F^*(G, A)$ such that $\partial f = b$. For an abelian group A, let $\langle A \rangle$ denote the family of graphs that are A-connected. It is observed in [6] that $G \in \langle A \rangle$ is independent of the orientation of G. This paper is mainly motivated by the following two conjectures.

Conjecture 1.1 (*Tutte [13]*). *Every* 4*-edge-connected graph admits a nowhere-zero* 3*-flow.*

Conjecture 1.2 (*Jaeger et al. [6]*). *Every* 5*-edge-connected graph is* Z_3*-connected.*

Conjecture 1.2 implies Conjecture 1.1 by a result of Kochol [7] that reduces Conjecture 1.1 to a consideration of 5-edge-connected graphs. So far, both conjecture are still open. Recently, degree conditions have been used to guarantee the existence of nowhere-zero 3-flows and Z_3-connectivity. For the literature, one can find the results in [4,5,11,15], a survey [9] and others. On the other hand, the concept of all generalized Tutte-orientations was introduced by Barát and Thomassen in [1]. Lai et al. in [10, Theorem 2.1] proved that a graph G admits all generalized Tutte-orientations if and only if G is Z_3-connected. Thus, the theorem in [1, Theorem 5.3] can be stated as follows: there exists a positive integer N such that every 2-edge-connected simple graph on $n \geq N$ vertices with the minimum degree at least $\frac{n}{4}$ is Z_3-connected. The result is, unfortunately, incorrect. An explicit counterexample was given in [9] as follows. Let n be an integer with $n \equiv 0 \pmod 3$. Denote $G(n)$ the graph obtained from K_3 by replacing each vertex of K_3 with a complete graph $K_{\frac{n}{3}}$. Then $G(n)$ is a 2-edge-connected simple graph with $\delta(G(n)) = \frac{n}{3} - 1 > \frac{n}{4}$ when $n \geq 15$. However, as $G(n)$ can be reduced to K_3 by contracting Z_3-connected subgraphs, $G(n)$ is not Z_3-connected by Lemma 2.3(4). Note that each of these counterexamples contains 3-cycles. Naturally, we consider the problem whether the above-mentioned Bará and Thomassen theorem would be valid when G has no 3-cycle. In particular, for bipartite graphs, what is the lower bound of the minimum degree for Barát and Thomassen's result? Thus, we investigate bipartite graphs and prove the following two theorems.

Theorem 1.3. *Let G be a simple bipartite graph on n vertices. If $\delta \geq \lceil \frac{n}{4} \rceil + 1$, then G admits a nowhere-zero* 3*-flow if and only if G is not isomorphic to G_1 shown in Fig. 1.*

Let $\mathcal{F}_{12}$ be a family of graphs such that $G \in \mathcal{F}_{12}$ if and only if G is a simple bipartite graph on 12 vertices and $\delta(G) = 4$.

Theorem 1.4. *Let G be a simple bipartite graph with bipartition (X, Y) with $G \notin \mathcal{F}_{12}$. If $\delta \geq \lceil \frac{n}{4} \rceil + 1$, then G is Z_3-connected if and only if $G \notin \{K_{2,2}, K_{3,3}, K_{3,4}, K_{3,5}, G_1, G_2\}$, where G_1 and G_2 are shown in Figs.* 1 and 2*, respectively.*

The bound is best possible in the sense that the lower bound for the minimum degree cannot be decreased. Let $n = 4l$ and $\delta = l \geq 2$. Let $G_1(n)$ denote the graph obtained by adding one edge between two copies of $K_{l,l}$. Since $G_1(n)$ has a cut edge, it does not admit a nowhere-zero 3-flow and so $G_1(n)$ is not Z_3-connected. On the other hand, so far, we have not determined whether G is Z_3-connected when $G \in \mathcal{F}_{12}$.

It is known that if G is Z_3-connected, then G admits a nowhere-zero 3-flow. Thus, in order to prove Theorem 1.3, we first prove Theorem 1.4. A subgraph K_4^-, which is obtained from K_4 by deleting one edge, has played a key role in investigation on nowhere-zero 3-flows and group connectivity in [4,5,11,15]. For bipartite graphs, it is easy to see that K_4^- does not work. In order to prove Theorem 1.4, some new techniques need to be developed.

We organize this paper as follows. We investigate Z_3-connectivity in bipartite graphs in Sections 2–4. In Section 5, we prove our main theorems.

2. Lemmas

The following two observations about the properties of bipartite graphs are straightforward.

Observation 2.1. *Let G be a simple bipartite graph with bipartition (X, Y). If $\delta \geq \lceil \frac{n}{4} \rceil + 1$, then G is 2-edge-connected.*

Observation 2.2. *Let $G = (X, Y; E)$ be a simple bipartite graph with bipartition (X, Y). If $|X| \leq |Y|$, then for every two distinct vertices $u, v \in Y$, $|N(u) \cap N(v)| \geq 2\delta - |X|$.*

Let G be a graph. For a subset $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting all loops generated by this process. Note that even if G is simple, G/X may have multiple edges. For simplicity, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then G/H denotes $G/E(H)$. For $S \subseteq V(G)$, $G - S$ denote the graph obtained from G by deleting all vertices of S together with all edges with at least one end in S. When $S = \{v\}$, we simplify this notation to $G - v$.

A *k-cycle* is a cycle of length k. For $k \geq 2$, a *wheel* W_k is the graph obtained from a k-cycle by adding a new vertex, called the center of the wheel, which is joined to every vertex of the k-cycle. We define W_k to be *odd* (even) if k is odd (or even, respectively). For technical reasons, we define the wheel W_1 to be a 3-cycle.

Some results in [3,8,9] on group connectivity are summarized as follows.

Lemma 2.3. *Let G be a graph and let A be an abelian group with $|A| \geq 3$. The following results are known.*
(1) *$K_{m,n}$ is A-connected if $m \geq n \geq 4$; none of $K_{2,t}$ and $K_{3,s}$ is Z_3-connected, where t and s are positive integers.*
(2) *If k is a positive integer, then $W_{2k} \in \langle Z_3 \rangle$ and $W_{2k+1} \notin \langle Z_3 \rangle$.*
(3) *If $G \notin \langle A \rangle$ and H is a spanning subgraph of G, then $H \notin \langle A \rangle$.*
(4) *If $H \subseteq G$, $H \in \langle A \rangle$, and $G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.*
(5) *If $e \in E(G)$ and if $G \in \langle A \rangle$, then $G/e \in \langle A \rangle$.*
(6) *If $d(v) \geq 2$ and $G - v \in \langle Z_3 \rangle$, then $G \in \langle Z_3 \rangle$.*
(7) *C_n is A-connected if and only if $|A| \geq n + 1$.*
(8) *K_1 is A-connected; K_n and K_n^- are A-connected if $n \geq 5$.*

For a graph G with $u, v, w \in V(G)$ such that $vu, wu \in E(G)$, let $G_{[uv,uw]}$ denote the graph obtained from G by deleting two edges uv and uw, and then adding edge wv, that is, $G_{[uv,uw]} = G \cup \{wv\} - \{uv, uw\}$.

Lemma 2.4 ([8]). *Let A be an abelian group, let G be a graph and let u, v, w be three vertices of G such that $d(u) \geq 4$ and $vu, wu \in E(G)$. If $G_{[uv,uw]}$ is A-connected, then so is G.*

For a simple bipartite graph G with $\delta \geq 4$ and a 4-cycle $C : x_1y_1x_2y_2x_1$ of G, let $G_{[x_1,x_2;(y_1y_2)]}$ denote the graph obtained from G by deleting four edges $x_1y_1, x_1y_2, x_2y_1, x_2y_2$ and adding two parallel edges y_1y_2. From Lemma 2.4, we obtain the following lemma immediately.

Lemma 2.5. *If $G_{[x_1,x_2;(y_1y_2)]}$ is Z_3-connected, then G is Z_3-connected.*

An orientation D of G is a *modular 3-orientation* if $|E^+(v)| - |E^-(v)| \equiv 0 \pmod 3$. It was proved [12] that a graph G admits a nowhere-zero 3-flow if and only if G admits a modular 3-orientation.

Lemma 2.6. *The graph G_1 shown in Fig. 1 does not admit a nowhere-zero 3-flow.*

Proof. Suppose otherwise that the graph G_1 admits a nowhere-zero 3-flow. Thus, it must admit a modular 3-orientation. For a vertex v of degree 3, $E^+(v) \equiv E^-(v) \pmod 3$ if and only if $E^+(v) = 3$ or $E^-(v) = 3$. We may assume, without loss of generality, that $E^+(x_1) = 3$. This leads $E^+(x_i) = 3$ and $E^-(y_i) = 3$, since $d(x_i) = 3$ and $d(y_i) = 3$, where $1 \leq i \leq 3$. Moreover, y_ix_4 is oriented from x_4 to y_i and x_jy_4 from x_j to y_4 for $i = 1, 2, 3$; $j = 1, 2, 3$. Since $d(x_4) = 4$, there is no orientation of $e = x_4y_4$ such that $E^+(y_4) - E^-(y_4) \equiv 0 \pmod 3$ and $E^+(x_4) - E^-(x_4) \equiv 0 \pmod 3$. This contradiction proves our lemma. □

Lemma 2.7 ([11]). *Let v be a vertex of degree three with $N_G(v) = \{v_1, v_2, v_3\}$. Let $b \in \mathbb{Z}(G, Z_3)$ and $b(v) \neq 0$. If $G_{(vv_1)}$ is Z_3-connected, then there exists an orientation D of G and $f \in F^*(G, Z_3)$ such that $\partial f = b$ under the orientation of D, where $G_{(vv_1)}$ is the resulting graph by removing vertex v together with all its incident edges from graph G and adding a new edge v_2v_3.*

From Lemma 2.7, we obtain the following lemma immediately.

Lemma 2.8. *Let v be a vertex of degree four with $N_G(v) = \{v_1, v_2, v_3, v_4\}$. Let $b \in \mathbb{Z}(G, Z_3)$ and $b(v) = 0$. If $G_{(v_1v_2,v_3v_4)}$ is Z_3-connected, then there exists an orientation D of G and $f \in F^*(G, Z_3)$ such that $\partial f = b$ under the orientation of D, where $G_{(v_1v_2,v_3v_4)}$ is the resulting graph by removing vertex v together with all its incident edges from graph G and adding two new edges v_1v_2 and v_3v_4.*

Lemma 2.9. *The two graphs depicted in Fig. 3 are Z_3-connected.*

Proof. Since G_3 is a spanning subgraph of G_4, by Lemma 2.3, it is sufficient to prove that G_3 is Z_3-connected. For simplicity, let G denote the graph G_3 depicted in Fig. 3. By the definition of Z_3-connectivity, we need to prove that for each $b \in \mathbb{Z}(G, Z_3)$, there is a function $f \in F^*(G, Z_3)$ such that $\partial f = b$. The proof is a routine job. For more detail, it can be seen in Appendix. □

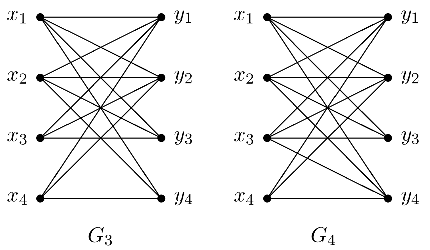

Fig. 3. Two Z_3-connected graphs.

3. Cases when *n* is small

In this section, we shows all bipartite graphs not in $\mathcal{F}_{12}$ on $n \leq 24$ vertices such that $\delta \geq \frac{n+1}{4}$ are Z_3-connected. This guarantees us to prove that the smaller graphs obtained in induction processing satisfy the hypothesis of Theorem 1.4. In particular, we will show that the minimum degree of each of such smaller graphs is at least 5 and so they are not in $\mathcal{F}_{12}$.

For this purpose, we need some notation as follows. In the rest of this section, assume that G is a simple bipartite graph with bipartition (X, Y) and that $|X| \leq |Y|$, $\delta \geq \lceil \frac{n}{4} \rceil + 1$, $X = \{x_1, x_2, \ldots, x_{n_1}\}$ and $Y = \{y_1, y_2, \ldots, y_{n_2}\}$, where $|X| = n_1$, $|Y| = n_2$. Relabeling the vertices if necessary, we may assume that $|N(y_1) \cap N(y_2)| = \max\{|N(y_i) \cap N(y_j)| : 1 \leq i < j \leq n_2\}$ and $N(y_1) \cap N(y_2) = \{x_1, x_2, \ldots, x_t\}$. By Observation 2.2, $t \geq 2$. Thus, we may assume that $C = x_1y_1x_2y_2x_1$ is a 4-cycle of G. Let H be a maximal Z_3-connected subgraph containing the 2-cycle (y_1, y_2) in $G_{[x_1,x_2;(y_1y_2)]}$, let $G^* = G/H$ and let v^* denote the new vertex which H is contracted to.

Note that $G^* - v^*$ is a subgraph of G. It is easy to see that $G^* - v^*$ is a simple bipartite graph with bipartition $(X - V(H), Y - V(H))$.

Lemma 3.1. *Suppose that $\delta \geq k \geq 5$. If one of the following holds,*

(i) $|Y - V(H)| \leq k - 2$;
(ii) $|X - V(H)| \leq k - 1$;
then G is Z_3-connected.

Proof. (i) Suppose that $|Y - V(H)| \leq k - 2$. If $X - (V(H) \cup \{x_1, x_2\}) \neq \emptyset$, let $x \in X - (V(H) \cup \{x_1, x_2\})$. Since $d(x) \geq k$, $e(x, V(H) \cap Y) \geq 2$. By Lemma 2.3(6), $x \in V(H)$, a contradiction. Thus, $X - V(H) = \{x_1, x_2\}$. Then $X - \{x_1, x_2\} \subset V(H)$. If there is a vertex $y \in Y - V(H)$, then $e(y, X \cap V(H)) \geq 3$ since $d(y) \geq 5$. By Lemma 2.3(6) again, $y \in V(H)$, a contradiction. Thus $Y \subset V(H)$. Since $d(x_1) \geq k$ and $d(x_2) \geq k$, each of x_1 and x_2 has $k - 2 \geq 3$ neighbors in Y in $G_{[x_1,x_2;(y_1y_2)]}$. By Lemma 2.3(6), both x_1 and x_2 in $V(H)$, which implies that $H = G_{[x_1,x_2;(y_1y_2)]}$ and so G is Z_3-connected by Lemma 2.5.

(ii) Suppose first that $|X - V(H)| \leq k - 2$. Since $d(y_j) \geq k$ for $j \in \{3, \ldots, n_2\}$, $e(y_j, X \cap V(H)) \geq 2$. By Lemma 2.3(6), $y_j \in V(H)$ and hence $Y - \{y_1, y_2\} \subset V(H)$. This means that H contains all vertices of Y. It follows by the minimum degree of $G_{[x_1,x_2;(y_1y_2)]}$ more than 2 that H contains all vertices of X. Thus, $H = G_{[x_1,x_2;(y_1y_2)]}$. Our lemma follows by Lemma 2.5.

