Frobenius categories

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1. Introduction

1.1. Let \( p \) be a prime number and \( G \) a finite group; since [8] we try to understand the meaning of what was vaguely called “\( p \)-local structure of \( G \)” and, on this way, in our Bourbaki’s lecture [9] we formally introduced the Frobenius category of \( G \) (at \( p \))—namely, the category \( \mathcal{F}_G \) formed by all the \( p \)-subgroups of \( G \), and all the group homomorphisms between them induced by the inner automorphisms of \( G \) and by their inclusions—a usual terminology in Chevalley’s Seminar. The fundamental Frobenius’ Criterion on the existence of a normal \( p \)-complement in \( G \) can be viewed as the conceptual origin of this approach.

1.2. More generally, since [1] the question switched from the “\( p \)-local structure of \( G \)” to the “\( b \)-local structure of \( G \)” for any Brauer block \( b \) of \( G \) and, coherently, in his Bourbaki’s lecture [2], Michel Broué introduces the Brauer category of \( b \), which already was our common language when considering the nilpotent blocks [5]. In the Brauer category of \( b \), the objects are all the so-called Brauer \( b \)-pairs \((P,e)\), formed by a \( p \)-subgroup \( P \) of \( G \) such that—in Brauer’s terms—\( b \) is induced from some block of \( C_G(P) \), and \( e \) is one of them; between these pairs it exists a suitable “inclusion,” and once again the morphisms are the group homomorphisms—between the \( p \)-groups—induced by the inner automorphisms of \( G \) and by these “inclusions.” Actually, we were quickly aware that many Brauer categories of blocks are ordinary Frobenius categories of (other) groups.
1.3. The existence of the localizing functor over the Brauer category of a block—proved in [10]—pushed us to look for an abstract description which could allow an eventual classification of the “$p$-local structures,” a purpose already hazarded in [8]. From this spirit, in 1991 we found the Frobenius categories over a finite $p$-group; our work, widely presented in Chevalley’s Seminar, remained unpublished waiting for a significant test on its interest. The recent interest of several people on the subject (see [4], for instance) motivates the present paper on the Frobenius categories and a forthcoming paper on the localizing functor and the localities [14].

1.4. As suggested above, the choice of the conditions in the abstract setting follows as near as possible the behavior of $F_G$; but, since it only makes sense to consider this category up to equivalences, we put a unique maximal object $P$ which is a finite $p$-group. As a matter of fact, we start with a very simple definition of the so-called category over $P$—or $P$-category in short—but are really interested on the $P$-categories fulfilling a suitable set of conditions, namely the Frobenius $P$-categories; however, it is handy to name the previous steps.\(^1\) The Frobenius categories of the normalizers and the centralizers of the $p$-subgroups of $G$ can be determined from $F_G$ and a first satisfactory result is that, in the abstract setting of a category over a finite $p$-group, they can be defined and inherit the set of conditions, determining new Frobenius categories.

1.5. The well-known Alperin’s Fusion Theorem is one of the typical statements on $G$ that can be formulated in $F_G$, and that it remains true in all the Frobenius $P$-categories $\mathcal{F}$. More precisely, as in [8, Chapter III], the most significant point about fusions is the emergency of the $\mathcal{F}$-essential objects; but, we have improved our formulation there by considering the corresponding additive category, where a fusion appears as a difference of two morphisms. A first consequence of this kind of results is that, we get an analogous of the Frobenius category of $O^p(G)$: for any Frobenius $P$-category $\mathcal{F}$ we prove that the intersection of all the Frobenius subcategories containing $O^p(\mathcal{F}(Q))$, for any subgroup $Q$ of $P$, is a Frobenius $P$-category too—called the adjoint Frobenius $P$-subcategory of $\mathcal{F}$.

1.6. From the translation of Alperin’s Fusion Theorem, it makes sense to consider the $\mathcal{F}$-focal or, better, the $\mathcal{F}$-hyperfocal subgroup of $P$, which corresponds to a Sylow $p$-subgroup of $O^p(G)$ whenever $\mathcal{F}$ is the Frobenius category of a finite group $G$; as a matter of fact, for any Frobenius $P$-category there exists a suitable Frobenius category over the $\mathcal{F}$-hyperfocal subgroup—called the hyperfocal Frobenius subcategory of $\mathcal{F}$. At this point, the iteration of the adjoint and the hyperfocal constructions leads us to the definition of the ($p$-)solvable Frobenius categories and a significant result that we will prove in [14] is that any of them is just the Frobenius category of a finite $p$-solvable group.

1.7. The notation is standard and mainly concerns group theory—our standard reference being [6]—and homological algebra—our standard reference being [13]. In particular,

\(^1\) After our unsuccessful efforts towards an agreement on a common terminology, we come back to the original one.
if $G$ is a finite group and $p$ a prime number, recall that $O^p(G)$, $O^{p'}(G)$, $O_p(G)$ and $O_{p'}(G)$ respectively denote the minimal or the maximal normal subgroups of $G$ with their index or their order being a power of $p$ or prime to $p$. For any subgroup $H$ of $G$, we set $N_G(H) = N_G(H)/H$ and, for another subgroup $K$, we denote by $T_G(K, H)$ the set of $x \in G$ fulfilling $xKx^{-1} \subset H$; if $K \subset H$, we denote by $i^H_K$ the corresponding inclusion map.

2. Definition of the Frobenius $P$-categories

2.1. Let $P$ be a finite $p$-group and denote by $F_P$ its Frobenius category. A $P$-category $F$ is a subcategory of the category of groups $\mathcal{G}$, containing $F_P$ and with the same objects, where all the homomorphisms are injective; for any pair of subgroups $Q$ and $R$ of $P$, we denote by $F(Q, R)$ the set of morphisms in $F$ from $R$ to $Q$ and we write $F(Q)$ instead of $F(Q, Q)$. Note that the intersection $F \cap F'$ of two $P$-categories is a $P$-category too, and that there is a unique maximal $P$-category, namely the $P$-category containing all the injective group homomorphisms between the subgroups of $P$.

2.2. Obviously, for any finite group $G$ having $P$ as a Sylow $p$-subgroup, we have a $P$-category $F_{G,P}$ by considering all the group homomorphisms between the subgroups of $P$ induced by the inner automorphisms of $G$; this category is equivalent to the Frobenius category of $G$ at $p$ and usually we simply write $F_G$; note that $G/O_{p'}(G)$ determines the same $P$-category as $G$.

2.3. We say that $F$ is divisible whenever it fulfills

2.3.1 if $Q$, $R$ and $T$ are subgroups of $P$, for any $\varphi \in F(Q, R)$ and any group homomorphism $\psi : T \to R$ the composition $\varphi \circ \psi$ belongs to $F(Q, T)$ (if and) only if $\psi \in F(R, T)$.

or, equivalently, whenever for any subgroup $Q$ of $P$, the category $(F)_Q$ over $Q$ is a full subcategory of $(\mathcal{G})_Q$. Note that the maximal $P$-category is divisible, and that the intersection of two divisible $P$-categories is divisible too. Actually, all the $P$-categories we will consider are divisible; notice that, in this case if $\varphi \in F(Q, R)$ then the equality $|Q| = |R|$ implies that $\varphi^{-1}$ belongs to $F(R, Q)$; more generally, if $Q'$ and $R'$ are respective subgroups of $Q$ and $R$ such that $\varphi(R') \subset Q'$, the group homomorphism $R' \to Q'$ determined by $\varphi$ belongs to $F(Q', R')$, as it is easily checked.

2.4. In particular, a divisible $P$-category $F$ is determined by the sets $F(P, Q)$ where $Q$ runs over the set of all the subgroups of $P$; conversely, if $\mathcal{X}$ is a $P$-stable set of subgroups of $P$ and, for any $Q \in \mathcal{X}$, $H(P, Q) \subset \text{Hom}(Q, P)$ is a set of injective homomorphisms containing $F_P(P, Q)$ and fulfilling the following two conditions:

2.4.1 if $Q \in \mathcal{X}$ and $\varphi \in H(P, Q)$ then any subgroup $R$ of $P$ containing $\varphi(Q)$ belongs to $\mathcal{X}$.

\[\text{Presented in June 1991 at the “Warwick Algebra Symposium.”}\]
2.4.2 if $Q, R \in \mathcal{X}$ and $\theta \in \text{Hom}(Q, R)$ fulfill $\mathcal{H}(P, Q) \cap (\mathcal{H}(P, R) \circ \theta) \neq \emptyset$, then we have $\mathcal{H}(P, R) \circ \theta \subset \mathcal{H}(P, Q)$, it is straightforward to prove that there is a divisible $P$-category $\mathcal{F}'$ fulfilling $\mathcal{F}'(P, Q) = \mathcal{H}(P, Q)$ for any $Q \in \mathcal{X}$.

2.5. Let $\mathcal{F}$ be a divisible $P$-category and $Q$ a subgroup of $P$. If $\mathcal{F}$ is the $P$-category associated with a finite group $G$ having Sylow $p$-subgroup $P$, it is obvious that the centralizer or the normalizer of $Q$ in $P$ need not be a Sylow $p$-subgroup of the corresponding centralizer or normalizer in $G$, and in our abstract setting we will determine when they are. For dealing with both—centralizer and normalizer—simultaneously, we introduce the $K$-normalizer $N^K_P(Q)$ of $Q$ in $G$ for any subgroup $K$ of $\text{Aut}(Q)$, which is just the converse image of $K$ in $N_G(Q)$; actually, in the case of $N^K_P(Q)$ only the $p$-subgroups of $\text{Aut}(Q)$ will be really involved. Moreover, it is handy to introduce the following notation: if $Q, R$ and $T$ are subgroups of $P$ and $Q \subset T$, any injective group homomorphism $\psi : T \to R$ determines a group isomorphism $\text{Aut}(Q) \cong \text{Aut}(\psi(Q))$ and we simply denote by $\psi K$ and $\psi \chi$ the images of $K \subset \text{Aut}(Q)$ and $\chi \in \text{Aut}(Q)$, respectively.

2.6. Let $\mathcal{F}$ be a divisible $P$-category, $Q$ a subgroup of $P$ and $K$ a subgroup of $\text{Aut}(Q)$; it is quite clear that, for any $\psi \in \mathcal{F}(P, Q \cdot N^K_P(Q))$, we have $\psi(N^K_P(Q)) \subset N^K_P(\psi(Q))$; thus, we say that $Q$ is fully $K$-normalized in $\mathcal{F}$ whenever it fulfills

2.6.1 for any $\psi \in \mathcal{F}(P, Q \cdot N^K_P(Q))$, we have $\psi(N^K_P(Q)) = N^K_P(\psi(Q))$.

If $K = \{\text{id}_{Q}\}$ or $K = \text{Aut}(Q)$, we respectively say that $Q$ is fully centralized or fully normalized in $\mathcal{F}$; note that $K$ and $K \cdot \mathcal{F}_Q(Q)$ play the same role, so that we always may assume that $\mathcal{F}_Q(Q) \subset K$.

2.7. From the above inclusion, it is quite clear that if $R$ is a subgroup of $Q \cdot N^K_P(Q)$ containing $Q$ and $\varphi \in \mathcal{F}(P, R)$ fulfills the condition

2.7.1 for any $\varphi' \in \mathcal{F}(P, R)$, we have $|N^K_P(\varphi'(Q))| \leq |N^K_P(\varphi(Q))|$, then $Q' = \varphi(Q)$ is fully $\varphi K$-normalized in $\mathcal{F}$; indeed, since $\varphi(R)$ is contained in $Q' \cdot N^K_P(Q')$ and $\mathcal{F}$ is divisible, any $\psi \in \mathcal{F}(P, Q' \cdot N^K_P(Q'))$ determines $\varphi' \in \mathcal{F}(P, R)$ mapping $v \in R$ on $\psi(\varphi(v))$ and therefore we get

2.7.2 $|N^K_P(\varphi'(Q))| \leq |N^K_P(\varphi'(Q'))|$ and $\psi(N^K_P(\varphi'(Q')) \subset N^K_P(\varphi'(Q')).$

In particular, there is $\varphi \in \mathcal{F}(P, Q)$ such that $\varphi(Q)$ is both fully centralized and fully $\varphi K$-normalized in $\mathcal{F}$; indeed, there exists $\varphi' \in \mathcal{F}(P, Q)$ such that $Q' = \varphi'(Q)$ is fully centralized in $\mathcal{F}$, and then there is $\psi' \in \mathcal{F}(P, Q' \cdot C_P(Q'))$ such that $\psi'(Q')$ is fully $\psi'(\varphi K)$-normalized in $\mathcal{F}$; but $\psi'(Q')$ is fully centralized too.
2.8. For any subgroup \( P' \) of \( P \), a \( P' \)-subcategory \( \mathcal{F}' \) of \( \mathcal{F} \) has an obvious definition, and the conditions to define a Frobenius \( P \)-category we are looking for will be a suitable formulation of Sylow’s Theorem in the \( P' \)-subcategories, for suitable \( P' \) we define here. With the notation and the hypothesis above, assume that \( Q \) is fully \( K \)-normalized in \( \mathcal{F} \); the \( K \)-normalizer of \( Q \) in \( \mathcal{F} \) is the \( N^K_p(Q) \)-subcategory \( N^K_{\mathcal{F}}(Q) \) where, for any pair of subgroups \( R \) and \( T \) of \( N^K_p(Q) \), the set of morphisms from \( T \) to \( R \) is the set of \( \psi \in \mathcal{F}(R,T) \) fulfilling the following condition

2.8.1 there are \( \psi \in \mathcal{F}(Q \cdot R, Q \cdot T) \) and \( \chi \in K \) such that \( \chi(u) = \psi(u) \) for any \( u \in Q \), and that \( \psi(v) = \varphi(v) \) for any \( v \in T \).

It is quite clear that \( N^K_{\mathcal{F}}(Q) \) is a \( N^K_p(Q) \)-category; moreover, since \( \mathcal{F} \) is divisible, it is easy to see that the isomorphism \( T \cong \varphi(T) \) determined by \( \psi \in \mathcal{F}(R,T) \) belongs to \( (N^K_{\mathcal{F}}(Q))(R,T) \) and therefore \( N^K_{\mathcal{F}}(Q) \) is divisible too; actually, \( (N^K_{\mathcal{F}}(Q))(R,T) \) also coincides with the set of group homomorphisms \( \varphi : T \to R \) fulfilling condition 2.8.1. Note that, whenever \( \mathcal{F} \) is the Frobenius category associated with a finite group \( G \), \( N^K_{\mathcal{F}}(Q) \) is just the Frobenius category associated with the \( K \)-normalizer of \( Q \) in \( G \).

2.9. Strictly speaking, the definition of \( N^K_{\mathcal{F}}(Q) \) does not depend on the assumption that \( Q \) is fully \( K \)-normalized in \( \mathcal{F} \), but only in this case it could come from the Frobenius category of a finite group; moreover, whenever \( \mathcal{F} \) is the Frobenius category associated with a finite group \( G \), for any \( \varphi \in \mathcal{F}(P, Q) \) the \( p \)-subgroup \( N^K_p(\varphi(Q)) \) has a \( G \)-conjugate in the \( K \)-normalizer of \( Q \) in \( \mathcal{F} \) and, by Sylow’s Theorem, we may choose the conjugate in \( N^K_p(Q) \). Thus, we say that a \( P \)-category \( \mathcal{F} \) is a Frobenius \( P \)-category (or a Frobenius category over \( P \)) if it is divisible and fulfills the following two conditions:

2.9.1 The group \( \mathcal{F}_P(P) \) of inner automorphisms of \( P \) is a Sylow \( p \)-subgroup of \( \mathcal{F}(P) \).
2.9.2 For any subgroup \( Q \) of \( P \), any subgroup \( K \) of \( \text{Aut}(Q) \) and any element \( \varphi \in \mathcal{F}(P, Q) \) such that \( \varphi(Q) \) is fully \( \psi \)-\( K \)-normalized in \( \mathcal{F} \), there are elements \( \psi \in \mathcal{F}(P, Q \cdot N^K_p(Q)) \) and \( \chi \in K \) such that \( \psi(u) = \varphi(\chi(u)) \) for any \( u \in Q \).

2.10. Actually, in condition 2.9.2 we may assume that \( K \) contains \( \mathcal{F}_Q(Q) \) (cf. 2.5), and that \( Q' = \varphi(Q) \) is fully centralized too; indeed, in any case there is \( \varphi' \in \mathcal{F}(P, Q) \) such that \( Q'' = \varphi'(Q) \) is both fully centralized and fully \( \psi \)-\( K \)-normalized in \( \mathcal{F} \); thus, denoting by \( \varphi'' \in \mathcal{F}(Q'', Q') \) the element fulfilling \( \varphi''(\varphi(u)) = \varphi'(u) \) for any \( u \in Q \), if there are elements \( \psi' \in \mathcal{F}(P, Q \cdot N^K_p(Q')) \), \( \psi'' \in \mathcal{F}(P, Q' \cdot N^K_p(Q')) \) and \( \chi', \chi'' \in K \) fulfilling \( \psi'(u) = \varphi'(\chi'(u)) \) for any \( u \in Q \), and \( \psi''(u') = \varphi''(\varphi'(\chi''(u'))) \) for any \( u' \in Q' \), then, since \( \mathcal{F} \) is divisible and we have

2.10.1

\[
\psi''(N^K_p(Q')) = N^K_{\mathcal{F}}(\varphi''(Q'')) \quad \text{and} \quad \psi''(\psi K) = \varphi' K,
\]

there is \( \xi' \in \mathcal{F}(Q' \cdot N^K_p(Q'), Q'' \cdot N^K_{\mathcal{F}}(K''')) \) such that \( \varphi(u) = \xi'(\varphi''(\chi''(u))) \) for any \( u \in Q' \); that is to say, the elements \( \psi \in \mathcal{F}(P, Q \cdot N^K_p(Q)) \) such that \( \psi(w) = \xi'(\psi'(w)) \), for any \( w \in Q \cdot N^K_p(Q) \), and \( \chi''^{-1} \circ \chi'' \in K \) fulfill condition 2.9.2.
2.11. Moreover, condition 2.9.2 implies the following one.

2.11.1 For any subgroup $Q$ of $P$, any $\varphi \in \mathcal{F}(P, Q)$ such that $\varphi(Q)$ is fully centralized in $\mathcal{F}$, and any subgroup $R$ of $N_P(Q)$ such that $Q \subset R$ and $\varphi \mathcal{F}_R(Q) \subset \mathcal{F}_P(\varphi(Q))$, there is $\rho \in \mathcal{F}(P, R)$ such that $\rho(u) = \varphi(u)$ for any $u \in Q$.

Indeed, first of all note that if $Q$ is a subgroup of $P$ fully centralized in $\mathcal{F}$ and $R$ is a subgroup of $N_P(Q)$ containing $Q$, then we have $N^\mathcal{F}_R(Q)(Q) = R \cdot C_P(Q)$ and, for any $\eta \in \mathcal{F}(P, R \cdot C_P(Q))$, we get

$$\eta(R \cdot C_P(Q)) = \eta(R) \cdot C_P(\eta(Q)) = N^\eta \mathcal{F}_R(Q)(\eta(Q)),$$

so that $Q$ is also fully $\mathcal{F}_R(Q)$-normalized in $\mathcal{F}$; hence, in condition 2.11.1 the subgroup $\varphi(Q)$ is fully $\mathcal{F}_R'(Q)$-normalized in $\mathcal{F}$ where $R'$ is the converse image of $\varphi \mathcal{F}_R(Q)$ in $N_P(\varphi(Q))$; then, condition 2.9.2 implies the existence of $\psi \in \mathcal{F}(P, R)$ and $\chi \in \mathcal{F}_R(Q)$ such that $\psi(u) = \varphi(\chi(u))$ for any $u \in Q$, and it suffices to choose $w \in R$ lifting $\chi$ and to define $\rho \in \mathcal{F}(P, R)$ by $\rho(v) = \psi(v^w)$ for any $v \in R$. Conversely, in [3] Broto, Levi and Oliver show that condition 2.9.2 can be replaced by condition 2.11.1 and Radu Stancu has noticed that the same is true by replacing “fully centralized” by “fully normalized”$^3$; actually, condition 2.9.2 can be replaced by condition 2.12.1 below.

**Proposition 2.12.** Let $\mathcal{F}$ be a divisible $P$-category such that $\mathcal{F}_P(P)$ is a Sylow $p$-subgroup of $\mathcal{F}(P)$. Then, $\mathcal{F}$ is a Frobenius $P$-category if and only if it fulfills:

2.12.1 For any subgroup $Q$ of $P$, any $\varphi \in \mathcal{F}(P, Q)$ such that $\varphi(Q)$ is both fully centralized and fully normalized in $\mathcal{F}$, and any subgroup $R$ of $N_P(Q)$ such that $Q \subset R$ and $\varphi \mathcal{F}_R(Q) \subset \mathcal{F}_P(\varphi(Q))$, there is $\rho \in \mathcal{F}(P, R)$ such that $\rho(u) = \varphi(u)$ for any $u \in Q$.

In this case, for any subgroup $Q$ of $P$ and any subgroup $K$ of $\text{Aut}(Q)$, the following statements are equivalent:

2.12.2 The subgroup $Q$ is fully $K$-normalized in $\mathcal{F}$.
2.12.3 For any $\varphi \in \mathcal{F}(P, Q)$, we have $|N^\varphi K(Q)| \leq |N^K P(Q)|$.
2.12.4 The subgroup $Q$ is fully centralized in $\mathcal{F}$ and $K \cap \mathcal{F}_P(Q)$ is a Sylow $p$-subgroup of $K \cap \mathcal{F}(Q)$.

**Proof.** We already know that statement 2.12.3 implies statement 2.12.2. Assume that $\mathcal{F}$ fulfills statement 2.12.1 and let us prove that condition 2.9.2 holds; actually, it is easily checked from 2.7 that $\mathcal{F}$ fulfills statement 2.11.1 also. Let $Q$ be a subgroup of $P$, $K$ a subgroup of $\text{Aut}(Q)$ and $\varphi \in \mathcal{F}(P, Q)$ an element such that $Q' = \varphi(Q)$ is fully central-

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$^3$ In our terminology!
ized and fully \( ^{q}K \)-normalized in \( F \); by Lemma 2.13 below, setting \( K' = ^{q}K \) we know that \( K' \cap F_{p}(Q') \) is a Sylow \( p \)-subgroup of \( K' \cap F(Q') \) and therefore, for a suitable \( \chi \in K \), we have

\[
2.12.5 \quad \varphi(\phi(K \cap F_{p}(Q)) \subset K' \cap F_{p}(Q').
\]

Moreover, choose \( \varphi' \in F(P, Q') \) such that \( Q'' = \varphi'(Q') \) is fully centralized and fully normalized in \( F \) (cf. 2.7); once again, up to a modification of our choice, we may assume that \( \varphi'K' \cap F_{p}(Q'') \) is a Sylow \( p \)-subgroup of \( \varphi'K' \cap F(Q'') \). Then, it follows from statement 2.12.1 that \( \varphi' \) can be extended to \( \rho' \in F(P, Q' \cdot N_{P}^{K}(Q')) \) and that there is \( \rho'' \in F(P, Q \cdot N_{P}^{K}(Q)) \) fulfilling \( \rho''(u) = (\varphi' \circ \varphi \circ \chi)(u) \) for any \( u \in Q \); but, since \( Q' \) is fully \( K' \)-normalized in \( F \), we have \( \rho'(N_{P}^{K}(Q')) = N_{P}^{\varphi'K}(Q'') \) and, denoting by \( \rho \in F(P, Q \cdot N_{P}^{K}(Q)) \) the element fulfilling \( \rho'(\rho(v)) = \rho''(v) \) for any element \( v \in Q \cdot N_{P}^{K}(Q) \), it is clear that \( \rho \) extends \( \varphi \circ \chi \).

Now, assume that \( F \) is a Frobenius \( P \)-category; we firstly prove that statement 2.12.2 implies statement 2.12.1; indeed, since \( F \) is divisible, any element \( \psi \in F(P, Q \cdot C_{P}(Q)) \) determines \( \varphi' \in F(P, \psi(Q)) \) such that \( \varphi'(\psi(u)) = u \) for any \( u \in Q \) (cf. 2.3) and therefore, setting \( Q' = \psi(Q) \) and \( K' = \psi K \), by condition 2.9.2 there are \( \xi' \in F(P, N_{P}^{K}(Q')) \) and \( \chi' \in K' \) such that \( \xi'(u) = \varphi'(\chi'(u)) \) for any \( u \in Q' \); in particular, \( \xi'(C_{P}(Q')) \subset C_{P}(Q) \) and therefore \( \psi(C_{P}(Q')) = C_{P}(Q') \); that is to say, \( Q \) is fully centralized in \( F \) and then statement 2.12.4 follows from Lemma 2.13 below.

Moreover, if \( Q \) is a subgroup of \( P \) and \( \varphi \in F(P, Q) \) an element such that \( Q' = \varphi(Q) \) is fully centralized and fully normalized in \( F \) then, for any subgroup \( R \) of \( N_{P}(Q) \) such that \( Q \subset R \) and \( \varphi F_{R}(Q) \subset F_{R}(R') \), denoting by \( R' \) the converse image of \( \varphi F_{R}(Q) \) in \( N_{P}(Q) \), so that \( \varphi F_{R}(Q) = F_{R}(R') \), it is easily checked that \( Q' \) is also fully \( F_{R}(R') \)-normalized in \( F \); hence, condition 2.9.2 implies the existence of \( \psi \in F(P, R) \) and \( \chi \in F_{R}(Q) \) such that \( \psi(u) = \varphi(\chi(u)) \) for any \( u \in Q \); thus, in order to prove statement 2.12.1, it suffices to choose \( w \in R \) lifting \( \chi \) and to define \( \rho \in F(P, R) \) by \( \rho(v) = \psi(v^{w}) \) for any \( v \in R \).

