Mean square exponential stability of impulsive stochastic difference equations

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Abstract

This article proposes a method to deal with the mean square exponential stability of impulsive stochastic difference equations. By establishing a difference inequality, we obtain some sufficient conditions ensuring the exponential stability, in mean square, of systems under consideration. The results extend and improve earlier publications. Two examples are provided to show the effectiveness of the proposed approach.

Keywords: Impulsive; Stochastic; Difference equations; Difference inequality; Mean square exponential stability

1. Introduction

Difference equations usually appear in the investigation of systems with discrete time or in the numerical solution of systems with continuous time [1]. A lot of difference systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, etc. In recent years, the stability investigation of stochastic difference equations has been interesting to many investigators, and various advanced results on this problem have been reported [2–6]. In particular, the following difference system

\begin{equation}
    x_{i+1} = F(i, x_{i-h}, \ldots, x_{i}) + G(i, x_{i-h}, \ldots, x_{i})\xi_i, \quad i \in Z,
\end{equation}

with initial condition

\begin{equation}
    x_i = \varphi_i, \quad i \in Z_0,
\end{equation}

has been investigated widely. Here \(i\) is a discrete time, \(i \in Z_0 \cup Z, Z_0 = \{-h, \ldots, 0\}, Z = \{0, 1, 2, \ldots\}, h\) is a given nonnegative integral number, the sequence of real numbers \(\{x_i\}\) is a solution of Eq. \((1), F, G : Z \times R^{h+1} \rightarrow R.\) Let

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{Ω, P, σ} be a basic probability space, \( f_i \in σ, i \in Z \), be a sequence of \( σ \)-algebras, \( E \) be the mathematical expectation, \( ξ_0, ξ_1, \ldots \) be a sequence of mutually independent random variables, \( ξ_i \in R \), \( ξ_i \) be \( f_{i+1} \)-adapted and independent on \( f_i \), \( Eξ_i = 0 \), \( Eξ_i^2 = 1, i \in Z \). For the difference system (1), many sufficient conditions to guarantee the asymptotical mean square stability have been obtained by using the method of Lyapunov functionals construction [7–9].

However, besides the stochastic effect, an impulsive effect likewise exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics and telecommunications. Recently, the asymptotic behaviors of impulsive difference equations have attracted considerable attention. Many interesting results on impulsive effect have been obtained [10–15]. It is well known that the existence of both stochastic and impulsive effects is extensive in difference systems. Therefore, it is necessary to investigate the dynamical behaviors of impulsive stochastic difference equations. However, to the best of our knowledge, there is no investigation of this problem. Based on this, in this paper, we mainly consider the following impulsive stochastic difference equation

\[
\begin{align*}
    x_{i+1} &= F(i, x_{i-h}, \ldots, x_i) + G(i, x_{i-h}, \ldots, x_i)ξ_i, \quad i \neq i_k, \ i \in Z, \\
    x_{i+1} &= H_i(x_i), \quad i = i_k,
\end{align*}
\]

(2)

with initial condition

\[ x_i = φ_i, \quad i \in Z_0, \]

where \( H_i : R \rightarrow R \). The fixed moments of time \( i_k \in Z \), and satisfy \( 0 < i_1 < i_2 < \cdots, \lim_{k \rightarrow \infty} i_k = \infty \); other assumptions are the same as in (1).

The main difficulty for stability investigation of Eq. (2) comes from both stochastic and impulsive effects on the system since the corresponding theory for impulsive stochastic difference equations has not yet been fully developed. Many criteria on stability for stochastic difference equations [2–9] and impulsive difference equations [10–15] respectively may be difficult and even ineffective for impulsive stochastic difference equations. Therefore, techniques and methods for the stability of impulsive stochastic difference equations should be developed and explored. This paper presents one such method by establishing a difference inequality. Using this difference inequality, we shall give sufficient conditions to guarantee the mean square exponential stability of system (2). Satisfyingly, employing the difference inequality mentioned above, we can also obtain the mean square exponential stability of system (1). Furthermore, our results can extend and improve those of earlier publications. Two examples are also worked out to demonstrate the effectiveness of the proposed approach.

2. Preliminaries

Let \( N[a, b] \equiv \{a, a + 1, \ldots, b\} \), where \( a < b \) and \( a, b \) are integral numbers. As a standing hypothesis, we assume that Eq. (2) has a solution which is denoted by \{\( x_i \)\} in this paper and admits a zero solution. Next, we will introduce some basic definitions.