Thus, $|X - V(H)| = k - 1$. We assume, without loss of generality, that $\{x_1, x_2, x_{n_1-k+4}, \ldots, x_{n_1}\} \cap V(H) = \emptyset$. By (i), $|Y - V(H)| \geq k - 1$. Since $d(y_j) \geq k$ for $y_j \in Y - V(H)$, $e(y_j, \{x_1, x_2, x_{n_1-k+4}, \ldots, x_{n_1}\}) = k - 1$. This implies that the subgraph induced by $X - V(H)$ and $Y - V(H)$ is a complete bipartite graph. Since $k - 1 \geq 4$, this complete bipartite graph is Z_3-connected by Lemma 2.3(1). Since $|X - V(H)| = k - 1$ and $d(y_j) \geq k$, each $y_j \in Y - V(H)$ has at least one neighbor in $X \cap V(H)$ and $|Y - V(H)| \geq k - 1 \geq 4$. Thus, all vertices of this complete bipartite graph belong to H. It follows that $H = G_{[x_1,x_2;(y_1y_2)]}$. Our lemma follows by Lemma 2.5. □

Lemma 3.2. *Let $X_1 \subseteq X - V(H)$ and $Y_1 \subseteq Y - V(H)$ and let G' be the graph induced by $X_1 \cup Y_1$. If each of the following holds.*

(i) $\delta(G') \geq 2$;
(ii) *v^* is adjacent to each vertex of $X_1 \cup Y_1$.*
Then G^* contains a Z_3-connected subgraph H^* such that $H \subset H^*$ with $|V(H^*) \cap X| \geq |V(H) \cap X| + 2$ and $|V(H^*) \cap Y| \geq |Y \cap V(H)| + 2$.

Proof. It is easy to see that G' is a simple bipartite graph. By (i), G' has a cycle C. Since G' is bipartite, C is an even cycle. By (ii), the subgraph H' induced by $V(C) \cup \{v^*\}$ is an even wheel which is Z_3-connected by Lemma 2.3(2). By Lemma 2.3(4), $H \cup H'$ is Z_3-connected. □

Lemma 3.3. *Let G be a simple bipartite graph with bipartition (X, Y) on $9 \leq n \leq 11$ vertices. If $\delta \geq 4$, then G is Z_3-connected.*

Proof. Assume first that $n = 9$. Since $\delta \geq 4$, $|Y| \geq |X| \geq 4$. It follows that G is isomorphic to $K_{4,5}$. Hence G is Z_3-connected by Lemma 2.3(2).

Next, assume that $n = 10$. It follows that $|X| = 4$ and $|Y| = 6$ or $|X| = |Y| = 5$. In the former case, G is isomorphic to $K_{4,6}$, G is Z_3-connected by Lemma 2.3(1). In the latter case, since $\delta \geq 4$ and $|X| = 5$, by Observation 2.2, $t \geq 3$.

If $t = 3$, then $\{x_3, y_1, y_2\} \subseteq V(H)$. If one of x_4 and x_5, say x_4, belongs to H, then one of y_3, y_4 and y_5 must be in H since $\delta \geq 4$. Since $d(x_5) \geq 4$, it follows that $x_5 \in V(H)$ and thus y_3, y_4 and y_5 are in H. Moreover H contains all the vertices of G. By Lemma 2.5, G is Z_3-connected. Thus, we assume that neither x_4 nor x_5 is in H. In this case, since $d(y_1) \geq 4$ and $d(y_2) \geq 4$, we may assume that $y_1x_4, y_2x_5 \in E(G)$ and $e(x_i, \{y_1, y_2\}) = 1$ for $i \in \{4, 5\}$. Since $\delta \geq 4$, the subgraph induced by $\{x_4, x_5, y_3, y_4, y_5\}$ is a $K_{2,3}$, v^* is adjacent to both x_4 and x_5, and v^* has at least two neighbors, say y_3 and y_4, in $Y - \{y_1, y_2\}$. We obtain an even wheel W_4 in G^* induced by $\{v^*, y_3, y_4, x_4, x_5\}$ with the center at v^*. By Lemma 2.3(2), this wheel is Z_3-connected. Contracting this wheel and iteratively contracting 2-cycles generated in the processing leads eventually to a K_1, which is Z_3-connected. This means that $H = G_{[x_1,x_2;(y_1y_2)]}$. By Lemma 2.5, G is Z_3-connected.

If $t \geq 4$, then $\{x_3, x_4, y_1, y_2\} \subseteq V(H)$. Since $\delta \geq 4$, x_3 and x_4 has at least one common neighbor, say y_3, in $\{y_3, y_4, y_5\}$. By Lemma 2.3(6), H contains y_3. Since $\delta \geq 4$, x_5 has at least two neighbors in $\{y_1, y_2, y_3\}$, which implies that $x_5 \in V(H)$ by Lemma 2.3(6). Since $e(y_j, \{x_3, x_4, x_5\}) \geq 2$ for $j \in \{4, 5\}$, H contains all vertices of Y. Since $d(x_1) \geq 4$ and $d(x_2) \geq 4$, each of x_1 and x_2 has 2 neighbors in Y in $G_{[x_1,x_2;(y_1y_2)]}$. By Lemma 2.3(6), both x_1 and x_2 in $V(H)$. Thus $H = G_{[x_1,x_2;(y_1y_2)]}$. By Lemma 2.5, G is Z_3-connected.

Finally, assume that $n = 11$. It follows that $|X| = 4$ and $|Y| = 7$ or $|X| = 5$ and $|Y| = 6$. In the former case, G is a $K_{4,7}$, and hence G is Z_3-connected by Lemma 2.3(1). In the latter case, since $\sum_{v\in X} d(v) = \sum_{v\in Y} d(v) \geq 24$, at least four vertices of X have degree 5. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. Relabeling the subscripts if necessary, we may assume that $d(x_i) \geq 5$ for $i = 1, 2$ and $N(x_1) \cap N(x_2) = \{y_1, y_2, \ldots, y_s\}$, where $s \geq 4$. This implies that $C = x_1y_1x_2y_2x_1$ is a 4-cycle of G. Thus, $G_{[y_1,y_2;(x_1x_2)]}$ contains a 2-cycle (x_1, x_2). In the case, let H_1 be a maximal Z_3-connected subgraph containing the 2-cycle (x_1, x_2) in $G_{[y_1,y_2;(x_1x_2)]}$. Let $G^* = G/H_1$ and let v^* denote the new vertex which H_1 is contracted to (here H_1 is different from H defined above). Since $s \geq 4$, $y_3, y_4 \in V(H_1)$. Since $e(\{y_3, y_4\}, \{x_3, x_4, x_5\}) \geq 4$, there is a vertex $x \in \{x_3, x_4, x_5\}$ adjacent to both y_3, y_4. Thus, H contains at least three vertices of X. Moreover, since $\delta \geq 4$, each of y_5, y_6 has two neighbors in $X \cap V(H_1)$. This implies that $\{y_3, y_4, y_5, y_6\} \subset V(H_1)$. Since $\delta \geq 4$, each $x' \in \{x_3, x_4, x_5\} - x$ has at least two neighbors in $\{y_3, y_4, y_5, y_6\}$, H_1 contains all vertices of X. Since $\delta \geq 4$, $y_1, y_2 \in V(H_1)$ and hence $H_1 = G_{[y_1,y_2;(x_1x_2)]}$. By Lemma 2.5, G is Z_3-connected. □

Lemma 3.4. *Let G be a simple bipartite graph with bipartition (X, Y) on $n \leq 12$ vertices. If $\delta \geq 5$, then G is Z_3-connected.*

Proof. Assume that $|X| \leq |Y|$. Since $\delta \geq 5$, $|Y| \geq |X| \geq 5$. If $n = 10$ or 11, then G is a complete bipartite graph, our lemma follows from Lemma 2.3(1). If $n = 12$, then $|X| = 5$ and $|Y| = 7$ or $|X| = |Y| = 6$. In the former case, G is a $K_{5,7}$, our result follows by Lemma 2.3(1). In the latter case, let $x \in X$. Since $\delta \geq 5$, $\delta(G - x) \geq 4$. Since $|V(G - x)| \leq 11$, Lemma 3.3 shows that $G - x$ is Z_3-connected. Since $d(x) \geq 5$, G is Z_3-connected by Lemma 2.3(6). □

Lemma 3.5. *Suppose that G is a bipartite graph with $\delta \geq \lceil \frac{n}{4} \rceil + 1$ on $n \geq 8$ vertices. If G contains a nontrivial Z_3-connected subgraph, then G is Z_3-connected.*

Proof. We prove our lemma by induction on n. Assume that H_1 is a nontrivial Z_3-connected subgraph of G. By Lemma 2.3(1), $|V(H_1)| \geq 8$. When $n = 8$, H_1 is a spanning subgraph of G. Thus, $G/H_1 = K_1$ and so G is Z_3-connected by Lemma 2.3(4) and (8). Thus, assume that $n \geq 9$. Let $G^* = G/H_1$ and v^* be the new vertex which H_1 is contracted to. If $H_1 = G$, then we are done. Thus, we may assume that $H_1 \neq G$. This means that there is at least one vertex in $G - V(H_1)$. By Lemma 2.3 (7), $G^* - v^*$ is a simple bipartite graph. Since $n \geq 9$, $\delta \geq 4$. By Lemma 2.3(6), $e(v, H_1) \leq 1$ for $v \in G - V(H_1)$. Thus, $|V(G^* - v^*)| \geq 6$. By Lemma 2.3(1), $|V(H_1)| \geq 8$. This means that $n \geq 14$. This implies that $\delta \geq 5$. By the same argument, $n \geq 16$. When $n = 16$, $G - V(H_1)$ is a $K_{4,4}$ which is Z_3-connected by Lemma 2.3(1). By Observation 2.1, G is 2-edge-connected. Thus, G is Z_3-connected. Thus, assume that $n \geq 17$. In this case, $\delta \geq 6$. When $n \in \{17, 18, 19, 20\}$, $|V(G^* - v^*)| \leq 20 - 8 = 12$ and $\delta(G^* - v^*) \geq 5$. By Lemma 3.4, $G^* - v^*$ is Z_3-connected. Thus, assume that $n \geq 21$. Note that $|V(H_1)| \geq 8$ by Lemma 2.3(1). Since $e(v, H_1) \leq 1$ for $v \in G^* - v^*$, $\delta(G^* - v^*) \geq \lceil \frac{n}{4} \rceil \geq \lceil \frac{n-8}{4} \rceil + 1$. If $|V(G^* - v^*)| \leq 12$, then $\delta(G^* - v^*) \geq 5$ and G is Z_3-connected by Lemma 3.4. Thus, $|V(G^* - v^*)| \geq 12$. Since $\delta(G^* - v^*) \geq \max\{\lceil \frac{|V(G^*-v^*)|}{4} \rceil + 1, 5\}$, $G^* - v^* \notin \mathcal{F}_{12}$. Applying the induction hypothesis to $G^* - v^*$, $G^* - v^*$ is Z_3-connected. By Lemma 2.3(6), G^* is Z_3-connected and so is G by Lemma 2.3(3). □

Lemma 3.6. *Suppose that G is a bipartite graph with bipartition (X, Y) on $n \geq 13$ vertices. If $\delta \geq \lceil \frac{n}{4} \rceil + 1$ and $|X| \leq \lceil \frac{n}{4} \rceil + 2$, then G is Z_3-connected.*

Proof. Assume that $|X| \leq |Y|$. Since $\delta \geq \lceil \frac{n}{4} \rceil + 1 \geq 5$, $|X| \geq \lceil \frac{n}{4} \rceil + 1$. If $|X| = \lceil \frac{n}{4} \rceil + 1$, then G is a complete bipartite graph, G is Z_3-connected by Lemma 2.3(1). Suppose $|X| = \lceil \frac{n}{4} \rceil + 2$. By Observation 2.2, $t \geq \lceil \frac{n}{4} \rceil$. Thus, $|X - V(H)| \leq 4$. By Lemma 3.1, G is Z_3-connected. □

Lemma 3.7. *Let G be a simple bipartite graph with bipartition (X, Y) on $n \geq 16$ vertices such that $|X| = |Y|$ and $n \equiv 0 \pmod 4$. If $\delta \geq \lceil \frac{n}{4} \rceil + 1$, then $t \geq 3$.*

Proof. Suppose that $\lceil \frac{n}{4} \rceil = k$. It follows that $n = 4k$ and $|X| = |Y| = 2k$. By Observation 2.2, $t \geq 2$. We only need to prove that $t \neq 2$. Suppose otherwise that $t = 2$. Since $\delta \geq \lceil \frac{n}{4} \rceil + 1 = k + 1$, $N(y_1) \cup N(y_2) = X$. Note that $|N(y_3) \cap N(y_1)| = |N(y_3) \cap N(y_2)| = 2$. Since $N(y_1) \cup N(y_2) = X$ and $N(y_3) \subseteq X$, $d(y_3) \leq 4$. This contradicts $\delta \geq \lceil \frac{n}{4} \rceil + 1 \geq 5$. Thus $t \geq 3$. □

Lemma 3.8. *Let G be a simple bipartite graph with bipartition (X, Y) on $13 \leq n \leq 16$ vertices. If $\delta \geq \lceil \frac{n}{4} \rceil + 1$, then G is Z_3-connected.*

Proof. Assume that $|X| \leq |Y|$. By Lemma 3.6, we assume that $|X| \geq \lceil \frac{n}{4} \rceil + 3$. We consider the following two cases by the size of $|X|$.

Case 1. $n_1 = 7$ and $n_2 \in \{7, 8, 9\}$.

If $t \geq 5$, then $X - V(H) \subseteq \{x_1, x_2, x_{n_1-1}, x_{n_1}\}$. By Lemma 3.1, G is Z_3-connected. By Observation 2.2, $t \geq 3$. Thus, we assume that $3 \leq t \leq 4$.

Suppose first that $t = 4$. In this case, $X - V(H) \subseteq X - \{x_3, x_4\}$. By Lemma 3.1, $X - V(H) = X - \{x_3, x_4\} = \{x_1, x_2, x_5, x_6, x_7\}$. Since $\delta \geq \lceil \frac{n}{4} \rceil + 1 \geq 5$, $e(\{y_1, y_2\}, X - \{x_1, x_2, x_3, x_4\}) \geq 2$ and hence v^* has at least two neighbors, say x_5 and x_6, in X. For each $i \in \{3, 4\}$, $e(x_i, \{y_3, \ldots, y_{n_2}\}) \geq 3$.

Assume that $n_2 = 7$. In this case, $e(\{x_3, x_4\}, \{y_3, \ldots, y_7\}) \geq 6$. Since $|\{y_3, \ldots, y_7\}| = 5$, there is $y_j \in \{y_3, \ldots, y_7\}$ such that $e(y_j, \{x_3, x_4\}) \geq 2$ for some j. By Lemma 2.3(6), $y_j \in V(H)$ and hence $|Y - V(H)| \leq 4$. If $|Y - V(H)| \leq 3$, by Lemma 3.1, G is Z_3-connected. Thus, $|Y - V(H)| = 4$. In this case, since $e(\{x_3, x_4\}, Y - \{y_1, y_2, y_j\}) = 4$, v^* is adjacent to each vertex of $Y - \{y_1, y_2, y_j\}$. Since $\delta \geq 5$ and $e(x_i, \{y_1, y_2, y_j\}) \leq 1$ for $i \in \{5, 6, 7\}$, $e(x_i, Y - \{y_1, y_2, y_j\}) \geq 4$. Thus, the subgraph induced by $\{v^*, x_5, x_6, x_7\} \cup (Y - \{y_1, y_2, y_j\})$ is a $K_{4,4}$ which is Z_3-connected by Lemma 2.3(1). This means that $Y \subseteq V(H)$. By Lemma 3.1, G is Z_3-connected.

Claim 1. *If $n_2 \in \{8, 9\}$, then $|V(H) \cap X| \geq 3$.*

Proof of Claim 1. Suppose otherwise that $|V(H) \cap X| = 2$, that is, $V(H) \cap X = \{x_3, x_4\}$. In this case, $|V(H) \cap Y| \leq 3$. Suppose otherwise that $|V(H) \cap Y| \geq 4$. Then $e(V(H) \cap Y, \{x_5, x_6, x_7\}) \geq 4$ since $\delta \geq 5$. It follows that there is at least one vertex $x \in \{x_5, x_6, x_7\}$ such that $e(x, V(H) \cap Y) \geq 2$. Thus, $x \in V(H)$ by Lemma 2.3(3), and $|V(H) \cap X| \geq 3$. This contradiction proves that $|V(H) \cap Y| \leq 3$.