Finally, assume that statement 2.12.4 holds; according to the divisibility of \( F \) and to condition 2.9.2, for any \( \varphi \in F(P, Q) \), setting \( Q' = \varphi(Q) \) there is \( \psi \in F(P, Q' \cdot C_{P}(Q')) \) such that \( \psi(\varphi(u)) = u \) for any \( u \in Q \), and therefore we have \( \psi(C_{P}(Q')) \subset C_{P}(Q) \); hence, we get

\[
2.12.6 \quad |N_{P}^{\psi K}(Q')| = |C_{P}(Q')| \cdot |^{\psi}K \cap F_{p}(Q')| \leq |C_{P}(Q)| \cdot |K \cap F(Q)|_{p} = |N_{P}^{K}(Q)|
\]

which proves statement 2.12.3 (cf. 2.7). \( \square \)

**Lemma 2.13.** Let \( F \) be a divisible \( P \)-category such that \( F_{P}(P) \) is a Sylow \( P \)-subgroup of \( F(P) \), and \( X \) a nonempty set of subgroups of \( P \) such that if \( Q \in X \) then any subgroup \( R \) of \( P \) fulfilling \( F(R, Q) \neq \emptyset \) belongs to \( X \). Assume that for any subgroup \( Q \in X \), any \( \varphi \in F(P, Q) \) such that \( \varphi(Q) \) is fully centralized in \( F \), and any subgroup \( R \) of \( N_{P}(Q) \) such that \( Q \subset R \) and \( \varphi F_{R}(Q) \subset F_{R}(\varphi(Q)) \), there is \( \rho \in F(P, R) \) such that \( \rho(u) = \varphi(u) \) for any \( u \in Q \). Then, for any \( Q \in X \) and any subgroup \( K \) of \( \text{Aut}(Q) \) such that \( Q \) is fully centralized and fully \( K \)-normalized in \( F \), \( K \cap F_{P}(Q) \) is a Sylow \( p \)-subgroup of \( K \cap F(Q) \).
Proof. We may assume that \( Q \neq P \) and argue by induction on \(|P:Q|\); set \( R = N^K_P(Q) \). In the case where \( K = \text{Aut}(Q) \), denoting by \( J \) the set of automorphisms of \( R \) stabilizing \( Q \), it is clear that \( N^J_P(R) = R \) and therefore, since \( Q \) is fully normalized in \( F \), \( R \) is fully \( J \)-normalized in \( F \) so that, according to the induction hypothesis, \( J \cap F_P(R) = F_R(R) \) is a Sylow \( p \)-subgroup of \( J \cap F(R) \); but, since \( Q \) is fully centralized in \( F \), it follows from our hypothesis that any element of \( N_{F(Q)}(F_R(Q)) \) can be lifted to \( J \cap F(R) \); consequently, \( F_R(Q) \) is a Sylow \( p \)-subgroup of \( N_{F(Q)}(F_R(Q)) \), so it is a Sylow \( p \)-subgroup of \( F(Q) \).

In the general case, choose \( \phi \in F(P, Q \cdot C_P(Q)) \) such that \( Q' = \phi'(Q) \) is fully normalized in \( F \) (cf. 2.7); thus, since \( Q' \) is also fully centralized in \( F \), according to the above argument, \( F_P(Q') \) is a Sylow \( p \)-subgroup of \( F(Q') \) and therefore it contains a Sylow \( p \)-subgroup of \( F(Q') \) for a suitable \( \tau \in F(Q') \); that is to say, up to a modification of our choice of \( \phi' \), we may assume that \( \phi'K \cap F_P(Q') \) is a Sylow \( p \)-subgroup of \( \phi'K \cap F(Q') \) containing \( \phi'F_P(Q) \); now, according to our hypothesis, \( \phi' \) can be extended to \( \rho' \in F(P, Q \cdot R) \) and moreover, if \( Q \) is fully \( K \)-normalized in \( F \), we have \( \rho'(R) = N_{F(P)}^\phi K(Q') \), so that we get

\[
\phi'(K \cap F_P(Q)) = \phi'K \cap F_P(Q');
\]

hence, since \( \phi'(K \cap F(Q)) = \psi'K \cap F(Q') \), \( K \cap F_P(Q) \) is a Sylow \( p \)-subgroup of \( K \cap F(Q) \). \( \Box \)

Corollary 2.14. Let \( F \) be a divisible \( P \)-category such that \( F_P(P) \) is a Sylow \( p \)-subgroup of \( F(P) \). Then, \( F \) is a Frobenius \( P \)-category if and only if, for any pair of \( F \)-isomorphic subgroups \( Q \) and \( Q' \) of \( P \) fully normalized in \( F \), there is an \( F \)-isomorphism \( N_P(Q) \cong N_P(Q') \) mapping \( Q \) onto \( Q' \) and moreover, for any subgroup \( R \) of \( N_P(Q) \) containing \( Q \), denoting by \( F(R)_Q \) the stabilizer of \( Q \) in \( F(R) \), the group homomorphism

\[
F(R)_Q \to N_{F(Q)}(F_R(Q))
\]

induced by the restriction is surjective.

Proof. If \( F \) is a Frobenius \( P \)-category, it follows from condition 2.9.2 that, for any \( F \)-isomorphism \( \phi: Q \cong Q' \), there is \( \psi \in F(P, N_P(Q)) \) extending \( \phi \circ \chi \) for a suitable \( \chi \in F(Q) \) and, since \( Q \) is fully normalized in \( F \), we have \( \psi(N_P(Q)) = N_P(Q') \); moreover, the surjectivity of homomorphism 2.14.1 follows from Proposition 2.12.

Conversely, assume that \( F \) fulfills the condition above; first of all, we claim that \( F_P(Q) \) is a Sylow \( p \)-subgroup of \( F(Q) \). Indeed, let \( \xi: N_P(Q) \to P \) be an \( F \)-morphism such that \( N' = \xi(N_P(Q)) \) is fully normalized in \( F \); arguing by induction, we may assume that \( F_P(N') \) is already a Sylow \( p \)-subgroup of \( F(N') \) and therefore, up to a modification of our choice of \( \xi \), we still may assume that \( F_P(N')_Q' \) is a Sylow \( p \)-subgroup of \( F(N')_Q' \). Then, according to our hypothesis, the image of \( F_P(N')_Q' \) is a Sylow \( p \)-subgroup of \( N_{F(Q)}(F_R(Q')) \); but, this image is contained in \( F_P(Q') = F_{N'}(Q') \); consequently, \( F_P(Q') \) is a Sylow \( p \)-subgroup of its normalizer in \( F(Q') \), so that it is a Sylow \( p \)-subgroup of \( F(Q') \) and \( F_P(Q) \) is one of \( F(Q) \).
Now, let $Q$ be a subgroup of $P$, $\varphi \in \mathcal{F}(P, Q)$ such that $Q' = \varphi(Q)$ is fully normalized in $\mathcal{F}$ and $R$ a subgroup of $N_P(Q)$ such that $Q \subset R$ and $\varphi^{ \mathcal{F}_R(Q)} \subset \mathcal{F}(P, \varphi(Q))$; there is $\psi \in \mathcal{F}(P, N_P(Q))$ such that $Q'' = \psi(Q)$ is fully normalized in $\mathcal{F}$ (cf. 2.7) and then, according to our hypothesis, there is an $\mathcal{F}$-isomorphism $\zeta$ such that $\zeta(Q''') = Q'$ such that $\zeta(Q''') = Q'$; in particular, since $\mathcal{F}$ is divisible, there is $\sigma' \in \mathcal{F}(Q')$ fulfilling $\varphi(u) = \sigma'(\psi(\varphi(u)))$ for any $u \in Q$, whereas $Q' = \xi(Q)$ remains fully normalized in $\mathcal{F}$. That is to say, setting $R' = \xi(\psi(R))$, the groups $\mathcal{F}_{R'}(Q')$ and $\sigma' \circ \mathcal{F}_{R'}(Q') \circ \sigma'^{-1} = \mathcal{F}(P, Q)$ are contained in $\mathcal{F}(Q')$ which is a Sylow $p$-subgroup of $\mathcal{F}(Q')$.

At this point, it suffices to prove that there is $\theta' \in \mathcal{F}(P, R')$ fulfilling $\theta'(Q') = Q'$ and $\theta'(u') = \sigma'(u')$ for any $u' \in Q'$; indeed, in this case the $\mathcal{F}$-morphism $R \to P$ mapping $v \in R$ to $\theta'(\psi(v)))$ extends $\varphi$ and the corollary follows from Proposition 2.12. We apply Alperin’s Fusion Theorem [6, Chapter 7, Theorem 2.6] to the group $\mathcal{F}(Q')$ and argue by induction on the length of the decomposition of $\sigma'$ in Alperin’s statement. Thus, there are $T' \subset N_P(Q')$, $\eta' \in \mathcal{F}(T', R')$ and $\tau' \in N_{\mathcal{F}(Q')}(\mathcal{F}(T'))$ such that $\eta'(Q') = Q'$ and that the image of the restriction of $\eta'$ to $Q'$ coincides with $\tau'^{-1} \circ \sigma'$; but, according to our hypothesis, $\tau'$ can be lifted to some $\rho' \in \mathcal{F}(T')Q'$ and therefore the restriction of $\rho' \circ \eta'$ to $Q'$ lifts $\sigma'$. □

**Proposition 2.15.** Let $\mathcal{F}$ be a Frobenius $P$-category, $Q$ a subgroup of $P$ and $K$ a subgroup of $\text{Aut}(Q)$. If $Q$ is fully $K$-normalized in $\mathcal{F}$ then $N^K_\mathcal{F}(Q)$ is a Frobenius $N^K_P(Q)$-category.

**Proof.** Set $\mathcal{F}' = N^K_\mathcal{F}(Q)$ and $P' = N^K_P(Q)$; since $P'$ is obviously fully normalized in $\mathcal{F}'$, denoting by $N$ the subgroup of automorphisms of $Q \cdot P'$ which stabilize $Q$ and $P'$, and act on $Q$ via elements of $K$, it follows from Lemma 2.16 below that $Q \cdot P'$ is fully $N$-normalized in $\mathcal{F}$ and then, it follows from Proposition 2.12 that $N \cap \mathcal{F}(Q \cdot P')$ is a Sylow $p$-subgroup of $\mathcal{F}(Q \cdot P')$; but, by the very definition of $\mathcal{F}'$ (cf. 2.9.1), the restriction to $P'$ determines a surjective homomorphism $N \cap \mathcal{F}(Q \cdot P') \to \mathcal{F}'(P')$ mapping $N \cap \mathcal{F}(Q \cdot P')$ onto $\mathcal{F}'(P')$, so that $\mathcal{F}'$ fulfills condition 2.9.1.

Let $R$ be a subgroup of $P'$, $L$ a subgroup of $\text{Aut}(R)$, $\varphi$ and element of $\mathcal{F}'(P'$, $R)$ such that $\varphi(R)$ is fully $\mathcal{F}$-$L$-normalized in $\mathcal{F}'$, and $\psi$ and element of $\mathcal{F}(Q \cdot P', Q \cdot R)$ such that we have $\psi(v) = \varphi(v)$ for any $v \in R$ and that there is $\chi \in K$ fulfilling $\psi(u) = \chi(u)$ for any $u \in Q$ (cf. 2.8.1); set $T = Q \cdot R$ and denote by $M$ the subgroup of automorphisms of $T$ which stabilize $Q$ and $R$, and act on them via elements of $K$ and $L$, respectively. According to Lemma 2.16 below, $\psi(T)$ is fully $M$-normalized in $\mathcal{F}$ and therefore, according to condition 2.9.2, there are $\zeta \in \mathcal{F}(P, T \cdot N^M_P(T))$ and $\mu \in M$ such that $\zeta(w) = \varphi(\mu(w))$ for any $w \in T$; in particular, for any $u \in Q$ we get $\zeta(u) = \chi(\mu(u))$ and therefore the action of $\zeta$ on $Q$ determines an element of $K$; thus, $\zeta(R \cdot N^M_P(T))$ also normalizes $Q$ and acts on it via a subgroup of $K$. Consequently, since $N^M_P(T) = N^L_P(R)$ (cf. Lemma 2.16 below) and the action of $\mu$ on $R$ determines an element $\lambda$ of $L$, the restriction of $\zeta$ to $R \cdot N^L_P(R)$ determines an element of $\mathcal{F}'(P', R \cdot N^L_P(R))$, and, for any $v \in R$, we have $\zeta(v) = \psi(\mu(v)) = \varphi(\lambda(v))$. This proves that $\mathcal{F}'$ fulfills condition 2.9.2. □

**Lemma 2.16.** Let $\mathcal{F}$ be a Frobenius $P$-category, $Q$ a subgroup of $P$ and $K$ a subgroup of $\text{Aut}(Q)$. Assume that $Q$ is fully $K$-normalized in $\mathcal{F}$. Let $R$ be a subgroup of $N^K_\mathcal{F}(Q)$ and $L$ a subgroup of $\text{Aut}(R)$, and denote by $M$ the subgroup of automorphisms of $Q \cdot R$
which stabilize $Q$ and $R$, and act on them via elements of $K$ and $L$, respectively. Then, we have

$$N^M_P(Q \cdot R) = N^L_P(R) \cap N^K_P(Q)$$

and if $R$ is fully $L$-normalized in $N^K_F(Q)$ then $Q \cdot R$ is fully $M$-normalized in $F$.

**Proof.** Set $F' = N^K_F(Q)$, $P' = N^K_P(Q)$ and $T = Q \cdot R$; the equality $N^M_P(T) = N^L_P(R)$ is easily checked and needs no hypothesis on $F$. For any $\psi \in F(P, T \cdot N^M_P(T))$, consider the element of $F(P, \psi(Q))$ obtained from the composition of the inclusion map determined by $Q \subset P$ and the inverse of the isomorphism $Q \cong \psi(Q)$ determined by $\psi$; since $Q$ is fully $K$-normalized in $F$, setting $Q' = \psi(Q)$ and according to condition 2.9.2, there are $\zeta \in F(P, Q' \cdot N^K_P(Q'))$ and $\chi \in K$ such that $\zeta(\psi(u)) = \chi(u)$ for any $u \in Q$; in particular, we have $\zeta(N^K_P(Q')) \subset P'$ and, since $\psi(Q \cdot P')$ is contained in $Q' \cdot N^K_P(Q')$, the homomorphism

$$\eta: T \cdot N^M_P(T) = Q \cdot (R \cdot N^L_P(R)) \rightarrow Q \cdot P'$$

mapping $w \in T \cdot N^M_P(T)$ on $\zeta(\psi(w))$ belongs to $F(Q \cdot P', T \cdot N^M_P(T))$ and, since $\psi(R) \subset N^K_P(\psi(Q))$, it determines an element of $F'(P', R \cdot N^L_P(R))$ (cf. 2.8.1).

So, if $R$ is fully $L$-normalized in $F'$, we have $\eta(N^L_P(R)) = N^L_P(\eta(R))$; but, we have $N^M_P(T) = N^L_P(R)$ and, according to the same equality applied to $\eta(Q) = Q$, $\eta(K) = K$, $\eta(R)$ and $\eta(L)$, we still have $N^M_P(\eta(T)) = N^L_P(\eta(R))$, so that

$$\zeta(\psi(N^M_P(T))) = \eta(N^M_P(T)) = N^M_P(\eta(T)) \supset \zeta(N^M_P(\psi(T))),$$

which forces $\psi(N^M_P(T)) = N^M_P(\psi(T))$. □

**Corollary 2.17.** Let $F$ be a Frobenius $P$-category. For any subgroup $Q$ of $P$ and any chain $q$ of normal subgroups of $Q$ there is $\varphi \in F(P, Q)$ such that, for any $R \in q$, $\varphi(R)$ is fully centralized in $F$.

**Proof.** We may assume that $q$ contains $Q$ and does not contain $\{1\}$, and we argue by induction on $|q|$; if $T$ is the minimal element of $q$ then there is a morphism $\psi \in F(P, Q)$ such that $T' = \psi(T)$ is $F'Q(T)$-fully normalized in $F$ (cf. 2.7) and therefore $F' = N^{F_0(T)}_F(T')$ is a Frobenius $P'$-category where $P' = N^{F_0(T)}_F(T')$ (cf. Proposition 2.15); consequently, according to the induction hypothesis applied to the chain $\psi(q - \{T\})$, there is $\varphi' \in F'(P', Q)$ such that, for any $R \in q - \{T\}$, $\varphi'(\psi(R))$ is fully centralized in $F'$, so in $F$ by Lemma 2.16; hence, it suffices to consider $\varphi \in F(P, Q)$ mapping $u \in Q$ on $\varphi'(\psi(u))$, since $\varphi(T) = T'$. □
3. Nilcentralized and selfcentralizing subgroups

3.1. Let $P$ be a finite $p$-group and $\mathcal{F}$ a divisible $P$-category; in order to check whether or not $\mathcal{F}$ is a Frobenius $P$-category, we only need to control condition 2.12.1 over a restricted family of subgroups of $P$—the $\mathcal{F}$-selfcentralizing subgroups—that we introduce in this section (see Theorem 3.6 below). Actually, this family has an interest on its own; for instance, as we show in the next section, the finite sequence category of the exterior quotient of the full subcategory of $\mathcal{F}$ over this family holds a direct product; this mainly depends on Proposition 3.3 which is fulfilled by a larger family—the $\mathcal{F}$-nilcentralized subgroups of $P$—preserved by central quotients (see Section 6 below). We say that a subgroup $Q$ of $P$ is $\mathcal{F}$-nilcentralized if $\mathcal{F}(\varphi(Q)) = \mathcal{F}_{C_{P}(\varphi(Q))}$ for some $\varphi \in \mathcal{F}(P, Q)$ such that $\varphi(Q)$ is fully centralized in $\mathcal{F}$ (i.e. if it has a nilpotent centralizer, which motivates the terminology).

**Proposition 3.2.** If $\mathcal{F}$ is a Frobenius $P$-category, a subgroup $R$ of $P$ which contains a $\mathcal{F}$-nilcentralized subgroup $Q$ is $\mathcal{F}$-nilcentralized too.

**Proof.** We may assume that $Q \triangleleft R$ and then, according to Corollary 2.17, that $R$ is fully centralized in $\mathcal{F}$; moreover, it follows from 2.7 that there is $\psi \in \mathcal{F}(P, R \cdot C_{P}(R))$ such that $Q' = \psi(Q)$ is fully normalized in $\mathcal{F}$ and therefore, according to Proposition 2.12, it is fully centralized too; but, it is clear that $R' = \psi(R)$ is also fully centralized in $\mathcal{F}$; in this situation the statement follows since $C_{\mathcal{F}}(R')$ is a subcategory of $\mathcal{F}(Q') = \mathcal{F}_{C_{P}(Q')}$. □

**Proposition 3.3.** Assume that $\mathcal{F}$ is a Frobenius $P$-category. Let $Q$ be a subgroup of $P$ and $R$ a $\mathcal{F}$-nilcentralized subgroup of $Q$. If $\varphi, \varphi' \in \mathcal{F}(P, Q)$ fulfill $\varphi(v) = \varphi'(v)$ for any $v \in R$ and the subgroup $R' = \varphi(R) = \varphi'(R)$ is fully centralized in $\mathcal{F}$, then there is $u \in C_{P}(R')$ such that $\varphi'(v) = \varphi(v)^{u}$ for any $v \in Q$.

**Proof.** We argue by induction on $|Q : R|$ and may assume that $R \neq Q$; moreover, up to the replacement of $Q$ and $R$ by $\varphi'(Q)$ and $\varphi'(R)$, from the divisibility of $\mathcal{F}$ we may assume that $\varphi'$ is the inclusion map and that $R$ is fully centralized in $\mathcal{F}$. Set $N = N_{Q}(R)$ and $U = \mathcal{F}_{P}(R)$, so that $N_{U}^{U}(R) = N_{P}(R)$; then, according to Proposition 2.12, $R$ is also fully $U$-normalized in $\mathcal{F}$ and, since $R$ is $\mathcal{F}$-nilcentralized, it is not difficult to see that $N_{\mathcal{F}}^{U}(R) = \mathcal{F}_{N_{P}(R)}$. In this situation, since $\varphi(v) = v$ for any $v \in R$, the restriction of $\varphi$ to $N$ determines an element of $\mathcal{F}(N_{U}^{U}(R)) = (N_{U}^{U}(R), N_{P}(R), N)$ and therefore there is $w \in N_{P}(R)$ such that $\varphi(v) = v$ for any $v \in N$; but, $N$ is also $\mathcal{F}$-nilcentralized and, according to Lemma 3.4 below, is fully centralized in $\mathcal{F}$ too; consequently, it follows from the induction hypothesis that there is $v \in C_{P}(N) \subset C_{P}(R)$ such that $\varphi(u)^{w} = u$ for any $u \in Q$. □

**Lemma 3.4.** Assume that $\mathcal{F}$ is a Frobenius $P$-category. A subgroup $Q$ of $P$ containing an $\mathcal{F}$-nilcentralized subgroup $R$ fully centralized in $\mathcal{F}$, is fully centralized in $\mathcal{F}$ too.

**Proof.** We obviously may assume that $R \triangleleft Q$; choose $\varphi \in \mathcal{F}(P, Q)$ such that $Q' = \varphi(Q)$ is fully centralized in $\mathcal{F}$ and set $R' = \varphi(R)$; then, it is clear that $\mathcal{F}_{Q}(R) = \mathcal{F}_{Q' \cdot C_{P}(R')}$ and therefore, since $R$ is fully centralized in $\mathcal{F}$, it follows from statement 2.11.1 that
there is $\psi \in \mathcal{F}(P, Q' \cdot C_P(R'))$ such that $\psi(\varphi(v)) = v$ for any $v \in R$; in particular, setting $U = \mathcal{F}_Q(R)$, $R$ is also fully $U$-normalized in $\mathcal{F}$ and the composition of $\varphi$ with the restriction of $\psi$ to $Q'$ determines an element of $(\mathcal{N}_\mathcal{F}^U(R))(Q \cdot C_P(R), Q)$; thus, since $\mathcal{N}_\mathcal{F}^U(R) = \mathcal{F}_Q \cdot C_P(R)$, up to a modification of our choice of $\psi$, we may assume that $\psi(\varphi(u)) = u$ for any $u \in Q$ and then we have $\psi(C_P(Q')) \subset C_P(Q)$, which forces the equality and proves that $Q$ is fully centralized too.

3.5. We say that a subgroup $Q$ of $P$ is $\mathcal{F}$-selfcentralizing\(^4\) whenever

$$C_P(\varphi(Q)) \subset \varphi(Q)$$

3.5.1 for any $\varphi \in \mathcal{F}(P, Q)$; then, $Q$ is $\mathcal{F}$-nilcentralized and a subgroup $R$ of $P$ such that $\mathcal{F}(R, Q) \neq \emptyset$ is $\mathcal{F}$-selfcentralizing too. Note that a $\mathcal{F}$-selfcentralizing subgroup of $P$ is fully centralized in $\mathcal{F}$; conversely, if $Q$ is a subgroup of $P$ fully centralized in $\mathcal{F}$ then $Q \cdot C_P(Q)$ is clearly $\mathcal{F}$-selfcentralizing. Denote by $\tilde{\mathcal{F}}$ the exterior quotient of $\mathcal{F}$, namely the category over the same set of objects and where, for any pair of subgroups $Q$ and $R$ of $P$, the set of morphisms $\tilde{\mathcal{F}}(Q, R)$ from $R$ to $Q$ is the quotient of $\mathcal{F}(Q, R)$ by the inner automorphisms of $Q$ (and of $R$); for any $\varphi \in \mathcal{F}(Q, R)$, we denote by $\tilde{\varphi}$ the class of $\varphi$.

Corollary 3.6. Assume that $\mathcal{F}$ is a Frobenius $P$-category. For any $\mathcal{F}$-selfcentralizing subgroups $Q$, $R$ and $T$ of $P$ and any element $\varphi \in \mathcal{F}(R, Q)$, the map $\tilde{\mathcal{F}}(T, R) \rightarrow \tilde{\mathcal{F}}(T, Q)$ determined by the composition with $\tilde{\varphi}$ is injective. In particular, any morphism in $\tilde{\mathcal{F}}$ from a $\mathcal{F}$-selfcentralizing subgroup of $P$ is an epimorphism.

Proof. If two elements $\tilde{\psi}, \tilde{\psi}' \in \tilde{\mathcal{F}}(T, R)$ fulfill $\tilde{\psi} \circ \tilde{\varphi} = \tilde{\psi}' \circ \tilde{\varphi}$, we may choose representatives $\psi$ of $\tilde{\psi}$ and $\psi'$ of $\tilde{\psi}'$ such that $\psi \circ \varphi = \psi' \circ \varphi$, and then it follows from Proposition 3.3 that there is $z \in Z(Q)$ fulfilling

3.6.1 $\psi'(v) = \psi(v^z) = \psi(v)^{\psi(z)}$

for any $v \in R$, so that $\tilde{\psi}' = \tilde{\psi}$.

