The initial boundary value problem for quasi-linear wave equation with viscous damping

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Abstract

In this paper, the existence and uniqueness of the local generalized solution and the local classical solution for the initial boundary value problem of the quasi-linear wave equation with viscous damping are proved. The nonexistence of the global solution for this problem is discussed by an ordinary differential inequality. Finally, an example is given.

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1. Introduction

In this paper, we are concerned with the following initial boundary value problem:

\[ u_{tt} - \sigma(u_x)_x - u_{xxt} + \delta |u_t|^{p-1} u_t = \mu |u|^{q-1} u, \quad x \in \Omega, \ t > 0, \]  
\[ u(0, t) = 0, \quad u(1, t) = 0, \quad t \geq 0, \]  
\[ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad x \in \Omega, \]  

where \( \delta > 0, \mu > 0, p \geq 1, q > 1 \) are constants, \( \sigma(s) \) is a given nonlinear function, \( \varphi(x) \) and \( \psi(x) \) are given initial value functions, \( \Omega = (0, 1) \) and subscripts \( x \) and \( t \) indicate the partial
derivative with respect to $x$ and $t$, respectively. Equations of type (1.1) are a class of nonlinear evolution equations governing the motion of a viscoelastic solid composed of the material of the rate type; see [1,2,4,6]. It can also be seen as field equation governing the longitudinal motion of a viscoelastic bar obeying the nonlinear Voigt model; see [3]. When $\delta = \mu = 0$, there have been many impressive works on the global existence and other properties of solutions of Eq. (1.1); see [1,2,5,8]. In special, in [9] the authors have proved the global existence and uniqueness of many impressive works on the global existence and other properties of solutions of Eq. (1.1); a viscoelastic bar obeying the nonlinear V oigt model; see [3].

But about the blow-up of the solution for problem (1.1)–(1.3) there has not been any discussion.

In the present paper, under certain conditions we prove that problem (1.1)–(1.3) admits a unique local generalized solution and local classical solution. To study the blow-up of the solution for problem (1.1)–(1.3), we first establish an ordinary differential inequality (see Lemma 3.2), next we apply this inequality to give the sufficient conditions of blow-up of the solution for problem (1.1)–(1.3). To this end, we also need to prove the existence of the local solution of problem (1.1)–(1.3).

This paper is organized as follows. In Section 2 we prove the existence and uniqueness of local solution for problem (1.1)–(1.3). We establish an ordinary differential inequality (Lemma 3.2) and use it to study the blow-up of the solution for problem (1.1)–(1.3) in Section 3. An example is given in Section 4.

**Theorem 1.1.** Assume that the following conditions hold:

1. $\sigma \in C^1(R), \sigma'(s) \geq C_0$ and
   \[
   \left| \sigma(s) \right| \geq C_1 \left| s \right|^\alpha + 1, \quad \left| s \right| \geq M_1, \\
   \left| \sigma(s) \right| \leq C_2 \left( 1 + \left| s \right|^\alpha + 1 \right), \quad s \in R, \ i = 1, 2, \ldots, N,
   \]

   where $C_0$ is a constant, $\alpha \geq 0$, $M_1$ and $C_i$ ($i = 1, 2, \ldots$) are positive constants, specifically $C_1 > -k_0$ as $\alpha = 0$, $k_0 = \min\{C_0, 0\}$;

2. $f \in C(R)$ and one of the following two conditions holds:
   \[
   \begin{align*}
   (2.1) & \quad f(s) s \geq -C_3 (s^2 + 1), \quad |f(s)| \leq C_4 \left( 1 + |s|^{\beta + 1} \right), \quad s \in R, \quad 0 \leq \beta \leq 2 (\alpha + 2 \geq 1); \\
   (3) & \quad g \in C(R), \quad |g(s)| \leq C_5 \left( |s|^{\gamma + 1} + 1 \right), \quad s \in R, \quad 0 \leq \gamma < \alpha (\alpha + 2 \geq 1).
   \end{align*}
   \]

(i) If $f$, $g$ and $\sigma$ are locally Lipschitz continuous, and $\varphi, \psi \in H^2 \cap H^1_0$, for any $T > 0$, then problem (1.1)–(1.3) admits a unique generalized solution

\[
   u \in W^{1, \infty}([0, T]; H^2(\Omega) \cap H^1_0(\Omega)) \cap W^{2, \infty}([0, T]; L^2(\Omega)) \cap H^2([0, T]; H^1_0(\Omega)).
\]

(ii) If $\sigma \in C^3(R)$, $f \in C^2(R)$, $g \in C^2(R)$ and $\varphi, \psi \in H^4(\Omega) \cap H^1_0(\Omega)$, for any $T > 0$, then problem (1.1)–(1.3) admits a unique classical solution

\[
   u \in H^3([0, T]; H^1_0(\Omega)) \cap H^2([0, T]; H^3(\Omega) \cap H^1_0(\Omega)).
\]

But about the blow-up of the solution for problem (1.1)–(1.3) there has not been any discussion.

In the present paper, under certain conditions we prove that problem (1.1)–(1.3) admits a unique local generalized solution and local classical solution. To study the blow-up of the solution for problem (1.1)–(1.3), we first establish an ordinary differential inequality (see Lemma 3.2), next we apply this inequality to give the sufficient conditions of blow-up of the solution for problem (1.1)–(1.3). To this end, we also need to prove the existence of the local solution of problem (1.1)–(1.3).

This paper is organized as follows. In Section 2 we prove the existence and uniqueness of local solution for problem (1.1)–(1.3). We establish an ordinary differential inequality (Lemma 3.2) and use it to study the blow-up of the solution for problem (1.1)–(1.3) in Section 3. An example is given in Section 4.
2. The existence and uniqueness of local solution for problem (1.1)–(1.3)

In this section we are going to prove the existence and the uniqueness of the local generalized solution and the local classical solution for problem (1.1)–(1.3) by the Galerkin method and the compactness theorem.

Let \( \{y_i(x)\} \) be the orthonormal bases in \( L^2(\Omega) \) composed of the eigenvalue problem

\[
y'' + \lambda y = 0, \quad x \in \Omega, \\
y(0) = y(1) = 0
\]

corresponding eigenvalue \( \lambda_i \) (\( i = 1, 2, \ldots \)), where \( \frac{d}{dx} = \frac{d}{dx} \).

Let

\[
u_N(x, t) = \sum_{i=1}^{N} \alpha_{N_i}(t)y_i(x)
\]

be the Galerkin approximate solution of problem (1.1)–(1.3), where \( \alpha_{N_i}(t) \) are the undermined functions, \( N \) is a natural number. Assume that the initial value functions \( \varphi(x) \) and \( \psi(x) \) may be expressed

\[
\varphi(x) = \sum_{i=1}^{\infty} \rho_i y_i(x), \quad \psi(x) = \sum_{i=1}^{\infty} \xi_i y_i(x),
\]

where \( \rho_i \) and \( \xi_i \) (\( i = 1, 2, \ldots \)) are constants. Substituting the approximate solution \( u_N(x, t) \) into (1.1), multiplying both sides by \( y_s(x) \) and integrating over \((0, 1)\), we obtain

\[
\ddot{\alpha}_s + \lambda_s \dot{\alpha}_s = \mu \left( |u_N|^{q-1} u_N, y_s \right) - \delta \left( |u_N|^{p-1} u_N, y_s \right) + \left( \sigma(u_Nx) \right) y_s, \quad s = 1, 2, \ldots, N,
\]

(2.1)

where \( \dot{\alpha}_s = \frac{d}{dt} \alpha_s(t) \), \((\cdot, \cdot)\) denotes the inner product in \( L^2(\Omega) \).