Suppose that $|V(H) \cap Y| = 2$. Since $\delta \geq 5$, $e(\{x_3, x_4\}, \{y_3, y_4, \ldots, y_{n_2}\}) \geq 6$ and $N(x_3) \cap N(x_4) \cap \{y_3, y_4, \ldots, y_{n_2}\} = \emptyset$. On the other hand, since $\delta \geq 5$, $e(x_i, \{y_3, y_4, \ldots, y_{n_2}\}) \geq 4$ for $i \in \{5, 6, 7\}$. Since $e(y_j, \{x_3, x_4\}) \leq 1$, $e(y_j, \{x_5, x_6, x_7\}) \geq 2$ for $j \in \{3, \ldots, n_2\}$. Similarly, $e(\{y_1, y_2\}, \{x_5, x_6, x_7\}) \geq 2$. We assume, without loss of generality, that v^* is adjacent to both x_5 and x_6.

When $n_2 = 8$, v^* is adjacent to each vertex of $\{y_3, y_4, \ldots, y_8\}$. It is easy to verify that there are $y_j, y_k \in \{y_3, y_4, \ldots, y_8\}$ such that the subgraph induced by x_5, x_6, y_j, y_k is a 4-cycle. By Lemma 3.2, $|V(H) \cap X| \geq 4$, contrary to the assumption that $|V(H) \cap X| = 2$.

When $n_2 = 9$, there is at most one vertex, say y_9, in $\{y_3, y_4, \ldots, y_9\}$ such that $e(y_9, \{x_3, x_4\}) = 0$. If there are two vertices $y_i, y_j \in \{y_3, \ldots, y_8\}$ such that $e(y_i, \{x_1, x_2\}) = 2$ and $e(y_j, \{x_1, x_2\}) = 2$, then $G_{[y_i, y_j; (x_1 x_2)]}$ has a Z_3-connected subgraph containing $\{x_1, x_2, x_3, x_4\}$. Since $\delta \geq 5$, $e(y_k, \{x_1, x_2, x_3, x_4\}) \geq 2$ for $k \in \{3, 4, \ldots, n_2\} \setminus \{i, j\}$. Thus, $y_k \in V(H)$. It implies that $x_5, x_6, x_7 \in V(H)$. This means that $G_{[y_i, y_j; (x_1 x_2)]}$ can be Z_3-reduced to K_1. By Lemma 2.5, G is Z_3-connected. Thus, assume that there is at most one vertex of $\{y_3, \ldots, y_8\}$ has two neighbors in $\{x_1, x_2\}$. Since $e(x_3, \{y_3, \ldots, y_8\}) \geq 3$, we assume, without loss of generality, that $x_3y_3, x_3y_4, x_3y_5 \in E(G)$ and $e(y_k, \{x_1, x_2\}) \leq 1$ for $k \in \{4, 5\}$. By our assumption, $y_k x_4 \notin E(G)$, $e(y_k, \{x_5, x_6, x_7\}) \geq 3$. In this case, the subgraph induced by x_5, x_6, y_4, y_5 is a 4-cycle. By Lemma 3.2, $|V(H) \cap X| \geq 4$, contrary to the assumption that $|V(H) \cap X| = 2$.

Suppose that $|V(H) \cap Y| = 3$. In this case, assume that $V(H) \cap Y = \{y_1, y_2, y_3\}$. Since $\delta \geq 5$, $e(\{y_1, y_2, y_3\}, \{x_5, x_6, x_7\}) \geq 3$. If $e(\{y_1, y_2, y_3\}, \{x_5, x_6, x_7\}) \geq 4$ or there is one vertex $x_i \in \{x_5, x_6, x_7\}$ such that $e(x_i, \{y_1, y_2, y_3\}) \geq 2$, then $|V(H) \cap X| \geq 3$ by Lemma 2.3(6), a contradiction. Thus, v^* is adjacent to each vertex of $\{x_5, x_6, x_7\}$. Since $\delta \geq 5$, we may assume that v^* is adjacent to each of y_4, y_5, y_6, y_7. Observe the subgraph induced by $\{x_5, x_6, x_7, y_4, y_5, y_6, y_7\}$, and the minimum degree of this subgraph is at least 2. By Lemma 3.2, $|V(H) \cap X| \geq 3$, a contradiction. □

By Claim 1, $|X - V(H)| \leq 4 \leq \delta - 1$. By Lemma 3.1, G is Z_3-connected.

Suppose that $t = 3$. Since $\delta \geq 5$, $e(\{y_1, y_2\}, \{x_4, x_5, x_6, x_7\}) \geq 4$ and $e(x_i, \{y_1, y_2\}) \leq 1$ for $i \in \{4, 5, 6, 7\}$. This implies that $e(x_i, \{y_1, y_2\}) = 1$ for $i \in \{4, 5, 6, 7\}$. Since $d(x_3) \geq 5$, $e(x_3, \{y_3, y_4, \ldots, y_{n_2}\}) \geq 3$. We assume, without loss of generality, that $x_3y_3, x_3y_4, x_3y_5 \in E(G)$. Since $e(y_3, \{x_4, x_5, x_6, x_7\}) \geq 2$, we may assume that $y_3x_4, y_3x_5 \in E(G)$. Let $G^{**} = G^*_{[v^* y_3, y_3 x_4]}$, and let H^* be the maximum Z_3-connected subgraph of G^{**} and let v^{**} be the vertex obtained by contracting H^*. Since $d(x_4) \geq 5$, let $y_p, y_q, y_r \in \{y_4, y_5, \ldots, y_{n_2}\}$ be three neighbors of x_4.

Assume first that there exist at least two vertices of y_p, y_q and y_r which is adjacent to x_3. In this case, we may assume these two vertices are y_4, y_5 since $x_3y_3, x_3y_4, x_3y_5 \in E(G)$. It follows that $y_4, y_5 \in V(H^*)$. Since $d(y_j) \geq 5$ for $j \in \{4, 5\}$, $e(y_j, \{x_5, x_6, x_7\}) \geq 1$. Recall that $e(x_i, \{y_1, y_2\}) \geq 1$ for $i \in \{5, 6, 7\}$. By Lemma 2.3(6), there is one vertex, say x_5, of x_5, x_6 and x_7 such that $x_5 \in V(H^*)$.

If one of x_6 and x_7 is in H^*, then by Lemma 2.3(6), $y_j \in V(H^*)$ for $j \in \{6, \ldots, n_2\}$ since $e(y_j, V(H^*) \cap X) \geq 2$. Thus, since $\delta \geq 5$, $x_6, x_7 \in V(H^*)$. Iteratively contracting 2-cycles generated in the processing leads eventually to a K_1, which is Z_3-connected. By Lemma 2.3(4), G^{**} is Z_3-connected. By Lemma 2.4, G^* is Z_3-connected and so is G. Thus, assume that neither x_6 nor x_7 is in H^*.

We claim that $|V(H^*) \cap Y| \geq 5$. Suppose otherwise that $|V(H^*) \cap Y| = 4$, that is, $V(H^*) \cap Y = \{y_1, y_2, y_4, y_5\}$. In this case, since $\delta \geq 5$ and $e(y_j, \{x_3, x_4, x_5\}) \leq 1$ where $j \in \{6, 7, \ldots, n_2\}$, $e(y_j, \{x_1, x_2, x_6, x_7\}) = 4$. When $n_2 = 9$, the subgraph induced by x_1, x_2, x_6, x_7 and y_6, y_7, y_8, y_9 is a $K_{4,4}$ which is Z_3-connected by Lemma 2.3(1). Thus, H^* should contain these eight vertices, contrary to the assumption that $|V(H^*) \cap Y| = 4$. When $n_2 = 8$, the subgraph induced by x_1, x_2, x_6, x_7 and y_6, y_7, y_8 is a $K_{3,4}$. On the other hand, since $\delta \geq 5$, $e(y_j, \{x_3, x_4, x_5\}) = 1$, that is, v^{**} is adjacent to y_j for $j \in \{6, 7, 8\}$. Recall that v^{**} is adjacent to both x_6 and x_7. Thus, G^{**} contains a 4-wheel induced by x_6, x_7, y_6, y_7 with the center at v^{**}. It follows that H^* should contain x_6, x_7, y_6, y_7, contrary to the assumption that $|V(H^*) \cap Y| = 4$. So far, we have proved that $|V(H^*) \cap Y| \geq 5$.

Now we assume, without loss of generality, that $\{y_1, y_2, y_4, y_5, y_6\} \subseteq V(H^*) \cap Y$. When $n_2 = 8$, for $i \in \{6, 7\}$, $e(x_i, \{y_1, y_2, y_4, y_5, y_6\}) \geq 2$ since $\delta(G) \geq 5$. By Lemma 2.3(6), $x_i \in V(H^*)$. When $n_2 = 9$, as in the argument above, the subgraph induced by x_1, x_2, x_6, x_7 and y_7, y_8, y_9 is a $K_{3,4}$. In this case, $e(x_i, \{y_1, y_2\}) = 1$ for each $i \in \{6, 7\}$ and $e(y_j, \{x_3, x_4, x_5\}) = 1$ for $j \in \{7, 8, 9\}$. By Lemma 3.2, $x_6, x_7 \in V(H^*)$. In both cases, iteratively contracting 2-cycles generated in the processing leads eventually to a K_1, which is Z_3-connected. By Lemma 2.3(4), G^{**} is Z_3-connected. By Lemma 2.4, G^* is Z_3-connected and so is G.

Next, we assume that there is at most one vertex of y_p, y_q and y_r which is adjacent to x_3. Thus, we assume, without loss of generality, that $y_5x_4 \notin E(G)$, $x_4y_6, x_4y_7 \in E(G)$. Since $\delta \geq 5$, $e(y_j, \{x_5, x_6, x_7\}) \geq 2$ for $j \in \{5, 6, 7\}$. It follows that the subgraph induced by $x_5, x_6, x_7, y_5, y_6, y_7$ contains a 4-cycle or a 6-cycle. Moreover, v^{**} is adjacent to each vertex of $x_5, x_6, x_7, y_5, y_6, y_7$. Thus, G^{**} contains a 4-wheel or a 6-wheel with the center at v^{**}. By Lemma 2.3(2), each such wheel is Z_3-connected. Consequently, H^* contains at least four vertices of X. Since $\delta \geq 5$, each vertex of Y except y_3 has two neighbors in H^*. By Lemma 2.3(6), all vertices of Y except y_3 are in H^*. Iteratively contracting 2-cycles generated in the processing leads eventually to a K_1, which is Z_3-connected. By Lemma 2.3, G^{**} is Z_3-connected. By Lemma 2.4, G^* is Z_3-connected and so is G.

Case 2. $n_1 = 8$ and $n_2 = 8$.

If $t \geq 5$, then $X - V(H) \subseteq \{x_1, x_2, x_6, x_7, x_8\}$. By Lemma 3.1, $X - V(H) = \{x_1, x_2, x_6, x_7, x_8\}$. Since $\delta \geq 5$, $e(x_i, \{y_3, \ldots, y_8\}) \geq 3$ for $i \in \{3, 4, 5\}$. By the principle of pigeonhole, $V(H)$ contains at least two vertices of $\{y_3, \ldots, y_8\}$. This implies that $|V(H) \cap Y| \geq 4$. We claim that $|V(H) \cap Y| = 4$. Suppose otherwise that $|V(H) \cap Y| > 4$, that is, $|Y - V(H)| < 4$. Thus, G is Z_3-connected by Lemma 3.1. We assume, without loss of generality, that $Y - V(H) = \{y_5, y_6, y_7, y_8\}$. Since $\delta \geq 5$, the subgraph induced by $\{x_6, x_7, x_8, y_5, y_6, y_7, y_8\}$ contains a $K_{3,4}$. Since $\delta \geq 5$, $e(x_i, \{y_5, y_6, y_7, y_8\}) \geq 1$ for $i \in \{3, 4, 5\}$ and $e(v^*, \{y_5, y_6, y_7, y_8\}) \geq 3$. It follows that the subgraph induced by $\{v^*, x_6, x_7, x_8, y_5, y_6, y_7, y_8\}$ contains G_4 in Fig. 3. By Lemmas 2.9 and 2.3(4), $G_{[x_1,x_2;(y_1y_2)]}$ contains a Z_3-connected subgraph H' such that $|Y \cap V(H')| \geq 5$. By Lemma 3.1, G is Z_3-connected.

If $t = 4$, then $X - V(H) \subseteq \{x_1, x_2, x_5, x_6, x_7, x_8\}$. If $|X \cap V(H)| \geq 4$, by Lemma 3.1, G is Z_3-connected. Thus, assume that $|V(H) \cap X| = 2$ or 3. Assume first that $|V(H) \cap X| = 2$. In this case, $X - V(H) = \{x_1, x_2, x_5, x_6, x_7, x_8\}$. Since $\delta \geq 5$ and $e(\{y_1, y_2\}, \{x_1, x_2, x_3, x_4\}) = 8$, v^* has two neighbors in $\{x_5, x_6, x_7, x_8\}$, say x_5, x_6. Since $\delta \geq 5$, $e(\{x_3, x_4\}, Y - \{y_1, y_2\}) \geq 6$ and $e(\{x_5, x_6\}, Y - \{y_1, y_2\}) \geq 8$. If $|V(H) \cap Y| = 2$, then v^* is adjacent to every vertex of $Y - \{y_1, y_2\}$. Thus, we obtain an even wheel W_4 with the center at v^*, which is Z_3-connected by Lemma 2.3(2). This implies that $x_5, x_6 \in V(H)$. In this case, $|X - V(H)| \leq 4$. By Lemma 3.1, G is Z_3-connected. If $|V(H) \cap Y| = 3$, then $V(H)$ contains one vertex, say y_3, of $Y - \{y_1, y_2\}$. Since $e(\{x_3, x_4\}, \{y_4, y_5, \ldots, y_8\}) \geq 4$, v^* has four neighbors in $\{y_4, \ldots, y_8\}$. In this case, $e(\{y_1, y_2, y_3\}, \{x_5, x_6, x_7, x_8\}) \geq 3$ and there is no vertex $x \in \{x_5, x_6, x_7, x_8\}$ such that $e(x, \{y_1, y_2, y_3\}) \geq 2$. We assume, without loss of generality, that $y_1x_5, y_2x_6 \in E(G)$. On the other hand, $e(x_i, \{y_4, y_5, y_6, y_7, y_8\}) \geq 4$ for $i \in \{5, 6\}$. Thus, there are $y', y'', y''' \in \{y_4, y_5, y_6, y_7, y_8\}$ such that the subgraph induced by $\{x_5, x_6, y', y'', y'''\}$ is a $K_{2,3}$. Since $e(\{x_3, x_4\}, \{y_4, y_5, \ldots, y_8\}) \geq 4$, by the principle of pigeonhole, we may assume that $x_3y', x_4y'' \in E(G)$. This implies that G^* contains an even wheel W_4 induced by $\{y', y'', x_5, x_6, v^*\}$ with the center at v^*. In this case, $|X - V(H)| \leq 4$. By Lemma 3.1, G is Z_3-connected.