3.7. Conversely, a subgroup $Q$ of $P$ fully centralized in $\mathcal{F}$ is $\mathcal{F}$-selfcentralizing if and only if we have $C_P(Q) = Z(Q)$, namely if $Q$ is selfcentralizing in $P$. But, if $\mathcal{F}$ is a Frobenius $P$-category, a subgroup $Q$ of $P$ fully normalized in $\mathcal{F}$ is also fully centralized (cf. Proposition 2.12) and therefore it is $\mathcal{F}$-selfcentralizing if and only if it is selfcentralizing in $P$; moreover, in this case, according to statement 2.10.1 and Proposition 3.4, if $R$ is a subgroup of $N_P(Q)$ containing $Q$, then any $\varphi \in \mathcal{F}(P, Q)$ such that $\varphi \mathcal{F}_R(Q) \subset \mathcal{F}_P(\varphi(Q))$ can be extended to $R$, in a unique way up to conjugation by $Z(Q)$. Conversely, we have the following criterion.\(^5\)

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\(^4\) Although in a restricted sense, the term “self centralizing” already appears in 1963 in the fundamental paper “Solvability of groups of odd paper” by W. Feit and J. Thompson.

Theorem 3.8. The divisible $P$-category $F$ is a Frobenius $P$-category if and only if the following conditions hold:

3.8.1 $F_{p}(P)$ is a Sylow $p$-subgroup of $F(P)$.

3.8.2 If $Q$ is a $F$-selfcentralizing subgroup of $P$, $R$ is a subgroup of $N_{P}(Q)$ containing $Q$ and $\varphi \in F(P, Q)$ fulfills $\psi F_{R}(Q) \subset F_{p}(\varphi(Q))$, then there is $\psi \in F(P, R)$ extending $\varphi$.

3.8.3 Any divisible $P$-category $F'$ fulfilling $F'(P, Q) \supset F(P, Q)$ for every $F$-selfcentralizing subgroup $Q$ of $P$ contains $F$.

Proof. Clearly, conditions 3.8.1, 3.8.2 and 3.8.3 are necessary (the necessity of 3.8.3 follows from 2.11.1). For any pair $Q$ and $Q'$ of $F$-isomorphic subgroups of $P$, consider the set $F'(Q', Q)$ of $\varphi \in F(Q', Q)$ such that there are subgroups $U$ and $U'$ of $P$ $F$-isomorphic to $Q$ and $Q'$, which are fully centralized and fully normalized in $F$, and admit $F$-morphisms

\[
N_{P}(Q) \overset{\lambda}{\longrightarrow} N_{P}(U), \quad U \cdot C_{P}(U) \overset{\sigma}{\longrightarrow} U' \cdot C_{P}(U') \quad \text{and} \quad U' \overset{\lambda'}{\longleftarrow} N_{P}(Q')
\]

fulfilling

3.8.5 \[\lambda(Q) = U, \quad \sigma(U) = U', \quad U' = \lambda'(Q') \quad \text{and} \quad \lambda'(\varphi(u)) = \sigma(\lambda(u))\]

for any $u \in Q$. Note that $F_{p}(Q', Q) \subset F'(Q', Q)$; indeed, if $\varphi$ is the conjugation by some $u \in P$ then, choosing $\lambda \in F(P, N_{P}(Q))$ such that $U = \lambda(Q)$ is fully centralized and fully normalized in $F$ (cf. 2.7), it is clear that $U' = uUu^{-1} = u\lambda(Q')$ is also fully centralized and fully normalized in $F$, and it suffices to consider $\lambda' = u\lambda$ and the group isomorphism $U \cdot C_{P}(U) \cong U' \cdot C_{P}(U')$ determined by the conjugation by $u$. On the other hand, if $Q$ is $F$-selfcentralizing then $F'(Q', Q) = F(Q', Q)$; indeed, choosing $\xi \in F(P, N_{P}(Q))$ and $\xi' \in F(P, N_{P}(Q'))$ such that $U = \xi(Q)$ and $U' = \xi'(Q')$ are both fully normalized in $F$ (cf. 2.7), in this case we have $C_{P}(U) \subset U$ and $C_{P}(U') \subset U'$, and therefore the existence of $\sigma \in F(U', U)$ fulfilling 3.8.5 is clear.

If $F$ is a Frobenius $P$-category, we claim that $F'(Q', Q) = F(Q', Q)$; indeed, we can choose subgroups $U$ and $U'$ of $P$ $F$-isomorphic to $Q$ and $Q'$, and both fully centralized and fully normalized in $F$; then, by condition 2.9.2, there are morphisms $\lambda \in F(N_{P}(U), N_{P}(Q))$ and $\lambda' \in F(N_{P}(U'), N_{P}(Q'))$ fulfilling $\lambda(Q) = U$ and $\lambda'(Q') = U'$; now, any $\varphi \in F(Q', Q)$ induces and element $\sigma \in F(U, U')$ fulfilling $\lambda'(\varphi(u)) = \sigma(\lambda(u))$ for any $u \in Q$ and, since $U$ is fully centralized in $F$, $\sigma$ can be extended to $U \cdot C_{P}(U)$. In particular, if $F''$ is a divisible $P$-subcategory of $F$ and, for any $F$-selfcentralizing subgroup $R$ of $P$, it fulfills $F''(P, R) = F(P, R)$, we claim that $F'' = F$; it suffices to prove that, for any subgroup $Q$ of $P$, we have $F''(P, Q) = F(P, Q)$. We argue by induction on $|P : Q|$, consider $\varphi \in F(P, Q)$ and set $Q' = \varphi(Q)$; since we may assume that $Q$ is not $F$-selfcentralizing, we have $U \neq U \cdot C_{P}(U)$ above and, according to our induction
hypothesis, the $\mathcal{F}$-morphisms $\lambda$, $\lambda'$ and $\sigma$ already are $\mathcal{F}''$-morphisms, which proves that $\varphi$ is a $\mathcal{F}''$-morphism too.

From now on, we assume that $\mathcal{F}$ fulfills the conditions above; more generally, for any pair $Q$ and $R$ of subgroups of $P$ we set

$$3.8.6 \quad \mathcal{F}'(Q, R) = \bigcup_{\varphi \in \mathcal{F}(Q, R)} \iota^Q_{\varphi(R)} \circ \mathcal{F}'(\varphi(R), R),$$

where $\iota^Q_{\varphi(R)}$ denotes the corresponding inclusion map, and we will prove that $\mathcal{F}'$ is a Frobenius $P$-category, so that $\mathcal{F}' = \mathcal{F}$ by condition 3.8.3; that is to say, arguing by induction on $|P : R|$, we will prove that $\mathcal{F}'$ is a category and fulfills conditions 2.3.1 and 2.9.2; note that, according to 2.3, at each step $p^n$ of the induction there is a divisible $P$-subcategory $\mathcal{F}'' \subset \mathcal{F}$ which coincides with $\mathcal{F}'$ over all the subgroups $Q$ of $P$ either $\mathcal{F}$-selfcentralizing or fulfilling $|P : Q| \leq p^n$, and therefore $\mathcal{F}' = \mathcal{F}$ by condition 3.8.3.

First of all, with the notation above we claim that if $\varphi \in \mathcal{F}'(Q', Q)$ then $\varphi^{-1}$ belongs to $\mathcal{F}'(Q, Q')$; indeed, since $U$ is fully centralized in $\mathcal{F}$, we have $\sigma(U \cdot C_P(U)) = U' \cdot C_P(U')$ and it suffices to consider the triple $(\lambda', \sigma^{-1}, \lambda)$. Let $Q''$ be a third subgroup of $P$ $\mathcal{F}$-isomorphic to $Q$ and consider a morphism $\varphi' \in \mathcal{F}'(Q'', Q')$, so that mutatis mutandis we have subgroups $V$ and $V'$ of $P$ $\mathcal{F}$-isomorphic to $Q'$ and $Q''$, which are fully centralized and fully normalized in $\mathcal{F}$ and admit

$$3.8.7 \quad N_P(Q') \xrightarrow{\mu} N_P(V), \quad V \cdot C_P(V) \xrightarrow{\tau} V' \cdot C_P(V') \quad \text{and} \quad N_P(V') \xleftarrow{\mu'} N_P(Q''),$$

fulfilling

$$3.8.8 \quad \mu(Q') = V, \quad \tau(V) = V', \quad V' = \mu'(Q'') \quad \text{and} \quad \mu'(\varphi'(u')) = \tau(\mu(u'))$$

for any $u' \in Q'$; in particular, denoting by $\lambda'^*$ the inverse of the group isomorphism $N_P(Q') \cong \lambda'(N_P(Q'))$ induced by $\lambda'$, we have the $\mathcal{F}$-morphism

$$3.8.9 \quad \mu \circ \lambda'^*: \lambda'(N_P(Q')) \to N_P(V)$$

inducing an $\mathcal{F}$-isomorphism $U' \cong V$; consequently, it follows from the induction hypothesis and from Lemma 3.9 below that, since $V$ is fully normalized in $\mathcal{F}$, this $\mathcal{F}$-isomorphism can be extended to $\rho \in \mathcal{F}(N_P(V), N_P(U'))$; then, the existence of $U$, $V'$, $\lambda$, $\mu'$ and the $\mathcal{F}$-morphism $U \cdot C_P(U) \to V' \cdot C_P(V')$ mapping $u \in U$ on $\tau(\rho(\sigma(u)))$ proves that $\varphi' \circ \varphi$ belongs to $\mathcal{F}'(Q'', Q)$.

Let $R$ and $R'$ be subgroups of $P$ respectively containing $Q$ and $Q'$, and assume that $\psi \in \mathcal{F}'(R', R)$ fulfills $\psi(Q) = Q'$; we claim that the $\mathcal{F}$-isomorphism $\varphi: Q \cong Q'$ induced by $\psi$ belongs to $\mathcal{F}'(Q', Q)$. Arguing by induction on $|R : Q|$, we may assume that $|R : Q| \neq 1$ and that $Q \lhd R$ and $Q' \lhd R'$; we already know that there are $\xi \in \mathcal{F}(P, R)$ and $\zeta' \in \mathcal{F}(P, R')$ such that $V = \xi(Q)$ and $V' = \zeta'(Q')$ are fully centralized and fully normal-
ized in $\mathcal{F}$ (cf. 2.7). Then, it follows from the induction hypothesis and from Lemma 3.9 below that there are $\mathcal{F}$-morphisms

$$v : N_p(Q) \rightarrow N_p(V) \quad \text{and} \quad v' : N_p(Q') \rightarrow N_p(V')$$

fulfilling $v(Q) = V$ and $v'(Q') = V'$; once again, considering the $\mathcal{F}$-isomorphism $\rho : v(R) \cong v'(R')$ such that $\rho(v(v)) = v'(\psi(v))$ for any $v \in R$, it follows from the induction hypothesis and from Lemma 3.9 below that, since $V'$ is fully centralized in $\mathcal{F}$, there is $\eta \in \mathcal{F}(P, V \cdot CP(V))$ fulfilling $\eta(v(u)) = v'(\psi(u))$ for any $u \in Q$; now, the existence of $V, V', v, v'$ and $\eta$ proves that $\varphi$ belongs to $\mathcal{F}'(Q', Q)$.

Hence, if $R$ and $T$ are subgroups of $P$, $\varphi$ an element of $\mathcal{F}'(R, Q)$ and $\psi$ an element of $\mathcal{F}'(T, R)$ then $\psi \circ \varphi$ belongs to $\mathcal{F}'(T, Q)$; indeed, setting $Q' = \varphi(Q)$ and $Q'' = \psi(Q')$, and denoting by $\varphi' : Q \cong Q'$ and $\psi' : Q' \cong Q''$ the corresponding $\mathcal{F}$-isomorphisms, it follows from our definition that $\varphi'$ belongs to $\mathcal{F}'(Q', Q)$ and, by the arguments above, we already know that $\psi'$ and $\psi' \circ \varphi'$ respectively belong to $\mathcal{F}'(Q'', Q')$ and to $\mathcal{F}'(Q'', Q)$.

It remains to prove that $\mathcal{F}'$ fulfills condition 2.9.2; let $K$ be a subgroup of $\text{Aut}(Q)$ containing $\mathcal{F}_Q(Q)$ and $\varphi \in \mathcal{F}'(P, Q)$ such that $Q' = \varphi(Q)$ is fully $^\varphi K$-normalized in $\mathcal{F}'$; since the isomorphism $Q \cong \varphi(Q)$ induced by $\varphi$ belongs to $\mathcal{F}'(Q', Q)$, as in 3.8.4 we have subgroups $U$ and $U'$ of $P$ $\mathcal{F}$-isomorphic to $Q$ and $Q'$, which are fully centralized and fully normalized in $\mathcal{F}$ and admit

$$N_p(Q) \xrightarrow{\lambda} N_p(U), \quad U \cdot CP(U) \xrightarrow{\sigma} U' \cdot CP(U')$$

fulfilling equalities 3.8.5 for any $u \in Q$; set

$$R = U \cdot CP(U) \quad \text{and} \quad R' = U' \cdot CP(U')$$

and denote by $\lambda'^*$ the inverse of the $\mathcal{F}$-isomorphism $N_p(Q') \cong \lambda'(N_p(Q'))$ induced by $\lambda'$; since $U$ is fully centralized in $\mathcal{F}$, we have $\sigma(R) = R'$; moreover, since $Q' = \varphi(Q)$ is fully $^\varphi K$-normalized in $\mathcal{F}'$, we get

$$\lambda'(N_p^{^\varphi K}(Q')) = N_p^{^\varphi K}(U') \supset CP(U');$$

hence, there is $\psi \in \mathcal{F}(P, R)$ fulfilling $\psi(v) = \lambda'^*(\sigma(v))$ for any $v \in R$. Finally, since $U \neq R$, it follows from the induction hypothesis and from Lemma 3.9 below that there are $\xi \in \mathcal{F}(P, U \cdot N_p^{^\varphi K}(U))$ and $\chi \in K$ such that, for any $u \in Q$, we have

$$\xi(\lambda(u)) = \psi(\lambda(\chi(u))) = \lambda'^*(\sigma(\lambda(\chi(u)))) = \varphi(\chi(u)). \quad \square$$

**Lemma 3.9.** Let $\mathcal{X}$ be a nonempty set of subgroups $Q$ of $P$ such that any subgroup $R$ of $P$ fulfilling $\mathcal{F}(R, Q) \neq \emptyset$ belongs to $\mathcal{X}$ and that any $\varphi \in \mathcal{F}(P, Q)$ and any subgroup $K$ of $\text{Aut}(Q)$ fulfill the following condition:
3.9.1 If \( \varphi(Q) \) is fully \( ^\psi K \)-normalized in \( F \), there are \( \zeta \in F(P, Q \cdot N_P^K(Q)) \) and \( \chi \in K \) such that \( \zeta(u) = \varphi(\chi(u)) \) for any \( u \in Q \).

Then, if \( R \) is a subgroup of \( P \), any \( \psi \in F(P, R) \) which can be extended to an element \( Q \) of \( X \) normalizing \( R \) and any \( Q \)-stable subgroup \( J \) of \( \text{Aut}(R) \) fulfill condition 3.9.1.

**Proof.** Let \( Q \) be an element of \( X \) such that \( R \subset Q \subset N_P(R) \), \( \eta \) an element of \( F(P, Q) \) extending \( \psi \) and \( J \) a \( Q \)-stable subgroup of \( \text{Aut}(R) \); we argue by induction on \(|N_P(R) : Q|\).

We assume that \( \psi(R) \) is fully \( ^\psi J \)-normalized in \( F \) and may assume that \( N_P J(R) \subset Q \) or, equivalently, that \( Q \neq Q \cdot N_P J(R) \). Denote by \( K \) the converse image of \( J \cdot F_Q(R) \) in the stabilizer \( \text{Aut}(Q)_R \) of \( R \) in \( \text{Aut}(Q) \) and choose \( \zeta \in F(P, N_P^K(Q)) \) such that \( \zeta(R) \) is fully \( \zeta(J \cdot F_Q(R)) \)-normalized and \( \zeta(Q) \) is fully \( \zeta K \)-normalized in \( F \). This is possible since, applying 2.7 to \( Q \subset N_P^K(Q) \), there is \( \zeta' \in F(P, N_P^K(Q)) \) such that \( \bar{Q} = \zeta'(Q) \) is fully \( \zeta \)-\( K \)-normalized in \( F \) and, setting \( \bar{J} = \zeta(J \cdot F_Q(R)) \) and \( \bar{K} = \zeta K \), and applying 2.7 to \( \bar{R} = \zeta'(R) \subset N_P^K(\bar{Q}) \subset N_P J(\bar{R}) \), there is \( \zeta'' \in F(P, N_P^K(\bar{Q})) \) such that \( \zeta''(\bar{R}) \) is fully \( \zeta'' \bar{K} \)-normalized in \( F \); moreover, since we have \( \zeta''(N_P^K(\bar{Q})) = N_P^K(\zeta''(\bar{Q})) \), \( \zeta''(\bar{Q}) \) is also fully \( \zeta'' \bar{K} \)-normalized in \( F \).

Set \( R' = \psi(R), Q' = \eta(Q) \) and \( J' = \psi J \); it follows from condition 3.9.1 applied to \( Q' \) and \( K' = \eta K \) that there are \( \xi \in F(P, N_P^K(Q')) \) and \( \chi \in K \) fulfilling \( \xi(\eta(u)) = \zeta(\chi(u)) \) for any \( u \in Q' \); in particular, \( \xi(R') = \zeta(R) \) since \( \chi(R) = R \). On the other hand, we have

\[
Q \neq N_P(Q) \cap Q \cdot N_P J(R) = N_P^K(Q) \quad \text{and} \quad Q' \neq N_P^K(Q')
\]

and therefore, since \( R'' = \zeta(R) \) is \( \zeta(J \cdot F_Q(R)) \)-fully normalized in \( F \), it follows from the induction hypothesis applied to \( R \), \( N_P^K(Q) \) and the restriction of \( \zeta \), and to \( R' \), \( N_P^K(Q') \) and the restriction of \( \xi \), that there are \( \alpha \in F(P, R \cdot N_P^K(R)) \), \( \alpha' \in F(P, R' \cdot N_P^K(R')) \) and \( \theta, \theta' \in J \) such that, for any \( v \in R \), we have

\[
\alpha(v) = \zeta(\theta(v)) \quad \text{and} \quad \alpha'(\psi(v)) = \xi(\psi(\theta'(v))) = \zeta(\chi(\theta'(v))).
\]

Moreover, since \( R' \) is fully \( J' \)-normalized in \( F \), setting \( J'' = \zeta J \) we get

\[
\alpha(R \cdot N_P J(R)) \subset R'' \cdot N_P J''(R'') = \alpha'(R' \cdot N_P J'(R'))
\]

and therefore, denoting by \( \omega \in F(R' \cdot N_P J'(R'), R \cdot N_P J(R)) \) the element fulfilling \( \alpha' \circ \omega = \alpha \), for any \( v \in R \) we finally obtain

\[
\zeta(\theta(v)) = \alpha'(\omega(v)) = \zeta(\chi((\theta' \circ \psi^*)(\omega(v)))) = \zeta((\chi \circ \theta' \circ \omega^*)(\omega(v)^{\eta(u)}))
\]

where \( \psi^* \in F(R, R') \) is the inverse of the isomorphism \( R \cong R' \) determined by \( \psi \), and \( u \) is an element of \( Q \) such that the image of \( \chi \) in \( \text{Aut}(R) \) is the composition of the conjugation by \( u \) with a suitable \( \theta'' \in J \); that is to say, we have \( \psi((\theta''^{-1} \circ \chi \circ \theta'' \circ \theta)(v)) = \omega(v)^{\eta(u)} \) for any \( v \in R \).
4. On the exterior quotient of a Frobenius category

4.1. Let $P$ be a finite $p$-group and $\mathcal{F}$ a Frobenius $P$-category. Denote by $\mathcal{F}_{sc}$ the full subcategory of $\mathcal{F}$ over the set of all the $\mathcal{F}$-selfcentralizing subgroups of $P$ and, as in 3.5, consider the exterior quotient $\tilde{\mathcal{F}}_{sc}$ of $\mathcal{F}_{sc}$. In this section we give some useful properties of this category. We already know that, for any triple of $\mathcal{F}$-selfcentralizing subgroups $Q$, $R$ and $T$, any morphism $\tilde{\alpha} : Q \to R$ in $\tilde{\mathcal{F}}_{sc}$ induces an injective map from $\tilde{\mathcal{F}}_{sc}(T,R)$ to $\tilde{\mathcal{F}}_{sc}(T,Q)$ (cf. Corollary 3.6) and we will consider the elements of $\tilde{\mathcal{F}}_{sc}(T,Q)$ which, even partially, cannot be extended via $\tilde{\alpha}$; precisely, we set

$$\tilde{\mathcal{F}}_{sc}(T,Q)_{\tilde{\alpha}} = \tilde{\mathcal{F}}_{sc}(T,Q) - \bigcup_{\tilde{\theta}'} \tilde{\mathcal{F}}_{sc}(T,Q') \circ \tilde{\theta}'$$

where $\tilde{\theta}'$ runs over the set of nonisomorphisms $\tilde{\theta}' : Q \to Q'$ from $Q$ in $\tilde{\mathcal{F}}_{sc}$—the set of nonfinal objects in $(\tilde{\mathcal{F}}_{sc})^Q$—fulfilling $\tilde{\theta}' \circ \tilde{\alpha} = \tilde{\alpha}$ for some $\tilde{\alpha}' \in \tilde{\mathcal{F}}(R, Q')$, which then is unique and we simply say that $\tilde{\theta}'$ divides $\tilde{\alpha}$ setting $\tilde{\alpha}' = \tilde{\alpha} / \tilde{\theta}'$; thus, if $\tilde{\alpha}$ is an isomorphism we have $\tilde{\mathcal{F}}_{sc}(T,Q)_{\tilde{\alpha}} = \tilde{\mathcal{F}}_{sc}(T,Q)$ and note that the existence of $\tilde{\theta}'$ is equivalent to the existence of a subgroup of $R$ which is $\mathcal{F}$-isomorphic to $Q'$ and contains $\alpha(Q)$ for $\alpha \in \tilde{\alpha}$.

4.2. Actually, an element $\tilde{\beta}$ in $\tilde{\mathcal{F}}_{sc}(T,Q)$ which can be extended to $Q'$ via $\tilde{\theta}'$, a fortiori it can be extended to $N_{Q'}(\tilde{\theta}'(Q))$ for $\tilde{\theta}' \in \tilde{\theta}$; hence, it follows from condition 2.11.1 that $\tilde{\beta}$ belongs to $\tilde{\mathcal{F}}_{sc}(T,Q)_{\tilde{\alpha}}$ if and only if, for some $\beta \in \tilde{\beta}$, we have

$$\alpha^* \mathcal{F}_R(\alpha(Q)) \cap \beta^* \mathcal{F}_T(\beta(Q)) = \mathcal{F}_Q(Q),$$

where $\alpha^*$: $\alpha(Q) \cong Q$ and $\beta^*$: $\beta(Q) \cong Q$ are the inverses of the isomorphisms respectively induced by $\alpha$ and $\beta$—which is a symmetric condition. That is to say, with the notation above

4.2.2 $\tilde{\beta} \in \tilde{\mathcal{F}}_{sc}(T,Q)_{\tilde{\alpha}}$ is equivalent to $\tilde{\alpha} \in \tilde{\mathcal{F}}_{sc}(R,Q)_{\tilde{\beta}}$.

Note that, if $\tilde{\mathcal{F}}(P,Q)_{\tilde{\alpha}} = \tilde{\mathcal{F}}(P,Q)$ then $\tilde{\alpha}$ belongs to $\mathcal{F}(R, Q)_{\tilde{\beta}}$ via $\tilde{\beta}$, which forces $\tilde{\alpha}$ to be an isomorphism. Moreover, the quotient

4.2.3 $\tilde{N}_R(\alpha(Q)) \cong \alpha^* \tilde{\mathcal{F}}_R(\alpha(Q))$

clearly acts on $\tilde{\mathcal{F}}_{sc}(T,Q)_{\tilde{\alpha}}$ by composition and if we have $\tilde{\beta} \circ \tilde{\alpha}^* \circ \kappa_v \circ \tilde{\alpha} = \tilde{\beta}$ for some $\tilde{\beta} \in \tilde{\mathcal{F}}_{sc}(T,Q)_{\tilde{\alpha}}$ and some $v \in R$, we still have

4.2.4 $\alpha^* \circ \kappa_v \circ \alpha = \beta^* \circ \kappa_w \circ \beta$

for some $w \in T$, so that $\tilde{N}_R(\alpha(Q))$ acts freely on $\tilde{\mathcal{F}}_{sc}(T,Q)_{\tilde{\alpha}}$; in particular,

4.2.5 if $\tilde{\alpha}$ is not an $\tilde{\mathcal{F}}$-isomorphism then $p$ divides $|\tilde{\mathcal{F}}_{sc}(T,Q)_{\tilde{\alpha}}|$. 