Substituting the approximate solution \( u_N(x, t) \) and the approximations

\[
\varphi_N(x) = \sum_{i=1}^{N} \rho_i y_i(x), \quad \psi_N(x) = \sum_{i=1}^{N} \xi_i y_i(x)
\]

of the initial value functions \( \varphi(x) \) and \( \psi(x) \) into (1.3), we arrive at

\[
\alpha_{N_s}(0) = \rho_s, \quad \dot{\alpha}_{N_s}(0) = \xi_s, \quad s = 1, 2, \ldots, N.
\]

(2.2)

Lemma 2.1. Suppose that \( \sigma \in C^m(R) \), \( |\sigma(s)| \leq K|s|^v \), \( |\sigma'(s)| \leq K|s|^{v-1} \), etc., where \( q > 1 \), \( p \geq 1 \), \( 3 \leq m \leq \min\{p+2, q+2\} \) when \( m \) is an odd number, \( 2 \leq m \leq \min\{p+1, q+1\} \) when \( m \) is an even number, \( v \geq 2 \) is a natural number and \( K \) is a positive constant.

If

\[
\lim_{N \to \infty} E_N(0) = A = \sum_{s=1}^{\infty} \left\{ (1 + \lambda_s + \lambda_s^{m-1}) \xi_s^2 + (1 + \lambda_s + \lambda_s^2 + \lambda_s^m) \rho_s^2 \right\} + 1 < \infty,
\]

(2.3)

then the initial value problem (2.1), (2.2) for the system of the ordinary differential equations admits the classical solution \( \alpha(t) = (\alpha_{N1}(t), \alpha_{N2}(t), \ldots, \alpha_{NN}(t)) \) on \([0, t_1]\) and

\[
E_N(t) \leq \frac{A}{(1 - (\beta - 1) K_1 A^{\beta-1} t)^{\frac{1}{\beta - 1}}} = M
\]

(2.4)
is uniformly bounded, where \( t_1 > 0, K_1 > 0 \) are constants independent of the bound \( M \) and \( N, \beta = \max\{\frac{p+1}{2}, \frac{q+1}{2}, v, 1\} \) and

\[
E_N(t) = \sum_{s=1}^{N} \left\{ (1 + \lambda_s + \lambda_s^{m-1})\hat{u}_{N,s}^2(t) + (1 + \lambda_s + \lambda_s^{2} + \lambda_s^m)\alpha_{N,s}^2(t) \right\} + 1
= (u_N, u_N) + (u_{Nx}, u_{Nx}) + (u_{Nx^2}, u_{Nx^2}) + (u_{Nx^m}, u_{Nx^m})
+ (u_{Nt}, u_{Nt}) + (u_{Nxt}, u_{Nxt}) + (u_{Nx^{m-1}t}, u_{Nx^{m-1}t}) + 1.
\]

(2.5)

**Proof.** The initial value problem (2.1), (2.2) is the initial value problem for the system of the second order ordinary differential equations with respect to \( \alpha_{N_i}(t), i = 1, 2, \ldots, N \), and we may equivalently reduce problem (2.1), (2.2) to the initial value problem for the system of \( 2N \)-dimensional ordinary differential equations of first order. Since nonlinear term is smooth, there always exists the local solution of problem (2.1), (2.2). Let \([0, T_N)\) be the maximal interval. It is easy from the following estimations of the solution to see that \( T_N \) has the positive lower bound independent of \( N \).

Multiplying both sides of system (2.1) by \( 2(1 + \lambda_s + \lambda_s^{m-1})\hat{u}_{N,s}(t) \) and summing up for \( s = 1, 2, \ldots, N \), adding \( 2[(u_N, u_{Nt}) - (u_{Nxx}, u_{Nt}) + (u_{Nx}, u_{Nx,t}) - (-1)^{m-1}(u_{Nx^2}, u_{Nx^{2(m-1)}txt})] \)
to two sides and using integration by parts, we obtain

\[
\frac{d}{dt} E_N(t) + 2(\|u_{N,x^2,t}\|^2 + \|u_{Nx^m,t}\|^2 + \|u_{Nx^m}\|^2)
= 2(\mu \|u_N\|^{q-1}u_N - \delta \|u_{Nt}\|^{p-1}u_{Nt} + \sigma (u_{Nx}, x, u_{Nt}) - u_{Nx^2,t} + (-1)^{m-1}u_{Nx^{2(m-1)}txt})
+ 2[(u_{Nxx}, u_{Nt}) - (u_{Nxx}, u_{Nt}) + (u_{Nxx}, u_{Nx,t}) - (-1)^{m-1}(u_{Nx^2}, u_{Nx^{2(m-1)}txt})],
\]

(2.6)

where and in sequel \( \| \cdot \|_p (1 \leq p \leq \infty) \) and \( \| \cdot \|_{H^m} \) denote the norm of the space \( L^p(\Omega) \) and \( H^m(\Omega) \), respectively, specially \( \| \cdot \| = \| \cdot \|_2 \).

Using the Gagliardo–Nirenberg interpolation theorem and (2.5) we arrive at

\[
\|u_N\|_{W^m,\tilde{p}(\Omega)} \leq C_5 \|u_N\|_{H^m} \leq C_6 (E_N(t))^{\frac{1}{2}},
\]

(2.7)

\[
\|u_{Nt}\|_{W^m,\tilde{p}(\Omega)} \leq C_7 \|u_{Nt}\|_{H^{m-1}} \leq C_8 (E_N(t))^{\frac{1}{2}},
\]

(2.8)

where \( 0 \leq \tilde{m} \leq m - 1, 0 \leq \tilde{\tilde{m}} \leq m - 2, 2 \leq \tilde{p} \leq \infty, \| \cdot \|_{W^{m,\tilde{p}}(\Omega)} \) and \( \| \cdot \|_{\tilde{W}^{m,\tilde{p}}(\Omega)} \) denote the norm of the Sobolev spaces \( W^{m,\tilde{p}}(\Omega) \) and \( \tilde{W}^{m,\tilde{p}}(\Omega) \), respectively, the positive constants \( C_5-C_8 \) are independent of \( N \) and \( t \). Using the Hölder inequality, (2.7), (2.8) and the assumptions of the lemma, we obtain

\[
|2(\mu |u_N|^{q-1}u_N, u_{Nt} - u_{Nx^2t})|
\leq 2\mu |\|u_N\|_2^q \|u_{Nt}\| + q \|u_N\|_\infty^{q-1} \|u_{Nx}\| \|u_{Nx,t}\|) \leq C_9 (E_N(t))^{\frac{q+1}{2}},
\]

(2.9)

\[
|2(\delta |u_{Nt}|^{p-1}u_{Nt}, u_{Nt} - u_{Nx^2t})|
\leq 2\delta |\|u_{Nt}\|_2^p \|u_{Nt}\| + p \|u_{Nt}\|_\infty^{p-1} \|u_{Nx,t}\|^2) \leq C_{10} (E_N(t))^{\frac{p+1}{2}},
\]