If $|V(H) \cap Y| = 4$, then $V(H)$ contains two vertices of $Y - \{y_1, y_2\}$, say y_3, y_4. Since $\delta(G) \geq 5$, the subgraph induced by x_5, x_6, x_7, x_8 and y_5, y_6, y_7, y_8 is $K_{4,4}$ which is Z_3-connected, contrary to Lemma 3.5. If $|V(H) \cap Y| > 4$, then $|Y - V(H)| \leq 3$. By Lemma 3.1, G is Z_3-connected.

Next, assume that $|V(H) \cap X| = 3$. In this case, we may assume $V(H)$ contains x_5. Since $t = 4$ and $x_5 \in V(H)$, $V(H)$ contains at least one vertex of $Y - \{y_1, y_2\}$. By Lemma 3.1, $3 \leq |V(H) \cap Y| \leq 4$.

We claim that $|V(H) \cap Y| = 4$. Suppose otherwise that $|V(H) \cap Y| = 3$. We assume, without loss of generality, that $y_3 \in V(H)$. Since $\delta(G) \geq 5$, $e(x_i, \{y_1, y_2, y_3\}) \leq 3$ for $i \in \{3, 4, 5\}$. Thus, $e(x_i, \{y_4, y_5, y_6, y_7, y_8\}) \geq 2$ for $i \in \{3, 4, 5\}$ and $e(\{x_3, x_4, x_5\}, \{y_4, \ldots, y_8\}) \geq 6$. This implies that there is $y \in \{y_4, \ldots, y_8\}$ such that $e(y, \{x_3, x_4, x_5\}) \geq 2$. By Lemma 2.3(6), $y \in V(H)$, a contradiction. Thus, $|V(H) \cap Y| = 4$. It follows that the graph induced by x_6, x_7, x_8 and y_5, y_6, y_7, y_8 is a $K_{3,4}$ and $x_iv^* \in E(G^*)$ for $i = 6, 7, 8$. Since $\delta \geq 5$ and $e(\{x_3, x_4\}, V(H) \cap Y) \leq 8$, v^* has two neighbors in $Y - V(H)$. It follows that G^* contains an even wheel W_4 with the center at v^*, contrary to the choice of H.

Suppose that $t = 3$. We claim that $|V(H) \cap X| \geq 2$. Suppose otherwise that $V(H) \cap X = \{x_3\}$. In this case, $e(x_j, \{y_1, y_2\}) \leq 1$ for each $j \in \{4, 5, 6, 7\}$. Since $\delta(G) \geq 5$, $e(\{y_1, y_2\}, \{x_4, \ldots, x_8\}) \geq 4$ and $e(x_3, Y - \{y_1, y_2\}) \geq 3$. We assume, without loss of generality, that $e(x_j, \{y_1, y_2\}) = 1$ for $j \in \{4, 5, 6, 7\}$ and $x_3y_3, x_3y_4, x_3y_5 \in E(G)$.

If $e(\{y_3, y_4, y_5\}, \{x_4, x_5, x_6, x_7\}) \geq 7$, then either there are $y, y' \in \{y_3, y_4, y_5\}$ and $x, x' \in \{x_4, x_5, x_6, x_7\}$ such that the subgraph induced by x, x', y, y' is a 4-cycle or there are $x, x', x'' \in \{x_4, x_5, x_6, x_7\}$ such that the subgraph induced by x, x', x'', y_3, y_4 and y_5 is a 6-cycle. It follows that the such subgraph is an even wheel either W_4 or W_6, which is Z_3-connected by Lemma 2.3(2). Thus, $|V(H) \cap X| \geq 3$ and $|V(H) \cap Y| \geq 4$. If $e(\{y_3, y_4, y_5\}, \{x_4, x_5, x_6, x_7\}) \leq 6$, then $e(\{x_4, x_5, x_6, x_7\}, \{y_6, y_7, y_8\}) \geq 20 - 4 - 6 = 10$. If either $e(\{x_4, x_5, x_6, x_7\}, \{y_6, y_7, y_8\}) \geq 11$ or $e(x_i, \{y_6, y_7, y_8\}) \geq 2$ for each $i \in \{4, 5, 6, 7\}$ and $e(y_j, \{x_4, x_5, x_6, x_7\}) \geq 3$ for each $j \in \{6, 7, 8\}$, then the subgraph induced by $\{x_4, x_5, x_6, x_7, v^*, y_6, y_7, y_8\}$ contains G_3 with one part $\{x_4, x_5, x_6, x_7\}$ and the other part $\{v^*, y_6, y_7, y_8\}$, which is Z_3-connected by Lemma 2.9. Thus, $|V(H) \cap X| \geq 5$. By

Lemma 3.1, G is Z_3-connected. Thus, we assume that $e(\{x_4, x_5, x_6, x_7\}, \{y_6, y_7, y_8\}) = 10$ and that there is one vertex, say x_4, of $\{x_4, x_5, x_6, x_7\}$ such that $e(x_4, \{y_6, y_7, y_8\}) = 1$ or there is one vertex, say y_8, of $\{y_6, y_7, y_8\}$ such that $e(y_8, \{x_4, x_5, x_6, x_7\}) = 2$.

If $e(y_8, \{x_4, x_5, x_6, x_7\}) = 2$, without loss of generality, let $x_6y_8, x_7y_8 \in E(G)$. In this case, the subgraph induced by $\{x_4, x_5, x_6, x_7, y_6, y_7\}$ is a $K_{2,4}$. If $e(x_4, \{y_6, y_7, y_8\}) = 1$, let $y_6x_4 \in E(G)$. Then the subgraph induced by $\{x_5, x_6, x_7, y_6, y_7, y_8\}$ is a $K_{3,3}$. In both cases, let $G^{**} = G^{*}_{[y_7,y_8,(x_6x_7)]}$ and H' be the maximum Z_3-connected subgraph in G^{**}. Then $\{x_3, x_6, x_7, y_1, y_2, y_6\} \subseteq V(H')$. Moreover, by Lemma 2.3(6), $\{x_4, x_5\} \subset V(H')$. Since $\delta \geq 5$, by Lemma 2.3(6), all vertices are in $V(H')$. This leads that $H' = G^{**}$. By Lemmas 2.3 and 2.5, G is Z_3-connected.

We next claim that $|V(H) \cap X| \geq 3$. Suppose otherwise that $|V(H) \cap X| = 2$ and assume that $V(H) \cap X = \{x_3, x_4\}$. Since $t = 3$, $e(x_4, \{y_1, y_2\}) \leq 1$. Since $\delta \geq 5$, $e(\{x_3, x_4\}, \{y_3, y_4, \ldots, y_8\}) \geq 7$. Thus, there is one vertex in $\{y_3, y_4, \ldots, y_8\}$, say y_3, such that $e(y_3, \{x_3, x_4\}) \geq 2$. By Lemma 2.3(6), $y_3 \in V(H)$. In this case, $e(\{x_3, x_4\}, \{y_4, y_5, \ldots, y_8\}) \geq 10-3-2 = 5$. If there is some $j \in \{4, 5, \ldots, 8\}$ such that $e(y_j, \{x_3, x_4\}) \geq 2$, then by Lemma 2.3, $y_j \in V(H)$. Thus, $e(\{y_1, y_2, y_3, y_j\}, \{x_5, x_6, x_7, x_8\}) \geq 20 - 4 - 3 - 3 - 3 = 7$ since $t = 3$, and there is $x \in \{x_5, x_6, x_7\}$ such that $e(x, \{y_1, y_2, y_3, y_j\}) \geq 2$. So, $x \in V(H)$ and $|V(H) \cap X| \geq 3$, a contradiction. Therefore, for each $j \in \{4, 5, \ldots, 8\}$, $e(y_j, \{x_3, x_4\}) = 1$. Similarly, for each $i \in \{5, 6, 7, 8\}$, $e(x_i, \{y_1, y_2, y_3\}) = 1$. Since $\delta \geq 5$, it is easy to verify that each vertex of the subgraph induced by $\{x_5, x_6, x_7, x_8, y_4, \ldots, y_8\}$ has degree at least 2. By Lemma 3.2, $|V(H) \cap X| \geq 4$. By Lemma 3.1, G is Z_3-connected.

We now claim that $|V(H) \cap X| \geq 4$. Suppose otherwise that $|V(H) \cap X| = 3$ and let $V(H) \cap X = \{x_3, x_4, x_5\}$. Since $\delta \geq 5$ and $t = 3$, $e(\{x_3, x_4, x_5\}, \{y_3, y_4, \ldots, y_8\}) \geq 4+4+3 = 11$. Thus, there are three vertices in $\{y_3, y_4, \ldots, y_8\}$ each of which has two neighbors in $\{x_3, x_4, x_5\}$. By Lemma 2.3(6), $|V(H) \cap Y| \geq 5$. By Lemma 3.1, G is Z_3-connected.

Thus, $|V(H) \cap X| \geq 4$. In this case, by Lemma 3.1, G is also Z_3-connected. □

Lemma 3.9. *Suppose that G is a simple bipartite graph with bipartition (X, Y), $|X| \leq |Y|$ and $\delta \geq \lceil \frac{n}{4} \rceil + 1 \geq 6$. If $t = 3$, then G is Z_3-connected.*

Proof. Let $k = \lceil \frac{n}{4} \rceil + 1$. By Observation 2.2, $t = 3 \geq 2\delta - |X| \geq 2k - |X|$. Thus, $|X| \geq 2k - 3$. On the other hand, since $k \geq \frac{n}{4} + 1$, $4(k-1) \geq n \geq 2|X|$ and hence $n_1 = |X| \leq 2k - 2$. Thus, we consider the following two cases.

Case 1. $n_1 = 2k - 3$.

In this case, since $n_2 \geq n_1$ and $n_1 + n_2 = n$, $n_2 \in \{2k-3, 2k-2, 2k-1\}$. Since $\delta \geq k$, $e(\{y_1, y_2\}, \{x_4, x_5, \ldots, x_{n_1}\}) \geq 2k - 6$ and $e(x_i, \{y_1, y_2\}) \leq 1$ for $i \in \{4, 5, \ldots, n_1\}$. This implies that $e(x_i, \{y_1, y_2\}) = 1$ for $i \in \{4, 5, \ldots, n_1\}$. Since $d(x_3) \geq k$, $e(x_3, \{y_3, y_4, \ldots, y_{n_2}\}) \geq k - 2$. We assume, without loss of generality, that $x_3y_3, \ldots, x_3y_k \in E(G)$. Since $e(y_3, \{x_4, x_5, \ldots, x_{n_1}\}) \geq k - 3$, we may assume that $y_3x_4 \in E(G)$. Let $G^{**} = G^{*}_{[x_4v^*,x_4y_3]}$, and let H^* be the maximum Z_3-connected subgraph of G^{**} and let v^{**} be the vertex obtained by contracting H^*. In this case, $\{y_1, y_2, y_3\} \subseteq V(H^*) \cap Y$.

Note that $e(x_i, \{y_1, y_2\}) = 1$ for $i \in \{4, 5, \ldots, n_1\}$. Without loss of generality we assume that $x_4y_1 \in E(G)$. If $e(y_3, \{x_1, x_2\}) = 2$, then $N(y_1) \cap N(y_3) \supseteq \{x_1, x_2, x_3, x_4\}$ and $t \geq 4$, contrary to our assumption that $t = 3$. Thus, $e(y_3, \{x_1, x_2\}) \leq 1$. Since $\delta \geq k$, $e(y_3, \{x_5, \ldots, x_{n_1}\}) \geq k-3$. Thus, $V(H^*)$ contains at least $k-2$ vertices of X. Since $|X| = n_1 = 2k-3$, $|X - V(H^*)| \leq k-1$. When $|X - V(H^*)| \leq k-2$, $e(y_j, X \cap V(H^*)) \geq 2$ since $d(y_j) \geq k$ for $j \in \{4, \ldots, n_2\}$. By Lemma 2.3(6), $y_j \in V(H^*)$. This means that H^* contains all vertices of Y. It follows by the minimum degree of G^{**} more than 2 that H^* contains all vertices of X. Thus $H^* = G^{**}$. Now we consider $|X - V(H^*)| = k-1$. If $|Y - V(H^*)| \leq k-2$, then $e(x_i, Y \cap V(H^*)) \geq 2$ for $i = 5, \ldots, n_1$. It is easy check that $H^* = G^{**}$. Otherwise that $|Y - V(H^*)| \geq k-1$. Since $d(y_j) \geq k$ for $y_j \in Y - V(H^*)$, $e(y_j, X - V(H^*)) = k-1$. This implies that the subgraph induced by $X - V(H^*)$ and $Y - V(H^*)$ is a complete graph. Since $k - 1 \geq 5$, this complete bipartite graph is Z_3-connected by Lemma 2.3(1). This contradicts Lemma 3.5.

Case 2. $n_1 = 2k - 2$.

Since $k \geq \frac{n}{4} + 1$, $4(k-1) \geq n = n_1 + n_2 \geq 2n_1$. Thus, $n_2 \leq 2k - 2$. On the other hand, $n_2 \geq n_1 = 2k - 2$ and hence $n_2 = 2k - 2$. Since $\delta \geq k$, $e(\{y_1, y_2\}, \{x_4, x_5, \ldots, x_{n_1}\}) \geq 2k - 6$. Since $t = 3$, $e(x_i, \{y_1, y_2\}) \leq 1$ for $i \in \{4, 5, \ldots, n_1\}$. We assume, without loss of generality, that $v^*x_i \in E(G^*)$ for $i \in \{4, \ldots, n_1 - 1\}$. Since $d(x_3) \geq k$, we may assume $y_1x_3, y_2x_3, \ldots, y_kx_3 \in E(G)$. Since $e(y_3, X - \{x_1, x_2, x_{n_1}\}) \geq k - 3 \geq 3$, y_3 has a neighbor in $\{x_4, \ldots, x_{n_1-1}\}$. Assume that $x_4y_3 \in E(G)$. Let $G^{**} = G^{*}_{[v^*y_3,y_3x_4]}$ and let H^* be the maximum Z_3-connected subgraph of G^{**} and let v^{**} be the vertex obtained by contracting H^*. We are to prove that $H^* = G^{**}$.

Assume first that x_4 has more than one neighbors in $N(x_3) - \{y_1, y_2, y_3\}$. Then $V(H^*)$ contains at least two vertices of $Y - \{y_1, y_2, y_3\}$. We assume, without loss of generality, that $y_4, y_5 \in V(H^*)$. Since $t = 3$, $|N(y_4) \cap N(y_5)| \leq 3$. Thus, $|N(y_4) \cup N(y_5)| = |N(y_4) + |N(y_5)| - |N(y_4) \cap N(y_5)| \geq 2k - 3$ and $|X - N(y_4) \cup N(y_5)| \leq 1$. Combining the fact that $v^*x_i \in E(G^*)$ for $i \in \{4, \ldots, n_1 - 1\}$, $|X - V(H^*)| \leq 4$. Since $\delta \geq 6$, each vertex of Y except y_3 has two neighbors in H^*. By Lemma 2.3(6), all vertices of Y except y_3 are in H^*. On the other hand, for each vertex $x \in X$, $d_{G^{**}}(x) \geq 4$. By Lemma 2.3(6), all vertices in X are in $V(H^*)$. This implies that $H^* = G^{**}$.