Proposition 4.3. For any triple of \( F \)-selfcentralizing subgroups \( Q, R \) and \( T \) of \( P \) and any \( \bar{\alpha} \in \tilde{F}(R, Q) \), we have

\[
\tilde{F}(T, Q) = \bigsqcup_{\tilde{\theta}'} \tilde{F}(T, Q')_{\bar{\alpha}/\bar{\theta}'} \circ \tilde{\theta}'
\]

where \( \tilde{\theta}' : Q \to Q' \) runs over a set of representatives for the isomorphism classes of objects in \( (\tilde{F}_{sc})_Q \) dividing \( \bar{\alpha} \). In particular, we have

\[
|\tilde{F}(T, Q)| \equiv |\tilde{F}(T)| \pmod{p}.
\]

Proof. It is quite clear that, arguing by induction on \( |T|/|Q| \), we get

\[
\tilde{F}(T, Q) = \bigsqcup_{\tilde{\theta}'} \tilde{F}(T, Q')_{\bar{\alpha}/\bar{\theta}'} \circ \tilde{\theta}'
\]

where \( \tilde{\theta}' \) runs over the set of morphisms \( \tilde{\theta}' : Q \to Q' \) from \( Q \) in \( \tilde{F}_{sc} \) dividing \( \bar{\alpha} \); hence, it suffices to prove that, when for another such a \( \tilde{F} \)-morphism \( \tilde{\theta}'' : Q \to Q'' \), we have

\[
(\tilde{F}(T, Q')_{\bar{\alpha}/\bar{\theta}'} \circ \tilde{\theta}') \cap (\tilde{F}(T, Q'')_{\bar{\alpha}/\bar{\theta}''} \circ \tilde{\theta}'') \neq \emptyset,
\]

there is a \( \tilde{F} \)-isomorphism \( \tilde{\eta} : Q' \cong Q'' \) fulfilling \( \tilde{\eta} \circ \tilde{\theta}' = \tilde{\theta}'' \) and, in particular, we have

\[
\tilde{F}(T, Q')_{\bar{\alpha}/\bar{\theta}'} \circ \tilde{\theta}' = \tilde{F}(T, Q'')_{\bar{\alpha}/\bar{\theta}''} \circ \tilde{\theta}''.
\]

We argue by induction on \( |R|/|Q| \) and may assume that \( |R| \neq |Q| \), that \( Q' \) and \( Q'' \) are subgroups of \( R \) containing \( \hat{Q} = \alpha(Q) \) for some \( \alpha \in \bar{\alpha} \), and that the respective homomorphisms \( \theta' : Q \to Q' \) and \( \theta'' : Q \to Q'' \) determined by \( \alpha \) are representatives of \( \tilde{\theta}' \) and \( \tilde{\theta}'' \), so that

\[
\bar{\alpha}/\bar{\theta}' = \bar{\alpha}/\bar{\theta''} = \bar{i}_Q^R;
\]

then, if \( \tilde{\beta}' \in \tilde{F}(T, Q')_{\bar{\alpha}/\bar{\theta}'} \) and \( \tilde{\beta}'' \in \tilde{F}(T, Q'')_{\bar{\alpha}/\bar{\theta}''} \) fulfill \( \tilde{\beta}' \circ \tilde{\theta}' = \tilde{\beta}'' \circ \tilde{\theta}'' \), choosing \( \beta' \in \tilde{\beta} \) and \( \beta'' \in \tilde{\beta}'' \) such that \( \beta' \circ \theta' = \beta'' \circ \theta'' \), and denoting by \( \beta \in F(T, \hat{Q}) \) the element determined by \( \beta' \circ \theta' = \beta'' \circ \theta'' \), we have

\[
\beta_{\tilde{F}_{Q'}(\hat{Q})} \subset F_T(\beta(\hat{Q})) \quad \text{and} \quad \beta_{\tilde{F}_{Q''}(\hat{Q})} \subset F_T(\beta(\hat{Q}));
\]

hence, setting \( N' = N_{Q'}(\hat{Q}) \), \( N'' = N_{Q''}(\hat{Q}) \) and \( N = \langle N', N'' \rangle \), we still have

\[
\beta_{\tilde{F}_N(\hat{Q})} \subset F_T(\beta(\hat{Q})).
\]
and therefore, denoting by $\nu: Q \to N$ the homomorphism induced by $\alpha$, it follows from condition 2.11.1 that there is $\tilde{\xi} \in \hat{F}(T, N)$ such that

$$\tilde{\xi} \circ \tilde{\nu} = \tilde{\beta}' \circ \tilde{\theta}' = \tilde{\beta}'' \circ \tilde{\theta}''$$

moreover, it follows from equality 4.3.3 that there is a morphism $\tilde{\theta}''': N \to Q'''$ in $\hat{F}_{sc}$ dividing $\tilde{\iota}_N^R$ such that $\tilde{\xi} = \tilde{\beta}'''' \circ \tilde{\theta}''''$ for some $\tilde{\beta}'''' \in \hat{F}(T, Q''')_{\tilde{\iota}_N^R/\tilde{\iota}''''}$.

Consequently, respectively denoting by $\nu': Q \to N'$ and $\nu'': Q \to N''$ the homomorphisms induced by $\alpha$, we get

$$\tilde{\iota}_N^Q \circ \tilde{\nu}' = \tilde{\iota}_N^Q \circ \tilde{\nu} = \tilde{\iota}'''' \circ \tilde{\iota}_N^Q \circ \tilde{\nu}';$$

$$\tilde{\iota}_N^Q \circ \tilde{\nu}'' = \tilde{\iota}_N^Q \circ \tilde{\nu} = \tilde{\iota}'''' \circ \tilde{\iota}_N^Q \circ \tilde{\nu}''$$

and therefore, according to Corollary 3.6, we still get

$$\tilde{\iota}_N^Q \circ \tilde{\nu}' = \tilde{\iota}'''' \circ \tilde{\iota}_N^Q \circ \tilde{\nu}';$$

now, it follows from the induction hypothesis that there are $\tilde{\eta}': Q''' \cong Q'$ and $\tilde{\eta}'': Q''' \cong Q''$ fulfilling

$$\tilde{\eta}' \circ \tilde{\nu}' = \tilde{\iota}'''' \circ \tilde{\iota}_N^Q \circ \tilde{\nu}'$$

hence, setting $\tilde{\eta} = \tilde{\eta}'' \circ \tilde{\eta}'^{-1}$, we obtain

$$\tilde{\eta} \circ \tilde{\theta}' = \tilde{\theta}'' \circ \tilde{\theta}' = \tilde{\theta}'' \circ \tilde{\iota}'''' \circ \tilde{\iota}_N^Q \circ \tilde{\nu}';$$

and, in particular, we still obtain

$$\hat{F}(T, Q')_{\tilde{\alpha}/\tilde{\theta}'} \circ \tilde{\theta}' = \hat{F}(T, Q'')_{\tilde{\alpha}/\tilde{\theta}''} \circ \tilde{\theta}'' \circ \tilde{\theta}' = \hat{F}(T, Q'')_{\tilde{\alpha}/\tilde{\theta}''} \circ \tilde{\theta}''.$$
is the pair \(((\tilde{\alpha} \ast g) \circ \tilde{\beta}, f \circ g)\) where \((\tilde{\alpha} \ast g) \circ \tilde{\beta}\) is the \(L\)-family of morphisms

\[
\tilde{\alpha}_{g(\ell)} \circ \tilde{\beta}_{\ell} : T_{\ell} \to R_{g(\ell)} \to Q_{(f \circ g)(\ell)}
\]

for any \(\ell \in L\). In this category it is quite clear that any pair of objects \(Q = \{Q_i\}_{i \in I}\) and \(R = \{R_j\}_{j \in J}\) have a direct sum, namely \(Q \oplus R\) is the corresponding \(I \sqcup J\)-family; in particular, we can identify \(\tilde{\mathcal{F}}_{\text{sc}}\) to a full subcategory of \(\tilde{\text{ad}}(\tilde{\mathcal{F}}_{\text{sc}})\) and then we have a canonical isomorphism \(Q \cong \bigoplus_{i \in I} Q_i\).

4.5. Now, the decomposition 4.3.1 allows us to consider in \(\tilde{\text{ad}}(\tilde{\mathcal{F}}_{\text{sc}})\) the exterior intersection of two \(\mathcal{F}\)-selfcentralizing subgroups of \(P\). Explicitly, if \(R\) and \(T\) are two \(\mathcal{F}\)-selfcentralizing subgroup of \(P\), we consider the set \(Y_{R,T}\) of triples \((\tilde{\alpha}, Q, \tilde{\beta})\) where \(Q\) is an \(\mathcal{F}\)-selfcentralizing subgroup of \(P\), \(\tilde{\alpha}\) belongs to \(\tilde{\mathcal{F}}(R, Q)_{\tilde{\beta}}\) and \(\tilde{\beta}\) to \(\tilde{\mathcal{F}}(P, Q)_{\tilde{\alpha}}\); we say that two triples \((\tilde{\alpha}, Q, \tilde{\beta})\) and \((\tilde{\alpha}', Q', \tilde{\beta}')\) are equivalent if there is an \(\mathcal{F}\)-isomorphism \(\theta: Q \cong Q'\) fulfilling \(\tilde{\alpha}' \circ \theta = \tilde{\alpha}\) and \(\tilde{\beta}' \circ \theta = \tilde{\beta}\), and we denote by \(\tilde{Y}_{R,T}\) the set of equivalent classes of such triples. Note that such an \(\tilde{\mathcal{F}}\)-isomorphism \(\tilde{\theta}: Q \cong Q'\) is unique; indeed, we may assume that the triples coincide and, choosing \(\alpha \in \tilde{\alpha}, \beta \in \tilde{\beta}\) and \(\theta \in \theta\), it is easily checked that \(\theta\) belongs to both \(a^* \mathcal{F}_R(\alpha(Q))\) and \(b^* \mathcal{F}_T(\beta(Q))\), and therefore it belongs to \(\mathcal{F}_Q(Q)\), so that \(\tilde{\theta}\) is the identity in \(\tilde{\mathcal{F}}(Q)\). Then, in \(\tilde{\text{ad}}(\tilde{\mathcal{F}}_{\text{sc}})\) we set

\[
R \cap T = \bigoplus_{(\tilde{\alpha}, Q, \tilde{\beta}) \in \tilde{Y}_{R,T}} Q
\]

for a choice of a set of representatives \(\tilde{Y}_{R,T}\) of \(\tilde{Y}_{R,T}\) in \(Y_{R,T}\), and we have canonical morphisms

\[
4.5.2\quad R \leftarrow R \cap T \to T
\]

determined by \(\tilde{\alpha}: Q \to R\) and \(\tilde{\beta}: Q \to T\).

4.6. Note that, for another choice of the set of representatives, we get an isomorphic object and a unique isomorphism being compatible with the canonical morphisms; actually, we may assume that, for any \((\tilde{\alpha}, Q, \tilde{\beta}) \in \tilde{Y}_{R,T}\), we have \(Q \subset R\) and \(\tilde{\alpha} = i^R_Q\). In the case that there are \(\tilde{\gamma} \in \mathcal{F}(P, R)\) and \(\tilde{\delta} \in \mathcal{F}(P, T)\) fulfilling \(\tilde{\gamma} \circ \tilde{\alpha} = \tilde{\delta} \circ \tilde{\beta}\), respectively choosing representatives \(\alpha, \beta, \gamma\) and \(\delta\) of \(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\) and \(\tilde{\delta}\) fulfilling \(\gamma \circ \alpha = \delta \circ \beta\), it follows from 4.2.1 that

\[
4.6.1\quad N_{\gamma(R)}\left((\gamma \circ \alpha)(Q)\right) \cap N_{\delta(T)}\left((\gamma \circ \alpha)(Q)\right) = (\gamma \circ \alpha)(Q)
\]

and therefore we get \(\gamma(R) \cap \delta(T) = (\gamma \circ \alpha)(Q)\), which motivates our notation.

**Proposition 4.7.** With the notation above, the category \(\tilde{\text{ad}}(\tilde{\mathcal{F}}_{\text{sc}})\) admits a distributive direct product mapping any pair of selfcentralizing subgroups \(R\) and \(T\) of \(P\) on their exterior intersection \(R \cap T\).
**Proof.** With the notation above, in order to discuss the functorial nature of the exterior intersection, consider an $\tilde{\mathcal{F}}$-selfcentralizing subgroup $U$ of $P$ and two morphisms $\tilde{\psi} \in \tilde{\mathcal{F}}(R, U)$ and $\tilde{\eta} \in \tilde{\mathcal{F}}(T, U)$; it follows from Proposition 4.3 that $\tilde{\eta}$ determines an isomorphism class of objects $\tilde{\theta}' : U \rightarrow U'$ in $(\tilde{\mathcal{F}}_{sc})_U$ dividing $\tilde{\psi}$ such that, setting $\tilde{\psi}' = \tilde{\psi} / \tilde{\theta}'$, we have $\tilde{\eta} = \tilde{\eta}' \circ \tilde{\theta}'$ for a suitable $\tilde{\eta}' \in \tilde{\mathcal{F}}(T, U)_{\tilde{\psi}'}$ and, once again, $\tilde{\eta}'$ is uniquely determined; that is to say, the pair $(\tilde{\psi}', \tilde{\eta}')$ determines an equivalent class of triples in $\tilde{Y}_{R,T}$ and, once we have chosen a set of representatives $\tilde{Y}_{R,T}$, it determines a unique triple $(\tilde{\psi}', U', \tilde{\eta}')$ and a unique morphism $\tilde{\theta}' : U \rightarrow U'$ fulfilling $\tilde{\psi} = \tilde{\psi}' \circ \tilde{\theta}'$ and $\tilde{\eta} = \tilde{\eta}' \circ \tilde{\theta}'$, so that the following canonical map is bijective

4.7.1 \[
\bigsqcup_{(\tilde{a},Q,\tilde{b}) \in \tilde{Y}_{R,T}} \tilde{\mathcal{F}}(Q, U) \rightarrow \tilde{\mathcal{F}}(R, U) \times \tilde{\mathcal{F}}(T, U).
\]

In particular, considering two $\tilde{\mathcal{F}}_{sc}$-morphisms $\tilde{\psi} : R' \rightarrow R$ and $\tilde{\eta} : T' \rightarrow T$ and a triple $(\tilde{a}', Q', \tilde{b}') \in Y_{R',T'}$, we have the morphisms $\tilde{\psi} \circ \tilde{a}' : Q' \rightarrow R$ and $\tilde{\eta} \circ \tilde{b}' : Q' \rightarrow T$, and therefore we get a triple $(\tilde{a}, Q, \tilde{b})$ in $\tilde{Y}_{R,T}$ and a morphism $\tilde{\theta} : Q \rightarrow Q'$ in $\tilde{\mathcal{F}}_{sc}$ fulfilling 4.7.2

\[
\psi \circ \tilde{a}' = \tilde{a} \circ \tilde{\theta} \quad \text{and} \quad \eta \circ \tilde{b}' = \tilde{b} \circ \tilde{\theta}.
\]

In conclusion, we have obtained a map

4.7.3 \[
\tilde{\psi} \tilde{\eta} : \tilde{Y}_{R',T'} \rightarrow \tilde{Y}_{R,T}
\]

and, for any respective choices $\tilde{Y}_{R,T}$ and $\tilde{Y}_{R',T'}$ of sets of representatives for $\tilde{Y}_{R,T}$ and $\tilde{Y}_{R',T'}$, this map induces a new map $\tilde{\psi} \tilde{\eta} : \tilde{Y}_{R',T'} \rightarrow \tilde{Y}_{R,T}$ which, together with the morphisms $\tilde{\theta} : Q \rightarrow Q'$ in $\tilde{\mathcal{F}}_{sc}$ and the isomorphisms coming from the equivalences, determine a morphism in $\mathfrak{ad}(\tilde{\mathcal{F}}_{sc})$

4.7.4 \[
\tilde{\psi} \tilde{\eta} : R' \tilde{\cap} T' \rightarrow R \tilde{\cap} T.
\]

Finally, always from the bijection 4.7.1, it is not difficult to check that, for two other morphisms $\tilde{\psi}' : R'' \rightarrow R'$ and $\tilde{\eta}' : T'' \rightarrow T'$ in $\tilde{\mathcal{F}}_{sc}$, we get

4.7.5 \[
(\tilde{\psi} \tilde{\eta} \circ \tilde{\psi}' \tilde{\eta}') = (\tilde{\psi} \circ \tilde{\psi}') \tilde{\eta} (\tilde{\eta} \circ \tilde{\eta}').
\]

By distributivity, we can extend the exterior intersection to the category $\mathfrak{ad}(\tilde{\mathcal{F}}_{sc})$, namely if $R = \bigoplus_{i \in I} R_i$ and $T = \bigoplus_{j \in J} T_j$ are two objects in this category, where $R_i$ and $T_j$ are $\mathcal{F}$-selfcentralizing subgroups of $P$, then we set

4.7.6 \[
R \tilde{\cap} T = \bigoplus_{(i,j) \in I \times J} R_i \tilde{\cap} T_j.
\]

Similarly, if we have two morphisms in this category
(\tilde{\psi}, f) : R \to R' = \bigoplus_{i' \in I'} R'_{i'} \quad \text{and} \quad (\tilde{\eta}, g) : T \to T' = \bigoplus_{j' \in J'} T'_j$

where $f : I \to I'$ and $g : J \to J'$ are maps, $\tilde{\psi}$ is an $I$-family of morphisms $\tilde{\psi}_i : R_i \to R'_{f(i)}$ for any $i \in I$ and $\tilde{\eta}$ is a $J$-family of morphisms $\tilde{\eta}_j : T_j \to T'_{g(j)}$ for any $j \in J$, all in $\tilde{F}_{sc}$, then we have morphisms

$\tilde{\psi}_i \cap \tilde{\eta}_j : R_i \cap T_j \to R'_{f(i)} \cap T'_{g(j)}$

in $\alpha(\tilde{F}_{sc})$ which clearly define a new morphism

$\tilde{\psi} \cap \tilde{\eta} : R \cap T \to R' \cap T'$.

Finally, it is quite clear that the bijections 4.7.1 imply the bijections

$\text{Mor}_{\alpha(\tilde{F}_{sc})}(R \cap T, U) \cong \text{Mor}_{\alpha(\tilde{F}_{sc})}(R, U) \times \text{Mor}_{\alpha(\tilde{F}_{sc})}(T, U),$

for any object $U$ in $\alpha(\tilde{F}_{sc})$, which proves that the exterior intersection is a direct product in this category. □

4.8. Actually, for any object $Q$ in $\alpha(\tilde{F}_{sc})$, the category $(\alpha(\tilde{F}_{sc}))_Q$ still admits a direct product or, equivalently, $\alpha(\tilde{F}_{sc})$ admits pull-backs; in order to show it, let us introduce the relative exterior intersection. For any pair of morphisms $R \xrightarrow{\tilde{\beta}} T \xleftarrow{\tilde{\beta}'} R'$ in $\tilde{F}_{sc}$, denote by $Y_{\tilde{\beta}, \tilde{\beta}'}$ the set of triples $(\tilde{\alpha}, Q, \tilde{\alpha}') \in Y_{R, R'}$ fulfilling $\tilde{\beta} \circ \tilde{\alpha} = \tilde{\beta}' \circ \tilde{\alpha}'$; it is clear that $Y_{\tilde{\beta}, \tilde{\beta}'}$ is a union of equivalent classes and, choosing a set of representatives $\tilde{Y}_{\tilde{\beta}, \tilde{\beta}'}$, we set

$R \cap_T R' = R \cap_{\tilde{\beta} \cap \tilde{\beta}'} R' = \bigoplus_{(\tilde{\alpha}, Q, \tilde{\alpha}') \in \tilde{Y}_{\tilde{\beta}, \tilde{\beta}'}} Q$

endowed with the morphisms $R \leftarrow R \cap_T R' \to R'$ determined by $\tilde{\alpha}$ and $\tilde{\alpha}'$ for any $(\tilde{\alpha}, Q, \tilde{\alpha}') \in \tilde{Y}_{\tilde{\beta}, \tilde{\beta}'}$.

**Proposition 4.9.** With the notation above, choosing representatives $\beta$ of $\tilde{\beta}$ and $\beta'$ of $\tilde{\beta}'$, we have

$R \cap_T R' \cong \bigoplus_u \beta(R)^u \cap \beta'(R')$

where $u \in T$ runs over the set of elements such that $\beta(R)^u \cap \beta'(R')$ is $\mathcal{F}$-selfcentralizing in a set of representatives for $\beta(R)/T/\beta'(R')$, and we consider the morphisms from $\beta(R)^u \cap \beta'(R')$ to $R$ and to $R'$ induced by $\beta$, $u$ and $\beta'$. In particular, the category $\alpha(\tilde{F}_{sc})$ admits pull-backs.
Proof. With the notation above, for any triple \((\tilde{\alpha}, Q, \tilde{\alpha}') \in Y_{\tilde{\beta}, \tilde{\beta}'}\) it follows from 4.6 that 
\[\beta(R)u \cap \beta'(R') = (\beta \circ \alpha)(Q)\]
where \(\alpha \in \tilde{\alpha}, \alpha' \in \tilde{\alpha}'\) and \(u \in T\) fulfills \(\beta' \circ \alpha' = \beta'' \circ \alpha\).
Conversely, for any \(w \in T\) such that \(\beta(R)w \cap \beta'(R')\) is \(F\)-selfcentralizing, we already know that the morphisms from \(\beta(R)w \cap \beta'(R')\) to \(R\) and to \(R'\) induced by \(\beta, w\) and \(\beta'\) determine a morphism to \(R \cap R'\) and therefore it determines a triple which clearly belongs to \(Y_{\tilde{\beta}, \tilde{\beta}'}\); consequently, by the first argument, there are a suitable \(u \in T\) and a morphism

4.9.2 \[\beta(R)w \cap \beta'(R') \rightarrow \beta(R)u \cap \beta'(R')\]
compatible with the morphisms induced by \(\beta, w, u\) and \(\beta'\); actually, we may assume that \(\beta(R)w \cap \beta'(R') \subset \beta(R)u \cap \beta'(R')\) and that morphism 4.9.2 is determined by this inclusion; hence, there is \(v \in R\) fulfilling

4.9.3 \[\beta(v)wtw^{-1}\beta(v)^{-1} = utu^{-1},\]
for any \(t \in \beta(R)w \cap \beta'(R')\), and therefore the element \(u^{-1}\beta(v)w \in T\) centralizes \(\beta(R)w \cap \beta'(R')\); in particular, \(u^{-1}\beta(v)w = \beta'(v')\) for some \(v' \in R'\), so that \(u\) and \(w\) determine the same class in \(\beta(R) \backslash T / \beta'(R')\) and we have

4.9.4 \[\beta(R)w \cap \beta'(R') = \beta(R)u \cap \beta'(R').\]

As above, by distributivity we can extend the relative exterior intersection to the category \(\mathfrak{ad}(\tilde{\mathcal{F}}_{\text{sc}})\) and it follows from bijections 4.7.10 and from isomorphisms 4.9.1 that the relative exterior intersection proves the existence of pull-backs in the category \(\mathfrak{ad}(\tilde{\mathcal{F}}_{\text{sc}})\). \(\square\)

4.10. Denote by \(\mathbb{Z}(p)\)-mod the category of finitely generated \(\mathbb{Z}(p)\)-modules and consider a contravariant functor \(m : \tilde{\mathcal{F}}_{\text{sc}} \to \mathbb{Z}(p)\)-mod; we will apply Jackowski and McClure’s method, namely [7, Corollary 5.16], to show that if \(\tilde{\mathcal{F}}(Q)\) acts trivially on \(m(Q)\) for any \(F\)-selfcentralizing subgroup \(Q\) of \(P\), then the \(n\)-cohomology groups of \(\tilde{\mathcal{F}}_{\text{sc}}\) over \(m\) vanish for \(n \geq 1\). Note that \(m\) can be extended to a unique contravariant functor \(\mathfrak{ad}(m) : \mathfrak{ad}(\tilde{\mathcal{F}}_{\text{sc}}) \to \mathbb{Z}(p)\)-mod compatible with direct sums.

4.11. Our result depends on the following lemma, where \(k\) is a field of characteristic \(p\) and we call Mackey complement of \(m\) any (covariant) functor \(m^\circ : \tilde{\mathcal{F}}_{\text{sc}} \to \mathbb{Z}(p)\)-mod such that the pair \((\mathfrak{ad}(m), \mathfrak{ad}(m^\circ))\) is a Mackey functor in Jackowski and McClure’s sense [7]; that is to say, \(m^\circ\) is a Mackey complement of \(m\) if it coincides with \(m\) over the objects and for any pull-back in \(\mathfrak{ad}(\tilde{\mathcal{F}}_{\text{sc}})\)

4.11.1

\[
\begin{array}{c}
\vdash \quad Q \\
\beta \quad R \cap Q \quad R' \\
\alpha \quad \alpha' \quad \beta'
\end{array}
\]
we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{(a\tilde{d}(m))(Q)} & \rightarrow & \text{(a\tilde{d}(m))(Q')} \\
\text{(a\tilde{d}(m))(R)} & \rightarrow & \text{(a\tilde{d}(m))(R')} \\
\text{(a\tilde{d}(m))(\beta)} & \rightarrow & \text{(a\tilde{d}(m))(R \cap Q R')} \\
\text{(a\tilde{d}(m))(\beta')} & \rightarrow & \text{(a\tilde{d}(m))(R')} \\
\end{array}
\]

**Lemma 4.12.** Let \( m : \mathcal{F}_{sc} \rightarrow k\text{-mod} \) be a contravariant functor and assume that, for every \( \mathcal{F} \)-selfcentralizing subgroup \( Q \) of \( P \) and every subgroup \( \bar{R} \) of order \( p \in \bar{N}_P(Q) \), the \( k\bar{R} \)-module \( m(Q) \) has no nonzero projective factors. Then, \( m \) admits a Mackey complement \( m^0 \) mapping any morphism \( \bar{\alpha} : R \rightarrow Q \) in \( \mathcal{F}_{sc} \) on \( m(\bar{\alpha})^{-1} \) whenever \( \bar{\alpha} \) is an isomorphism and on zero otherwise.