**Definition 1.** The zero solution of Eq. (1) or (2) is called mean square stable if for any \( ε > 0 \) there exists a \( δ > 0 \) such that \( Ex_i^2 < ε, i \in Z \), when the initial condition \( φ = (φ_{-h}, \ldots, φ_0)^T \) satisfies \( ∥φ∥^2 = \sup_{i \in Z_0} Ex_i^2 < δ \). If, besides, \( \lim_{i \rightarrow \infty} Ex_i^2 = 0, i \in Z \), for all initial condition \( φ \), then the zero solution of Eq. (1) or (2) is called asymptotically mean square stable.

**Definition 2.** The zero solution of Eq. (1) or (2) is called mean square exponential stable if there are positive constants \( λ \) and \( M \) such that for any initial condition \( φ \),

\[ Ex_i^2 \leq M∥φ∥^2e^{-λi}, \quad i \in Z.\]  

(3)

Here \( λ \) is called the exponential convergence rate.

To establish the main results of system (2), we will employ the following assumptions.

(A_{1}) For any \( i \in Z \), there exist positive constants \( a_j(i) \) and \( b_j(i) \) such that

\[ |F(i, x_{i-h}, \ldots, x_i)| \leq \sum_{j=0}^{h} a_j(i)|x_{i-j}|, \]
\[ |G(i, x_{i-h}, \ldots, x_i)| \leq \sum_{j=0}^{h} b_j(i)|x_{i-j}|. \]

(A2) \( \sup_{i \in \mathbb{Z}} \{a^2(i) + b^2(i)\} = \mu < 1 \), where \( a(i) = \sum_{j=0}^{h} a_j(i) \) and \( b(i) = \sum_{j=0}^{h} b_j(i) \).

(A3) There exist constants \( d_k \geq 1 \) such that \( |H_{ik}(x_{ik})| \leq d_k|x_{ik}|, \quad k = 1, 2, \ldots \).

(A4) There exists constant \( \alpha \geq 0 \) such that

\[
\frac{2 \ln d_k}{i_k - i_{k-1}} \leq \alpha < \lambda, \quad k = 1, 2, \ldots, \tag{4}
\]

where \( i_0 = 0 \) and \( \lambda \) satisfies

\[
0 < \lambda \leq \frac{1}{h+1} \ln \frac{1}{\eta}. \tag{5}
\]

3. Main results

It is well known that difference inequalities are very important tools for studying difference equations [16,17]. In order to obtain the mean square exponential stability of Eq. (2), we need to estimate every part of Eq. (2) on \([i_k, i_{k+1}]\) with its initial conditions on \([i_k - h, i_k]\) for \( k = 1, 2, \ldots \). It is therefore difficult to obtain the estimate (3) by using the existing difference inequalities ([16,17], etc.). To overcome these difficulties, we introduce the following difference inequality.

**Theorem 1.** Suppose \( c_j(i) \in \mathbb{R}^+, i \in \mathbb{Z}, j \in \mathbb{N}[0, h] \) and \( \sup_{i \in \mathbb{Z}} \{\sum_{j=0}^{h} c_j(i)\} = \eta < 1 \). Let \( \{u_i\} \) be a sequence of real numbers satisfying the following difference inequality:

\[
u_{i+1} \leq \sum_{j=0}^{h} c_j(i)u_{i-j}, \quad i \geq i', i \in \mathbb{Z}. \tag{6}\]

Then

\[
u_i \leq de^{-\lambda i}, \quad i \geq i', i \in \mathbb{Z}, \tag{7}\]

provided that the initial condition satisfies

\[
u_i \leq de^{-\lambda i}, \quad i \in \mathbb{N}[i' - h, i'], \tag{8}\]

where \( i' \in \mathbb{Z}, d \in \mathbb{R}^+ \) and \( \lambda \) satisfies

\[
0 < \lambda \leq \frac{1}{h+1} \ln \frac{1}{\eta}. \tag{9}\]

**Proof.** Since \( \eta < 1 \), there exists a constant \( \lambda \) satisfying the inequality (9). Then,

\[
e^{\lambda(h+1)\eta} \leq 1. \tag{10}\]

Let

\[y_i = u_ie^{\lambda i}. \tag{11}\]

Then from (8), we have

\[y_i \leq d, \quad i \in \mathbb{N}[i' - h, i']. \tag{12}\]
If this is not true, then there must be a positive integral number \(i^* \geq i'\) such that
\[
y_{i^*+1} > d \quad \text{and} \quad y_i \leq d, \quad i \in N[i' - h, i^*]. \tag{13}
\]