(2.10)
where $\ell$ is an even number, from (2.12)–(2.14) we get

\[
\frac{\partial}{\partial x^\ell}\left[\sigma(u_{Nx})_x\right]_{x=0} = \frac{\partial}{\partial x^\ell}\left[|u_N|^{q-1}u_N\right]_{x=0} = 0,
\]

\[
\frac{\partial}{\partial x^\ell}\left[|u_N|^{q-1}u_N\right]_{x=0} = \frac{\partial}{\partial x^\ell}\left[|u_{Nt}|^{p-1}u_{Nt}\right]_{x=0} = 0,
\]

where $\ell = 0, 2, 4, \ldots, m-3$ when $m \geq 3$ is an odd number; $\ell = 0, 2, 4, \ldots, (m-2)$ when $m \geq 2$ is an even number, from (2.12)–(2.14) we get

\[
2\left|\sigma(u_{Nx})_x, (-1)^{m-1}u_{Nx^2(m-1)t}\right| \leq 2(-1)^{2m-3}\int_\Omega \sigma(u_{Nx})_x u_{Nx^2(m-1)t} \, dx \leq C_{15} \left\| u_N \right\|_{H^m}^\nu \left\| u_{Nx}\right\|_{L^2} \]

\[
\leq C_{16}(E_N(t))^{\nu} + \left\| u_{Nx}\right\|_{L^2}^2,
\]

(2.15)

\[
2\left|\mu|u_N|^{q-1}u_N, (-1)^{m-1}u_{Nx^2(m-1)t}\right| \leq 2\mu\int_\Omega (|u_N|^{q-1}u_N)_x u_{Nx^2(m-1)t} \, dx \leq C_{17} \left\| u_N \right\|_{H^m-1}^q \left\| u_{Nx^m-1t}\right\| \leq C_{18}(E_N(t))^{\frac{q+1}{2}},
\]

(2.16)

\[
2\left|\delta|u_{Nt}|^{p-1}u_{Nt}, (-1)^{m-1}u_{Nx^2(m-1)t}\right| \leq 2\delta\int_\Omega (|u_{Nt}|^{p-1}u_{Nt})_x u_{Nx^2(m-1)t} \, dx \leq C_{19} \left\| u_{Nt}\right\|_{H^m-1}^p \left\| u_{Nx^m-1t}\right\| \leq C_{20}(E_N(t))^{\frac{p+1}{2}},
\]

(2.17)

where $C_{16}$–$C_{20}$ are constants independent of $N$.

We apply (2.5) to obtain

\[
2\left[|u_N - u_{Nx}, u_{Nt} + (u_{Nx}, u_{Nt})| - (-1)^{m-1}|(u_{Nx^2}, u_{Nx^2(m-1)t})|\right] \leq 2\left(\left\| u_N \right\| + \left\| u_{Nt}\right\| + \left\| u_{Nx}\right\| + \left\| u_{Nxt}\right\| + \left\| u_{Nx^2}\right\| + \left\| u_{Nx^2t}\right\| + \left\| u_{Nx^m}\right\| + \left\| u_{Nx^mt}\right\| \right) \leq C_{21}E_N(t) + \left\| u_{Nxt}\right\|^2 + \left\| u_{Nx^2t}\right\|^2 + \left\| u_{Nx^mt}\right\|^2.
\]

(2.18)
Substituting (2.9)–(2.11) and (2.15)–(2.18) into (2.6) and taking 
$\beta = \max\{\frac{q+1}{2}, \frac{q+1}{2}, \nu, 1\}$, we infer 
\[
\frac{d}{dt} E_N(t) \leq K_1 (E_N(t))^\beta, 
\]
where $K_1 > 0$ is a constant independent of $N$.

For any $t \in (0, T_N)$ it follows from (2.19) that
\[
E_N(T) \leq E_N(0) \left[1 - (\beta - 1) K_1 (E_N(0))^{\beta - 1} t \right]^{\frac{1}{\beta - 1}} \leq A \left[1 - (\beta - 1) K_1 A^{\beta - 1} t \right]^{\frac{1}{\beta - 1}}. 
\]

If we take $t_1$ which satisfies
\[
B > 1 - (\beta - 1) K_1 A^{\beta - 1} t_1 > 0, 
\]
where $0 < B < 1$, then (2.4) holds on $[0, t_1]$. It follows from the above formula that
\[
0 < B (\beta - 1) K_1 A^{\beta - 1} t_1 < (\beta - 1) K_1 A^{\beta - 1}, 
\]
where $\frac{1-B}{(\beta-1)K_1A^{\beta-1}} > 0$ is a constant. This shows that $T_N$ has the positive lower bound. This completes the proof of Lemma 2.1. \qed

It is easy from Lemma 2.1 to see that the following lemma is valid.

**Lemma 2.2.** Under the conditions of Lemma 2.1, the approximate solution $u_N(x, t)$ of problem (1.1)–(1.3) satisfies

\[
\|u_N\|_{H^m} + \|u_N\|_{H^{m-1}} \leq C_{22}, \quad t \in [0, t_1], 
\]

where $C_{22}$ is a constant independent of $N$.

**Lemma 2.3.** Suppose that the conditions of Lemma 2.1 hold and $m \geq 5$, the approximate solution $u_N(x, t)$ of problem (1.1)–(1.3) has the estimation

\[
\|u_{Ntt}\|_{H^{m-3}} + \|u_{N}\|_{H^{m-5}} \leq C_{23}, \quad t \in [0, t_1], 
\]

where $C_{23}$ is a constant independent of $N$.

**Proof.** Multiplying both sides of (2.1) by $(1 + \lambda_{m-3})^s \tilde{u}_{N,s}(t)$ and summing up for $s = 1, 2, \ldots, N$, we get

\[
\|u_{Ntt}\|^2 + \|u_{Nx^{m-3}tt}\|^2 = (u_{Ntt}, u_{Nxx}) + (u_{N_{x^{m-1}t}}, u_{N_{x^{m-3}tt}}) 
+ (\mu |u_N|^{q-1} u_N - \delta |u_N|^p u_Nt + \sigma (u_{Nx})_x, u_{Ntt} + (-1)^{m-3} u_{N_x x^{2(m-3)tt}}). 
\]

Using the Hölder inequality, the Cauchy inequality, (2.12)–(2.14) and (2.22), from (2.24) we conclude

\[
\|u_{Ntt}\|^2 + \|u_{Nx^{m-3}tt}\|^2 \leq C_{24} (\|u_{Ntt}\|^2 + \|u_{N_{x^{m-1}t}}\|^2 + \|u_N|^{q-1} u_N\|^2 + \|u_N|^p u_Nt\|^2 + \|\sigma (u_{Nx})_x\| 
+ \left(\|u_N|^{q-1} u_N\|_{x^{m-3}}\right)^2 + \left(\|u_N|^p u_Nt\|_{x^{m-3}}\right)^2 + \|\sigma (u_{Nx})_x\|^2 \|^2 \leq C_{24}, \quad t \in [0, t_1], 
\]

where $C_{24}$ is a constant independent of $N$. 

\[
(2.25) 
\]
Differentiating (2.1) with respect to \( t \), multiplying it by \((1 + \lambda_s^{m-5})\tilde{\alpha}_N(t)\) and summing up for \( s = 1, 2, \ldots, N \), we have
\[
\|u_{Nt^3}\|^2 + \|u_{Nx^{m-5}t^3}\|^2 = (u_{Nx^2tt}, u_{Nt^3} + (-1)^{m-5}u_{Nx^{2(m-5)}t^3})
+ (\mu q|u_N|^{q-1}u_{Nt} - \delta p|u_N|^{p-1}u_{Ntt} + \sigma(u_{Nx})_{xt}, u_{Nt^3}
+ (-1)^{m-5}u_{Nx^{2(m-5)}t^3}).
\]
(2.26)

Using the Hölder inequality and the Cauchy inequality, (2.22) and (2.25), from (2.26) we assert
\[
\|u_{Nt^3}\|^2 + \|u_{Nx^{m-5}t^3}\|^2
\leq C_{25}(\|u_{Nx^2t^2}\|^2 + \|u_{Nx^{m-3}t}t^3\|^2 + \|u_N|^{q-1}u_{Nt}\|^2 + \|u_{Nt}|^{p-1}u_{Ntt}\|^2
+ \|\sigma(u_{Nx})_{xt}\|^2 + \|(|u_N|^{q-1}u_{Nt})_{x^{m-5}}\|^2 + \|(|u_{Nt}|^{p-1}u_{Ntt})_{x^{m-5}}\|^2
\]
\[
+ \|\sigma(u_{Nx})_{x^{m-4}t}\|^2) \leq C_{25}, \quad t \in [0, t_1].
\]
(2.27)

It follows from (2.25) and (2.27) that the estimation (2.23) holds. The lemma is proved.