**Proof.** We have indeed a functor \( m^0 : \mathcal{F}_{sc} \rightarrow k\text{-mod} \) which coincides with \( m \) over the objects and maps any morphism \( \bar{\alpha} : R \rightarrow Q \) on \( m(\bar{\alpha})^{-1} \) or 0 according to \( \bar{\alpha} \) is an isomorphism or not. Then, it suffices to prove the commutativity of diagram 4.11.2 and, by distributivity, we may assume that \( Q, R \) and \( R' \) are objects in \( \mathcal{F}_{sc} \); moreover, if \( \bar{\alpha} \) is an isomorphism then \( \bar{\beta}' \) is an isomorphism too and the commutativity is clear.

Thus, assuming that \( \bar{\alpha} \) is not an isomorphism, we have to prove that \( (a\tilde{d}(m^0))(\bar{\beta}') \circ (a\tilde{d}(m))(\beta) = 0 \); but, choosing representatives \( \alpha \) of \( \bar{\alpha} \) and \( \alpha' \) of \( \bar{\alpha}' \), we know that (cf. Proposition 4.9)

\[
R \cap Q R' \cong \bigoplus_u \alpha(R)^u \cap \alpha'(R')
\]

where \( u \in Q \) runs over the set of elements such that \( \alpha(R)^u \cap \alpha'(R') \) is \( \mathcal{F} \)-selfcentralizing in a set of representatives for \( \alpha(R) \backslash Q / \alpha'(R') \) and we consider the morphisms from \( \alpha(R)^u \cap \alpha'(R') \) to \( R \) and to \( R' \) induced by \( \alpha \), \( u \) and \( \alpha' \); hence, since \( (a\tilde{d}(m^0))(\bar{\beta}') \) vanish over the terms of the sum such that \( \alpha(R)^u \nsubseteq \alpha'(R') \), for any \( a \in m(R) \) we have to prove that

\[
\sum_u (m^0(\bar{\alpha}')^{-1} \circ m(\tilde{\kappa}_{\alpha(R),\alpha'(R')}(u)) \circ m(\bar{\alpha}_s)^{-1})(a) = 0,
\]

where \( u \in Q \) runs over a set of representatives for \( \alpha(R) \backslash Q / T_Q(\alpha'(R'), \alpha(R)) \) and we denote by \( \tilde{\kappa}_{\alpha(R),\alpha'(R')}(u) \) the group exomorphism \( \alpha'(R') \rightarrow \alpha(R) \) determined by the conjugation by \( u \), and by \( \alpha_s \) and \( \alpha'_s \) the respective isomorphisms \( R \cong \alpha(R) \) and \( R' \cong \alpha'(R') \) induced by \( \alpha \) and \( \alpha' \).

But, since we assume that the \( k\bar{N}_Q(\alpha(R)) \)-module \( m(\alpha(R)) \) is projective-free, we have \( \text{Tr}_{\alpha(R)}^{N_Q(\alpha(R))}(m(\bar{\alpha}_s)(a)) = 0 \) and therefore, choosing a set of representatives \( U \) for \( N_Q(\alpha(R)) \backslash T_Q(\alpha'(R'), \alpha(R)) \), the sum in 4.12.2 coincides with
4.13.4 have the commutative diagram

\[ H_{\alpha(R)} \circ \alpha'_{(R')} (u) \circ \alpha'_*= (\text{Tr}^\kappa_{\alpha(R)}) (m(\alpha'_*)^{-1}(a))) = 0. \]

\[ \square \]

**Theorem 4.13.** Let \( m: \tilde{\mathcal{F}}_{\text{sc}} \to \mathbb{Z}_{(p)} \)-mod be a contravariant functor and assume that, for every \( \mathcal{F} \)-selfcentralizing subgroup \( Q \) of \( P \) and every subgroup \( \tilde{R} \) of order \( p \) in \( \tilde{N}_P(Q) \), the \( k\tilde{R} \)-module \( k \otimes \mathbb{Z}_{(p)} m(Q) \) has no nonzero projective factors. Then, we have \( \mathbb{H}^n(\tilde{\mathcal{F}}_{\text{sc}}, m) = \{0\} \) for any \( n \geq 1 \).

**Proof.** Since any quotient of any subfunctor of \( m \) still fulfill our hypothesis, from the filtration by powers of \( p \) we are easily reduced to the case \( p \cdot m = 0 \). Moreover, by Corollary 4.4 in [7], for any \( n \geq 1 \) we have

\[ \mathbb{H}^n(\tilde{\mathcal{F}}_{\text{sc}}, m) \cong \mathbb{H}^n(\tilde{\mathcal{F}}_{\text{sc}}, \tilde{\mathcal{F}}_{\text{sc}}). \]

Consider the functor \( f_P : \tilde{\mathcal{F}}_{\text{sc}} \to \tilde{\mathcal{F}}_{\text{sc}} \) defined by the exterior intersection with \( P \); the structural morphism \( Q \tilde{\cap} P \to P \) for any object \( Q \) in \( \tilde{\mathcal{F}}_{\text{sc}} \) shows that \( f_P \) factorizes via the forgetful functor \( f_P : \tilde{\mathcal{F}}_{\text{sc}} \to \tilde{\mathcal{F}}_{\text{sc}} \); explicitly, it suffices to consider the functor \( \tilde{\mathcal{F}}_{\text{sc}} \to \tilde{\mathcal{F}}_{\text{sc}} : P \mapsto f_P(P) \) mapping \( Q \) on the structural morphism above and any morphism \( \tilde{\alpha} : R \to Q \) on \( \tilde{\mathcal{F}}_{\text{sc}} \). But, since the category \( \tilde{\mathcal{F}}_{\text{sc}} \) shows the existence of a natural map

\[ \mathbb{H}^n(\tilde{\mathcal{F}}_{\text{sc}}, m) \cong \mathbb{H}^n(\tilde{\mathcal{F}}_{\text{sc}}, m) \]

for any \( n \geq 1 \); hence, we still have \( \mathbb{H}^n(f_P, m) = 0 \) for any \( n \geq 1 \).

Moreover, the existence of the structural morphism \( \tilde{\pi}_Q : Q \tilde{\cap} P \to Q \) for any object \( Q \) in \( \tilde{\mathcal{F}}_{\text{sc}} \) shows the existence of a natural map

\[ \mathbb{H}^n(\tilde{\mathcal{F}}_{\text{sc}}) \to \mathbb{H}^n(\tilde{\mathcal{F}}_{\text{sc}}) \]

sending \( Q \) to \( \tilde{\pi}_Q \); then, it is more or less well known (see [13, Proposition 8.2]) that we have the commutative diagram

\[ \mathbb{H}^n(\tilde{\mathcal{F}}_{\text{sc}}) \]

consequently, we get \( \mathbb{H}^n(\tilde{\mathcal{F}}_{\text{sc}}) = 0 \).

On the other hand, we claim that \( \tilde{\mathcal{F}}_{\text{sc}} \) defines a natural map

\[ \mathbb{H}^n(\tilde{\mathcal{F}}_{\text{sc}}) \to \mathbb{H}^n(\tilde{\mathcal{F}}_{\text{sc}}) \]

sending any object \( Q \) in \( \tilde{\mathcal{F}}_{\text{sc}} \) to \((\tilde{\mathcal{F}}_{\text{sc}})^{(\tilde{\pi}_Q)} \); indeed, for any morphism \( \tilde{\alpha} : R \to Q \) in \( \tilde{\mathcal{F}}_{\text{sc}} \), it is easily checked that the corresponding commutative diagram
is a pull-back and therefore our claim follows from Lemma 4.12.

Moreover, we claim that the composition \((\text{ad}(m) \ast \nu_P) \circ \mu_P\) coincides with \(|\tilde{F}(P)| \cdot \text{id}_{\tilde{a}(m)}\); indeed, for any \(\mathcal{F}\)-selfcentralizing subgroup \(Q\) of \(P\), the image of \(a \in m(Q)\) on throughout \(\mu_P(Q) = (\text{ad}(m^\circ))(\tilde{\eta}_Q)\) has a zero component in the terms where \(\tilde{\eta}\) is not an isomorphism, whereas if \(\tilde{\eta}\) is an isomorphism then we may assume that \(T = Q\) and \(\tilde{\eta} = \text{id}_Q\); hence, \((\text{ad}(m) \ast \nu_P)(\mu_P(Q))\) maps \(a \in m(Q)\) on

\[
\sum_{\tilde{\phi} \in \tilde{F}(P, Q)} a = |\tilde{F}(P)| \cdot a
\]

which proves the claim. But, we have \(H^n(\text{ad}(\tilde{F}_{\text{sc}}), (\text{ad}(m) \ast \nu_P) \circ \mu_P) = 0\). Consequently, we get \(H^n(\text{ad}(\tilde{F}_{\text{sc}}), \text{ad}(m)) = \{0\}\). \(\square\)

5. Alperin fusions in a Frobenius category

5.1. Let \(P\) be a finite \(p\)-group and \(\mathcal{F}\) a Frobenius \(P\)-category; in this section, we prove that the contents of the appendix in [12] can be translated in this abstract setting; as we explain there, the origin of the concepts and the results below goes back to [8], where we formulate the first systematic treatment of Alperin's Fusion Theorem. As a matter of fact, when dealing with contravariant functors \(\alpha : \mathcal{F} \to \text{Ab}\) to the category of abelian groups \(\text{Ab}\), it is handy to consider the category \(\mathbb{Z}\mathcal{F}\) where the objects are once again the subgroups of \(P\) and, for any pair of subgroups \(Q\) and \(R\) of \(P\), the set of morphisms from \(R\) to \(Q\) is the free \(\mathbb{Z}\)-module \(\mathbb{Z}\mathcal{F}(Q, R)\) over \(\mathcal{F}(Q, R)\), with the distributive composition which extends the composition in \(\mathcal{F}\); then, we consider the evident augmentation \(\mathbb{Z}\)-linear map

\[
\varepsilon_{Q, R} : \mathbb{Z}\mathcal{F}(Q, R) \to \mathbb{Z}
\]

sending any \(\varphi \in \mathcal{F}(Q, R)\) to 1. Moreover, let us say that a family \(S\) of subsets \(S_Q \subset \alpha(Q)\), where \(Q\) runs over the set of proper subgroups of \(P\), is a generator family of \(\alpha\) whenever, for any proper subgroup \(Q\) of \(P\), we have

\[
\alpha(Q) = \sum_R \alpha(\mathbb{Z}\mathcal{F}(R, Q))(S_R),
\]

where \(R\) runs over the set of subgroups of \(P\) (such that \(|R| \geq |Q|\)).
5.2. In particular, for any two different elements \( \varphi, \varphi' \in \mathcal{F}(Q, R) \), we call \( \mathcal{F} \)-dimorphism or Alperin \( \mathcal{F} \)-fusion from \( R \) to \( Q \) the difference \( \varphi' - \varphi \); it is clear that the set of \( \mathcal{F} \)-dimorphisms is stable by left and right composition with \( \mathcal{F} \)-morphisms; note that, for any \( \varphi \in \mathcal{F}(Q, R) \), the family

\[
\{ \varphi' - \varphi \}_{\varphi' \in \mathcal{F}(Q, R) - \{ \varphi \}}
\]

is a \( \mathbb{Z} \)-basis of \( \text{Ker}(\varepsilon_{Q,R}) \). The next elementary lemma relates any decomposition of an Alperin \( \mathcal{F} \)-fusion, as a sum of some of them, with Alperin’s original formulation [6, Chapter 7] and with the partially defined linear combinations introduced in [8, Chapter III] and in [7, (2.15.3)].

**Lemma 5.3.** With the notation above, let \( \{ Q_i \}_{i \in I} \) and \( \{ R_i \}_{i \in I} \) be finite families of subgroups of \( P \) and, for any \( i \in I \), \( \varphi' - \varphi \) a \( \mathcal{F} \)-dimorphism from \( R_i \) to \( Q_i \), \( \mu_i \) an element of \( \mathcal{F}(Q, Q_i) \) and \( \nu_i \) an element of \( \mathcal{F}(R_i, R) \). If we have

\[
\varphi' - \varphi = \sum_{i \in I} \mu_i \circ (\varphi'_i - \varphi_i) \circ \nu_i
\]

then there are \( n \in \mathbb{N} \) and an injective map \( \sigma : \Delta_n \rightarrow I \) fulfilling

\[
\varphi = \mu_{\sigma(0)} \circ \varphi_{\sigma(0)} \circ \nu_{\sigma(0)},
\]

\[
\mu_{\sigma(\ell-1)} \circ \varphi'_{\sigma(\ell-1)} \circ \nu_{\sigma(\ell-1)} = \mu_{\sigma(\ell)} \circ \varphi_{\sigma(\ell)} \circ \nu_{\sigma(\ell)} \quad \text{for any } 1 \leq \ell \leq n,
\]

\[
\mu_{\sigma(n)} \circ \varphi'_{\sigma(n)} \circ \nu_{\sigma(n)} = \varphi'.
\]

**Proof.** Equality 5.3.1 is obviously equivalent to

\[
\varphi' + \sum_{i \in I} \mu_i \circ \varphi_i \circ \nu_i = \varphi + \sum_{i \in I} \mu_i \circ \varphi'_i \circ \nu_i
\]

and therefore, since \( \varphi' \neq \varphi \), there are \( i', i'' \in I \) (the possibility \( i' = i'' \) is not excluded!) and a bijection

\[
\pi : I - \{ i'' \} \rightarrow I - \{ i' \}
\]

such that, for any \( i \in I - \{ i' \} \), we have

\[
\mu_i \circ \varphi_i \circ \nu_i = \varphi, \quad \varphi' = \mu_{i''} \circ \varphi'_{i''} \circ \nu_{i''}
\]

and

\[
\mu_i \circ \varphi'_i \circ \nu_i = \mu_{\pi(i)} \circ \varphi_{\pi(i)} \circ \nu_{\pi(i)};
\]

then, we inductively define \( \sigma \) setting \( \sigma(0) = i' \) and \( \sigma(\ell + 1) = \pi(\sigma(\ell)) \) for any \( \ell \in \mathbb{N} \) such that \( \sigma(\ell) \) is already defined and different from \( i'' \), and we denote by \( n \in \mathbb{N} \) the maximal \( \ell \) where \( \sigma \) is defined (so that we get \( \sigma(n) = i'' \)). Indeed, arguing by contradiction, assume
that there are $0 \leq \ell < \ell' \leq n$ such that $\sigma(\ell) = \sigma(\ell')$; that is to say, we have $\pi^\ell(i') = \pi^{\ell'}(i')$ and therefore we get $i' = \pi^{\ell'-\ell}(i')$, a contradiction.  

5.4. Note that, by the so-called Yoneda’s Lemma, the contravariant functor $\mathfrak{h}_\mathcal{F}: \mathcal{F} \to \text{Ab}$ mapping any subgroup $Q$ of $P$ on $\mathbb{Z}\mathcal{F}(P, Q)$, and any morphism $\alpha : R \to Q$ in $\mathcal{F}$ on the group homomorphism $\mathfrak{h}_\mathcal{F}(Q) \to \mathfrak{h}_\mathcal{F}(R)$ defined by the composition with $\alpha$, is projective in the category of contravariant functors $\text{Fct}(\mathcal{F}, \text{Ab})$; moreover, denoting by $\mathbb{Z}: \mathcal{F} \to \text{Ab}$ the trivial contravariant functor mapping all the objects on $\mathbb{Z}$ and all the morphisms on $\text{id}_\mathbb{Z}$, the augmentation maps define a surjective natural map

$$\varepsilon_\mathcal{F}: \mathfrak{h}_\mathcal{F} \to \mathbb{Z}$$

and therefore the kernel $\mathfrak{w}_\mathcal{F} = \text{Ker}(\varepsilon_\mathcal{F})$ is the Heller translated of the trivial functor $\mathbb{Z}$; in particular, we have the exact sequence

$$0 \to \text{Nat}(\mathbb{Z}, a) \to \text{Nat}(\mathfrak{h}_\mathcal{F}, a) \to \text{Nat}(\mathfrak{w}_\mathcal{F}, a) \to H^1(\mathcal{F}, a) \to 0.$$ 

Now, according to Lemma 5.3, it is quite clear that, in the Frobenius category $\mathcal{F}_G$ associated with a finite group $G$, the genuine purpose of Alperin’s Fusion Theorem [6, Chapter 7] is to describe some generator families of the contravariant functor $\mathfrak{w}_\mathcal{F}_G: \mathcal{F}_G \to \text{Ab}$.

5.5. With the analogous purpose in $\mathcal{F}$, we set $\mathfrak{r}_\mathcal{F}(P) = \mathfrak{w}_\mathcal{F}(P)$ and, for any proper subgroup $Q$ of $P$, set

$$\mathfrak{r}_\mathcal{F}(Q) = \sum_R \mathfrak{w}_\mathcal{F}(R) \circ \mathbb{Z}\mathcal{F}(R, Q),$$

where $R$ runs over the set of subgroups of $P$ such that $|R| > |Q|$; note that, since there is $\psi \in \mathcal{F}(P, R)$ such that $\psi(R)$ is fully normalized in $\mathcal{F}$ (cf. 2.7) and the inverse of the isomorphism $R \cong \psi(R)$ determined by $\psi$ belongs to $\mathcal{F}(R, \psi(R))$, in definition 5.5.1 it suffices to restrict the sum to the subgroups $R$ which are fully normalized in $\mathcal{F}$; moreover, if $Q'$ is a subgroup of $P$ and $\theta \in \mathcal{F}(Q, Q')$ is an isomorphism then clearly we have

$$\mathfrak{r}_\mathcal{F}(Q) \circ \theta = \mathfrak{r}_\mathcal{F}(Q').$$

We say that $Q$ is $\mathcal{F}$-essential when $\mathfrak{r}_\mathcal{F}(Q) \neq \mathfrak{w}_\mathcal{F}(Q)$ and call $\mathcal{F}$-irreducible the elements of $\mathfrak{w}_\mathcal{F}(Q) - \mathfrak{r}_\mathcal{F}(Q)$.

5.6. Coherently, the elements of $\mathfrak{r}_\mathcal{F}(Q)$ are called $\mathcal{F}$-reducible; actually, any element of $\mathfrak{r}_\mathcal{F}(Q)$ is a sum of a family of $\mathcal{F}$-reducible Alperin $\mathcal{F}$-fusions from $Q$ to $P$. Denoting by

$$\rho_Q : \mathfrak{h}_\mathcal{F}(Q) \to \mathfrak{h}_\mathcal{F}(Q)/\mathfrak{r}_\mathcal{F}(Q)$$
the canonical map, it is clear that $\mathcal{F}(Q)$ acts on the image of $\mathcal{F}(P, Q)$ in $\mathcal{F}(Q)/\tau_{\mathcal{F}}(Q)$ by multiplication on the right and, as a matter of fact, this action is transitive as we prove below; we denote by $\mathcal{F}(Q)_{\rho_Q(\varphi)}$ the stabilizer of the image of $\varphi \in \mathcal{F}(P, Q)$ and, with the notation above, we clearly have

$$\theta \mathcal{F}(Q')_{\rho_Q'(\varphi \theta)} = \mathcal{F}(Q)_{\rho_Q(\varphi)}.$$  

Note that the correspondence mapping $Q$ on $\tau_{\mathcal{F}}(Q)$ defines a subfunctor $\tau_{\mathcal{F}}$ of $\varpi_{\mathcal{F}}$.

**Proposition 5.7.** Let $S = \{S_Q\}_Q$ be a generator family of $\varpi_{\mathcal{F}}$, where $Q$ runs over the set of proper subgroups of $P$. The family of $\mathcal{F}$-irreducible elements of $S$ still is a generator family of $\varpi_{\mathcal{F}}$ and, for any $\mathcal{F}$-essential subgroup $Q$ of $P$, there is $\varphi \in \mathcal{F}(P, Q)$ such that $S_{\varphi(Q)}$ contains an $\mathcal{F}$-irreducible element of $\varpi_{\mathcal{F}}(\varphi(Q))$.

**Proof.** If $\omega \in \varpi_{\mathcal{F}}(Q)$ is $\mathcal{F}$-irreducible and we have $\omega = \sum R \sum_{\sigma \in S_R} \sigma \circ \alpha_{R, \sigma}$ for suitable $\alpha_{R, \sigma} \in \mathbb{Z} \mathcal{F}(R, Q)$, where $R$ runs over the set of subgroups of $P$ setting $S_P = \varpi_{\mathcal{F}}(P)$, then necessarily there are a suitable subgroup $R$ of $P$ such that $|R| = |Q|$ and an $\mathcal{F}$-irreducible element $\sigma \in S_R$ such that $0 \neq \alpha_{R, \sigma}$; in particular, we have $R = \varphi(Q)$ for some $\varphi \in \mathcal{F}(P, Q)$.

On the other hand, if $\tau \in S_Q$ is an $\mathcal{F}$-reducible element then either $Q = P$ or we have $\tau = \sum R \sum_{\theta \in \varpi_{\mathcal{F}}(R)} \theta \circ \beta_{R, \theta}$ for suitable $\beta_{R, \theta} \in \mathbb{Z} \mathcal{F}(R, Q)$, where $R$ runs over the set of subgroups of $P$ such that $|R| > |Q|$; in the second case, considering a $S$-decomposition of any $\theta \in \varpi_{\mathcal{F}}(R)$, we still have $\tau = \sum R \sum_{\sigma \in S_R} \sigma \circ \gamma_{R, \sigma}$ for suitable $\gamma_{R, \sigma} \in \mathbb{Z} \mathcal{F}(R, Q)$, where $R$ runs over the set of subgroups of $P$ such that $|R| > |Q|$; so that the new family where we replace $S_Q$ by $S_Q - \{\tau\}$ is a generator family of $\varpi_{\mathcal{F}}$ too.

**Theorem 5.8.** A subgroup $Q$ of $P$ is $\mathcal{F}$-essential if and only if it fulfills the following two conditions:

5.8.1 $Q$ is $\mathcal{F}$-selfcentralizing.

5.8.2 $\mathcal{F}(Q)$ has a proper subgroup $M$ containing $\mathcal{F}_Q(Q)$ such that $p$ divides $|M/\mathcal{F}_Q(Q)|$ and does not divide $|(M \cap M^{\sigma})/\mathcal{F}_Q(Q)|$ for any $\sigma \in \mathcal{F}(Q) - M$.

In this case, the groups $\mathcal{F}(Q)_{\rho_Q(\varphi)}$, when $\varphi$ runs over $\mathcal{F}(P, Q)$, are the minimal proper subgroups of $\mathcal{F}(Q)$ fulfills condition 5.8.2 and they contain Sylow $p$-subgroups of $\mathcal{F}(Q)$. Moreover, $\rho_Q(\mathcal{F}(P, Q))$ is a $\mathbb{Z}$-basis of $\mathcal{F}_Q(Q)/\tau_{\mathcal{F}}(Q)$ and $\mathcal{F}(Q)$ acts transitively on this set.

**Proof.** Let $\varphi$ be an element of $\mathcal{F}(P, Q)$ such that $Q' = \varphi(Q)$ is fully normalized in $\mathcal{F}$ (cf. 2.7); if $\psi' \in \mathcal{F}(P, Q)$, set $R' = N_P(\psi'(Q))$ and consider the isomorphism $\varphi'(Q) \cong Q'$ determined by $\varphi$ and $\psi'$; it follows easily from condition 2.9.2 that there are $\rho' \in \mathcal{F}(P, R')$ and $\sigma \in \mathcal{F}(Q)$ such that $\rho'(\varphi'(u)) = \varphi(\sigma(u))$ for any $u \in Q$; consequently, denoting by $\psi': Q \to R'$ the group homomorphism determined by $\varphi'$, we get

$$\varphi' - \varphi \circ \sigma = (\iota_{R'}^P - \rho') \circ \psi'.$$
In particular, assuming that $Q$ is $\mathcal{F}$-essential, $\varphi'$ and $\varphi \circ \sigma$ have the same image in $h_{\mathcal{F}(Q)/\mathcal{F}(Q)}$, so that the $\mathbb{Z}$-linear map

$$5.8.5 \quad \mathbb{Z}\mathcal{F}(Q) \rightarrow h_{\mathcal{F}(Q)/\mathcal{F}(Q)}$$

sending $\sigma \in \mathcal{F}(Q)$ to the class of $\varphi \circ \sigma$ is surjective, and $\mathcal{F}(Q)$ acts transitively on the image of $\mathcal{F}(P, Q)$; moreover, from the very definition of $\mathcal{F}(Q)_{\rho Q(\varphi)}$, we get the factorization

$$5.8.6 \quad \mathbb{Z}(\mathcal{F}(Q)_{\rho Q(\varphi}) \setminus \mathcal{F}(Q)) \rightarrow h_{\mathcal{F}(Q)/\mathcal{F}(Q)}.$$ 

Furthermore, if we assume that $\sigma/\in \mathcal{F}(Q)_{\rho Q(\varphi)}$ then the Alperin $\mathcal{F}$-fusion $\varphi - \varphi \circ \sigma$ is not $\mathcal{F}$-reducible; but, setting $U = Q' \cdot CP(Q')$ and considering the element of $\mathcal{F}(Q')$ determined by $\sigma$, it follows from statement 2.11.1 that there is $\rho \in \mathcal{F}(U)$ such that, for any $u \in Q$, we have $\rho(\varphi(u)) = \varphi(\sigma(u))$; hence, denoting by $\psi : Q \rightarrow U$ the group homomorphism determined by $\varphi$, we get

$$5.8.7 \quad \varphi - \varphi \circ \sigma = i_p^U \circ (\text{id}_U - \rho) \circ \psi,$$

which forces $Q' = \varphi(Q) = U$; since we have $\rho'(C_P(\varphi'(Q))) \subset C_P(Q')$, it is clear that $Q$ fulfills condition 5.8.1.