Assume then that x_4 has only one neighbor in $N(x_3) - \{y_1, y_2, y_3\}$, say y_4. In this case, since $e(x_4, \{y_1, y_2\}) \leq 1$, $|N(x_3) \cap N(x_4)| \leq 3$. Note that $y_4x_4, y_4x_3 \in E(G)$, $x_4y_1 \in E(G)$ or $x_4y_2 \in E(G)$. Since $t = 3$, $e(y_4, \{x_1, x_2\}) \leq 1$. For otherwise, $|N(y_1) \cap N(y_4)| \geq 4$ or $|N(y_2) \cap N(y_4)| \geq 4$, which implies that $t \geq 4$, contrary to our assumption that $t = 3$. Thus, $e(y_4, \{x_5, \ldots, x_{n_1-1}\}) \geq k - 4 \geq 2$, $V(H^*)$ contained at least two vertices of $\{x_5, \ldots, x_{n_1-1}\}$, say x_5 and x_6. Since $\delta \geq k$ and $e(\{x_5, x_6\}, \{y_1, y_2\}) = 2$, $|N(x_5) \cup N(x_6)| \geq k + 1$. Note that $|N(x_3) \cup N(x_4)| \geq 2k - 3$ and $n_2 = 2k - 2$, $V(H^*) \cap Y$ contains at least k vertices, that is, $|Y - V(H^*)| \leq k - 2$. Since $\delta \geq k$, by Lemma 2.3(6), H^* contains all vertices of $X - \{x_1, x_2\}$. This implies H^* contains all vertices of Y. Keeping this procedure, $H^* = G^{**}$.

Finally, assume that x_4 has no neighbor in $N(x_3) - \{y_1, y_2, y_3\}$. In this case, $Y = N(x_3) \cup N(x_4)$. Since $\delta \geq k \geq 6$, $e(x_i, \{y_5, y_6, \ldots, y_{n_2-1}\}) \geq 2$ for $i \in \{5, 6, \ldots, n_1 - 1\}$ and $e(y_j, \{x_5, x_6, \ldots, x_{n_1-1}\}) \geq 2$ for $j \in \{5, 6, \ldots, n_2 - 1\}$. The subgraph induced by $\{y_5, y_6, \ldots, y_{n_2-1}\} \cup \{x_5, x_6, \ldots, x_{n_1-1}\}$ contains a cycle of length even since such subgraph is bipartite. This implies G^{**} contains an even wheel, which is Z_3-connected by Lemma 2.3(2). Contracting this wheel, H^* contains at least two vertices of $\{x_5, x_6, \ldots, x_{n_1-1}\}$. As in the proof of the case when x_4 has only one neighbor of $N(x_3) - \{y_1, y_2, y_3\}$, we can prove $H^* = G^{**}$.

So far, we have proved $H^* = G^{**}$. By Lemma 2.3(4), G^{**} is Z_3-connected. By Lemma 2.4, G^* is Z_3-connected and so is G. □

Lemma 3.10. *Let $k \geq 5$ and let $G = (X, Y)$ be a simple bipartite graph on $4k$ vertices such that $|X| = |Y| = 2k$. If $\delta \geq k + 1$, then G is Z_3-connected.*

Proof. By Lemmas 3.7 and 3.9, $t \geq 4$. Assume first that $t \geq 6$. Since $\delta(G) \geq k + 1$, $e(\{x_3, x_4, x_5, x_6\}, Y - \{y_1, y_2\}) \geq 4k - 4$. By Lemma 3.5, G does not a contain Z_3-connected subgraph. By Lemma 2.3(1), at most one vertex of $Y - \{y_1, y_2\}$ has four neighbors in $\{x_3, x_4, x_5, x_6\}$. It follows that $V(H) \cap Y$ contains at least $2 + 1 + \lfloor \frac{4k-4-4}{3} \rfloor = k + \lfloor \frac{k+1}{3} \rfloor$ vertices. Thus, $|Y - V(H)| \leq k - \lfloor \frac{k+1}{3} \rfloor \leq k - 1$. By Lemma 3.1, G is Z_3-connected. Thus, $4 \leq t \leq 5$.

Claim. $|V(H) \cap X| \geq 4$ *and* $|V(H) \cap Y| \geq 4$.

Proof of Claim. Assume first that $t = 5$. We now prove that $|V(H) \cap Y| \geq 4$. Suppose otherwise that $|V(H) \cap Y| \leq 3$. Then each vertex of $Y - V(H)$ has at most one neighbor in $\{x_3, x_4, x_5\}$. Thus, $e(Y - V(H), \{x_3, x_4, x_5\}) \leq |Y - V(H)| \leq 2k - 2$. On the other hand, $e(\{x_3, x_4, x_5\}, Y - V(H)) \geq 3(k+1) - 9 = 3k - 6$. It implies that $k \leq 3$, a contradiction. Thus, $|V(H) \cap Y| \geq 4$.

We assume, without loss of generality, that $y_1, y_2, y_3, y_4 \in V(H) \cap Y$. By Lemmas 3.5 and 2.3(1), $e(\{y_1, y_2, y_3, y_4\}, \{x_1, x_2, x_3, x_4, x_5\}) \leq 18$. Thus, $e(\{y_1, y_2, y_3, y_4\}, X - \{x_1, x_2, x_3, x_4, x_5\}) \geq 4(k+1) - 18$. If $|V(H) \cap X| = 3$, then $e(\{y_1, y_2, y_3, y_4\}, X - \{x_1, x_2, x_3, x_4, x_5\}) \leq 2k - 5$. This implies that $2k \leq 9$ and $k \leq 4$, a contradiction. Thus, $|V(H) \cap X| \geq 4$.

Next, we assume that $t = 4$. In this case, $\{x_3, x_4\} \subseteq V(H)$. Since $|X| = |Y|$, by symmetry, we assume that $|N(x_3) \cap N(x_4)| \leq 4$ (If $|N(x_3) \cap N(x_4)| \geq 5$, then we replace X with Y and obtain $|N(y_3) \cap N(y_4)| \geq 5$. This implies the case $t \geq 5$ which we have proved.) Thus, by Lemma 3.7, $|N(x_3) \cap N(x_4)| = 4$ or 3. When $|N(x_3) \cap N(x_4)| = 4$, $V(H) \cap Y$ contains two vertices of $Y - \{y_1, y_2\}$. We assume, without loss of generality, that $y_3, y_4 \in V(H)$. By Lemma 2.9, $e(\{y_1, y_2, y_3, y_4\}, X - \{x_1, x_2, x_3, x_4\}) \geq 4(k+1) - 16 + 1 = 4k - 11 \geq 2k - 1$. Note that $e(x_i, \{y_1, y_2\}) \leq 1$ for $x_i \in X - \{x_1, x_2, x_3, x_4\}$. Thus, $X - \{x_1, x_2, x_3, x_4\}$ contains two vertices, say x_5, x_6, such that $e(x_i, \{y_1, y_2, y_3, y_4\}) \geq 2$ for $i \in \{5, 6\}$. By Lemma 2.3, $x_5, x_6 \in V(H)$. Thus, $|V(H) \cap X| \geq 4$ and $|V(H) \cap Y| \geq 4$. When $|N(x_3) \cap N(x_4)| = 3$, $V(H) \cap Y$ contains one vertex of $Y - \{y_1, y_2\}$. We assume, without loss of generality, that $y_3 \in V(H)$. Since $\delta \geq k + 1$, $e(\{y_1, y_2, y_3\}, X - \{x_1, x_2, x_3, x_4\}) \geq 3(k+1) - 12 = 3k - 9$. If $3k - 9 > 2k - 4$, then $k \geq 6$ and there is one vertex, say x_5, in $X - \{x_1, x_2, x_3, x_4\}$ such that $ex_5, \{y_1, y_2, y_3\} \geq 2$. By Lemma 2.3(6), $x_5 \in V(H)$ and $|V(H) \cap X| \geq 3$. If $3k - 9 = 2k - 4$, then $k = 5$. In this case, if there is a vertex x_i such that $e(x_i, \{y_1, y_2, y_3\}) \geq 2$, then $|V(H) \cap X| \geq 3$ and $|V(H) \cap Y| \geq 3$. Thus, we assume that $e(x_i, \{y_1, y_2, y_3\}) \leq 1$ for $i \in \{5, 6, \ldots, 10\}$. Since $\delta \geq 6$, $e(x_i, \{y_1, y_2, y_3\}) = 1$ for $i \in \{5, 6, \ldots, 10\}$. On the other hand, if there is a vertex $y_j \in \{y_4, \ldots, y_{10}\}$, say y_4, such that $e(y_4, \{x_3, x_4\}) \geq 2$. Then $|V(H) \cap Y| \geq 4$. Since $d(y_4) \geq 6$, y_4 must be adjacent to a vertex $x \in \{x_5, \ldots, x_{10}\}$. Hence $x \in V(H)$ and $|V(H) \cap X| \geq 3$. Thus, we assume that $e(y_j, \{x_3, x_4\}) = 1$ for $j \in \{4, 5, \ldots, 9\}$. It is easy to verify that the subgraph induced by $\{x_5, x_6, \ldots, x_{10}, y_4, \ldots, y_9\}$ has the minimum degree 2 and each vertex of the subgraph is adjacent to v^*. By Lemma 3.2, $|V(H) \cap X| \geq 4$ and $|V(H) \cap Y| \geq 5$. Thus, in each case we have $|V(H) \cap X| \geq 3$ and $|V(H) \cap Y| \geq 3$. Using the argument above, we can obtain $|V(H) \cap X| \geq 4$ and $|V(H) \cap Y| \geq 4$. □

By Claim, we may assume that $\{x_3, x_4, x_5, x_6\} \subseteq V(H) \cap X$. By the argument above as in the proof for case when $t \geq 6$, G is Z_3-connected. □

Lemma 3.11. *Let $G = (X, Y)$ be a simple bipartite graph on $17 \leq n \leq 20$ vertices. If $\delta \geq \lceil \frac{n}{4} \rceil + 1$, then G is Z_3-connected.*

Proof. Recall that $n_1 \leq n_2$. By Lemma 3.9, $t \geq 4$. By Lemma 3.6, $n_1 \geq 8$. Assume first that $n_1 = 8$ and $n_2 \in \{9, 10, 11, 12\}$. If $t \geq 5$, then $\{x_3, x_4, x_5\} \subseteq V(H)$. In this case, $\delta \geq 6$. By Lemma 3.1, G is Z_3-connected. Now suppose that $t = 4$. If $|X \cap V(H)| \geq 3$, then G is Z_3-connected by Lemma 3.1. Thus, assume that $|X \cap V(H)| = 2$. Since $\delta \geq 6$, $e(\{y_1, y_2\}, \{x_5, x_6, x_7, x_8\}) \geq 4$ and $e(x_i, \{y_1, y_2\}) \leq 1$ for $i \in \{5, 6, 7, 8\}$. This implies that $e(x_i, \{y_1, y_2\}) = 1$ for each $i \in \{5, 6, 7, 8\}$. If $V(H)$ contains a vertex of $Y - \{y_1, y_2\}$, say y_3, then $V(H)$ contains at least two vertices of $X - \{x_1, x_2, x_3, x_4\}$ since $e(y_3, \{x_5, x_6, x_7, x_8\}) \geq 2$. It follows that $|X - V(H)| \leq 8 - 4 \leq 5$. By Lemma 3.1, G is Z_3-connected. Thus, $V(H) \cap (Y - \{y_1, y_2\}) = \emptyset$. Since $e(\{x_3, x_4\}, Y - \{y_1, y_2\}) \geq 8$, v^* has at least 8 neighbors in $Y - \{y_1, y_2\}$. Let $y_{j_1}, y_{j_2} \in N_{G^*}(v^*)$. Since $\delta \geq 6$, $e(y_j, \{x_5, x_6, x_7, x_8\}) \geq 3$ for $j = j_1, j_2$. Hence there exist two vertices $x_{i_1}, x_{i_2} \in \{x_5, x_6, x_7, x_8\}$ such that $e(\{x_{i_1}, x_{i_2}\}, \{y_{j_1}, y_{j_2}\}) = 4$. Thus, we get an even wheel W_4 with the center at v^*, which is Z_3-connected by Lemma 2.3(2). Therefore, $V(H)$ contains at least 4 vertices of X and $|X - V(H)| \leq 5$. Thus, G is Z_3-connected by Lemmas 3.1 and 2.5.

Next, assume that $n_1 = 9$ and $n_2 \in \{9, 10, 11\}$. If $t \geq 6$, then $\{x_3, x_4, x_5, x_6\} \subseteq V(H)$. By Lemma 3.1, G is Z_3-connected. If $t = 5$, then $\{x_3, x_4, x_5\} \subseteq V(H)$. We claim that $|V(H) \cap X| \geq 4$. Suppose otherwise that $V(H) \cap X = \{x_3, x_4, x_5\}$. Since $\delta \geq 6$, $e(\{x_3, x_4, x_5\}, Y - \{y_1, y_2\}) \geq 12$. Since $|Y - \{y_1, y_2\}| \leq 9$, $V(H)$ contains at least two vertices of $Y - \{y_1, y_2\}$, say $y_3, y_4 \in V(H)$. This implies that $e_{G^*}(\{y_1, y_2, y_3, y_4\}, X - \{x_3, x_4, x_5\}) \geq 2 + 6 = 8$. Since $|X - \{x_3, x_4, x_5\}| \leq 6$, $V(H)$ contains at least one vertex of $X - \{x_3, x_4, x_5\}$, a contradiction. Hence $|V(H) \cap X| \geq 4$ and $|V(H) - X| \leq 5$. Thus, G is Z_3-connected by Lemma 3.1.

If $t = 4$, then $\{x_3, x_4\} \subseteq V(H)$. We claim that $|V(H) \cap X| \geq 4$. Suppose otherwise that $|V(H) \cap X| \leq 3$. Since $\delta \geq 6$, $e(\{y_1, y_2\}, \{x_5, x_6, x_7, x_8, x_9\}) \geq 4$. We assume, without loss of generality, that $v^*x_i \in E(G^*)$ for $i \in \{5, 6, 7, 8\}$. If $V(H)$ contains two vertices, say y_3, y_4, of $Y - \{y_1, y_2\}$, then $|N(y_3) \cap N(y_4)| \leq 4$ since $t = 4$. Thus, $|N(y_3) \cup N(y_4)| = |N(y_3)| + |N(y_4)| - |N(y_3) \cap N(y_4)| \geq 12 - 4 = 8$ and $|X - N(y_3) \cup N(y_4)| \leq 1$. Since $v^*x_i \in E(G^*)$ for $i \in \{5, 6, 7, 8\}$, $V(H)$ contains at least three vertices of $X - \{x_3, x_4\}$. Hence $|V(H) \cap X| \geq 4$. Otherwise $V(H)$ contains at most one vertex y of $Y - \{y_1, y_2\}$. In this case, v^* has at least six neighbors in $Y - \{y_1, y_2, y\}$, say, $y_3, y_4, y_5, y_6, y_7, y_8$. Since $e(y_j, \{x_5, x_6, x_7, x_8\}) \geq 2$ for $j \in \{3, 4, 5, 6, 7, 8\}$ and $e(x_i, \{y_3, y_4, y_5, y_6, y_7, y_8\}) \geq 2$ for $i \in \{3, 4, 5, 6\}$, the minimum degree of the subgraph induced by $\{x_5, x_6, x_7, x_8, y_3, y_4, y_5, y_6, y_7, y_8\}$ is at least 2. By Lemma 3.2, $|V(H) \cap X| \geq 4$. This implies that $|X - V(H)| \leq 5$. By Lemma 3.1, G is Z_3-connected.

We are left to the case when $n_1 = 10$. Since $n_1 \leq n_2$ and $n \leq 20$, $n_2 = 10$. By Lemma 3.10, G is Z_3-connected. □

Lemma 3.12. *Let $G = (X, Y)$ be a simple bipartite graph on $21 \leq n \leq 24$ vertices. If $\delta \geq \lceil \frac{n}{4} \rceil + 1$, then G is Z_3-connected.*

Proof. Since $21 \leq n \leq 24$, $\delta \geq 7$. By Lemma 3.9, $t \geq 4$. Recall that $n_1 \leq n_2$. By Lemma 3.6, $n_1 \geq 9$.