**Theorem 2.1.** Suppose that

1. \( \sigma \in C^m(R), |\sigma(s)| \leq K|s|^v, |\sigma'(s)| \leq K|s|^{v-1} \), etc., where \( v \geq 2 \);
2. \( \varphi \in H^m(\Omega) \) and \( \psi \in H^{m-1}(\Omega) \).

If \( 4 \leq m \leq \min\{p + 1, q + 1\} \) (if \( m \) is an odd number, \( m \leq \min\{p + 2, q + 2\} \); when \( p = 1, 4 \leq m \leq q + 1 \)), then problem (1.1)–(1.3) admits a local generalized solution \( u(x, t) \) which satisfies the following identity:
\[
\int_0^{t_1} \int_\Omega \left\{ u_{tt} - \sigma(u_x)_{xt} - u_{xxt} + \delta|u_t|^{p-1}u_t - \mu|u|^{q-1}u \right\} h(x, t) \, dx \, dt = 0,
\]
\( \forall h \in L^2(Q_{t_1}), \)
(2.28)

and the initial boundary conditions in the classical sense, where \( Q_{t_1} = \Omega \times (0, t_1) \). The solution has the continuous derivatives \( u_{x^s}(x, t) \) (\( 0 \leq s \leq m - 2 \)), \( u_{x^s}t(x, t) \) (\( 0 \leq s \leq m - 4 \)) and the generalized derivatives \( u_{x^s}(x, t) \) (\( 0 \leq s \leq m \)), \( u_{x^s}t(x, t) \) (\( 0 \leq s \leq m - 1 \)) and \( u_{x^s}tt(x, t) \) (\( 0 \leq s \leq m - 3 \)). If \( m \geq 5 \), then the solution of problem (1.1)–(1.3) is unique.

If \( 6 \leq m \leq \min\{p + 1, q + 1\} \), then problem (1.1)–(1.3) admits a unique local classical solution \( u(x, t) \) and the solution has the continuous derivatives \( u_{x^s}(x, t) \) (\( 0 \leq s \leq m - 2 \)), \( u_{x^s}t(x, t) \) (\( 0 \leq s \leq m - 4 \)), \( u_{x^s}tt(x, t) \) (\( 0 \leq s \leq m - 6 \)) and the generalized derivatives \( u_{x^s}(x, t) \) (\( 0 \leq s \leq m \)), \( u_{x^s}t(x, t) \) (\( 0 \leq s \leq m - 1 \)), \( u_{x^s}tt(x, t) \) (\( 0 \leq s \leq m - 3 \)) and \( u_{x^s}tt(x, t) \) (\( 0 \leq s \leq m - 5 \)).

**Proof.** From (2.22) and (2.23) we know that when \( m = 4 \), using the Sobolev embedding theorem we infer
\[
\|u_N\|_{C^{2,2}_{\lambda_4}(\mathcal{D})} + \|u_{Nt}\|_{C^{2,2}_{\lambda_4}(\mathcal{D})} + \|u_{Ntt}\|_{C^{0,2}_{\lambda_4}(\mathcal{D})} \leq C_{26}, \quad t \in [0, t_1],
\]
(2.29)
where \( 0 < \lambda \leq \frac{1}{2} \). If \( m = 4 \), it follows from (2.29) and Ascoli–Arzelá theorem that there exists a function \( u(x, t) \) and a subsequence of \( \{u_N(x, t)\} \), still denoted by \( \{u_N(x, t)\} \), such that
when \( N \to \infty \), \( \{u_{N_{x^i}}(x,t)\} (i = 0, 1, 2) \) and \( \{u_{N_{t}}(x,t)\} \) uniformly converge to \( u_{x^i}(x,t) \) (i = 0, 1, 2) and \( u_{t}(x,t) \) on \( \overline{\Omega}_{t_1} \), respectively. The subsequences \( \{u_{N_{x^i_{t}}}^{\ast}(x,t)\} (i = 3, 4) \), \( \{u_{N_{x^i_{t}}}^{\ast}(x,t)\} \) (i = 1, 2, 3) and \( \{u_{N_{x^i_{t}}}^{\ast}(x,t)\} (i = 0, 1) \) weakly converge to \( u_{x^i}(x,t) \) (i = 3, 4), \( u_{x^i_{t}}(x,t) \) (i = 1, 2, 3) and \( u_{x^i_{t}}^{\ast}(x,t) \) (i = 0, 1) in \( L^2(\Omega_{t_1}) \), respectively. Thus when \( m \geq 4 \), the initial boundary value problem (1.1)–(1.3) has a local generalized solution. This solution has the regularities as those stated in Theorem 2.1 and satisfies (2.28) and the initial boundary conditions in the classical sense.

We now prove the uniqueness of the solution. Suppose that \( u(x,t) \) and \( v(x,t) \) are two solutions of the initial boundary value problem (1.1)–(1.3). Let

\[
\begin{align*}
\dot{w}(x,t) &= u(x,t) - v(x,t).
\end{align*}
\]

Then \( \dot{w}(x,t) \) satisfies the initial boundary value problem

\[
\begin{align*}
\dot{w}_{tt} - \left[ \sigma(u_x) - \sigma(v_x) \right] - w_{xx} = \dot{w}_{tt} + \delta|\dot{w}|^{p-1}\dot{w} - \delta|\dot{v}|^{p-1}\dot{v} = \mu|\dot{u}|^{q-1}\dot{u} - \mu|\dot{v}|^{q-1}\dot{v},
\end{align*}
\]

\( (x,t) \in \Omega \times (0,t_1), \quad (2.30) \)

\[
\begin{align*}
w(0,t) &= 0, \quad w(1,t) = 0, \quad 0 \leq t \leq t_1, \\
w(x,0) &= 0, \quad w_t(x,0) = 0, \quad x \in \overline{\Omega}.
\end{align*}
\]

Multiplying both sides of (2.30) by \( 2w_t(x,t) \), adding \( 2w t_t - 2w_{xx} w_t \) to the both sides and integrating on \( \Omega \), we obtain by calculation

\[
\begin{align*}
\frac{d}{dt} \left[ \|w\|^2 + \|w_t\|^2 + \|w_x\|^2 \right] + 2\|w_{xt}\|^2 &= 2 \int_{\Omega} w w_t \, dx + 2\mu q \int_{\Omega} |\ddot{u}|^{q-1}\ddot{u}_x w w_t \, dx - 2\delta p \int_{\Omega} |\ddot{v}|^{p-1}\ddot{v}_x w_t^2 \, dx \\
- 2 \int_{\Omega} \sigma'(\ddot{u}_x) w_x w_{xt} \, dx - 2 \int_{\Omega} w_{xx} w_t \, dx
\leq \|w\|^2 + \|w_t\|^2 + C_{27} \max_{\overline{Q}_{t_1}} \left\{|\ddot{u}|^{q-1}|\ddot{u}_x| + |\ddot{v}|^{p-1}|\ddot{v}_{xt}|\right\}\{\|w\|^2 + \|w_t\|^2\} \quad \\
+ \max_{\overline{Q}_{t_1}} |\sigma'(\ddot{u}_x)|^2 \|w_x\|^2 + \|w_{xt}\|^2 + \|w_{xx}\|^2 + \|w_t\|^2.
\end{align*}
\]

(2.33)

Since \( \dddot{u}, \dddot{u}_x, \dddot{u}_{xt} \) and \( \dddot{u}_x \) take the median between \( u \) and \( v, u_x \) and \( v_x, u_t \) and \( v_t, u_{xt} \) and \( v_{xt}, u_x \) and \( v_x \), respectively, and they are bounded, it follows from (2.33) that

\[
\begin{align*}
\|w\|^2 + \|w_t\|^2 + \|w_x\|^2 \leq C_{28} \int_0^t \left\{\|w\|^2 + \|w_t\|^2 + \|w_x\|^2\right\} \, dt.
\end{align*}
\]

The Gronwall inequality yields

\[
\|w\|^2 + \|w_t\|^2 + \|w_x\|^2 = 0.
\]

Therefore \( u(x,t) = v(x,t) \).