By (6), (10), (11) and (13), we have
\[
y_{i^*+1} = u_{i^*+1} e^{\lambda(i^*+1)}
\]
\[
\leq e^{\lambda(i^*+1)} \sum_{j=0}^{h} c_j(i^*) u_{i^*-j}
\]
\[
= e^{\lambda(i^*+1)} \sum_{j=0}^{h} c_j(i^*) y_{i^*-j} e^{-\lambda(i^*+j)}
\]
\[
\leq e^{\lambda(h+1)} d \sum_{j=0}^{h} c_j(i^*)
\]
\[
\leq e^{\lambda(h+1)} d \eta
\]
\[
\leq d,
\]
which contradicts the first inequality of (13). Thus (12) holds for any \(i \geq i'\). Therefore, we have
\[
u_i \leq de^{-\lambda i}, \quad i \geq i'.
\]
The proof is completed. \(\square\)

**Remark 1.** In the next context, we can obtain the estimate (3) by using Theorem 1 since \(d\) in the estimate (7) is an arbitrary nonnegative constant, provided the initial conditions satisfy the same exponential estimate.

**Theorem 2.** Assume that conditions \((A_1)\)–\((A_4)\) hold, then the zero solution of Eq. (2) is square exponential stable and the exponential convergence rate is equal to \(\lambda - \alpha\).

**Proof.** From (2), condition \((A_1)\) and the Hölder inequality [18], we have
\[
E_{x_i}^2 \leq E F^2(i, x_{i-h}, \ldots, x_i) + EG^2(i, x_{i-h}, \ldots, x_i)
\]
\[
\leq E \left( \sum_{j=0}^{h} a_j(i) |x_{i-j}| \right)^2 + E \left( \sum_{j=0}^{h} b_j(i) |x_{i-j}| \right)^2
\]
\[
\leq \sum_{j=0}^{h} a_j(i) \sum_{j=0}^{h} a_j(i) E |x_{i-j}|^2 + \sum_{j=0}^{h} b_j(i) \sum_{j=0}^{h} b_j(i) E |x_{i-j}|^2
\]
\[
= \sum_{j=0}^{h} [a(i)a_j(i) + b(i)b_j(i)] E x_{i-j}^2, \quad i \neq i_k, \quad k = 1, 2, \ldots. \tag{14}
\]

From condition \((A_2)\), we obtain
\[
\sup_{i \in Z} \left\{ \sum_{j=0}^{h} [a(i)a_j(i) + b(i)b_j(i)] \right\} = \sup_{i \in Z} \{a^2(i) + b^2(i)\} = \mu < 1. \tag{15}
\]

For the initial condition \(x_i = \varphi_i, \quad i \in Z_0 = N[-h, 0]\), we have
\[
E x_i^2 \leq \|\varphi\|^2 e^{-\lambda i}, \quad i \in N[-h, 0], \tag{16}
\]
where \(\|\varphi\|^2 = \sup_{i \in Z_0} E \varphi_i^2 < \delta\). Then, all the conditions of Theorem 1 are satisfied by (14)–(16). So, we can obtain
\[
E x_i^2 \leq \|\varphi\|^2 e^{-\lambda i}, \quad i \in N[0, i_1].
\]
Suppose for all \( q = 1, 2, \ldots, k \), the inequalities
\[
Ex_i^2 \leq d_0^2 d_1^2 \cdots d_{q-1}^2 \|\varphi\|^2 e^{-\lambda i}, \quad i \in N[i_{q-1}, i_q].
\] (17)
hold, where \( d_0 = 1 \) and \( i_0 = 0 \). Then from condition (A3) and (17), we have
\[
Ex_{i_k+1}^2 = E|H_{i_k}(x_{i_k})|^2
\leq d_k^2 Ex_{i_k}^2
\leq d_0^2 d_1^2 \cdots d_{k-1}^2 d_k^2 \|\varphi\|^2 e^{-\lambda i}.
\]
This, together with (17) and \( d_k \geq 1, k = 1, 2, \ldots \), leads to
\[
Ex_i^2 \leq d_0^2 d_1^2 \cdots d_k^2 \|\varphi\|^2 e^{-\lambda i}, \quad i \in N[i_k + 1 - h, i_k + 1].
\] (18)
It follows from (14), (15) and (18) and Theorem 1 that
\[
Ex_i^2 \leq d_0^2 d_1^2 \cdots d_k^2 \|\varphi\|^2 e^{-\lambda i}, \quad i \in N[i_k + 1, i_{k+1}],
\]
yielding, together with (17), that
\[
Ex_i^2 \leq d_0^2 d_1^2 \cdots d_k^2 \|\varphi\|^2 e^{-\lambda i}, \quad i \in N[i_k, i_{k+1}].
\]
By mathematical induction, we can conclude that
\[
Ex_i^2 \leq d_0^2 d_1^2 \cdots d_{k-1}^2 \|\varphi\|^2 e^{-\lambda i}, \quad i \in N[i_{k-1}, i_k], \quad k = 1, 2, \ldots.
\] (19)
Noticing that \( d_k^2 \leq e^{a(i-k-1)} \) by (4), we can use (19) to conclude that
\[
Ex_i^2 \leq e^{a(i-1)} \cdots e^{a(i-k-2)} \|\varphi\|^2 e^{-\lambda i},
\leq e^{a i} \|\varphi\|^2 e^{-\lambda i},
= \|\varphi\|^2 e^{-(\lambda - a)i}, \quad i \in N[i_{k-1}, i_k], \quad k = 1, 2, \ldots.
\]
This implies that the conclusion of Theorem 2 holds. \( \square \)