Assume first that $n_1 = 9$ and $n_2 \in \{12, 13, 14, 15\}$. By Observation 2.2, $t \geq 5$. Thus, $|X - V(H)| \leq 9 - 3 = 6$. By Lemma 3.1, G is Z_3-connected.

Next, we assume that $n_1 = 10$ and $n_2 \in \{11, 12, 13, 14\}$. By Observation 2.2, $t \geq 4$. We claim that $|V(H) \cap X| \geq 4$. If $t \geq 6$, then $|V(H) \cap X| \geq 4$ and we are done. If $t = 5$, $e(\{x_3, x_4, x_5\}, Y - \{y_1, y_2\}) \geq 15$ and $|Y - \{y_1, y_2\}| \leq 12$. It follows that $V(H)$ contains at least one vertex, say y_3, of $Y - \{y_1, y_2\}$. Since $e(\{y_1, y_2, y_3\}, \{x_6, x_7, x_8, x_9, x_{10}\}) \geq 3\delta - 15 \geq 6$, there is one vertex $x \in \{x_6, x_7, x_8, x_9, x_{10}\}$ such that $e(x, \{y_1, y_2, y_3\}) \geq 2$. By Lemma 2.3(6), $x \in V(H)$. This shows that $|V(H) \cap X| \geq 4$. If $t = 4$, then $e(x_i, \{y_1, y_2\}) = 1$ for $i \in \{5, 6, \ldots, 10\}$. Since $e(\{x_3, x_4\}, Y - \{y_1, y_2\}) \geq 2\delta - 4 \geq 10$ and $|Y - \{y_1, y_2\}| \leq 12$, $e(y_j, \{x_3, x_4\}) \leq 1$ for each $j \in \{3, 4, \ldots, n_2\}$ or there is some $k \in \{3, 4, \ldots, n_2\}$ such that $e(y_k, \{x_3, x_4\}) \geq 2$. In the former case, $e(y_j, \{x_5, x_6, \ldots, x_{10}\}) \geq 4$ for $y_j \in N_{G^*}(v^*) \cap (Y - \{y_1, y_2\})$ and $e(x_i, N_{G^*}(v^*) \cap (Y - \{y_1, y_2\})) \geq 4$ for $i \in \{5, 6, \ldots, 10\}$. Thus, the subgraph induced by $x_5, x_6, \ldots, x_{10}$ and $N_{G^*}(v^*) \cap (Y - \{y_1, y_2\})$ contains an even cycle. By Lemma 3.2, $|V(H) \cap X| \geq 4$. In the latter case, by Lemma 2.3(6), $y_k \in V(H)$. Since $e(y_k, \{x_5, x_6, \ldots, x_{10}\}) \geq \delta - 4 \geq 3$ and $N(y_1) \cup N(y_2) = X$, $V(H)$ contains at least three vertices of $\{x_5, x_6, \ldots, x_{10}\}$. Thus, $|V(H) \cap X| \geq 4$. In both cases, $|X - V(H)| \leq 6$. By Lemma 3.1, G is Z_3-connected.

We now assume that $n_1 = 11$ and $n_2 \in \{11, 12, 13\}$. By Lemma 3.1, $t \leq 6$. If $t = 6$, then $e(\{x_3, x_4, x_5, x_6\}, Y - \{y_1, y_2\}) \geq 4\delta - 8 \geq 20$. Thus, there are at least two vertices, say y_3, y_4, of $Y - \{y_1, y_2\}$ such that $e_{G^*}(y_j, \{x_3, x_4, x_5, x_6\}) \geq 2$ for $j \in \{3, 4\}$. By Lemma 2.3(6), $\{y_3, y_4\} \subset V(H)$. Since $e(\{y_1, y_2, y_3, y_4\}, X - \{x_3, x_4, x_5, x_6\}) \geq 4\delta - 16 \geq 28 - 16 = 12$, similarly, $V(H)$ contains at least one vertex of $X - \{x_3, x_4, x_5, x_6\}$ and $|V(H) \cap X| \geq 5$. Thus $|X - V(H)| \leq 6$. By Lemma 3.1, G is Z_3-connected. If $t = 5$, then $\{x_3, x_4, x_5\} \subset V(H)$. Since $\delta \geq 7$, $e(\{y_1, y_2\}, X - \{x_1, x_2, x_3, x_4, x_5\}) \geq 2\delta - 10 \geq 4$ and $e_{G^*}(\{x_3, x_4, x_5\}, Y - \{y_1, y_2\}) \geq 3\delta - 6 \geq 15$. Since $|Y - \{y_1, y_2\}| \leq 11$, G^* contains at least two vertices of $Y - \{y_1, y_2\}$, say y_3, y_4, such that $e_{G^*}(y_j, \{x_3, x_4, x_5\}) \geq 2$ for $j \in \{3, 4\}$. By Lemma 2.3(6), $y_3, y_4 \in V(H)$. Since $t = 5$, $|X - N(y_3) \cup N(y_4)| = |X| - |N(y_3)| - |N(y_4)| + |N(y_3) \cap N(y_4)| \leq 11 - 2\delta + 5 \leq 2$. On the other hand, $e_{G^*}(\{y_1, y_2\}, X - \{x_1, x_2, x_3, x_4, x_5\}) \geq 4$ and there is no vertex in $X - \{x_1, x_2, x_3, x_4, x_5\}$ such that $e_{G^*}(x, \{y_1, y_2\}) \geq 2$ since $t = 5$. This implies that there are two vertices, say x_6, x_7, of $X - \{x_1, x_2, x_3, x_4, x_5\}$ such that $e_{G^*}(x_i, \{y_1, y_2\}) \geq 1$ and $e_{G^*}(x_i, \{y_3, y_4\}) \geq 1$ for $i \in \{6, 7\}$. Thus, by Lemma 2.3(6), $x_6, x_7 \in V(H)$ and $|V(H) \cap X| \geq 5$. Hence $|X - V(H)| \leq 6$. By Lemma 3.1, G is Z_3-connected.

If $t = 4$, then $\{x_3, x_4\} \subseteq V(H)$. Since $\delta \geq 7$, $e_{G^*}(\{y_1, y_2\}, \{x_5, x_6, \ldots, x_{11}\}) \geq 2\delta - 8 \geq 6$. We assume, without loss of generality, that $v^*x_i \in E(G^*)$ for $i \in \{5, 6, \ldots, 10\}$. If H contains at least two vertices, say y_3 and y_4, of $Y - \{y_1, y_2\}$, then $|N(y_3) \cup N(y_4)| = |N(y_3)| + |N(y_4)| - |N(y_3) \cap N(y_4)| \geq 2\delta - 4 \geq 10$. Thus, H contains at least 7 vertices of X. By Lemma 3.1, G is Z_3-connected. Otherwise H contains at most one vertex y of $Y - \{y_1, y_2\}$. In this case, $e(\{x_3, x_4\}, Y - \{y_1, y_2, y\}) \geq 14 - 6 = 8$. Thus, v^* has at least six neighbors in $Y - \{y_1, y_2\}$, say, $y_4, y_5, \ldots, y_{10}$. Since $e(y_j, \{x_5, x_6, \ldots, x_{10}\}) \geq 3$ for $j \in \{4, 5, \ldots, 10\}$ and $e(x_i, \{y_4, y_5, \ldots, y_{10}\}) \geq 2$ for $i \in \{5, 6, \ldots, 10\}$, the minimum degree of the subgraph induced by $x_5, x_6, \ldots, x_{10}$ and $y_4, y_5, \ldots, y_{10}$ is at least 2. By Lemma 3.2, H contains at least 4 vertices of Y and four vertices of X. With the argument above, we conclude that G is Z_3-connected.

We are left to the case when $n_1 = 12$. Since $n_1 \leq n_2$ and $n \leq 24$, $n_2 = 12$. By Lemma 3.10, G is Z_3-connected. □

4. Proof of Theorem 1.4

By Lemmas 2.3 and 2.6, if $G \in \{K_{2,2}, K_{3,3}, K_{3,4}, K_{3,5}, G_1, G_2\}$, then G is not Z_3-connected.

Conversely, we prove our theorem by induction on $n = |V(G)|$. By the hypothesis of Theorem 1.4, when $n \leq 8$, $G \in \{K_{4,4}, G_3, G_4\}$, then G is Z_3-connected by Lemmas 2.3 and 2.9. When $9 \leq n \leq 24$, our theorem follows by Lemmas 3.3, 3.7, 3.8, 3.11 and 3.12. Suppose that $n \geq 25$ and our theorem follows for every graph with the number of vertices less than n.

By Lemma 3.5, we may assume that G does not contain a nontrivial Z_3-connected subgraph. We further assume that $|X| \leq |Y|$. Take two vertices y_1 and y_2 such that $|N(y_1) \cap N(y_2)|$ is as large as possible. Assume that $N(y_1) \cap N(y_2) = \{x_1, x_2, \ldots, x_t\}$. It follows from Observation 2.2 that $t \geq 2$. Thus, $C = x_1y_1x_2y_2x_1$ is a 4-cycle in G. Let H be a maximal Z_3-connected subgraph containing the 2-cycle (y_1, y_2) in $G_{[x_1,x_2;(y_1y_2)]}$. Let $G^* = G/H$ and let v^* denote the new vertex which H is contracted to. When $|X| \leq \lceil \frac{n}{4} \rceil + 2$, G is Z_3-connected by Lemma 3.6. Thus, we assume that $|X| \geq \lceil \frac{n}{4} \rceil + 3$.

Case 1. $\lceil \frac{n}{4} \rceil + 3 \le |X| \le 2\lceil \frac{n}{4} \rceil - 4$.

By Observation 2.2, $t \ge 2\delta - |X| \ge 2\lceil \frac{n}{4} \rceil + 2 - 2\lceil \frac{n}{4} \rceil + 4 = 6$. If $t = 6$, then $\{x_3, x_4, x_5, x_6\} \subseteq V(H) \cap X$. Since $e_G(\{y_1, y_2\}, X - \{x_3, x_4, x_5, x_6\}) \ge 2\delta - 12 \ge 2\lceil \frac{n}{4} \rceil + 2 - 12 = 2\lceil \frac{n}{4} \rceil - 4 - 6 \ge |X - \{x_1, x_2, x_3, x_4, x_5, x_6\}|$, $|X| = 2\lceil \frac{n}{4} \rceil - 4$ and v^* is adjacent to each vertex of $X - \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Note that $e_G(\{x_3, x_4, x_5, x_6\}, Y - \{y_1, y_2\}) \ge 4\delta - 8 \ge 4(\lceil \frac{n}{4} \rceil - 1) \ge n - 8$ and $|Y - \{y_1, y_2\}| \le n - (\lceil \frac{n}{4} \rceil + 3) - 2 \le \frac{3n}{4} - 5$. Since $n \ge 25$, $n - 8 > \frac{3n}{4} - 5$ and $V(H)$ contains at least one vertex, say y_3, of $Y - \{y_1, y_2\}$. Since $\delta \ge \lceil \frac{n}{4} \rceil + 1$, $e_G(y_3, X - \{x_3, x_4, x_5, x_6\}) \ge \lceil \frac{n}{4} \rceil - 3$. This implies that $V(H)$ contains at least $\lceil \frac{n}{4} \rceil - 3 + 4 - 2 = \lceil \frac{n}{4} \rceil - 1$ vertices of X. Since $\lceil \frac{n}{4} \rceil + 3 \le |X| \le 2\lceil \frac{n}{4} \rceil - 4$, $|X - V(H)| \le 2\lceil \frac{n}{4} \rceil - 4 - (\lceil \frac{n}{4} \rceil - 1) = \lceil \frac{n}{4} \rceil - 3 \le \delta - 4$. Thus, by Lemma 3.1, G is Z_3-connected.

If $t \ge 7$, then $\{x_3, x_4, x_5, x_6, x_7\} \subseteq V(H)$. Since $\delta(G) \ge \lceil \frac{n}{4} \rceil + 1$, $e_{G^*}(\{x_3, x_4, x_5, x_6, x_7\}, Y - \{y_1, y_2\}) \ge 5(\lceil \frac{n}{4} \rceil - 1)$. By the principle of pigeonhole, $V(H)$ contains at least two vertices, say y_3 and y_4, of $Y - \{y_1, y_2\}$. Let $N' = N(y_3) \cap N(y_4)$. By Observation 2.2, $|N'| \ge 6$. By Lemma 3.5, $|N' \cap \{x_1, x_2, \ldots, x_7\}| \le 3$, for otherwise the subgraph induced by $\{y_1, y_2, y_3, y_4, x_1, x_2, \ldots, x_6\}$ contains a $K_{4,4}$ which is Z_3-connected. It follows that $V(H)$ contains at least $5 + 6 - 3 = 8$ vertices of X. Thus, $|V(H) \cap V(G)| \ge 12$. If G^* is a K_1, then G is Z_3-connected by Lemmas 2.3(4) and 2.5 and so we are done. Thus we assume that $G^* \ne K_1$. In this case, $|V(G^* - v^*)| \le n - 12$ and $\delta(G^* - v^*) \ge \lceil \frac{n}{4} \rceil - 2 \ge \lceil \frac{n-12}{4} \rceil + 1 \ge \lceil \frac{|V(G^*) - v^*|}{4} \rceil + 1$. Moreover, note that $n \ge 25$, $\delta(G) \ge 8$ and for each vertex $u \in V(G^* - v^*) - \{x_1, x_2\}$, $d_{G^* - v^*}(u) \ge d_G(u) - 1$, for $u \in \{x_1, x_2\}$, $d_{G^*}(u) = d_G(u) - 2$. This means that for each vertex $u \in V(G^* - v^*)$, $d_{G^* - v^*}(u) \ge 5$. Thus $G^* - v^* \notin \mathcal{F}_{12}$ and $G^* - v^*$ is not one of $K_{2,2}, K_{3,3}, K_{3,4}, K_{3,5}, G_2$ and G_1. By the induction hypothesis, $G^* - v^*$ is Z_3-connected. Hence G^* is Z_3-connected by Lemma 2.3(6). By Lemmas 2.3(4) and 2.5, G is Z_3-connected.

Case 2. $2\lceil \frac{n}{4} \rceil - 3 \le |X| \le \lfloor \frac{n}{2} \rfloor$.

By Observation 2.2, $t \ge 2\delta - |X| \ge 2$. Let $k = \lceil \frac{n}{4} \rceil$. We claim that $t \ge 3$. Suppose otherwise that $t = 2$. It follows that $|N(y_i) \cap N(y_j)| \le 2$ for $i, j \in \{1, 2, 3\}$. Thus, $2k \ge |X| \ge |N(y_1) \cup N(y_2) \cup N(y_3)| = |N(y_1)| + N(y_2)| + |N(y_3)| - |N(y_1) \cap N(y_2)| - |N(y_1) \cap N(y_3)| - |N(y_2) \cap N(y_3)| + |N(y_1) \cap N(y_2) \cap N(y_3)| \ge 3(k + 1) - 6 = 3k - 3$, a contradiction. By Lemma 3.9, $t \ge 4$. Since $n \ge 25$, we may assume that $n \in \{4k - 3, 4k - 2, 4k - 1, 4k\}$, where $k \ge 7$. Recall that $n_1 \le n_2$. If $n_1 = n_2 = 2k$, by Lemma 3.10, G is Z_3-connected. Thus, we assume that $n_1 \le 2k - 1$. In this case, since $|X| \ge 2k - 3$, $|Y| \le 4k - (2k - 3) = 2k + 3$.