When \( m \geq 6 \), it is easy to prove that problem (1.1)–(1.3) admits a unique local classical solution \( u(x,t) \). This solution has the regularities as those stated in Theorem 2.1. This completes the proof of the theorem.
Remark 2.1. Under the conditions of Theorem 2.1, if \(3 = m \leq \min\{p + 2, q + 2\}\), then problem (1.1)–(1.3) admits local generalized solution \(u(x, t)\) which satisfies (2.28), the boundary value condition (1.2) in the classical sense and the initial value condition (1.3) in the generalized sense are fulfilled.

3. An ordinary differential inequality and blow-up of solution

In this section, we are going to discuss the blow-up of the solution for problem (1.1)–(1.3). To this end, we first establish an ordinary differential inequality and use it to study the blow-up of the solution for problem (1.1)–(1.3).

To prove Lemma 3.2 we quote the following lemma.

**Lemma 3.1.** [7] Assume that \(\dot{u} = H(t, u), \dot{v} \geq H(t, v)\), \(H \in C([0, \infty) \times (-\infty, \infty))\) and \(u(t_0) = v(t_0), t_0 \geq 0\), then when \(t \geq t_0\), \(v(t) \geq u(t)\).

**Lemma 3.2.** Suppose that a positive differentiable function \(M(t)\) satisfies the inequality

\[
\dot{M}(t) + M(t) \geq Ct^{\frac{r-2}{r}} (M(t))^{\frac{r+3}{4}}, \quad t \geq t_1 > 0,
\]

with

\[
M(t) \geq -Ft^2 + \dot{M}(0)t + M(0), \quad t \geq t_1 > 0,
\]

where \(M(0), \dot{M}(0), r > 1, C > 0\) are constants and

\[
F \leq -\left[\frac{2}{C(1 - e^{-\frac{r+1}{4}})}\right]^\frac{4}{r-1} < 0.
\]

Then there is a constant \(\tilde{T}\) such that \(M(t) \to \infty\) as \(t \to \tilde{T}^−\).

**Proof.** We consider the following initial value problem of the Bernoulli equation:

\[
\dot{W}(t) + W(t) = Ct^{\frac{r-2}{r}} (W(t))^{\frac{r+3}{4}}, \quad t > t_1,
\]

\[
W(t_1) = M(t_1).
\]

Solving problem (3.3), (3.4), we obtain the solution

\[
W(t) = e^{-(t-t_1)} \left\{\left(M(t_1)\right)^{\frac{1-r}{4}} - \frac{C(r-1)}{4} \int_{t_1}^{t} \tau^{\frac{1-r}{r}} e^{-\frac{r+1}{4}(\tau-t_1)} d\tau\right\}^{\frac{4}{r+3}}
\]

\[
= e^{-(t-t_1)} M(t_1) Z^\frac{4}{r+3}(t), \quad t \geq t_1,
\]

where

\[
Z(t) = 1 - \frac{C(r-1)}{4} \left(M(t_1)\right)^{\frac{1-r}{4}} \int_{t_1}^{t} \tau^{\frac{1-r}{r}} e^{-\frac{r+1}{4}(\tau-t_1)} d\tau.
\]

Clearly, \(Z(t_1) = 1\),
\[ Q(t) = \frac{C(r - 1)}{4} \left( M(t_1) \right)^{-\frac{r-1}{4}} \int_{t_1}^{t} \tau^{-\frac{r}{4}} e^{-\frac{r-1}{4} (\tau-t_1)} d\tau \]\[ \geq \frac{C(r - 1)}{4} \left( M(t_1) \right)^{-\frac{r-1}{4}} (t_1 + 1)^{-\frac{r}{4}} \int_{t_1}^{t+1} e^{-\frac{r-1}{4} (\tau-t_1)} d\tau \]

\[ = C \left( M(t_1) \right)^{-\frac{r-1}{4}} (t_1 + 1)^{-\frac{r}{4}} (1 - e^{-\frac{r-1}{4}}), \quad t \geq t_1 + 1. \quad (3.6) \]

It follows from (3.2) that
\[ \left( M(t) \right)^{-\frac{r-1}{4}} (t+1)^{-\frac{r}{4}} \geq \left\{ \frac{-F t^2 + \dot{M}(0) t + M(0)}{(t+1)^2} \right\} \left( M(t_1) \right)^{-\frac{r-1}{4}} \rightarrow (-F)^{-\frac{r-1}{4}} \]

as \( t \to \infty \). Take \( t_1 \) sufficiently large such that
\[ \left( M(t_1) \right)^{-\frac{r-1}{4}} (t_1 + 1)^{-\frac{r}{4}} \geq \frac{1}{2} (-F)^{-\frac{r-1}{4}}. \]

We assert from (3.6) and the assumption of \( F \) that
\[ Q(t) \geq \frac{C}{2} (-F)^{-\frac{r-1}{4}} (1 - e^{-\frac{r-1}{4}}) \geq 1, \quad t \geq t_1 + 1. \quad (3.7) \]

Therefore,
\[ Z(t) = 1 - Q(t) \leq 0, \quad t \geq t_1 + 1. \quad (3.8) \]

By virtue of the continuity of \( Z(t) \) and the theorem of intermediate values, there is a constant \( \tilde{T} \) \((t_1 < \tilde{T} \leq t_1 + 1)\) such that \( Z(\tilde{T}) = 0 \). Hence, \( W(t) \to \infty \) as \( t \to \tilde{T}^- \). We conclude from Lemma 3.1 that \( M(t) \geq W(t), t \geq t_1 \). Thus \( M(t) \to \infty \) as \( t \to \tilde{T}^- \). Lemma 3.2 is proved. \( \square \)

**Theorem 3.1.** Assume that:

1. \( p = 1 \) and \( q > 1 \);
2. \( \sigma(s) \in C^1(R), s \sigma(s) \leq K \int_0^s \sigma(y) dy, \int_0^s \sigma(y) dy \leq -\alpha|s|^{\gamma+1}, \) where \( K > 2, \alpha > 0 \) and \( \gamma > 1 \) are constants;
3. \( \varphi \in H_0^1(\Omega) \cap L^{q+1}(\Omega), \psi \in H_0^1(\Omega) \) and

\[ E(0) + \frac{q-1}{2(q+1)} \left[ \frac{\mu(q-1)}{2} \right]^{-\frac{2}{q-1}} \left( \frac{\delta^2}{2} \right)^{\frac{q+1}{q-1}} \leq -\left[ \frac{2}{A_3 (1 - e^{-\frac{\gamma+1}{4}})} \right]^4, \]

where
\[ E(0) = \| \psi \|^2 - \frac{2\mu}{q+1} \| \varphi \|_{q+1}^{q+1} + 2 \int_0^{\varphi(x)} \int_0^\varphi \sigma(s) ds dx, \]
\[ A_3 = \sqrt{A_2} = \left\{ (K - 2)\alpha \frac{2^3 - \gamma}{\gamma + 3} \right\}^{\frac{1}{2}}. \]
Then the generalized solution \( u(x, t) \) or the classical solution \( u(x, t) \) of problem (1.1)–(1.3) blows-up in finite time \( \tilde{T} \), i.e.

\[
\|u(\cdot, t)\|^2 + \int_0^t \int_{\Omega} |u_x(x, \tau)|^2 \, dx \, d\tau + \int_0^t \int_0^\tau \int_{\Omega} |u_x(x, s)|^2 \, dx \, ds \, d\tau \to \infty
\]
as \( t \to \tilde{T}^- \).