Remark 2. It follows from the proof of Theorem 2 that the above difference inequality plays an important role in obtaining the estimate (3). Satisfyingly, employing Theorem 1, we also can obtain the mean square exponential stability of system (1).

Theorem 3. Assume that conditions (A1) and (A2) hold; then the zero solution of Eq. (1) is square exponential stable and the exponential convergence rate \( \lambda \) satisfies (5).

Proof. Similarly to (14), we can obtain
\[
Ex_{i+1}^2 \leq \sum_{j=0}^h \{a(i)a_j(i) + b(i)b_j(i)\} Ex_{i-j}^2, \quad i \in Z.
\] (20)
It follows from condition (A2) that (15) holds. And the initial conditions also satisfy (16). Then, all the conditions of Theorem 1 are satisfied by (15), (16) and (20). So, we have
\[
Ex_i^2 \leq \|\varphi\|^2 e^{-\lambda i}, \quad i \in Z.
\]
The proof is completed. \( \square \)

4. Examples

In this section, we shall discuss two examples in order to illustrate the effectiveness of our results.
Example 1. Consider the following impulsive stochastic difference equation:

\[
\begin{aligned}
    x_{i+1} &= \frac{1}{4} \sin(x_i) - \frac{1}{3} x_{i-1} + \frac{1}{2} x_i \xi_i, \quad i \neq i_k, \ i \in \mathbb{Z}, \\
    x_{i_k+1} &= e^{0.05 k} x_{i_k},
\end{aligned}
\]

(21)

where \( i_k = i_{k-1} + k \). Thus,

\[
    h = 1, \quad F(i, x_{i-h}, \ldots, x_i) = \frac{1}{4} \sin(x_i) - \frac{1}{3} x_{i-1},
\]

\[
    G(i, x_{i-h}, \ldots, x_i) = \frac{1}{2} x_i, \quad H_i(x_i) = e^{0.05 k} x_i,
\]

yielding

\[
    |F(i, x_{i-h}, \ldots, x_i)| \leq \frac{1}{4} |x_i| + \frac{1}{3} |x_{i-1}|,
\]

\[
    |G(i, x_{i-h}, \ldots, x_i)| \leq \frac{1}{2} |x_i|, \quad |H_i(x_i)| = e^{0.05 k} |x_i|.
\]

So, the parameters of conditions (A1), (A2) and (A3) are as follows:

\[
    a_0(i) = \frac{1}{4}, \quad a_1(i) = \frac{1}{3}, \quad b_0(i) = \frac{1}{2}, \quad b_1(i) = 0, \quad i \in \mathbb{Z},
\]

\[
    \mu = \frac{85}{144} < 1, \quad d_k = e^{0.05 k} > 1, \quad k = 1, 2, \ldots.
\]

Let \( \alpha = \frac{2 \ln d_k}{i-t_{i-1}} = 0.1 \) and \( \lambda = 0.26 \leq \frac{1}{\pi + 1} \ln \frac{1}{\mu} = \frac{1}{2} \ln \frac{144}{85} \). Then condition (A4) is satisfied. So, by Theorem 2, we can get that the zero solution of Eq. (21) is square exponential stable and the exponential convergence rate is equal to 0.16.

Example 2. Consider the following stochastic difference equation:

\[
    x_{i+1} = ax_i + b \sum_{j=1}^{k} (k+1-j)x_{i-j} + \sigma \sum_{j=0}^{m} (m+1-j)x_{i-j} \xi_i, \quad i \in \mathbb{Z},
\]

(22)

where \( k, m \in \mathbb{Z} \). Kolmanovskii and Shaikhet have investigated this stochastic difference equation (see (2) in Ref. [9]) and obtained the following sufficient condition:

\[
    \left( |a| + |b| \frac{k(k+1)}{2} \right)^2 + \frac{\sigma^2}{4} (m+1)^2 (m+2)^2 < 1,
\]

(23)

to guarantee the asymptotical mean square stability by using the method of Lyapunov functionals construction. On the other hand, by Theorem 3 above, we can obtain the following property.