When $t \ge 6$, $e(\{x_3, x_4, x_5, x_6\}, Y - \{y_1, y_2\}) \ge 4\delta - 8 \ge 4k - 4$. By Lemma 3.5, G has no Z_3-connected subgraphs. Thus, G contains neither $K_{4,4}$ nor G_3 or G_4. Let $d_i^* = e(y_i, \{x_3, x_4, x_5, x_6\})$. We relabel vertices of Y if necessary so that $d_3^* \ge d_4^* \ge \cdots \ge d_n^*$. Since G contains neither $K_{4,4}$ nor G_3 or G_4. It follows that $(d_3^*, d_4^*) \ne (4, 4), (4, 3)$. This means that $d_3^* \le 3$ and $d_i^* \le 2$ for $i = 4, 5, \ldots, n_2$. On the other hand, note that $|Y - \{y_1, y_2\}| \le 2k + 1$. We claim that $Y - \{y_1, y_2\}$ has 4 vertices, each of which has at least 2 neighbors in $\{x_3, x_4, x_5, x_6\}$. Suppose otherwise that $Y - \{y_1, y_2\}$ has only 3 vertices each of which has at least 2 neighbors in $\{x_3, x_4, x_5, x_6\}$. It follows that $e(Y - \{y_1, y_2\}, \{x_3, x_4, x_5, x_6\}) \le 3 + 2 \times 2 + (2k + 1 - 3)$. Thus, $7 + 2k - 2 \ge 4k - 4$, which implies $k \le 3$, contrary to that $k \ge 7$. So, we assume that $\{y_3, y_4, y_5, y_6\}$ are such four vertices in $Y - \{y_1, y_2\}$. Let $X_1 \subseteq X - \{x_3, x_4, x_5, x_6\}$ such that each $x \in X_1$ has at most one neighbor in $\{y_1, y_2, \ldots, y_5, y_6\}$ and $\ell = |X_1|$. By Lemmas 3.5 and 2.3, G has no $K_{4,4}$ as a subgraph. This means that $X - \{x_3, x_4, x_5, x_6\}$ contains at most three vertices each of which is adjacent to all vertices in $\{y_3, y_4, y_5, y_6\}$. Thus $e(\{y_3, y_4, y_5, y_6\}, X - \{x_3, x_4, x_5, x_6\}) \le 4 \times 3 + 3(n_1 - 4 - 3 - \ell) + \ell = 3n_1 - 2\ell - 9$. On the other hand, since G has no $K_{4,4}$ nor G_3 or G_4 as a subgraph, $e(\{y_3, y_4, y_5, y_6\}, \{x_3, x_4, x_5, x_6\}) \le 14$. Thus, $e(\{y_3, y_4, y_5, y_6\}, X - \{x_3, x_4, x_5, x_6\}) \ge 4\delta - 14 \ge 4k - 10$. This means that $4k - 10 \le 3n_1 - 2\ell - 9$. Since $n_1 \le 2k - 1$, $\ell \le k - 1$. By Lemma 3.1, G is Z_3-connected. Thus, we assume that $t \le 5$.

Assume first that $t = 5$. Since $2k - 1 \ge n_1 \ge 2k - 3$, $|Y - \{y_1, y_2\}| \le 2k + 1$. We claim that $V(H)$ contains at least two vertices of $Y - \{y_1, y_2\}$. Suppose otherwise that $V(H)$ contains at most one vertex y of $Y - \{y_1, y_2\}$. It follows that y is adjacent to at most three vertices of $\{x_3, x_4, x_5\}$ and for each vertex $y' \in Y - \{y_1, y_2, y\}$, $e(y', \{x_3, x_4, x_5\}) \le 1$. Thus, $e(\{x_3, x_4, x_5\}, Y - \{y_1, y_2\}) \le 3 + (2k + 1) - 1 = 2k + 3$. On the other hand, since $\delta(G) \ge \lceil \frac{n}{4} \rceil + 1 = k + 1$, $e_G(\{x_3, x_4, x_5\}, Y - \{y_1, y_2\}) \ge 3(k - 1)$. It leads to that $3k - 3 \le 2k + 3$, which implies that $k \le 6$, contrary to our assumption that $k \ge 7$. We assume, without loss of generality, that $y_3, y_4 \in V(H)$. Since $t = 5$, $|N(y_3) \cup N(y_4)| \ge |N(y_3)| + |N(y_4)| - t \ge 2k + 2 - 5 = 2k - 3$. On the other hand, $e(\{y_1, y_2\}, \{x_6, \ldots, x_{n_1}\}) \ge 2\delta - 10 \ge 2(k + 1) - 10 = 2k - 8$. Since $2k - 3 \le n_1 \le 2k - 1$, we may assume that $e(x_i, \{y_1, y_2\}) = 1$ for $i \in \{6, 7, \ldots, n_1 - 2\}$. Let $A = N(y_3) \cup N(y_4) \setminus \{x_1, x_2, \ldots, x_5\}$ and $B = \{x_i : x_iy_1 \in E(G) \text{ or } x_iy_2 \in E(G), i \in \{6, 7, \ldots, n_1\}\}$. Thus, $x \in V(H)$ if and only if either $x \in \{x_3, x_4, x_5\}$ or $x \in A \cap B$. Note that $|A \cap B| = |A| + |B| - |A \cup B| \ge 2k - 8 + (n_1 - 2 - 5) - (n_1 - 5) = 2k - 10$. Thus, $|V(H) \cap X| \ge 2k - 10 + 3 = 2k - 7$. This means that $V(H)$ contains at least $2k - 7$ vertices of X. Thus, since $2k - 3 \le n_1 \le 2k - 1$, $|X - V(H)| \le 2k - 1 - (2k - 7) = 6$. By Lemma 3.1, G is Z_3-connected.

Next, assume $t = 4$. By Observation 2.2, $n_1 \in \{2k - 2, 2k - 1\}$. In this case, $\{x_3, x_4\} \subseteq V(H)$ and $\delta \ge k + 1$, $e(\{y_1, y_2\}, X - \{x_1, x_2, x_3, x_4\}) \ge 2k - 6$. When $n_1 = 2k - 2$, $e(x_i, \{y_1, y_2\}) \le 1$ for $i = 5, 6, \ldots, n_1$. This implies that $e(x_i, \{y_1, y_2\}) = 1$ for $i = 5, 6, \ldots, n_1$. When $n_1 = 2k - 1$, we assume, without loss of generality, that $e(x_i, \{y_1, y_2\}) = 1$ for $i \in \{5, 6, \ldots, n_1 - 1\}$.

We claim that $Y - \{y_1, y_2\}$ contains at least one vertex y' such that $e(y', \{x_3, x_4\}) = 2$. Suppose otherwise that for each y_j, where $j \in \{3, 4, \ldots, n_2\}$, $e(y_j, \{x_3, x_4\}) \le 1$. In this case, since $e(\{x_3, x_4\}, Y - \{y_1, y_2\}) \ge 2k + 2 - 4 = 2k - 2$. We may assume that $e(y_j, \{x_3, x_4\}) = 1$ for $j \in \{3, 4, \ldots, n_2 - 2\}$. Let Γ be the subgraph induced by $x_5, x_6, \ldots, x_{n_1 - 1}, y_3, y_4, \ldots, y_{n_2 - 3}$ and $y_{n_2 - 2}$. Note that $e(y_j, \{x_5, x_6, \ldots, x_{n_1 - 1}\}) \ge k + 1 - 4 \ge 4$ for $j \in \{3, 4, 5, \ldots, n_2 - 2\}$ and $e(x_i, \{y_3, y_4, \ldots, y_{n_2 - 2}\}) \ge k + 1 - 3 \ge 5$ for $i \in \{5, 6, \ldots, n_1 - 1\}$. This means that $\delta(\Gamma) \ge 4$. By Lemma 3.2, H contains at least two vertices of $Y - \{y_1, y_2\}$, a contradiction. Therefore, we assume, without loss of generality, that $y_3 \in V(H)$ is adjacent to both x_3, x_4. Repeating the procedure above, we get $Y - \{y_1, y_2, y_3\}$ contains at least one vertex, say y_4, such that $e(y_4, \{x_3, x_4\}) = 2$.

Thus, $y_3, y_4 \in V(H)$. Since $e(\{y_3, y_4\}, X - \{x_1, x_2, x_3, x_4\}) \geq 2(k+1) - 8 = 2k - 6$, $V(H)$ contains at least $2k - 6 + 2 - 1$ vertices of X. Thus, $|X - V(H)| \leq (2k - 1) - (2k - 5) = 4$. By Lemma 3.1, G is Z_3-connected. □

5. Proof of Theorem 1.3

Lemma 5.1. *Suppose that G is a simple bipartite graph with $n \leq 8$. If $\delta \geq \lceil \frac{n}{4} \rceil + 1$ and G is not G_1, then G admits a nowhere-zero 3-flow.*

Proof. Suppose that $n \leq 4$. Since G is a simple bipartite graph with $\delta \geq 2$, G must be $K_{2,2}$. Thus G admits a nowhere-zero 3-flow.

Suppose that $5 \leq n \leq 8$. When $n = 5$, there is not bipartite graph for $\delta \geq 3$. When $n = 6, 7$, clearly G admits a nowhere-zero 3-flow since G is a complete bipartite graph. When $n = 8$, by the hypothesis, $G \in \{K_{4,4}, G_2, G_3, G_4\}$. If G is $K_{4,4}$, then G admits a nowhere-zero 3-flow by Lemma 2.3(1). If G is G_2, then G admits a nowhere-zero 3-flow since G is a cubic bipartite graph. If $G \in \{G_3, G_4\}$, then G is Z_3-connected by Lemma 2.9. Thus G admits a nowhere-zero 3-flow. □

Lemma 5.2. *Suppose that $G = (X, Y)$ is a simple bipartite graph on 12 vertices. If $\delta \geq 4$, then G admits a nowhere-zero 3-flow.*

Proof. Assume that $|X| \leq |Y|$. Since $\delta \geq 4$, $|Y| \geq |X| \geq 4$. When $|X| = 4$, G is a complete bipartite graph. By Lemma 2.3(1), G is Z_3-connected, and so G admits a nowhere-zero 3-flow. When $|X| = 5$ and $|Y| = 7$, $\sum_{v \in X} d(v) = \sum_{v \in Y} d(v) \geq 28$. It follows that at least two vertices of X have degree more than 5. Assume $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, \ldots, y_7\}$. We assume, without loss of generality, that $d(x_i) \geq 6$ for $i \in \{1, 2\}$. Let $N(x_1) \cap N(x_2) = \{y_1, y_2, \ldots, y_t\}$. By Observation 2.2, $t \geq 5$. Note that $C = x_1 y_1 x_2 y_2 x_1$ is a 4-cycle of G, and $G_{[y_1, y_2; (x_1 x_2)]}$ contains a 2-cycle (x_1, x_2). Iteratively contracting 2-cycles leads eventually to a K_1. By Lemmas 2.3 and 2.5, G is Z_3-connected, and so G admits a nowhere-zero 3-flow.

Suppose $|X| = |Y| = 6$. Assume first that G has two vertices of degree more than 4 in the same partition. We further assume, without loss of generality, that $y_j \in Y$ such that $d(y_j) \geq 5$ for $j \in \{1, 2\}$. In this case, $|N(y_1) \cap N(y_2)| \geq 4$ by Observation 2.2. Let $x_1, x_2, x_3, x_4 \in N(y_1) \cap N(y_2)$. Let $G^* = G_{[x_1, x_2; (y_1 y_2)]}$, H be the maximum Z_3-connected subgraph containing 2-cycle (y_1, y_2) and v^* be the new vertex which H is contracted to. We claim that $|V(H) \cap X| \geq 4$. Suppose otherwise that $|V(H) \cap X| \leq 3$. Let $|V(H) \cap X| = 2$. If $|V(H) \cap Y| = 2$, then $e(x_i, \{y_1, y_2\}) = 1$ for $i \in \{5, 6\}$ and $e(y_j, \{x_3, x_4\}) = 1$ for $j \in \{3, 4, 5, 6\}$. Since $\delta \geq 4$, $e_{G^*}(x_i, \{y_3, y_4, y_5, y_6\}) \geq 3$ for each $i \in \{5, 6\}$. Thus, there are $y', y'' \in \{y_3, y_4, y_5, y_6\}$ such that the subgraph induced by $\{x_5, x_6, y', y''\}$ is a $K_{2,2}$. By Lemma 3.2, $|V(H) \cap X| \geq 4$. If $|V(H) \cap Y| = 3$, then the subgraph induced by $\{x_5, x_6, y_4, y_5, y_6\}$ is $K_{2,3}$, $e(x_i, \{y_1, y_2\}) = 1$ for $i \in \{5, 6\}$, and $e(\{y_4, y_5, y_6\}, \{x_3, x_4\}) = 2$. By Lemma 3.2, $|V(H) \cap X| \geq 4$. Let $|V(H) \cap X| = 3$ and $V(H) \cap X = \{x_3, x_4, x_5\}$. If $|V(H) \cap Y| = 2$, then $e(\{x_3, x_4, x_5\}, \{y_3, y_4, \ldots, y_6\}) \geq 6$. Thus, there is one vertex $y \in \{y_3, y_4, \ldots, y_6\}$ such that $e(y, \{x_3, x_4, x_5\}) \geq 2$. By Lemma 2.3(6), $y \in V(H)$ and $|V(H) \cap Y| = 3$. Let $V(H) \cap Y = \{y_1, y_2, y_3\}$. Since $e(y_j, \{x_3, x_4, x_5\}) \leq 1$, the subgraph induced by $\{x_1, x_2, x_6, y_4, y_5, y_6\}$ is $K_{3,3}$. Let $G' = G_{[y_5, y_6, (x_1, x_2)]}$ and H' be the maximum Z_3-connected subgraph. Then $\{x_1, x_2, x_3, x_4, x_5\} \subseteq V(H')$. Iteratively contracting 2-cycles generating in the processing leads eventually to a K_1. By Lemmas 2.3(4) and 2.5, G is Z_3-connected, and so G admits a nowhere-zero 3-flow. Note that when $|V(H) \cap X| \geq 4$, each vertex of Y has at least two neighbors in $V(H) \cap X$. By Lemma 2.3(6), $Y \subseteq V(H)$ and so $G^* = H$. Similarly, G is Z_3-connected, and so G admits a nowhere-zero 3-flow.

Now we assume that there are only two vertices of degree 5 with one in X and the other in Y. We claim that G is 3-edge-connected. Suppose otherwise that G_1 and G_2 are two components of a 2-edge-cut and (X_i, Y_i) is the bipartition of G_i, where $i \in \{1, 2\}$. If $|X_i| < 4$ or $|Y_i| < 4$, then it contradicts that $\delta \geq 4$. Thus, $|X_i| \geq 4$ and $|Y_i| \geq 4$ for $i \in \{1, 2\}$, which implies $|V(G)| \geq 16$, a contradiction. Let P_1, P_2, P_3 be 3 edge-disjoint paths between the two vertices of degree 5 in G. Clearly the graph H induced by $E(P_1) \cup E(P_2) \cup E(P_3)$ admits a nowhere-zero 3-flow f_1. Since $G - E(H)$ is an even graph and admits a nowhere-zero 2-flow f_2. Therefore $f = f_1 + f_2$ is a nowhere-zero 3-flow of G.

Finally, G has no odd vertex and is Eulerian, and so G admits a nowhere-zero 2-flow. □

Proof of Theorem 1.3. Suppose that G is not G_1. When $n \leq 8$, by Lemma 5.1, G admits a nowhere-zero 3-flow. When $9 \leq n \leq 11$, by Lemma 3.3, G is Z_3-connected and so G admits a nowhere-zero 3-flow. When $n = 12$, G admits a nowhere-zero 3-flow by Lemma 5.2. When $n \geq 13$, G is Z_3-connected by Theorem 1.4 and so G admits a nowhere-zero 3-flow. Conversely, the result follows by Lemma 2.6. □

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Appendix

Here we give the detail of the proof of Lemma 2.9. Recall that G denotes the graph G_3 depicted in Fig. 3. For this purpose, we first establish four claims.