**Proof.** Multiplying both sides of (1.1) by 2\( u_t \), integrating over \((0, 1)\), we arrive at

\[
E(t) = E(0), \quad t > 0,
\]
where

\[
E(t) = \|u_t(\cdot, t)\|^2 + 2 \int_0^t \int_{\Omega} |u_{xt}(\cdot, \tau)|^2 \, d\tau + 2 \int_0^t \int_{\Omega} \sigma(s) \, ds
\]

\[
\quad + 2\delta \int_0^t \|u_t(\cdot, t)\|^2 \, d\tau - \frac{2\mu}{q+1} \|u(\cdot, t)\|^{q+1}_{q+1}.
\]

Let

\[
M(t) = \|u(\cdot, t)\|^2 + \int_0^t \int_{\Omega} |u_x(x, t)|^2 \, dx \, d\tau + \int_0^t \int_0^\tau \int_{\Omega} |u_x(x, s)|^2 \, dx \, ds \, d\tau. \tag{3.9}
\]

We have

\[
\dot{M}(t) = 2 \int_{\Omega} u(x, t) u_t(x, t) \, dx + \int_{\Omega} |u_x(x, t)|^2 \, dx + \int_0^t \int_{\Omega} |u_x(x, \tau)|^2 \, dx \, d\tau. \tag{3.10}
\]

Using the assumption (2) of Theorem 3.1, integrating by parts and observing

\[
K \int_{\Omega} \int_0^{u_x(x, t)} \sigma(s) \, ds \, dx
\]

\[
= E(0) - \|u_t(\cdot, t)\|^2 - 2 \int_0^t \|u_{xt}(\cdot, \tau)\|^2 \, d\tau
\]

\[
\quad + \frac{2\mu}{q+1} \|u(\cdot, t)\|^{q+1}_{q+1} - 2\delta \int_0^t \|u_t(\cdot, \tau)\|^2 \, d\tau + (K - 2) \int_0^t \int_{\Omega} \sigma(s) \, ds \, dx, \tag{3.11}
\]

further we infer by the assumptions of Theorem 3.1 that

\[
\dot{M}(t) = 2 \int_{\Omega} \left\{ u_t^2(x, t) + u(x, t)u_{tt}(x, t) + u_x(x, t)u_{xt}(x, t) + \frac{1}{2} u_x^2(x, t) \right\} \, dx
\]
\[
\begin{align*}
&= 2 \int_{\Omega} \left\{ u_t^2(x, t) + u(x, t) \left[ u_{xx}(x, t) + \sigma(u_x(x, t))_x - \delta u_t(x, t) \right] \\
&\quad + \mu |u(x, t)|^{q-1} u_x(x, t) \right\} dx \\
&= 2 \int_{\Omega} \left\{ u_t^2(x, t) - \sigma(u_x(x, t)) u_x(x, t) - \delta u_t(x, t) u(x, t) + \mu |u(x, t)|^{q+1} \\
&\quad + \frac{1}{2} u_x^2(x, t) \right\} dx \\
&\geq 2 \left\{ 2 \left\| u_t(\cdot, t) \right\|^2 - \left( E(0) + 2 \int_0^T \left\| u_{x\tau}(\cdot, \tau) \right\|^2 d\tau + \frac{\mu(q-1)}{q+1} \left\| u(\cdot, t) \right\|_{q+1}^{q+1} \\
&\quad + \frac{2}{q+1} \int_0^T \left\| u_x(\cdot, \tau) \right\|^2 d\tau + \alpha(K-2) \left\| u_x(\cdot, t) \right\|_{\gamma+1}^{\gamma+1} - \delta \int_{\Omega} u_t(x, t) u(x, t) \right\} \\
&\quad + \frac{1}{2} \left\| u_x(\cdot, t) \right\|^2 \right\}. \tag{3.12}
\end{align*}
\]

By use of the Cauchy inequality, the Hölder inequality and the Young inequality with \( \varepsilon = \frac{\mu(q-1)}{(q+1)} \), we deduce
\[
\begin{align*}
&\left| 2\delta \int_{\Omega} u_t(x, t) u(x, t) \right| dx \\
&\leq 2 \left\| u_t(\cdot, t) \right\|^2 + \frac{\delta^2}{2} \left\| u(\cdot, t) \right\|^2 \leq 2 \left\| u_t(\cdot, t) \right\|^2 + \frac{\delta^2}{2} \left\| u(\cdot, t) \right\|_{q+1}^{q+1} \\
&\leq 2 \left\| u_t(\cdot, t) \right\|^2 + \frac{\mu(q-1)}{q+1} \left\| u(\cdot, t) \right\|_{q+1}^{q+1} + \frac{q-1}{q+1} \left( \frac{\mu(q-1)}{2} \right)^{-\frac{q}{2}} \left( \frac{\delta^2}{2} \right)^{\frac{q+1}{q+1}}. \tag{3.13}
\end{align*}
\]

Substituting (3.13) into (3.12), we find
\[
\dot{M}(t) \geq 2 \left\| u_t(\cdot, t) \right\|^2 + \left( \frac{q-1}{q+1} \right) \mu^{q+1} + 2(K-2) \alpha \int_{\Omega} \left\| u_x(\cdot, t) \right\|_{\gamma+1}^{\gamma+1} dx \\
+ \left\| u_x(\cdot, t) \right\|^2 - \left[ 2E(0) + B \right] > 0, \quad t > 0, \tag{3.14}
\]

where
\[
B = \frac{q-1}{q+1} \left( \frac{\mu(q-1)}{2} \right)^{-\frac{q}{2}} \left( \frac{\delta^2}{2} \right)^{\frac{q+1}{q+1}}.
\]

It follows from (3.14) that
\[
\dot{M}(t) \geq \left[ -2E(0) - B \right] t + 2\alpha(K-2) \int_0^T \int_{\Omega} \left\| u_x(\cdot, \tau) \right\|_{\gamma+1}^{\gamma+1} dx \ d\tau
\]
\[ + \int_0^t \int_\Omega |u_x(x, \tau)|^2 \, dx \, d\tau + \dot{M}(0) \]  
\hspace{1cm} (3.15) 

and

\[ M(t) \geq 2\alpha(K - 2) \int_0^t \int_0^\tau \int_\Omega |u_x(x, s)|^{\gamma+1} \, dx \, ds \, d\tau - \frac{1}{2} \left[ 2E(0) + B \right] t^2 \]
\[ + \dot{M}(0)t + M(0), \] \hspace{1cm} (3.16)

where

\[ \dot{M}(0) = 2 \int_\Omega \varphi(x) \psi(x) \, dx + \int_\Omega |\varphi_x(x)|^2 \, dx, \quad M(0) = \| \varphi \|^2. \]