Property 1. If (23) holds, the zero solution of (22) is mean square exponentially stable and the exponential convergence rate \( \lambda \) satisfies

\[
    0 < \lambda \leq \frac{1}{h+1} \ln \frac{1}{\mu},
\]

where \( h = \max\{k, m\} \) in (22) and \( \mu \) is equal to the left-hand side of the inequality (23).

Proof. For system (22), it is easy to obtain that the parameters of condition (A1) are as follows:

\[
    a_0(i) = |a|, \quad a_j(i) = (k+1-j)|b|, \quad j = 1, 2, \ldots, k,
\]

\[
    b_j(i) = (m+1-j)|\sigma|, \quad j = 0, 1, \ldots, m, \ i \in \mathbb{Z}
\]
yielding
\[\mu = \sup_{i \in \mathbb{Z}} \{a^2(i) + b^2(i)\}\]
\[= \sup_{i \in \mathbb{Z}} \left\{ \left( \sum_{j=0}^{k} a_j(i) \right)^2 + \left( \sum_{j=0}^{m} b_j(i) \right)^2 \right\}\]
\[= \sup_{i \in \mathbb{Z}} \left\{ \left( |a| + |b| \sum_{j=1}^{k} (k + 1 - j) \right)^2 + \left( \sigma \sum_{j=0}^{m} (m + 1 - j) \right)^2 \right\}\]
\[= \left( |a| + |b| \frac{k(k + 1)}{2} \right)^2 + \frac{\sigma^2}{4} (m + 1)^2 (m + 2)^2.\]

It follows from (23) that condition (A2) is satisfied. Thus, by Theorem 3, Property 1 holds. □

Remark 3. Theorem 1 above can extend and improve the following Discrete Halanay’s Lemma established by Liz and Ferreiro; see Theorem 1 in [19].

**Discrete Halanay’s Lemma.** Let \( r > 0 \) be a natural number, and let \( \{x_n\}_{n \geq -r} \) be a sequence of real numbers satisfying the inequality
\[\Delta x_n \leq -ax_n + b \max\{x_n, x_{n-1}, \ldots, x_{n-r}\}, \quad n \geq 0.\] (24)

If \( 0 < b < a \leq 1 \), then there exists a constant \( \lambda_0 \in (0, 1) \) such that
\[x_n \leq \max\{0, x_0, x_{-1}, \ldots, x_{-r}\} \lambda_0^n, \quad n \geq 0.\] (25)

Moreover, \( \lambda_0 \) can be chosen as the smallest root in the interval \( (0, 1) \) of equation
\[\lambda^{r+1} + (a - 1)\lambda^r - b = 0,\]
where \( \Delta x_n = x_{n+1} - x_n.\)

However, by Theorem 1 above, we can obtain the following property.

**Property 2.** For the inequality (24), if \( 0 < b < a \leq 1 \), then
\[x_n \leq \max\{0, x_0, x_{-1}, \ldots, x_{-r}\} e^{-\lambda_0 n}, \quad n \geq 0.\] (26)

where \( \lambda_0 \) satisfies
\[0 < \lambda_0 \leq \frac{1}{r + 1} \ln \frac{1}{1 - a + b}.\]

Proof. Let \( x_{n-\tau(n)} = \max\{x_n, x_{n-1}, \ldots, x_{n-r}\}, \quad \tau(n) \in \mathbb{Z}[0, r], \quad n \in \mathbb{Z}. \) Then from (24), we get
\[x_{n+1} \leq (1 - a)x_n + bx_{n-\tau(n)}, \quad n \geq 0.\] (27)

It follows from \( 0 < b < a \leq 1 \) that
\[1 - a \geq 0, \quad b > 0, \quad 0 < 1 - a + b < 1.\] (28)

Furthermore, the initial condition satisfies
\[x_n \leq \max\{0, x_0, x_{-1}, \ldots, x_{-r}\} e^{-\lambda_0 n}, \quad n \in \mathbb{N}[-r, 0].\] (29)

Then, all the conditions of Theorem 1 are satisfied by (27)–(29). So, Property 2 is true. □

Remark 4. In particular, the difference inequality established above is also effective for investigating the stability of difference equations with variable coefficients or the “maximum” functional.
References