Now, set $R = N_P(Q')$; according to Proposition 2.12, $\mathcal{F}_R(Q')$ is a Sylow $p$-subgroup of $\mathcal{F}(Q')$ and it is nontrivial by the argument above; moreover, if $v \in R$ and $v$ is the image of $v$ in $\mathcal{F}(Q)$ by the isomorphism determined by $\varphi$, it is easily checked that

$$5.8.8 \quad \varphi - \varphi \circ v = (\text{id}_p - \kappa_P(v)) \circ \varphi,$$

where $\kappa_P(v)$ is the image of $v$ in $\mathcal{F}(P)$, so that $v$ belongs to $\mathcal{F}(Q)_{\rho Q(\varphi)}$; that is to say, $\mathcal{F}(Q)_{\rho Q(\varphi)}$ contains $\mathcal{F}_R(Q')$ and, in particular, $\mathcal{F}(Q)_{\rho Q(\varphi)}$ contains $\mathcal{F}(Q)$ and $p$ divides $|\mathcal{F}(Q)_{\rho Q(\varphi)}/\mathcal{F}(Q)|$.

Now, consider

$$5.8.9 \quad F = \mathcal{F}(Q)_{\rho Q(\varphi')} \cap \mathcal{F}(Q)_{\rho Q(\varphi)}$$

and assume that $p$ divides $|F/\mathcal{F}(Q)|$, choosing a $p$-subgroup $V$ of $F$ strictly containing $\mathcal{F}(Q)$; thus, since $\mathcal{F}(Q)_{\rho Q(\varphi')} = (\mathcal{F}(Q)_{\rho Q(\varphi)})^\sigma$ and $\mathcal{F}_R(Q')$ is a Sylow $p$-subgroup of $\mathcal{F}(Q)_{\rho Q(\varphi)}$, there are elements $\tau \in \mathcal{F}(Q)_{\rho Q(\varphi)}$ and $\tau' \in \mathcal{F}(Q)_{\rho Q(\varphi')}$ such that

$$5.8.10 \quad \varphi \circ \tau V \subset \mathcal{F}_R(Q') \triangleright \varphi \circ \sigma \circ \tau' V$$

and therefore, since we already have $C_P(Q') = Z(Q')$, it follows from statement 2.11.1 that, denoting by $T$ the converse image of $\varphi \circ \tau V$ in $P$, there is $\zeta \in \mathcal{F}(R, T)$ fulfilling $\zeta(\varphi(u)) = \varphi((\sigma \circ \tau' \circ \tau^{-1})(u))$ for any $u \in Q$; in conclusion, denoting by $\xi : T \rightarrow P$ the inclusion map, by $\xi' : T \rightarrow P$ the composition of $\zeta$ with the corresponding inclusion map,
and by $\eta \in \mathcal{F}(T, Q)$ the group homomorphism determined by $\varphi \circ \tau$ and by the inclusion $\varphi(Q) \subset T$, we have

5.8.11 \[ \xi \circ \eta = \varphi \circ \tau \quad \text{and} \quad \xi' \circ \eta = \varphi \circ \sigma \circ \tau' \]

and therefore the elements

5.8.12 \[ \varphi - \xi \circ \eta, \quad (\xi - \xi') \circ \eta \quad \text{and} \quad \xi' \circ \eta - \varphi \circ \sigma \]

are $\mathcal{F}$-reducible, so that we get $\mathcal{F}(Q)_{\rho_Q(\varphi)} = \mathcal{F}(Q)_{\rho_Q(\varphi \circ \sigma)} = \mathcal{F}(Q)_{\rho_Q(\varphi')}$. Conversely, assume that $Q$ fulfills conditions 5.8.1 and 5.8.2, and denote by $M$ a proper subgroup of $\mathcal{F}(Q)$ as in condition 5.8.2; in particular, for any $p$-subgroup $V$ of $M$ strictly containing $\mathcal{F}(Q)$, $M$ contains $N_{\mathcal{F}(Q)}(V)$ and therefore a Sylow $p$-subgroup of $M$ is a Sylow $p$-subgroup of $\mathcal{F}(Q)$ too; hence, with the notation above, there is $\sigma' \in \mathcal{F}(Q)$ such that $\varphi' \circ \sigma'$ contains the image of $R'$ in $\mathcal{F}(\varphi'(Q))$; but, since $CP(\varphi'(Q)) = \varphi'(Z(Q))$, this image strictly contains $\mathcal{F}(\varphi'(Q))(\varphi'(Q))$ and therefore such elements $\sigma'$ determine a unique class in $\mathcal{F}(Q)/M$.

Consider the map $\mathcal{F}(P, Q) \to \mathcal{F}(Q)/M$ sending $\varphi'$ to the class of $\sigma'$; we claim that the corresponding $\mathbb{Z}$-linear map

5.8.13 \[ \mathfrak{h}_{\mathcal{F}(Q)} \to \mathbb{Z}(\mathcal{F}(Q)/M) \]

annihilates $\mathfrak{r}_{\mathcal{F}(Q)}$; according to definition 5.5.1, it suffices to prove that, for any subgroup $T$ of $P$ such that $|T| > |Q|$, any $\xi, \xi' \in \mathcal{F}(P, T)$ and any $\eta \in \mathcal{F}(T, Q)$, this map annihilates $(\xi' - \xi) \circ \eta$; moreover, it is clear that we may assume that $\eta(Q)$ is normal in $T$. Since $M$ contains a Sylow $p$-subgroup of $\mathcal{F}(Q)$, for suitable $\tau, \tau' \in \mathcal{F}(Q)$ the automorphism groups $\xi \circ \eta \circ \tau M$ and $\xi' \circ \eta \circ \tau' M$ contain the respective images of $\xi(T)$ in $\mathcal{F}((\xi \circ \eta)(Q))$ and of $\xi'(T)$ in $\mathcal{F}((\xi' \circ \eta)(Q))$; consequently, the image of $T$ in $\mathcal{F}(\eta(Q))$ is contained in the intersection $\eta \circ \tau M \cap \eta \circ \tau' M = \eta(\tau M \cap \tau' M)$ and strictly contains $\mathcal{F}_{\eta(Q)}(\eta(Q))$ by condition 5.8.1, so that we have $\tau M = \tau' M$ which forces $\tau$ and $\tau'$ to be in the same class.

In conclusion, $Q$ is $\mathcal{F}$-essential and, from the $\mathbb{Z}$-linear maps 5.8.6 and 5.8.13, we get the composed $\mathbb{Z}$-linear map

5.8.14 \[ \mathbb{Z}(\mathcal{F}(Q)/\mathcal{F}(Q)_{\rho_Q(\varphi)}) \to \mathfrak{h}_{\mathcal{F}(Q)}/\mathfrak{r}_{\mathcal{F}(Q)} \to \mathbb{Z}(\mathcal{F}(Q)/M) \]

sending $\sigma \mathcal{F}(Q)_{\rho_Q(\varphi)}$ to $\sigma M$; in particular, this proves that $\mathcal{F}(Q)_{\rho_Q(\varphi)} \subset M$ and, applying it to $M = \mathcal{F}(Q)_{\rho_Q(\varphi)}$, that the $\mathbb{Z}$-linear map 5.8.6 is injective too. \(\square\)

**Remark 5.9.** As in [8, Chapter II], if a subgroup $Q$ of $P$ is $\mathcal{F}$-essential then the following two statements hold:

5.9.1 *For any $\varphi \in \mathcal{F}(P, Q)$, any proper subgroup $M$ of $\mathcal{F}(Q)$ containing $\mathcal{F}(Q)_{\rho_Q(\varphi)}$ fulfills condition 5.8.2.*
Indeed, since any $p$-subgroup of $\mathcal{F}(Q)$ which strictly contains $\mathcal{F}_Q(Q)$ fixes a unique element on the image of $\mathcal{F}(P,Q)$ in $h\mathcal{F}(Q)/\mathcal{F}(Q)$, and the index $|M/\mathcal{F}(Q)_{\rho_Q(\varphi)}|$ is prime to $p$, any $p$-subgroup of $M$ strictly containing $\mathcal{F}_Q(Q)$ is not contained in $M'$ for any $\sigma \in \mathcal{F}(Q) - M$.

5.9.2 The set of normal subgroups $X$ of $\mathcal{F}(Q)$ such that $\mathcal{F}_Q(Q)$ is contained in $X$ and $p$ divides $|X/\mathcal{F}_Q(Q)|$ has a unique minimal element $X_{\mathcal{F}(Q)}$ and then we have

$$\mathcal{F}(Q) = \mathcal{O}^p(X_{\mathcal{F}(Q)}) \cdot \mathcal{F}(Q)_{\rho_Q(\varphi)}.$$ 

Indeed, arguing by contradiction, we may assume that there are two normal subgroups $\bar{X}$ and $\bar{X}'$ of $\bar{\mathcal{F}(Q)} = \mathcal{F}(Q)/\mathcal{F}_Q(Q)$ such that $p$ divides $|\bar{X}|$ and $|\bar{X}'|$, but does not divide $|\bar{X} \cap \bar{X}'|$; in particular, if $\bar{R}$ is a Sylow $p$-subgroup of $\mathcal{F}(Q)_{\rho_Q(\varphi)}/\mathcal{F}_Q(Q)$, setting $\bar{T} = \bar{X} \cap \bar{R}$ and $\bar{T}' = \bar{X}' \cap \bar{R}$, they are not trivial and we have

5.9.3

$$[\bar{T}, \bar{T}'] \subset \bar{X} \cap \bar{X}' \cap \bar{R} = \{1\}$$

and then, since $\bar{T} \cdot \bar{T}'$ contains a noncyclic subgroup of order $p^2$, it is well known (cf. [6, Theorem 3.16]) that

5.9.4

$$\bar{X} \cap \bar{X}' = \{ C_{\bar{X} \cap \bar{X}'}(\bar{t}) \mid \bar{t} \in \bar{T} \cdot \bar{T}' - \{1\} \}$$

which proves that $\mathcal{F}(Q)_{\rho_Q(\varphi)}/\mathcal{F}_Q(Q)$ contains $\bar{X} \cap \bar{X}'$; but, in the quotient $\bar{\mathcal{F}(Q)}/(\bar{X} \cap \bar{X}')$, the images of $\bar{X}$ and $\bar{X}'$ centralize each other and, in particular, the image of $\bar{T}$ centralizes the image of $\bar{X}'$, so that this last image is contained in the image of $\mathcal{F}(Q)_{\rho_Q(\varphi)}$, a contradiction.

Corollary 5.10. Let $\mathcal{E}$ be a $\mathcal{F}(P)$-stable set of $\mathcal{F}$-essential subgroups of $P$ containing at least a representative for each $\mathcal{F}$-isomorphism class. For any subgroup $Q$ of $P$ and any $\varphi \in \mathcal{F}(P,Q)$, there are $\sigma \in \mathcal{F}(P)$, a finite family $\{Q_i\}_{i \in I}$ of elements of $\mathcal{E}$ and, for any $i \in I$, a $p'$-element $\sigma_i \in X_{\mathcal{F}(Q_i)}$ not fixing $\rho_{Q_i}(t_{Q_i}^P)$ and an element $v_i$ of $\mathcal{F}(Q_i, Q)$ fulfilling

5.10.1

$$\varphi = \sigma \circ t_Q^P + \sum_{i \in I} t_{Q_i}^P \circ (\sigma_i - \text{id}_{Q_i}) \circ v_i.$$ 

Proof. Setting $\mathcal{E}' = \mathcal{E} \cup \{P\}$ and $X_{\mathcal{F}(P)} = \mathcal{F}(P)$, we firstly prove that, for any $\psi, \psi' \in \mathcal{F}(P, Q)$, there are a finite family $\{Q_j\}_{j \in J}$ of elements of $\mathcal{E}'$ and, for any $j \in J$, a $p'$-element $\eta_j \in X_{\mathcal{F}(Q_j)}$ not fixing $\rho_{Q_j}(t_{Q_j}^P)$ and $\mu_j \in \mathcal{F}(Q_j, Q)$ fulfilling

5.10.2

$$\psi - \psi' = \sum_{j \in J} t_{Q_j}^P \circ (\eta_j - \text{id}_{Q_j}) \circ \mu_j.$$
It is clear that, arguing by induction on \(|P:Q|\), we may assume that \(Q \neq P\) and that \(\psi - \psi'\) is \(\mathcal{F}\)-irreducible; but, in this case, \(Q\) is \(\mathcal{F}\)-essential and thus there is \(Q' \in \mathcal{E}\) which admits an isomorphism \(\theta \in \mathcal{F}(Q', Q)\); moreover, there are \(\tau, \tau' \in \mathcal{O}^p(X_{\mathcal{F}}(Q'))\) such that

\[
\rho = \psi \circ \theta^{-1} - t_{Q'}^p \circ \tau \quad \text{and} \quad \rho' = \psi' \circ \theta^{-1} - t_{Q'}^p \circ \tau'
\]

are \(\mathcal{F}\)-reducible (cf. Theorem 5.8 and 5.9.2); thus, since we have

\[
\psi - \psi' = (\rho - \rho') \circ \theta + t_{Q'}^p \circ (\tau - \text{id}_{Q'}) \circ \theta + t_{Q'}^p \circ (\tau' - \text{id}_{Q'}) \circ (\tau' \circ \theta)
\]

and since \(\tau\) and \(\tau'\) can be decomposed as products of \(p'\)-elements of \(X_{\mathcal{F}}(Q')\), it suffices to apply the induction hypothesis again.

Now, we set \(\psi = \varphi\) and \(\psi' = t_{Q'}^p\) and argue by induction on \(|J|\); we may assume that \(J \neq \emptyset\) and, according to Lemma 3.3, there is \(j \in J\) such that \(\varphi = t_{Q_j}^p \circ \eta_j \circ \mu_j\) and then, according to the induction hypothesis, \(\varphi' = t_{Q_j}^p \circ \mu_j\) admits the announced decomposition 3.9.1. If \(Q_j \in \mathcal{E}\) then \(\varphi = \varphi' + t_{Q_j}^p \circ (\eta_j - \text{id}_{Q_j}) \circ \mu_j \) gives the announced decomposition for \(\varphi\). If \(Q_j = P\) then \(t_{Q_j}^p = \text{id}_P\), \(\eta_j\) belongs to \(\mathcal{F}(P)\) and it is easy to check that \(\varphi = \eta_j \circ \varphi'\) still gives the announced decomposition for \(\varphi\).  

\[\Box\]

5.11. With the notation of the corollary above, for the inductive purposes it is handy to introduce the \(\mathcal{E}\)-length of \(\varphi\): it is the smallest integer \(\ell_{\mathcal{E}}(\varphi)\) such that we have a decomposition 5.10.1 with \(|J| = \ell_{\mathcal{E}}(\varphi)\); it is clear that if \(\ell_{\mathcal{E}}(\varphi) \geq 1\) then there are \(R \in \mathcal{E}\), \(\eta \in \mathcal{F}(R, Q)\) and a \(p'\)-element \(\tau \in X_{\mathcal{F}}(R)\) not fixing \(\rho_R(t_{Q}^p)\) such that \(\varphi = t_{Q}^p \circ \tau \circ \eta\) and \(\ell_{\mathcal{E}}(t_{Q}^p \circ \eta) = \ell_{\mathcal{E}}(\varphi) - 1\). When \(\mathcal{E}\) is the set of all the \(\mathcal{F}\)-essential subgroups of \(P\) fully normalized in \(\mathcal{F}\), we simply write \(\ell(\varphi)\) and call it the length of \(\varphi\).

6. Quotients and normal subcategories of a Frobenius category

6.1. Let \(P\) be a finite \(p\)-group and \(\mathcal{F}\) a \(P\)-category; if \(P'\) is a second finite \(p\)-group and \(\mathcal{F}'\) a \(P'\)-category, we say that a group homomorphism \(\alpha : P \to P'\) is \((\mathcal{F}, \mathcal{F}')\)-functorial whenever, for any pair of subgroups \(Q\) and \(R\) of \(P\) and any \(\varphi \in \mathcal{F}(Q, R)\), we have \(\varphi(R \cap \text{Ker}(\alpha)) \subset \text{Ker}(\alpha)\) and the group homomorphism \(\varphi' : \alpha(R) \to \alpha(Q)\) determined by \(\varphi\) belongs to \(\mathcal{F}'(\alpha(Q), \alpha(R))\). In this case, \(\alpha\) determines an evident functor

\[\mathfrak{F}_\alpha : \mathcal{F} \to \mathcal{F}'\]

that we call Frobenius functor; denote by \(\text{Fb}(\mathcal{F}, \mathcal{F}')\) the set of \((\mathcal{F}, \mathcal{F}')\)-functorial homomorphisms from \(P\) to \(P'\); clearly, the composition of Frobenius functors is a Frobenius functor. If \(\mathcal{F}' = \mathcal{F}\) then \(\text{id}_P\) is obviously \((\mathcal{F}, \mathcal{F})\)-functorial and \(\alpha\) still determines a natural isomorphism \(\text{id}_\mathcal{F} \cong \mathfrak{F}_\alpha\); note that if \(\mathcal{F}\) is divisible then

\[\mathcal{F}(P) \subset \text{Fb}(\mathcal{F}, \mathcal{F})\]

and we call inner Frobenius functors the Frobenius functors determined by \(\mathcal{F}(P)\).
6.2. We say that a subgroup $U$ of $P$ is $\mathcal{F}$-stable$^6$ if $\varphi(Q \cap U) \subset U$ for any subgroup $Q$ of $P$ and any element $\varphi$ of $\mathcal{F}(P, Q)$; note that, in particular, $U$ is normal in $P$, and that the $\mathcal{F}$-stability is a necessary condition to guarantee that $U$ is the kernel of some $(\mathcal{F}, \mathcal{F}')$-functorial homomorphism from $P$ to a finite $p$-group $P'$, where $\mathcal{F}'$ is a $P'$-category; actually, the next result states that, in the Frobenius categories, it is also a sufficient condition. In this section, from now on we assume that $\mathcal{F}$ is a Frobenius $P$-category.

**Proposition 6.3.** Let $U$ be a $\mathcal{F}$-stable subgroup of $P$ and set $\bar{P} = P / U$. We have a Frobenius $\bar{P}$-category $\bar{\mathcal{F}}$ such that, for any pair of subgroups $\bar{Q} = Q / U$ and $\bar{R} = R / U$ of $\bar{P}$, $\bar{\mathcal{F}}(\bar{Q}, \bar{R})$ is the set of group homomorphisms $\varphi : \bar{R} \to \bar{Q}$ induced by the homomorphism in $\mathcal{F}(Q, R)$. Moreover, the canonical homomorphism $P \to \bar{P}$ is $(\mathcal{F}, \bar{\mathcal{F}})$-functorial.

**Proof.** It is clear that the above correspondence defines a $\bar{P}$-category; moreover, whenever $\bar{Q} = Q / U$, $\bar{R} = R / U$ and $\bar{T} = T / U$ are subgroups of $\bar{P}$, $\bar{\varphi}$ is an element of $\bar{\mathcal{F}}(\bar{Q}, \bar{R})$ and $\bar{\theta} : \bar{T} \to \bar{R}$ is a group homomorphism such that $\bar{\varphi} \circ \bar{\theta}$ belongs to $\bar{\mathcal{F}}(\bar{Q}, \bar{T})$, then $\bar{\varphi} \circ \bar{\theta}$ can be lifted to some $\psi \in \mathcal{F}(Q, T)$ and in particular $\psi(T) \subset \varphi(R)$, so that there is a group homomorphism $\theta : R \to T$ fulfilling $\varphi \circ \theta = \psi$, which implies that $\theta$ belongs to $\mathcal{F}(R, T)$ since $\mathcal{F}$ is divisible, and therefore $\theta$ belongs to $\bar{\mathcal{F}}(\bar{R}, \bar{T})$; thus, $\bar{\mathcal{F}}$ is divisible too.

It is clear that $\bar{\mathcal{F}}$ fulfills condition 2.9.1. On the other hand, let $Q$ be a subgroup of $P$ containing $U$ and $\varphi \in \mathcal{F}(P, Q)$, set $\bar{Q} = Q / U$ and denote by $\bar{\varphi} : \bar{Q} \to \bar{P}$ the group homomorphism determined by $\varphi$; moreover, let $\bar{K}$ be a subgroup of Aut($\bar{Q}$) and denote by $K$ the converse image of $\bar{K}$ in the stabilizer Aut($Q)_U$ of $U$ in Aut($Q$); although $K$ need not map onto $\bar{K}$, it is clear that $N_{\bar{P}}^K(\bar{Q})$ is the image of $N_P^K(Q)$. Set $Q' = \varphi(Q)$, $K' = \varphi K$, $\bar{Q}' = \bar{\varphi}(\bar{Q})$ and $\bar{K}' = \bar{\varphi}K$, and assume that $\bar{Q}'$ is fully $\bar{K}'$-normalized in $\bar{\mathcal{F}}$; since $\bar{\mathcal{F}}(\bar{P}, \bar{Q}' \cdot N_{\bar{P}}^{\bar{K}'}(\bar{Q}'))$ is the image of $\mathcal{F}(P, Q' \cdot N_P^{K'}(Q'))$, it is not difficult to check that $Q'$ is fully $K'$-normalized in $\mathcal{F}$, and therefore, since $\mathcal{F}$ is a Frobenius category, there are $\zeta \in \mathcal{F}(P, Q \cdot N_P^K(Q))$ and $\chi \in K$ fulfilling $\zeta(u) = \varphi(\chi(u))$ for any $u \in Q$; in particular, since $\chi(U) = U$, we get $\zeta(U) = U$, so that $\zeta$ determines an element $\bar{\zeta}$ of $\bar{\mathcal{F}}(\bar{P}, \bar{Q} \cdot N_{\bar{P}}^{\bar{K}'}(\bar{Q}))$ fulfilling $\bar{\zeta}(\bar{u}) = \bar{\varphi}(\bar{\chi}(\bar{u}))$ for any $\bar{u} \in \bar{Q}$.

It remains to prove that the canonical homomorphism $P \to \bar{P}$ is $(\mathcal{F}, \bar{\mathcal{F}})$-functorial; since $\bar{\mathcal{F}}$ is divisible, it suffices to prove that, for any subgroup $Q$ of $P$ and any $\varphi \in \mathcal{F}(P, Q)$, setting $\bar{Q} = Q \cdot U / U$, the homomorphism $\bar{\varphi} : \bar{Q} \to \bar{P}$ induced by $\varphi$ belongs to $\bar{\mathcal{F}}(\bar{P}, \bar{Q})$, namely to the image of $\mathcal{F}(P, Q \cdot U)$ in the set of group homomorphisms from $\bar{Q}$ to $\bar{P}$. We argue by induction on $|P : Q|$ and on the length $\ell$ of $\varphi$ (cf. 5.11), and it is clear that we may assume that $U \not\subset Q$; if $\ell = 0$ then $\varphi = \sigma \circ i_Q^P$, where $\sigma \in \mathcal{F}(P)$, and it suffices to consider the group homomorphism $Q \cdot U \to P$ induced by $\sigma$.

Assume that $\ell \geq 1$, so that we have $\varphi = i_R^P \circ \sigma \circ \nu$, where $R$ is a $\mathcal{F}$-essential subgroup of $P$ fully normalized in $\mathcal{F}$, $\sigma$ is a $p'$-element of $\mathcal{F}(R)$ and $\nu$ is an element of $\mathcal{F}(R, Q)$ such that $i_R^P \circ \nu$ has length $\ell - 1$ (cf. 5.11); thus, according to the induction hypothesis,

---

$^6$ We borrow this term from Cartan and Eilenberg; the term "strongly closed" which has been somewhat employed in local theory is inadequate since there is no "strongly closure"!
there is \( \psi \in \mathcal{F}(P, Q \cdot U) \) inducing the same homomorphism as \( i_R^P \circ \nu \) from \( \bar{Q} \) to \( \bar{P} \); in particular, we have \( \bar{\psi} \in \bar{R} \) and therefore we still have \( \psi(Q \cdot U) \subset R \cdot U \); since \( \mathcal{F} \) is divisible, there is \( \eta \in \mathcal{F}(R \cdot U, Q \cdot U) \) such that \( \bar{\eta} \bar{\psi}(\bar{u}) = \bar{\psi}(\bar{u}) = \bar{v}(\bar{u}) \) for any \( \bar{u} \in \bar{Q} \). If \( U \subset R \) then \( \bar{\sigma} \) belongs to \( \bar{\mathcal{F}}(\bar{R}) \) and therefore \( \bar{\phi} = i_R \circ \bar{\sigma} \circ \bar{\eta} \) belongs to \( \bar{\mathcal{F}}(\bar{P}, \bar{Q}) \).