From (3.14)–(3.16) we have

\[ \ddot{M}(t) + \dot{M}(t) + M(t) \]
\[ \geq 2 \left\| u_t(\cdot, t) \right\|^2 + \frac{(q - 1)\mu}{q + 1} \left\| u(\cdot, t) \right\|_{q+1}^{q+1} + \left\| u_x(\cdot, t) \right\|^2 \]
\[ + 2\alpha(K - 2) \left\{ \int_\Omega |u_x(\cdot, t)|^{\gamma+1} \, dx + \int_0^t \int_\Omega |u_x(\cdot, \tau)|^{\gamma+1} \, dx \, d\tau \right\} \]
\[ + \int_0^t \int_\Omega |u_x(\cdot, s)|^{\gamma+1} \, dx \, ds \, d\tau \]
\[ + \int_0^t \int_\Omega |u_x(\cdot, \tau)|^{\gamma+1} \, dx \, d\tau \]
\[ - \left[ 2E(0) + B \right] \left( \frac{t^2}{2} + t + 1 \right) + \dot{M}(0)(t + 1) + M(0). \] \hspace{1cm} (3.17)

Substituting (3.10) into (3.17), we arrive at

\[ \ddot{M}(t) + 2 \int_\Omega u(x, t) u_t(x, t) \, dx + \int_\Omega |u_x(x, t)|^2 \, dx + \int_0^t \int_\Omega |u_x(x, \tau)|^2 \, dx \, d\tau + M(t) \]
\[ \geq 2 \left\| u_t(\cdot, t) \right\|^2 + \frac{(q - 1)\mu}{q + 1} \left\| u(\cdot, t) \right\|_{q+1}^{q+1} + \left\| u_x(\cdot, t) \right\|^2 \]
\[ + \int_0^t \int_\Omega |u_x(x, \tau)|^2 \, dx \, d\tau + 2(K - 2)\alpha \left\{ \int_\Omega |u_x(\cdot, t)|^{\gamma+1} \, dx \right\} \]
\[ + \int_0^t \int_\Omega |u_x(x, \tau)|^{\gamma+1} \, dx \, d\tau \]
\[ + \int_0^t \int_\Omega |u_x(x, s)|^{\gamma+1} \, dx \, ds \, d\tau \]
\[ + \int_0^t \int_\Omega |u_x(x, \tau)|^{\gamma+1} \, dx \, d\tau \]
\[ - \left[ 2E(0) + B \right] \left( \frac{t^2}{2} + t + 1 \right) + \dot{M}(0)(t + 1) + M(0). \] \hspace{1cm} (3.18)
Since $\dot{M}(t) > 0$, $M(t) \geq 0$ and
\[
2 \int_{\Omega} u(x, t) u_x(x, t) \, dx \leq \|u(\cdot, t)\|^2 + \|u_t(\cdot, t)\|^2,
\]
from (3.18) we deduce
\[
\dot{M}(t) + M(t) \geq \frac{(q - 1)\mu}{2(q + 1)} \int_{\Omega} |u(x, t)|^{q+1} \, dx + (K - 2)\alpha \int_{\Omega} |u_x(\cdot, t)|^{q+1} \, dx
\]
\[
+ \int_{0}^{t} \int_{\Omega} |u_x(x, \tau)|^{q+1} \, dx \, d\tau + \int_{0}^{t} \int_{0}^{\tau} \int_{\Omega} |u_x(x, s)|^{q+1} \, dx \, ds \, d\tau
\]
\[
- \left[ E(0) + \frac{B}{2} \right] \left( t^2 + t + 1 \right) + \frac{1}{2} \dot{M}(0)(t + 1) + \frac{1}{2} M(0).
\] (3.19)

Using the Hölder inequality and the Poincaré inequality, we assert
\[
\int_{\Omega} |u_x(x, t)|^{q+1} \, dx \geq \left( \int_{\Omega} |u(x, t)|^2 \, dx \right)^{\frac{q+1}{2}}, \quad (3.20)
\]
\[
\int_{0}^{t} \int_{\Omega} |u_x(x, t)|^{q+1} \, dx \, dt \geq t^{\frac{1-\gamma}{2}} \left( \int_{0}^{t} \int_{\Omega} |u_x(x, \tau)|^2 \, dx \, d\tau \right)^{\frac{q+1}{2}}, \quad (3.21)
\]
\[
\int_{0}^{t} \int_{0}^{\tau} \int_{\Omega} |u_x(x, s)|^{q+1} \, dx \, ds \, d\tau \geq 2^{\frac{q+1}{2}} t^{1-\gamma} \left( \int_{0}^{t} \int_{0}^{\tau} \int_{\Omega} |u_x(x, s)|^2 \, dx \, ds \, d\tau \right)^{\frac{q+1}{2}}. \quad (3.22)
\]

Substituting (3.20)–(3.22) into (3.19) and using the inequality
\[(a_1 + b_1 + c_1)^n \leq 2^{2(n-1)}(a_1^n + b_1^n + c_1^n), \quad a_1, b_1, c_1 > 0, \quad n > 1,
\]
we find
\[
\dot{M}(t) + M(t) \geq A_1 \left\{ \left( \int_{\Omega} |u(x, t)|^2 \, dx \right)^{\frac{q+1}{2}} + t^{\frac{1-\gamma}{2}} \left( \int_{0}^{t} \int_{\Omega} |u_x(x, \tau)|^2 \, dx \, d\tau \right)^{\frac{q+1}{2}} \right.
\]
\[
+ 2^{\frac{q+1}{2}} t^{1-\gamma} \left( \int_{0}^{t} \int_{0}^{\tau} \int_{\Omega} |u_x(x, s)|^2 \, dx \, ds \, d\tau \right)^{\frac{q+1}{2}} \left[ E(0) + \frac{B}{2} \right] \left( t^2 + t + 1 \right)
\]
\[
+ \frac{1}{2} \dot{M}(0)(t + 1) + \frac{1}{2} M(0)
\]
\[
\geq A_1 2^{1-\gamma} t^{1-\gamma} (M(t))^{\frac{q+1}{2}} - \left[ E(0) + \frac{B}{2} \right] \left( t^2 + t + 1 \right)
\]
\[
+ \frac{1}{2} \dot{M}(0)(t + 1) + \frac{1}{2} M(0), \quad t \geq 1.
\] (3.23)
where $A_1 = (K - 2)\alpha$.

It follows from (3.15), (3.16) that $\dot{M}(t) \to \infty$ and $M(t) \to \infty$ as $t \to \infty$. Therefore, there is a $t_0 \geq 1$ such that when $t \geq t_0$, $\dot{M}(t) > 0$ and $M(t) > 0$. We multiply both sides of (3.23) by $2\dot{M}(t)$ and use (3.15). Then we infer

$$
\frac{d}{dt} \left[ (\dot{M}(t))^2 + (M(t))^2 \right] \geq A_2 t^{1-\gamma} \frac{d}{dt} \left( M(t) \right)^{\frac{\gamma+3}{2}} + D(t), \quad t \geq t_0,
$$

(3.24)

where

$$
A_2 = \frac{A_1 2^{3-\gamma}}{\gamma + 3},
$$

$$
D(t) = \left\{ \left[ -4E(0) - 2B \right] t + 2\dot{M}(0) \right\} \times \left\{ \left[ -E(0) - \frac{B}{2} \right] \left( \frac{t^2}{2} + t + 1 \right) + \frac{\dot{M}(0)}{2} (t + 1) + \frac{1}{2} M(0) \right\}.
$$