Claim A.1. *If $b \in \mathbb{Z}(G, Z_3)$ such that $b(x_3) \neq 0$, $b(x_4) \neq 0$, then there is an $f \in F^*(G, Z_3)$ such that $\partial f = b$.*

Proof of Claim A.1. Assume that $b \in \mathbb{Z}(G, Z_3)$ such that $b(x_3) \neq 0$, $b(x_4) \neq 0$. Let $H = G_{(x_3y_3)}$. Define $b' : V(G) \setminus \{x_3\} \to Z_3$ as follows: $b'(y_3) = b(x_3) + b(y_3)$ and $b'(u) = b(u)$ for any other vertex u. Then $b' \in \mathbb{Z}(H, Z_3)$. It is easy to see that $H_{(x_4y_4)}$ contains a 2-cycle (y_1, y_2). Iteratively contracting 2-cycles generated in the processing leads eventually to a K_1, which is Z_3-connected. By Lemma 2.3(4), $H_{(x_4y_4)}$ is Z_3-connected. Thus, by Lemma 2.7, there exists an $f' \in F^*(H, Z_3)$ with $\partial f' = b'$. We now extend such an $f' \in F^*(H, Z_3)$ to an $f \in F^*(G, Z_3)$ such that $\partial f = b$. We assume, without loss of generality, that the new edge y_1y_2 is oriented from y_1 to y_2 and assume that the edge y_1x_3 is oriented from y_1 to x_3, the edge y_2x_3 from x_3 to y_2 and the edge x_3y_3 from x_3 to y_3. Define $f(x_3y_3) = b(x_3)$, $f(y_1x_3) = f(y_2x_3) = f'(y_1y_2)$ and for any other $e \in E(G)$, let $f(e) = f'(e)$. It is easy to check that $f \in F^*(G, Z_3)$ and $\partial f = b$. □

Claim A.2. *If $b(x_4) \neq 0$ and $0 \in \{b(x_1), b(x_2)\}$, then there is an $f \in F^*(G, Z_3)$ such that $\partial f = b$.*

Proof of Claim A.2. By symmetry, we assume that $b(x_1) = 0$. Let H denote the graph from G by removing x_1 and adding two edges y_1y_2 and y_3y_4. Define $b' : V(G) \setminus \{x_1\} \to Z_3$ by $b'(v) = b(v)$ for $v \in V(G) - \{x_1\}$. Clearly, $b' \in \mathbb{Z}(H, Z_3)$. It is easy to see that $H_{(x_4y_4)}$ contains a 2-cycle (y_1, y_2). Iteratively contracting 2-cycles generated in the processing leads eventually to a K_1, which is Z_3-connected. By Lemma 2.3(4), $H_{(x_4y_4)}$ is Z_3-connected. Thus, by Lemma 2.7, there exists an $f' \in F^*(H, Z_3)$ with $\partial f' = b'$. We now extend such an $f' \in F^*(H, Z_3)$ to an $f \in F^*(G, Z_3)$ as follows. We assume, without loss of generality, that y_1y_2 is oriented from y_1 to y_2, y_3y_4 from y_3 to y_4, y_1x_1 from y_1 to x_1, x_1y_2 from x_1 to y_2, y_3x_1 from y_3 to x_1 and x_1y_4 from x_1 to y_4. Define $f(y_1x_1) = f(x_1y_2) = f'(y_1y_2)$, $f(y_3x_1) = f(x_1y_4) = f'(y_3y_4)$ and $f(e) = f'(e)$ for all other edges of G. It is easy to verify that $\partial f = b$. □

Claim A.3. *If two of $\{b(y_1), b(y_2), b(x_1), b(x_2)\}$ are zero, then there is an $f \in F^*(G, Z_3)$ such that $\partial f = b$.*

Proof of Claim A.3. By symmetry, assume first that $b(y_1) = b(y_2) = 0$. Let H denote the graph from G by removing y_1 and adding two edges x_1x_2 and x_3x_4. Define $b' : V(G) \setminus \{y_1\} \to Z_3$ by $b'(v) = b(v)$ for $v \in V(G) - \{y_1\}$. Clearly, $b' \in \mathbb{Z}(H, Z_3)$. It is easy to see that $H_{(x_1x_2, x_3x_4)}$ contains two 2-cycles (x_1, x_2) and (x_3, x_4). Iteratively contracting 2-cycles generated in the processing leads eventually to a K_1, which is Z_3-connected. By Lemma 2.3(4), $H_{(x_1x_2, x_3x_4)}$ is Z_3-connected. Thus, by Lemma 2.8, there exists an $f' \in F^*(H, Z_3)$ with $\partial f' = b'$. We now extend such an $f' \in F^*(H, Z_3)$ to an $f \in F^*(G, Z_3)$ as follows. We assume, without loss of generality, that x_1x_2 is oriented from x_1 to x_2, x_3x_4 from x_3 to x_4, x_1y_1 from x_1 to y_1, y_1x_2 from y_1 to x_2, x_3y_1 from x_3 to x_1 and y_1x_4 from y_1 to x_4. Define $f(x_1y_1) = f(y_1x_2) = f'(x_1x_2)$, $f(x_3y_1) = f(y_1x_4) = f'(x_3x_4)$ and $f(e) = f'(e)$ for all other edges of G. It is easy to verify that $\partial f = b$.

Next assume that $b(x_1) = b(y_1) = 0$. Let H be the graph from G by removing x_1 and y_1 and adding edges x_2y_2, x_3x_4 and y_3y_4. Then H contains a 2-cycle. By contracting this 2-cycle, we obtain an even wheel W_4, which is Z_3-connected by Lemma 2.3(2). By Lemma 2.8, there are an orientation D and an $f \in F^*(G, Z_3)$ such that $\partial f = b$. □

Claim A.4. *If $b(x_4) \neq 0$ and $b(x_4) + b(y_4) = 0$, then there is an $f \in F^*(G, Z_3)$ such that $\partial f = b$.*

Proof of Claim A.4. Assume that $b \in \mathbb{Z}(G, Z_3)$ such that $b(x_4) \neq 0$ and $b(x_4) + b(y_4) = 0$. It follows that $b(y_4) \neq 0$. Let $H = G_{(y_4x_4)}$. In this case, let $b' : V(H) \to Z_3$ by $b'(v) = b(v)$ if $v \notin \{x_4, y_4\}$ and $b'(x_4) = b(x_4) + b(y_4) = 0$ otherwise. Let H_1 be the graph from H by removing x_4 and adding y_1y_2. On other word, H_1 consists of K_4 and an edge with one end vertex adjacent to two vertices of the K_4 and the other end vertex adjacent to the other two vertices of the K_4. Let $b'' : V(H_1) \to Z_3$ by $b''(v) = b(v)$. It is easy to verify that $b'' \in \mathbb{Z}(H_1, Z_3)$. By Theorem [11, Theorem 1.8], H_1 is Z_3-connected. Thus, there is a function $f_1 \in F^*(H_1, Z_3)$ such that $\partial f_1 = b''$. As the argument of Claim A.1, there is a function $f \in F^*(G, Z_3)$ such that $\partial f = b$. □

By Claims A.1–A.4 and by symmetry, we only need to verify 25 different cases for $b \in \mathbb{Z}(G, Z_3)$. For each case, the reader can find a function $f \in F^*(G, Z_3)$ such that $\partial f = b$.

Case 1. $b(x_4) \neq 0$ and $b(y_4) \neq 0$.

By Claims A.1 and A.2, $0 \notin \{b(x_1), b(x_2), b(y_1), b(y_2)\}$ and $b(x_3) = b(y_3) = 0$. If $b(x_4) = b(y_4) = 1$, then by symmetry either $b(x_1) = b(x_2) = b(y_1) = 2$ and $b(y_2) = 1$ or $b(x_1) = b(x_2) = b(y_1) = b(y_2) = 1$. The former case is Case 1 and the latter case is Case 2 in Table 1. If $b(x_4) = 2$ and $b(y_4) = 2$, then by symmetry $b(x_1) = 2$ and $b(x_2) = b(y_1) = b(y_2) = 1$ or $b(x_1) = b(y_1) = b(x_2) = b(y_2) = 2$. The former case is Case 3 and the latter case is Case 4 in Table 1.

Case 2. $b(x_3) \neq 0$ and $b(y_4) \neq 0$.

By Claims A.1, A.2 and A.4, $0 \notin \{b(x_1), b(x_2), b(y_1), b(y_2)\}$ and $b(x_4) = b(y_3) = 0$. If $b(x_3) = b(y_4) = 1$, then by symmetry either $b(x_1) = b(x_2) = b(y_1) = 2$ and $b(y_2) = 1$ or $b(x_1) = b(x_2) = b(y_1) = b(y_2) = 1$. The former case is Case 5 and the latter case is Case 6 in Table 1. If $b(x_3) = 2$ and $b(y_4) = 1$, then by symmetry $b(x_1) = b(x_2) = 2$ and $b(y_1) = b(y_2) = 1$ or $b(x_1) = b(x_2) = 1$ and $b(y_1) = b(y_2) = 2$ or $b(x_1) = b(y_1) = 2$ and $b(x_2) = b(y_2) = 1$. They are Cases 7–9 in Table 1. If $b(x_3) = 2$ and $b(y_4) = 2$, then by symmetry $b(x_1) = 2$ and $b(x_2) = b(y_1) = b(y_2) = 1$ or $b(x_1) = b(y_1) = b(x_2) = b(y_2) = 2$. They are Cases 10 and 11 in Table 1.

Case 3. $b(x_3) \neq 0$ and $b(y_4) = b(x_4) = b(y_3) = 0$.

In this case, $0 \notin \{b(x_1), b(x_2)\}$. By Claim A.3, at most one of $\{b(y_1), b(y_2)\}$ is zero. If $b(x_3) = 1$, then by symmetry $b(x_1) = 2$ and $b(x_2) = b(y_1) = b(y_2) = 1$ or $b(y_1) = 2$ and $b(x_2) = b(x_1) = b(y_2) = 1$ or $b(x_1) = b(x_2) = b(y_1) = b(y_2) = 2$ or $b(x_1) = 2$, $b(x_2) = 1$, $b(y_1) = 2$ and $b(y_2) = 0$ or $b(x_1) = 2$, $b(x_2) = 2$, $b(y_1) = 1$ and $b(y_2) = 0$. They are

Table 1
25 cases.

Case	b	f
1	(2, 2, 0, 1, 2, 1, 0, 1)	(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 2, 1)
2	(1, 1, 0, 1, 1, 1, 0, 1)	(2, 2, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2)
3	(2, 1, 0, 2, 1, 1, 0, 2)	(2, 2, 2, 2, 2, 2, 2, 1, 2, 2, 2, 2, 2, 1)
4	(2, 2, 0, 2, 2, 2, 0, 2)	(1, 1, 2, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1)
5	(2, 2, 1, 0, 2, 1, 0, 1)	(1, 1, 1, 2, 2, 1, 1, 1, 2, 1, 1, 2, 2, 2)
6	(1, 1, 1, 0, 1, 1, 0, 1)	(1, 2, 2, 2, 2, 1, 2, 2, 1, 1, 2, 1, 1, 1)
7	(2, 2, 2, 0, 1, 1, 0, 1)	(1, 1, 1, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1, 1)
8	(1, 1, 2, 0, 2, 2, 0, 1)	(2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 1, 1, 1)
9	(2, 1, 2, 0, 2, 1, 0, 1)	(2, 2, 2, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2)
10	(2, 1, 2, 0, 1, 1, 0, 2)	(1, 1, 2, 1, 1, 2, 2, 2, 2, 1, 2, 1, 1, 1)
11	(2, 2, 2, 0, 2, 2, 0, 2)	(2, 1, 1, 1, 1, 2, 1, 1, 2, 2, 1, 2, 2, 2)
12	(2, 1, 1, 0, 1, 1, 0, 0)	(2, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1)
13	(1, 1, 1, 0, 2, 1, 0, 0)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1)
14	(2, 2, 1, 0, 2, 2, 0, 0)	(2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 1)
15	(2, 1, 1, 0, 2, 0, 0, 0)	(2, 2, 2, 2, 2, 1, 2, 2, 1, 1, 2, 2, 2, 2)
16	(2, 2, 1, 0, 1, 0, 0, 0)	(1, 1, 1, 2, 1, 1, 1, 2, 1, 2, 1, 2, 2, 2)
17	(1, 2, 2, 0, 2, 2, 0, 0)	(2, 1, 2, 2, 2, 2, 2, 2, 1, 2, 2, 2, 2, 2)
18	(2, 2, 2, 0, 1, 2, 0, 0)	(2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 2, 2, 2, 2)
19	(1, 1, 2, 0, 1, 1, 0, 0)	(2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2)
20	(2, 1, 2, 0, 1, 0, 0, 0)	(1, 2, 1, 1, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1)
21	(1, 1, 2, 0, 2, 0, 0, 0)	(2, 2, 2, 1, 2, 2, 2, 1, 2, 1, 2, 1, 1, 1)
22	(1, 1, 0, 0, 2, 2, 0, 0)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
23	(1, 2, 0, 0, 1, 2, 0, 0)	(2, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)
24	(2, 2, 0, 0, 2, 0, 0, 0)	(1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1)
25	(1, 1, 0, 0, 1, 0, 0, 0)	(2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 2, 2, 2, 2)

Cases 12–16 in Table 1. If $b(x_3) = 2$, then by symmetry $b(x_1) = 1$ and $b(x_2) = b(y_1) = b(y_2) = 2$ or $b(y_1) = 1$ and $b(x_2) = b(x_1) = b(y_2) = 2$ or $b(x_1) = b(x_2) = b(y_1) = b(y_2) = 1$ or $b(x_1) = 2, b(x_2) = 1, b(y_1) = 1$ and $b(y_2) = 0$ or $b(x_1) = 1, b(x_2) = 1, b(y_1) = 2$ and $b(y_2) = 0$. They are Cases 17–21 in Table 1.

Case 4. $b(x_3) = b(y_4) = b(x_4) = b(y_3) = 0$.

In this case, there are Cases 22–25 in Table 1: $b(x_1) = b(x_2) = 1$ and $b(y_1) = b(y_2) = 2$; $b(x_1) = b(y_1) = 1$ and $b(x_2) = b(y_2) = 2$; $b(x_1) = b(x_2) = b(y_1) = 2$ and $b(y_2) = 0$; $b(x_1) = b(x_2) = b(y_1) = 1$ and $b(y_2) = 0$.

For each b in above four cases, we want to find an $f \in F^*(G, Z_3)$ such that $\partial f = b$. For this purpose, we assume the edges are oriented from X to Y in G and we use vectors to represent a $b \in \mathbb{Z}(G, Z_3)$ and an $f \in F^*(G, Z_3)$, respectively, where $b = (b(x_1), b(x_2), b(x_3), b(x_4), b(y_1), b(y_2), b(y_3), b(y_4))$ and $f = (f(x_1y_1), f(x_1y_2), f(x_1y_3), f(x_1y_4), f(x_2y_1), f(x_2y_2), f(x_2y_3), f(x_2y_4), f(x_3y_1), f(x_3y_2), f(x_3y_3), f(x_4y_1), f(x_4y_2), f(x_4y_4))$. Then f is responding to the b in each row in the following table. Thus, G_3 is Z_3-connected.

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