Otherwise, denote by \( K \) the subgroup of elements of \( \mathcal{F}(R) \) acting trivially on \( \bar{R} \) and set \( T = N_{R}(R) \); since \( \mathcal{F}_{P}(R) \) is a Sylow \( p \)-subgroup of \( \mathcal{F}(R) \), \( \mathcal{F}_{T}(R) \) is a Sylow \( p \)-subgroup of \( K \) and, by the Frattini argument, we get

\[
\mathcal{F}(R) = K \cdot N_{\mathcal{F}(R)}(\mathcal{F}_{T}(R));
\]

but, since \( R \) is fully centralized in \( \mathcal{F} \), it follows from statement 2.11.1 that any element in \( N_{\mathcal{F}(R)}(\mathcal{F}_{T}(R)) \) can be extended to \( T \) and actually determines an element of the stabilizer \( \mathcal{F}(T)_{R} \) of \( R \) in \( \mathcal{F}(T) \), which clearly stabilizes \( U \cap T = N_{U}(R) \); consequently, \( N_{\mathcal{F}(R)}(\mathcal{F}_{T}(R)) \) normalizes \( \mathcal{F}_{U}(R) \) and therefore, setting \( S = N_{R \cdot U}(R) = R \cdot (U \cap T) \), we still get

\[
\mathcal{F}(R) = K \cdot N_{\mathcal{F}(R)}(\mathcal{F}_{S}(R));
\]

and, again by statement 2.11.1, any element in \( N_{\mathcal{F}(R)}(\mathcal{F}_{S}(R)) \) can be extended to \( S \) and actually determines an element of \( \mathcal{F}(S)_{R} \). Thus, there are \( \chi \in K \) and \( \tau \in \mathcal{F}(P, S) \) such that \( i_R^P \circ \sigma = \tau \circ i_R^S \circ \chi \); but, since \( U \not\subset R \), \( R \) is properly contained in \( S \); hence, \( \bar{\tau} \) belongs to \( \bar{\mathcal{F}}(\bar{P}, \bar{S}) \) and therefore \( \bar{\phi} = \bar{\tau} \circ \bar{i}_R^S \circ \bar{\eta} \) belongs to \( \bar{\mathcal{F}}(\bar{P}, \bar{Q}) \).  

\[
\square
\]

6.4. With the notation of Proposition 6.3, we call \( \bar{\mathcal{F}} \) the \( U \)-quotient of \( \mathcal{F} \) and denote it by \( \mathcal{F}/U \). On the other hand, if \( P' \) is a \( \mathcal{F} \)-stable subgroup of \( P \), we say that a divisible \( P' \)-subcategory \( \mathcal{F}' \) of \( \mathcal{F} \) is normal in \( \mathcal{F} \) if \( \mathcal{F}(P') \) stabilizes \( \mathcal{F}' \) and, for any subgroup \( Q \) of \( P' \) and any \( \varphi \in \mathcal{F}(P, Q) \), we have

\[
\mathcal{F}'(\varphi(Q)) = \varphi(\mathcal{F}'(Q));
\]

in particular, in this case \( \mathcal{F}'(Q) \) is a normal subgroup of \( \mathcal{F}(Q) \); note that

6.4.2 a subgroup \( Q \) of \( P' \) which is fully \( K \)-normalized in \( \mathcal{F} \), for a subgroup \( K \) of Aut\( (Q) \), is fully \( K \)-normalized in \( \mathcal{F}' \) too.

Indeed, for any \( \psi \in \mathcal{F}'(P', Q \cdot N_{P'}^{K}(Q)) \), setting \( Q' = \psi(Q) \) and \( K' = \psi K \), the homomorphism \( \psi' : Q' \to P \) mapping \( \psi(u) \) on \( u \), for any \( u \in Q \), composed with a suitable \( \chi \in K' \), can be extended to some \( \xi \in \mathcal{F}(P, Q' \cdot N_{P'}^{K'}(Q')) \) and, since \( P' \) is \( \mathcal{F} \)-stable, we get

\[
\xi((Q' \cdot N_{P'}^{K'}(Q')) \subset (Q \cdot N_{P}^{K}(Q)) \cap P' = Q \cdot N_{P}^{K}(Q)
\]

which forces the equality.
6.5. In this section, we prove the existence of a normal Frobenius $P$-subcategory $\mathcal{F}^a$ of $\mathcal{F}$—called the adjoint subcategory of $\mathcal{F}^a$—which is the analogous of $\mathcal{O}^p(G)$ in a finite group $G$ (but $(\mathcal{F}_G)^a$ need not to coincide with $\mathcal{F}_{\mathcal{O}^p(G)}$). We start by proving a general criterion on normality, which corresponds to the so-called “Frattini argument” in order to iterate the “Frattini argument,” we say that a subgroup $Q$ of $P$ is fully highnormalized in $\mathcal{F}$ whenever $N_0 = Q$ and $N_i = N_P(N_{i-1})$ for any $i \geq 1$ are fully normalized in $\mathcal{F}$; it is clear that, for any subgroup $Q$ of $P$ there is $\varphi \in \mathcal{F}(P, Q)$ such that $\varphi(Q)$ is fully highnormalized in $\mathcal{F}$.

Proposition 6.6. Let $P'$ be an $\mathcal{F}$-stable subgroup of $P$ and $\mathcal{F}'$ a divisible $P'$-subcategory of $\mathcal{F}$. Then, $\mathcal{F}'$ is normal in $\mathcal{F}$ if and only if $\mathcal{F}(P')$ stabilizes $\mathcal{F}'$ and, for any subgroup $Q$ of $P'$ we have

6.6.1 $\mathcal{F}(P', Q) = \mathcal{F}(P') \circ \mathcal{F}'(P', Q)$.

Proof. Firstly, assume that $\mathcal{F}'$ is normal in $\mathcal{F}$; we already know that $\mathcal{F}(P')$ stabilizes $\mathcal{F}'$. We argue by induction on $|P : Q|$ and may assume that $Q \neq P'$; let $\varphi \in \mathcal{F}(P', Q)$, set $Q' = \varphi(Q)$ and choose $\psi \in \mathcal{F}(P, Q')$ such that $Q'' = \psi(Q')$ is fully $\mathcal{F}'(Q'')$-normalized in $\mathcal{F}$ (cf. 2.7). Consequently, since $\mathcal{F}$ is a Frobenius $P$-category, setting $R = N_P(Q)$, it is easily checked that there exist $\zeta \in \mathcal{F}(P', R)$ and $\chi^1 \in \mathcal{F}'(Q)$ such that $\psi(\varphi(u)) = \zeta(\chi^1(u))$ for any $u \in Q$; then, by the induction hypothesis, there are $\zeta' \in \mathcal{F}'(P', R)$ and $\mu \in \mathcal{F}(P')$ such that $\zeta = \mu \circ \zeta'$, and therefore, denoting by $\varphi_* \in \mathcal{F}(Q', Q)$ the element determined by $\varphi$, we get

6.6.3 $\psi \circ \varphi_* = \iota_{P'}^P \circ \mu \circ (\zeta' \circ \iota_{Q'}^Q) \circ \chi'$.

Mutatis mutandis, we still get $\psi = \iota_{P'}^P \circ v \circ \xi^1$ for suitable $v \in \mathcal{F}(P')$ and $\xi^1 \in \mathcal{F}'(P', Q')$, so that $\xi' = \iota_{v^{-1}(Q''')}^P \circ \xi^1$ for some $\xi^1 \in \mathcal{F}'(v^{-1}(Q'''), Q')$ since $\mathcal{F}'$ is divisible, and therefore we have

6.6.4 $\mu \circ (\zeta' \circ \iota_{Q'}^{Q''}) \circ \chi' = v \circ \xi' \circ \varphi_* = v \circ \iota_{v^{-1}(Q''')}^P \circ \xi^1 \circ \varphi_*;

that is to say, setting $\lambda = v^{-1} \circ \mu$ and denoting by $\alpha' \in \mathcal{F}'(\mu^{-1}(Q''), Q)$ the isomorphism mapping $u \in Q$ on $\zeta'(\chi'(u))$ and by $\delta \in \mathcal{F}(v^{-1}(Q'''), \mu^{-1}(Q''))$ and $\varepsilon \in \mathcal{F}(Q', \mu^{-1}(Q'))$ the isomorphisms determined by $\lambda$, we finally obtain $\delta \circ \alpha' = \xi^1 \circ \varphi_*$ and therefore we get

6.6.5 $\varphi = \iota_Q^P \circ \varphi_* = \iota_Q^P \circ \xi^1 \circ \delta \circ \alpha' = \lambda \circ \iota_{\lambda^{-1}(Q')}^P \circ (\varepsilon^{-1} \circ \xi^{-1} \circ \delta) \circ \alpha'$, where $\varepsilon^{-1} \circ \xi^{-1} \circ \delta$ belongs to $\mathcal{F}'(\lambda^{-1}(Q'), \mu^{-1}(Q''))$ since $\mathcal{F}'$ is normal in $\mathcal{F}$.

The terminology comes from the Chevalley groups. All this part has been presented in the Chevalley Seminar in February 1992.
Conversely, if $F(P')$ stabilizes $F'$ and equality 6.6.1 holds then, for any subgroup $Q$ of $P'$ and any $\varphi \in F(P, Q)$, we already know that $\varphi = t^P_Q \circ \psi$ where $\psi \in F(P', Q)$ and therefore we have $\varphi = v \circ \psi'$ for suitable $v \in F(P')$ and $\psi' \in F'(P', Q)$; thus, we get

$$F'(\varphi(Q)) = F'(v(\psi'(Q))) = v(F'(\psi'(Q))) = \varphi'(F'(Q)). \quad \Box$$

**Proposition 6.7.** Let $P'$ be an $F$-stable subgroup of $P$ and $F'$ a normal Frobenius $P'$-subcategory of $F$. For any subgroup $Q$ of $P'$ fully highnormalized in $F'$, the restriction determines a group isomorphism

$$F(P')_Q/F'(P')_Q \cong F(Q)/F'(Q).$$

Moreover, if $Q'$ is a subgroup of $P'$ fully highnormalized in $F'$ and $\theta$: $Q \cong Q'$ is an $F$-isomorphism then there is $\sigma \in F(P')$ inducing with $\theta$ the commutative diagram

$$\begin{array}{ccc}
F(P')_Q/F'(P')_Q & \cong & F(Q)/F'(Q) \\
\downarrow & & \downarrow \\
F(P')_Q/F'(P')_Q & \cong & F(Q)/F'(Q').
\end{array}$$

**Proof.** First of all note that, by Proposition 6.6, we have $t^P_Q \circ \theta = \tau \circ \varphi'$, for suitable $\tau \in F(P')$ and $\varphi' \in F'(P', Q)$, and then $Q'' = \varphi'(Q) = \tau^{-1}(Q')$ still remains fully high-normalized in $F'$; so, replacing $Q'$ by $Q''$ and $\theta$ by the isomorphism $\theta'$: $Q \cong Q''$ determined by $\varphi'$, we may assume that $\theta$ is an $F'$-isomorphism.

Set $N_0 = Q$, $N'_0 = Q'$ and $N_i = N_{P'}(N_{i-1})$, $N'_i = N_{P'}(N'_{i-1})$ for any $i \geq 1$; since $F'$ is a Frobenius $P'$-category, for any $F'$-isomorphism $\theta_i$: $N_i \cong N'_i$ it follows from our hypothesis and from condition 2.9.2 that there are a $F'$-isomorphism $\theta_{i+1}$: $N_{i+1} \cong N'_{i+1}$ and an element $\chi_i \in F'(N_i)$ fulfilling $\theta_{i+1}(u) = \theta_i(\chi_i(u))$ for any $u \in N_i$. In the case where $Q' = Q$, we have $N'_i = N_i$ and this proves the surjectivity of the group homomorphism

$$F(N_{i+1})_{N_i}/F'(N_{i+1})_{N_i} \rightarrow F(N_i)/F'(N_i);$$

moreover, if $\theta_i \in F'(N_i)$ then $\theta_i \circ \chi_i$ can be extended to an element of $F'(N_{i+1})$ since it can be extended to one of $F(N_{i+1})$ (cf. 2.11.1), which proves the injectivity. In the general case $\theta_i$ and $\theta_{i+1}$ induce the commutative diagram

$$\begin{array}{ccc}
F(N_{i+1})_{N_i}/F'(N_{i+1})_{N_i} & \cong & F(N_i)/F'(N_i) \\
\downarrow & & \downarrow \\
F(N'_{i+1})_{N'_i}/F'(N'_{i+1})_{N'_i} & \cong & F(N'_i)/F'(N'_i).
\end{array}$$

Now, the proposition follows from the composition of isomorphisms 6.7.3 for all $i \geq 0$. \quad \Box
Proposition 6.8. If $\mathcal{F}'$ is a divisible $P$-category contained in $\mathcal{F}$ and fulfilling $X_{\mathcal{F}'}(R) \subset \mathcal{F}'(R)$ for any $\mathcal{F}$-essential subgroup $R$ of $P$ then, for any subgroup $Q$ of $P$, we have

6.8.1

$\mathcal{F}(P, Q) = \mathcal{F}(P) \circ \mathcal{F}'(P, Q)$.

In particular, for any subgroup $K$ of Aut($Q$), $Q$ is fully $K$-normalized in $\mathcal{F}'$ if and only if it is fully $K$-normalized in $\mathcal{F}$; moreover, $Q$ is $\mathcal{F}'$-selfcentralizing if and only if it is $\mathcal{F}$-selfcentralizing.

Proof. Clearly $\mathcal{F}(P) \circ \mathcal{F}'(P, Q) \subset \mathcal{F}(P, Q)$; conversely, we will prove that any $\varphi \in \mathcal{F}(P, Q)$ belongs to $\mathcal{F}(P) \circ \mathcal{F}'(P, Q)$ arguing by induction on $\ell(\varphi)$ introduced in 5.11; since the inclusion $\iota_Q^P : Q \to P$ belongs to $\mathcal{F}(P, Q)$, we may assume that $\ell(\varphi) \geq 1$ and therefore there are $\psi \in \mathcal{F}(P, Q)$, an $\mathcal{F}$-essential subgroup $R$ of $P$ fully normalized in $\mathcal{F}$, an element $\eta$ of $\mathcal{F}(R, Q)$ and a $p'$-element $\tau$ of $X_{\mathcal{F}}(R)$ (cf. 5.9.2) such that

6.8.2

$\ell(\psi) = \ell(\varphi) - 1$, $\varphi = \iota_R^P \circ \tau \circ \eta$ and $\psi = \iota_R^P \circ \eta$;

by the induction hypothesis, we get $\psi = \sigma \circ \psi'$ for suitable $\sigma \in \mathcal{F}(P)$ and $\psi' \in \mathcal{F}'(P, Q)$ and thus, setting $R' = \sigma^{-1}(R)$ and denoting by $\theta \in \mathcal{F}(R', R)$ the isomorphism determined by $\sigma^{-1}$, we have $\psi' = \iota_{R'}^P \circ \theta \circ \eta$ which implies that $\theta \circ \eta \in \mathcal{F}'(R', Q)$ since $\mathcal{F}'$ is divisible; but, denoting by $\tau'$ the image of $\tau$ in $X_{\mathcal{F}}(R')$ throughout $\theta$, we have $\sigma^{-1} \circ \varphi = \iota_{R'}^P \circ \tau' \circ (\theta \circ \eta)$ and therefore we get $\sigma^{-1} \circ \varphi \in \mathcal{F}'(P, Q)$.

If $Q$ is fully $K$-normalized in $\mathcal{F}$ and $\psi$ an element of $\mathcal{F}'(P, Q \cdot N^K_P(Q))$, we have $\psi(N^K_P(Q)) = N^K_P(\psi(Q))$ since $\psi \in \mathcal{F}(P, Q \cdot N^K_P(Q))$. Conversely, if $Q$ is fully $K$-normalized in $\mathcal{F}'$ and $\psi \in \mathcal{F}(P, Q \cdot N^K_P(Q))$, by the above argument, we get $\psi = \sigma \circ \psi'$ where $\sigma \in \mathcal{F}(P)$ and $\psi' \in \mathcal{F}'(P, Q \cdot N^K_P(Q))$, and therefore we still get

6.8.3

$\psi(N^K_P(Q)) = \sigma(N^K_P(\psi'(Q))) = N^K_P(\psi(Q))$.

Moreover, if $Q$ is fully centralized in both $\mathcal{F}$ and $\mathcal{F}'$, then $Q$ is either $\mathcal{F}$- or $\mathcal{F}'$-selfcentralizing if and only if $C_P(Q) \subset Q$. $\Box$

Corollary 6.9. If $\mathcal{F}'$ is a Frobenius $P$-category contained in $\mathcal{F}$ which, for any $\mathcal{F}$-essential subgroup $R$ of $P$, fulfills $X_{\mathcal{F}'}(R) \subset \mathcal{F}'(R)$ then any $\mathcal{F}$-essential subgroup $Q$ of $P$ is $\mathcal{F}'$-essential and we have

6.9.1

$X_{\mathcal{F}'}(Q) = X_{\mathcal{F}'}(Q)$.

Further, a $\mathcal{F}'$-essential subgroup $Q$ of $P$ fulfilling $\mathcal{F}'(Q) \triangleleft \mathcal{F}(Q)$ is $\mathcal{F}$-essential.

Proof. If $Q$ is $\mathcal{F}$-essential (cf. 5.5), it is $\mathcal{F}$-selfcentralizing (cf. 5.8.1), and we have (cf. 5.9.2)

6.9.2

$\mathcal{F}'(Q) = X_{\mathcal{F}'}(Q) \cdot \mathcal{F}'(Q)_{\tau(\psi)}$.
thus, $\mathcal{F}'(Q)_{\xi'(Q)}$ is a proper subgroup of $\mathcal{F}'(Q)$ fulfilling the corresponding condition 5.8.2; moreover, $Q$ is $\mathcal{F}'$-selfcentralizing by Proposition 6.8; hence, according to Theorem 5.8 and statement 5.9.2, $Q$ is $\mathcal{F}'$-essential too and we have $X_{\mathcal{F}'}(Q) = X_{\mathcal{F}}(Q)$.

Conversely, assume that $Q$ is $\mathcal{F}'$-essential and that $\mathcal{F}'(Q) \triangleleft \mathcal{F}(Q)$; then, $Q$ is $\mathcal{F}'$-selfcentralizing (cf. 5.8.1), so that it is $\mathcal{F}$-selfcentralizing by Proposition 6.8 again. Moreover, let $M'$ be a proper subgroup of $\mathcal{F}'(Q)$ fulfilling condition 5.8.2; then the Frattini argument proves that $N_{\mathcal{F}'(Q)}(M')$ is a proper subgroup of $\mathcal{F}(Q)$ and it is easily checked that it fulfills the corresponding condition 5.8.2. $\square$

**Proposition 6.10.** Let $H$ be a subgroup of $\text{Aut}(P)$ containing $\mathcal{F}(P)$ and stabilizing $\mathcal{F}$. If $\bar{H} = H/\mathcal{F}(P)$ is a $p'$-group then there is a unique Frobenius $P$-category $\mathcal{F}_{\bar{H}}$ fulfilling

$$\mathcal{F}_{\bar{H}}(P, Q) = H \circ \mathcal{F}(P, Q)$$

for any subgroup $Q$ of $P$.

**Proof.** For any pair of subgroups $Q$ and $R$ of $P$, we define $\mathcal{F}_{\bar{H}}(Q, R)$ as the set of group homomorphisms $\varphi : R \to Q$ such that $\iota_Q^P \circ \varphi$ belongs to $H \circ \mathcal{F}(P, R)$; thus, if $T$ is a subgroup of $P$ and $\psi \in \mathcal{F}_{\bar{H}}(R, T)$, we have $\iota_T^P \circ \psi = \chi^{-1} \circ \eta$ for suitable $\chi \in H$ and $\eta \in \mathcal{F}(P, T)$, so that $\iota_T^P \circ (\chi_R \circ \psi) = \eta$ where $\chi_R : R \to \chi(R)$ denote the group isomorphism determined by $\chi$, and therefore $\chi_R \circ \psi$ belongs to $\mathcal{F}(\chi(R), T)$; hence, since $\chi \circ \iota_Q^P \circ (\varphi \circ \psi) = \chi \circ (\iota_Q^P \circ \varphi) \circ (\chi_R)^{-1} \circ (\chi_R \circ \psi)$

and $\chi \circ (\iota_Q^P \circ \varphi) \circ (\chi_R)^{-1}$ belongs to $H \circ \mathcal{F}(P, \chi(R))$, $\iota_Q^P \circ (\varphi \circ \psi)$ belongs to $H \circ \mathcal{F}(P, T)$; that is to say, $\mathcal{F}_{\bar{H}}$ is a $P$-category and we claim that it is a Frobenius $P$-category too.

Indeed, if $\varphi \circ \theta \in \mathcal{F}_{\bar{H}}(Q, T)$ for some group homomorphism $\theta : T \to R$ then, for some $\zeta \in H$ and $v \in \mathcal{F}(P, T)$, we have $\iota_T^P \circ \varphi \circ \theta = \zeta^{-1} \circ v$; moreover, since we have $\iota_Q^P \circ \varphi = \xi \circ \mu$ for suitable $\xi \in H$ and $\mu \in \mathcal{F}(P, R)$, denoting by $\rho : R \to (\zeta \circ \xi)(R)$ the group isomorphism determined by $\zeta \circ \xi$, we get

$$(\zeta \circ \xi) \circ \mu \circ \rho^{-1} \circ (\rho \circ \theta) = v$$

and therefore, since $\mathcal{F}$ is divisible, $\rho \circ \theta$ belongs to $\mathcal{F}((\zeta \circ \xi)(R), T)$, so that the group homomorphism $\iota_{(\zeta \circ \xi)(R)}^P \circ \rho \circ \theta = (\zeta \circ \xi) \circ \iota_R^P \circ \theta$ belongs to $\mathcal{F}(P, T)$. Consequently, $\mathcal{F}_{\bar{H}}$ is divisible too.

Since $\mathcal{F}_{\bar{H}}(P) = H$, $\mathcal{F}_{\bar{H}}$ fulfills condition 2.9.1. On the other hand, let $Q$ be a subgroup of $P$, $K$ a subgroup of $\text{Aut}(Q)$ and $\varphi$ an element of $\mathcal{F}_{\bar{H}}(P, Q)$ such that $\varphi(Q)$ is fully $\psi_K$-normalized in $\mathcal{F}_{\bar{H}}$; thus, there are $\eta \in H$ and $\psi \in \mathcal{F}(P, Q)$ such that $\varphi = \eta \circ \psi$, and it is quite clear that $\psi(Q)$ is fully $\psi_K$-normalized in $\mathcal{F}_{\bar{H}}$, and a fortiori in $\mathcal{F}$ (since $\mathcal{F}_{\bar{H}}$ contains $\mathcal{F}$); consequently, since $\mathcal{F}$ is a Frobenius $P$-category, there are $\zeta \in \mathcal{F}(P, Q \cdot N_P^K(Q))$ and $\chi \in K$ such that $\zeta(u) = \psi(\chi(u))$ for any $u \in Q$; hence, setting $\xi = \eta \circ \zeta$, we get $\xi(u) = \varphi(\chi(u))$ for any $u \in Q$. $\square$
Theorem 6.11. The set of Frobenius $P$-categories $\hat{F}$ contained in $F$ which fulfill $X_F(Q) \subset \hat{F}(Q)$ for any $F$-essential subgroup $Q$ of $P$ has a smallest element $F^a$. Moreover, if $P'$ is a $F$-stable subgroup of $P$ and $F'$ is a normal Frobenius $P'$-subcategory of $F$ then $F^a$ contains $F'^a$.

Proof. First of all we will prove that if $\hat{F}$ and $\hat{F}'$ are Frobenius $P$- and $P'$-categories which are contained in $F$ and $F'$, respectively, and fulfill

$$X_F(Q) \subset \hat{F}(Q) \quad \text{and} \quad X_{F'}(Q') \subset \hat{F}(Q') \cap \hat{F}'(Q')$$

for any $F$-essential subgroup $Q$ of $P$ and any $F'$-essential subgroup $Q'$ of $P'$, then the smallest element of the set of divisible $P'$-categories $F''$ which are contained in $F$ and, for any $F'$-selfcentralizing subgroup $Q$ of $P'$, fulfill

$$F''(P', Q) = \hat{F}(P', Q) \cap \hat{F}'(P', Q)$$

is a Frobenius $P'$-category.

Denote by $F''$ this smallest element; we will prove that $F''$ fulfills the four conditions of Theorem 3.8. By the very definition of $F''$, it fulfills condition 3.8.3. Since $F_{P'}(P')$ is a Sylow $p$-subgroup of $F'(P')$, a fortiori it is a Sylow $p$-subgroup of $F''(P')$. Moreover, by Proposition 6.8, a selfcentralizing subgroup $Q'$ of $P'$ fully normalized in $F''$ is fully normalized in $F'$ too and therefore it is $F''$-selfcentralizing, so that it is $F''$-selfcentralizing.