From (3.24) we obtain

$$
\frac{d}{dt} \left\{ t^{\gamma-1} \left[ (\dot{M}(t))^2 + (M(t))^2 \right] - A_2 (M(t))^{\frac{\gamma+3}{2}} \right\} \geq t^{\gamma-1} D(t), \quad t \geq t_0.
$$

(3.25)

Integrating (3.25) over $(t_0, t)$, we arrive at

$$
t^{\gamma-1} \left[ (\dot{M}(t))^2 + (M(t))^2 \right] - A_2 (M(t))^{\frac{\gamma+3}{2}} \geq \int_{t_0}^{t} t^{\gamma-1} D(\tau) d\tau + t_0^{\gamma-1} \left[ (\dot{M}(t_0))^2 + (M(t_0))^2 \right] - A_2 (M(t_0))^{\frac{\gamma+3}{2}}, \quad t \geq t_0.
$$

(3.26)

We see that when $t \to \infty$, the right-hand side of (3.26) approach to positive infinity, hence there is a $t_1 \geq t_0$ such that when $t \geq t_1$, the right-hand side of (3.26) is large than or equal to zero. We thus have

$$
t^{\gamma-1} \left[ (\dot{M}(t))^2 + (M(t))^2 \right] \geq A_2 (M(t))^{\frac{\gamma+3}{2}}, \quad t \geq t_1.
$$

(3.27)

Extracting the square root of both sides of (3.27), we conclude that

$$
\dot{M}(t) + M(t) \geq A_3 t^{\frac{1-\gamma}{2}} (M(t))^{\frac{\gamma+3}{2}}, \quad t \geq t_1,
$$

(3.28)

where $A_3 = \sqrt{A_2}$.

It follows from (3.16) that

$$
M(t) \geq - \left[ E(0) + \frac{B}{2} \right] t^2 + \dot{M}(0) t + M(0).
$$

(3.29)

By virtue of Lemma 3.2 there is a constant $\tilde{T}$, such that

$$
\|u(\cdot, t)\|^2 + \int_0^t \int_{\Omega} |u_x(x, \tau)|^2 dx d\tau + \int_0^t \int_0^\tau \int_{\Omega} |u_x(x, s)|^2 dx ds d\tau \to \infty
$$

as $t \to \tilde{T}^{-}$. This completes the proof. □

Similarly to Theorem 3.1 we can prove
Theorem 3.2. Assume that:

1. \( 1 \leq p < 2 \) and \( q < \frac{p}{2} \);
2. \( \sigma(s) \in C^1(R) \), \( s \sigma(s) \leq K \int_0^s \sigma(y) \, dy \), \( \int_0^s \sigma(y) \, dy \leq -\alpha |s|^{\gamma+1} \), \( K > 2 \), \( \alpha > 0 \) and \( \gamma > 1 \) are constants;
3. \( \varphi \in H^1_0(\Omega) \cap L^{q+1}(\Omega) \), \( \psi \in H^1_0(\Omega) \) and
   \[
   E(0) + \frac{1}{2d} \left( \frac{a}{\zeta} \right)^d \leq -\left[ \frac{2}{A_3(1-e^{-\frac{\gamma-1}{4}})} \right]^{\frac{d}{\gamma+1}},
   \]
where
   \[
   E(0) = \|\psi\|^2 - \frac{2\mu}{q+1} \|\varphi\|_{q+1}^{q+1} + 2 \int_0^{\varphi(x)} \int_0^\sigma(s) \, ds \, dx,
   \]
   \[
   d = \frac{(2-p)(q+1)}{(2-p)q-p}, \quad a = 2 - p \left( \frac{\delta^2 p^p}{4p-1} \right)^{\frac{1}{\gamma-1}},
   \]
   \[
   \zeta = \left[ \frac{\mu(q-1)(2-p)}{2} \right]^{\frac{2}{(\gamma-pq+1)}},
   \]
   \[
   A_3 = \sqrt{A_2} = \left\{ \frac{2}{(k-2)\alpha} \frac{\delta^2 p^p}{\gamma+3} \right\}^{\frac{1}{2}}.
   \]

Then the generalized solution \( u(x,t) \) or the classical solution \( u(x,t) \) of problem (1.1)–(1.3) blows-up in finite time \( \tilde{T} \), i.e.

\[
\|u(\cdot,t)\|^2 + \int_0^t \int_\Omega |u_x(x,\tau)|^2 \, dx \, d\tau + \int_0^\tau \int_0^\tau \int_\Omega |u_x(x,s)|^2 \, dx \, ds \, d\tau \rightarrow \infty
\]
as \( t \rightarrow \tilde{T}^- \).

Remark 3.1.

1. The method used in the proof of Theorems 3.1 and 3.2 may be used in the case when \( \delta|u_t|^{p-1}u_t \) and \( \mu|u|^{q-1}u \) are replaced by \( f(u_t) \) and \( g(u) \) in Eq. (1.1), respectively.
2. Lemma 3.2 may be used to study blow-up of solutions for many nonlinear wave equations.

4. An example

In this section we take an example to illustrate that function \( \sigma(s) \) satisfying assumptions (2) and (3) of Theorem 3.1 and functions \( \varphi(x) \) and \( \psi(x) \) satisfying the conditions of Theorem 3.1 there exist. For example, \( \sigma(s) = -s^{2k+1} \) \((k = 1, 2, \ldots)\). Indeed, if we take \( \sigma(s) = -s^3 \), \( K = \frac{7}{2} \), \( q = 5 \), \( p = 1 \), \( \gamma = 3 \), \( \alpha = \frac{9}{40} \), then \( \sigma(s) \in C^1(R) \),

\[
s\sigma(s) = -s^4 < K \int_0^s \sigma(y) \, dy = -\frac{7}{8}s^4,
\]
\[ \int_0^s \sigma(y) \, dy \left( = -\frac{1}{4} s^4 \right) \leq -\alpha |s|^{\gamma+1} \left( = -\frac{9}{40} s^4 \right), \quad \text{and} \]

\[ p \quad (\ = 1) < q \quad (\ = 5) \]

hold. If we take \( \mu = 1, \delta = 2, \varphi(x) = \frac{17}{2} x(x-1), \psi(x) = x(x-1), \) then \( \varphi \in H_0^1(\Omega) \cap L^{q+1}(\Omega), \)

\[ E(0) = \| \psi \|^2 + 2 \int_\Omega \int_0^1 \sigma(s) \, ds \, dx - \frac{2\mu}{q+1} \| \varphi \|_{q+1}^{q+1} = \int_0^1 \left[ x(x-1) \right]^2 \, dx + \]

\[ + 2 \int_0^1 \int_0^{x-1} (-s^3) \, ds \, dx - \frac{1}{3} \int_0^1 \left( \frac{17}{2} \right)^6 x^6(x-1)^6 \, dx \approx -584.8313, \]

\[ B = \frac{q-1}{q+1} \left[ \frac{\mu(q-1)}{2} \right]^{-\frac{2}{q-1}} \left( \frac{\delta^2}{2} \right)^{\frac{q+1}{q-1}} \approx 1.3333, \]

\[ A_3 = \left\{ (K-2)\alpha \frac{2^{3-\gamma}}{\gamma+3} \right\}^\frac{1}{2} \approx 0.2372. \]

Furthermore,

\[ E(0) + \frac{B}{2} \approx -584.1646 \leq -\left[ \frac{2}{A_3(1 - e^{-\frac{\gamma+1}{\gamma+3}})} \right]^\frac{4}{\gamma-1} (\approx -459.5118) \]

holds. Thus the conditions of Theorem 3.1 are satisfied. In the light of Theorem 3.1 the solution \( u(x,t) \) of problem (1.1)–(1.3) in the above case blows-up in finite time \( \tilde{T} \).

References