If $Q$ is a $F''$-selfcentralizing subgroup of $P'$ then it is also $F'$- and $\hat{F}'$-selfcentralizing by Proposition 6.8; let $R$ be a subgroup of $N_{P'}(Q)$ containing $Q$, and $\varphi' \in F''(P', Q)$ such that $\varphi' F_R(Q) \subset F_{P'}(Q')$, where $Q' = \varphi'(Q)$; then, it follows from statement 2.11.1 that there is $\psi' \in F'(P', R)$ extending $\varphi'$. Moreover, setting $R' = \psi'(R)$, let $\theta$ be an element of $F(P, R')$ such that $Q'' = \theta(\hat{Q})$ is $\theta F_{R'}(Q')$-fully normalized in $\hat{F}$; according to condition 2.9.2, there are $\zeta \in F(P, R)$ and $w \in R$ fulfilling $\zeta(uw) = \theta(\varphi'(u))$ for any $u \in Q$; but, since $P'$ is $F'$-stable, we have

$$\theta = \iota_P^P \circ \omega \quad \text{and} \quad \zeta \circ \kappa_R(w)^{-1} = \iota_P^P \circ \eta$$

for suitable $\omega \in \hat{F}(P', R)$ and $\eta \in \hat{F}(P', R)$; thus, $\eta$ and the group homomorphism $R \to P'$ mapping $v \in R$ on $\omega(\varphi'(v))$ coincide over $Q$ and therefore, since $Q$ is $\hat{F}'$-selfcentralizing, we have $\eta(R) = R'' = \omega(\varphi'(R))$ and there is $\sigma'' \in F(R'')$ fulfilling $\sigma''(\eta(v)) = \omega(\varphi'(v))$ for any $v \in R$, and $\sigma''(u) = u$ for any $u \in Q$.

On the one hand, by Thompson’s Lemma (cf. Theorem 3.4 in [6, Chapter 5]), the group $K''$ of automorphisms of $R''$ acting trivially on $Q''$ is a $p$-group. Moreover, it follows from Proposition 6.6 that, since $F'$ is normal in $F$, $Q''$ is $F'$-selfcentralizing too and therefore we have $P' \cap (C_P(Q'') \cdot R'') = R''$; in particular, we get $N_{K''}^{K''}(R'') = C_P(Q'')$ and therefore, since $Q''$ is fully centralized in $F$ by Propositions 2.12 and 6.8, $R''$ is fully $K''$-normalized in $F'$; then, by Proposition 2.12 again, $K'' \cap F_{P''}(R'') = K'' \cap F(R'')$ and thus we obtain $z \in C_P(Q'')$ fulfilling $\sigma''(\eta(v)) = \eta(v)^2$ for any $v \in R$. Hence, $\sigma'' \circ \eta$ also belongs to $\hat{F}(P', R)$ and therefore, since $\hat{F}$ is divisible, the group isomorphism $R \cong R'$
determined by $\psi'$ belongs to $\hat{\mathcal{F}}(R', R)$ too, so that $\psi'$ belongs to $\mathcal{F}''(P', R)$ (cf. 6.8.1). Consequently, $\mathcal{F}''$ is a Frobenius $P'$-category.

Now, applying this result to the case $P' = P$ and $\mathcal{F}' = \mathcal{F}$, we get the smallest element $\mathcal{F}^a$ in the set above. In the general case note that, for any subgroup $Q$ of $P'$, we have $\mathcal{O}^p(\mathcal{F}'(Q)) \subset \mathcal{O}^p(\mathcal{F}(Q))$; according to inclusion 6.12.1 below, we can apply the above result to the Frobenius $P$- and $P'$-categories $\hat{\mathcal{F}} = \mathcal{F}^a$ and $\hat{\mathcal{F}}' = \mathcal{F}'^a$; then, the minimality of $\mathcal{F}^a$ forces $\mathcal{F}^a = \mathcal{F}^a \cap \mathcal{F}'^a$, so that $\mathcal{F}^a$ contains $\mathcal{F}'^a$. □

6.12. It follows from Corollary 6.3 that $(\mathcal{F}^a)^a = \mathcal{F}^a$. Moreover, from Propositions 6.6 and 6.8 it is not difficult to prove that if $\mathcal{F}'$ is a normal Frobenius $P'$-category of $\mathcal{F}$ then $\mathcal{F}'^a$ is normal in $\mathcal{F}$ too. In particular, for any subgroup $Q$ of $P$, $\mathcal{F}^a(Q)$ is a normal subgroup of $\mathcal{F}(Q)$; but, choosing $\varphi \in \mathcal{F}(P, Q)$ such that $\varphi(Q)$ is fully normalized in $\mathcal{F}$, it follows from Propositions 6.12 and 6.8 that $|\mathcal{F}^a(Q)|_p = |\mathcal{F}(Q)|_p$; consequently, we get

6.12.1 \[ \mathcal{O}^p(\mathcal{F}'(Q)) \subset \mathcal{F}^a(Q). \]

Furthermore, if $U$ is a $\mathcal{F}$-stable subgroup of $P$ then, setting $\tilde{P} = P/U$ and denoting by $\tilde{\mathcal{F}}$ the $U$-quotient of $\mathcal{F}$, the image $\tilde{\mathcal{F}}^a$ of $\mathcal{F}^a$ in $\tilde{\mathcal{F}}$ contains $\mathcal{F}$; indeed, on one hand it is clear that $U$ is also $\mathcal{F}^a$-stable and this image is just the $U$-quotient of $\mathcal{F}^a$; on the other hand, for any subgroup $Q$ of $P$, $\mathcal{O}^p(\mathcal{F}'(Q))$ maps onto $\mathcal{O}^p(\tilde{\mathcal{F}}(\tilde{Q}))$, where $\tilde{Q} = Q \cdot U/U$, and therefore, according to inclusion 6.12.1, $\tilde{\mathcal{F}}^a(\tilde{Q})$ contains $X_{\tilde{\mathcal{F}}}(\tilde{Q})$ whenever $\tilde{Q}$ is $\tilde{\mathcal{F}}$-essential.

7. The hyperfocal subcategory of a Frobenius category

7.1. Let $P$ be a finite $p$-group and $\mathcal{F}$ a Frobenius $P$-category; denoting by $i_{\mathcal{F}}$ the inclusion functor from $\mathcal{F}$ to the category $\mathfrak{S}$ of groups, we call $\mathcal{F}$-focal subgroup of $P$ the kernel $F_{\mathcal{F}}$ of the surjective canonical homomorphism

7.1.1 \[ P \to \lim\limits_{\rightarrow} i_{\mathcal{F}}; \]

actually, it follows easily from Corollary 5.10 that $F_{\mathcal{F}}$ is generated by the union of the sets $\{u^{-1}\sigma(u)\}_{u \in Q}$ where $Q$ runs over the set of subgroups of $P$ and $\sigma \in \mathcal{F}(Q)$. More generally, the $\mathcal{F}$-hyperfocal subgroup is the subgroup $H_{\mathcal{F}}$ of $P$ generated by the union of the sets $\{u^{-1}\sigma(u)\}_{u \in Q}$ where $Q$ runs over the set of subgroups of $P$ and $\sigma$ over the set of $p'$-elements of $\mathcal{F}(Q)$; if $\mathcal{F}$ is the Frobenius category associated with a finite group $G$ then $H_{\mathcal{F}} = P \cap \mathcal{O}^p(G)$ as it is explained, for instance, in the Introduction of [11]; clearly, it follows from Proposition 2.12 that

7.1.2 $H_{\mathcal{F}} = \{1\}$ if and only if $\mathcal{F} = \mathcal{F}_p$.

It is clear that $\mathcal{F}(P)$ stabilizes $H_{\mathcal{F}}$ and that $\mathcal{O}^p(\mathcal{F}(P))$ acts trivially on the quotient $P/H_{\mathcal{F}}$; more precisely, we have the following result.
Lemma 7.2. For any subgroup $Q$ of $P$ and any $\varphi \in \mathcal{F}(P, Q)$, there is $w \in P$ such that, for any $u \in Q$, we have

$$7.2.1 \quad \varphi(u) \equiv uw \mod H_F.$$ 

In particular, we get $F_F = H_F \cdot [P, P]$. 

Proof. We argue by induction on the length $\ell$ of $\varphi$ (cf. 5.11); since $O_p(\mathcal{F}(P))$ acts trivially on the quotient $P/H_F$, we may assume that $\ell \geq 1$ and therefore that we have $\varphi = \iota_P^R \circ \sigma \circ v$, where $R$ is a $F$-essential subgroup of $P$, $\sigma$ is a $p'$-element of $\mathcal{F}(R)$ and $v$ is an element of $\mathcal{F}(R, Q)$ such that $\iota_P^R \circ v$ has length $\ell - 1$ (cf. 5.11); consequently, there is $w \in P$ such that, for any $u \in Q$, we get

$$7.2.2 \quad \varphi(u) \equiv \sigma(v(u)) \equiv v(u) \equiv uw \mod H_F.$$ 

7.3. Thus, any normal subgroup $P'$ of $P$ containing the hyperfocal subgroup $H_F$ is $\mathcal{F}$-stable and the corresponding $P'$-quotient $\bar{F}$ clearly is the Frobenius category associated with the group $\bar{P} = P/P'$. We will prove the existence of a Frobenius $H_F$-subcategory $\mathcal{F}^h$ of $\mathcal{F}$—called the hyperfocal subcategory of $\mathcal{F}$—which will correspond to $O^p(G)$ whenever $\mathcal{F}$ is the Frobenius category associated with a finite group $G$. First of all note that

7.3.1 for any subgroup $Q$ of $P$ fully normalized in $\mathcal{F}$, $\mathcal{F}_{HF}(Q)$ contains a Sylow $p$-subgroup of $O^p(\mathcal{F}(Q))$. 

Indeed, we already know that $\mathcal{F}_P(Q)$ is a Sylow $p$-subgroup of $\mathcal{F}(Q)$ (cf. Proposition 2.12) and that the intersection $\mathcal{F}_P(Q) \cap O^p(\mathcal{F}(Q))$ is generated by the union of the sets $[\mathcal{F}_R(Q), \sigma]$ where $R$ runs over the set of subgroups of $N_P(Q)$ and $\sigma$ over the set of $p'$-elements of $N_{\mathcal{F}(Q)}(\mathcal{F}_R(Q))$ (cf. 7.1); but such a $\sigma$ can be lifted to a $p'$-element $\tau$ of $\mathcal{F}(R)_Q$ (cf. 2.11.1) and then $[\mathcal{F}_R(Q), \sigma]$ is the image of the set $\{v^{-1} \tau^{-1}(v)\}_{v \in R}$ which is contained in $H_F$. 

Theorem 7.4. Let $P'$ be a normal subgroup of $P$ containing $H_F$. Then, we have a normal Frobenius $P'$-subcategory $\mathcal{F}'$ of $\mathcal{F}$ such that

$$7.4.1 \quad \mathcal{F}'(Q') = \mathcal{F}_{P'}(Q') \circ O^p(\mathcal{F}(Q'))$$

for any subgroup $Q'$ of $P'$ fully normalized in $\mathcal{F}$. 

Proof. For any subgroup $Q$ of $P'$, choose $\eta \in \mathcal{F}(P, Q)$ such that $Q' = \eta(Q)$ is fully normalized in $\mathcal{F}$ and denote by $\mathcal{F}'(Q)$ the subgroup of $\mathcal{F}(Q)$ fulfilling

$$7.4.2 \quad \eta \mathcal{F}'(Q) = \mathcal{F}_{P'}(Q') \circ O^p(\mathcal{F}(Q'));$$

note that, since $\mathcal{F}_{HF}(Q')$ contains a Sylow $p$-subgroup of $O^p(\mathcal{F}(Q'))$, $\mathcal{F}_{P'}(Q')$ is a Sylow $p$-subgroup of $\mathcal{F}'(Q')$, which will guarantee condition 2.9.1 in 7.4.9 below, and we claim
that \( \mathcal{F}'(Q) \) does not depend on the choice of \( \eta \). Indeed, for another choice \( \eta' \in \mathcal{F}(P,Q) \) of \( \eta \), setting \( Q'' = \eta'(Q) \) and denoting by \( \sigma : Q' \cong Q'' \) the corresponding group isomorphism, we obviously have

\[
7.4.3 \quad \sigma \mathcal{O}^p(\mathcal{F}(Q')) = \mathcal{O}^p(\mathcal{F}(Q''))
\]

and there is \( \zeta' \in \mathcal{F}(P,\mathcal{N}_P(Q')) \) extending \( \sigma \circ \chi' \) for a suitable \( \chi' \in \mathcal{F}(Q') \); thus, since \( P' \) is \( \mathcal{F} \)-stable, we get \( \zeta'(\mathcal{N}_{P'}(Q')) = \mathcal{N}_{P'}(Q'') \) and, since \( \mathcal{F}'(Q') \) is normal in \( \mathcal{F}(Q') \), we still get

\[
7.4.4 \quad \sigma (\mathcal{F}_{P'}(Q') \circ \mathcal{O}^p(\mathcal{F}(Q'))) = \mathcal{F}_{P'}(Q'') \circ \mathcal{O}^p(\mathcal{F}(Q'')).
\]

Consequently, it is easily checked that, for any \( \varphi \in \mathcal{F}(P,Q) \), we have

\[
7.4.5 \quad \varphi \mathcal{F}'(Q) = \mathcal{F}'(\varphi(Q)).
\]

Moreover, if \( R \) is a subgroup of \( Q \) then the action over \( R \) of the stabilizer \( \mathcal{F}'(Q)_R \) of \( R \) is contained in \( \mathcal{F}'(R) \); indeed, we may assume that \( Q \) is fully normalized in \( \mathcal{F} \) and then \( \mathcal{F}_{P'}(Q) \) is a Sylow \( p \)-subgroup of \( \mathcal{F}'(Q) \); thus, up to \( \mathcal{F}'(Q) \)-conjugation, we may assume that \( \mathcal{F}_{P}(Q)_R \) is a Sylow \( p \)-subgroup of \( \mathcal{F}(Q)_R \) and then \( \mathcal{F}_{P'}(Q)_R \) is one of \( \mathcal{F}'(Q)_R \), so that we have \( \mathcal{F}'(Q)_R = \mathcal{F}_{P'}(Q)_R \circ \mathcal{O}^p(\mathcal{F}'(Q)_R) \); but, the actions over \( R \) of \( \mathcal{F}_{P'}(Q)_R \) and of \( \mathcal{O}^p(\mathcal{F}'(Q)_R) \) are contained in \( \mathcal{F}_{P'}(R) \) and in \( \mathcal{O}^p(\mathcal{F}(R)) \), so in \( \mathcal{F}'(R) \).

Let \( \hat{\mathcal{F}}' \) be the minimal divisible \( P' \)-category such that \( \mathcal{F}'(Q) \subset \hat{\mathcal{F}}'(Q) \) for any subgroup \( Q \) of \( P' \) (cf. 2.3), so that \( \mathcal{F}'(P') = \hat{\mathcal{F}}'(P') \); it is quite clear that \( \hat{\mathcal{F}}' \) is a normal \( P' \)-subcategory of \( \mathcal{F} \). For any pair \( Q \) and \( Q' \) of \( \hat{\mathcal{F}}' \)-isomorphic subgroups of \( P' \), consider the set \( \mathcal{F}'(Q) \) of \( \varphi \in \hat{\mathcal{F}}'(Q',Q) \) such that there is a subgroup \( U \) of \( P' \) \( \hat{\mathcal{F}}' \)-isomorphic to \( Q \) and \( Q' \), which is fully \( \mathcal{F}'(U) \)-normalized in \( \hat{\mathcal{F}}' \) and admits \( \hat{\mathcal{F}}' \)-morphisms

\[
7.4.6 \quad \lambda : N_{P'}(Q) \to N_{P'}(U) \quad \text{and} \quad \lambda' : N_{P'}(Q') \to N_{P'}(U)
\]

and \( \sigma \in \mathcal{F}'(U) \) fulfilling

\[
7.4.7 \quad \lambda(Q) = U = \lambda'(Q') \quad \text{and} \quad \lambda'(\varphi(u)) = \sigma(\lambda(u))
\]

for any \( u \in Q \); note that we have \( \varphi^{-1} \in \mathcal{F}'(Q, Q') \). Moreover, we have

\[
7.4.8 \quad \mathcal{F}'(Q) \subset \mathcal{F}'(Q, Q) \quad \text{and} \quad \mathcal{F}_{P'}(Q',Q) \subset \mathcal{F}'(Q',Q);
\]

indeed, choosing \( \lambda \in \hat{\mathcal{F}}'(P', N_{P'}(Q)) \) such that \( U = \lambda(Q) \) is fully \( \mathcal{F}'(U) \)-normalized in \( \hat{\mathcal{F}}' \) (cf. 2.7), if \( \varphi \in \mathcal{F}'(Q) \) then it suffices to choose \( \lambda' = \lambda \) and \( \sigma = \lambda' \varphi \), while if \( \varphi \) is the conjugation by some \( u \in P' \) then we consider \( \sigma = \text{id}_U \) and \( \lambda' \) equal to the composition of \( \lambda \) with the isomorphism \( N_{P'}(Q') \cong N_{P'}(Q) \) induced by \( u^{-1} \).

More generally, for any pair of subgroups \( Q \) and \( R \) of \( P' \) we set
7.4.14 \[ F' (Q, R) = \bigcup_{\varphi \in \hat{F}'(Q, R)} t^Q_{\varphi(R)} \circ \hat{F}'(\varphi(R), R), \]

where \( t^Q_{\varphi(R)} \) denotes the corresponding inclusion map, and, arguing by induction on \(|P' : R|\), we will prove that \( F' \) is \( P' \)-category fulfilling conditions 2.3.1 and 2.9.2, that it coincides with \( \hat{F}' \) and that, for any subgroup \( Q \) of \( P' \), it fulfills \( F'(Q, Q) = F'(Q) \). First of all note that, according to 2.3, at each step \( p^n \) of the induction there is a divisible \( P' \)-category \( F'' \) which is contained in \( \hat{F}' \), fulfills \( F'(Q, Q) \subset F''(Q, Q) \) for any subgroup \( Q \) of \( P \) and coincides with \( F' \) over all the pairs of subgroups \( R \) of \( P' \) fulfilling \(|P' : R| \leq p^n\), so that we have \( F'' = \hat{F}' \) by minimality. Moreover, if \( Q \) is fully \( F'(Q) \)-normalized in \( \hat{F}' \), we have \(|P' : Q| = p^{n+1}\), \( \varphi \) is an element of \( F'(Q, Q) \) and we choose \( U, \lambda, \lambda' \) and \( \sigma \) as above then, denoting by \( \lambda^* \) the inverse of the isomorphism \( N_{P'}(Q) \cong \lambda(N_{P'}(Q)) = N_{P'}(U) \) induced by \( \lambda \), we have

7.4.10 \[ \lambda^* \circ \lambda' \in \hat{F}'(N_{P'}(Q)) \quad \text{and} \quad (\lambda^* \circ \lambda')(Q) = Q; \]

but, since \( Q \neq N_{P'}(Q) \), we have \( \hat{F}'(N_{P'}(Q)) = F'(N_{P'}(Q)) \) by the induction hypothesis; hence, the restriction of \( \lambda^* \circ \lambda' \) to \( Q \) belongs to \( \hat{F}'(Q) \) and, since for any \( u \in Q \) we have (cf. 7.4.7)

7.4.11 \[ (\lambda^* \circ \lambda')(\varphi(u)) = \lambda^*(\sigma(\lambda(u))), \]

\( \varphi \) belongs to \( \hat{F}'(Q) \). In conclusion, since \( \hat{F}' \) is a normal \( P' \)-subcategory of \( F \), for any subgroup \( Q \) of \( P' \) having index \( p^{n+1} \) we get

7.4.12 \[ \hat{F}'(Q) = \hat{F}'(Q, Q). \]

With the notation above, if \( Q'' \) is a third subgroup of \( P' \) \( \hat{F}' \)-isomorphic to \( Q \) and \( Q' \), and \( \psi \) is an element of \( \hat{F}'(Q'', Q') \) obtained from a corresponding choice \( U', \mu, \mu' \) and \( \sigma' \), then, denoting by \( \lambda'^* \) the inverse of the isomorphism \( N_{P'}(Q') \cong \lambda'(N_{P'}(Q')) \) induced by \( \lambda' \) and considering the composition

7.4.13 \[ \mu \circ \lambda'^* \in \hat{F}'(\mu(N_{P'}(Q')), \lambda'(N_{P'}(Q'))), \]

it follows from the induction hypothesis and from Lemma 3.9 that there are \( \zeta \in \hat{F}'(N_{P'}(U'), N_{P'}(U)) \) and \( \chi \in \hat{F}'(U) \) fulfilling \( \zeta(u) = (\mu \circ \lambda'^*)(\chi(u)) \); then, considering \( U', \zeta \circ \lambda, \mu' \) and \( \sigma' \circ \xi(\chi^{-1} \circ \sigma), \) for any \( u \in Q \) we have

7.4.14 \[ \mu'( (\psi \circ \varphi)(u) ) = \sigma'( (\mu \circ \lambda'^*)(\lambda'(\varphi(u))) ) = \sigma'( (\xi(\chi^{-1} \circ \sigma)(\lambda(u))) ) = (\sigma' \circ \xi(\chi^{-1} \circ \sigma))(\zeta \circ \lambda)(u) \]

which proves that \( \psi \circ \varphi \) belongs to \( \hat{F}'(Q'', Q) \).
Let $R$ and $R'$ be subgroups of $P'$ respectively containing $Q$ and $Q'$, and assume that
\[ \psi \in \mathcal{F}(R, R) \text{ fulfills } \psi(Q) = Q'; \]
we claim that the $\mathcal{F}'$-isomorphism $\varphi: Q \cong Q'$ induced by $\psi$ belongs to $\mathcal{F}'(Q', Q)$. Arguing by induction on $|R: Q|$, we may assume that $|R: Q| \neq 1$ and that both $Q$ is normal in $R$ and $Q'$ is normal in $R'$; we already know that there is $\xi \in \mathcal{F}'(P', R)$ and $\xi' \in \mathcal{F}'(P', R')$ such that $V = \xi(Q)$ and $V' = \xi'(Q')$ are fully $\mathcal{F}'(V)$- and $\mathcal{F}'(V')$-normalized in $\mathcal{F}'$, respectively (cf. 2.7). Then, it follows from the induction hypothesis and from Lemma 3.9 that there are $\mathcal{F}$-morphisms
\[ \mu: N_{P'}(Q) \to N_{P'}(V) \quad \text{and} \quad \mu': N_{P'}(Q') \to N_{P'}(V') \]
such that $\mu(Q) = V$ and $\mu'(Q') = V'$; once again, considering the $\mathcal{F}'$-isomorphism $\rho: \mu(R) \cong \mu'(R')$ such that $\rho(\mu(v)) = \mu'(\psi(v))$ for any $v \in R$, since $\rho(V) = V'$, it follows from the induction hypothesis and from Lemma 3.9 that there are $\xi \in \mathcal{F}'(N_{P'}(V'), N_{P'}(V))$ and $\theta \in \mathcal{F}'(V)$ fulfilling $\xi(v) = \rho(\theta(v))$ for any $v \in V$, and therefore for any $u \in Q$ we get
\[ \xi(\mu(u)) = \rho(\theta(\mu(u))) = \rho(\theta(\mu'(\psi(u)))); \]
now, considering the choice $V', \xi \circ \mu, \mu'$ and $\rho \theta^{-1}$, we see that $\varphi$ belongs to $\mathcal{F}'(Q', Q)$.

Consequently, if $R$ and $T$ are subgroups of $P'$, $\varphi$ is an element of $\mathcal{F}'(T, R)$ and $\psi$ an element of $\mathcal{F}'(T, R)$ then $\psi \circ \varphi$ belongs to $\mathcal{F}'(T, Q)$; indeed, setting $Q' = \psi(Q)$ and $Q'' = \psi(Q')$, and denoting by $\varphi_*$: $Q \cong Q'$ and by $\psi_*$: $Q' \cong Q''$ the corresponding $\mathcal{F}'$-isomorphisms, according to our definition, $\varphi_*$ belongs to $\mathcal{F}'(Q', Q)$ and we already know that $\psi_*$ and $\psi_* \circ \varphi_*$ respectively belong to $\mathcal{F}'(Q'', Q')$ and to $\mathcal{F}'(Q'', Q)$.

It remains to prove that $\mathcal{F}'$ fulfills condition 2.9.2; note that we already have proved that $\mathcal{F}'(Q) = \mathcal{F}'(Q, Q) = \mathcal{F}'(Q)$. Let $K$ be a subgroup of $\text{Aut}(Q)$ and $\varphi$ an element of $\mathcal{F}'(P', Q)$ such that $Q' = \varphi(Q)$ is fully $\psi\varphi(K)$-normalized in $\mathcal{F}'$; actually, we may assume that $K \subset \mathcal{F}'(Q)$ and, setting $R' = Q' \cdot N_{P'}(Q)$, we choose $\psi' \in \mathcal{F}(P, R')$ such that $Q'' = \psi'(Q')$ is both fully $\psi'\varphi(K)$-normalized and fully $\mathcal{F}'(Q'')$-normalized in $\mathcal{F}$ (cf. 2.7); then, since $\mathcal{F}$ is a Frobenius $P$-category, there are $\zeta \in \mathcal{F}(P, Q \cdot N_{P'}^{K}(Q))$ and $\chi \in K$ fulfilling $\zeta(u) = \psi'(\varphi(\chi(u)))$ for any $u \in Q$; but, since $P'$ is $\mathcal{F}$-stable, we have
\[ \zeta(Q \cdot N_{P'}^{K}(Q)) \subset P' \supset \psi'(R') \]
and, by Proposition 6.6, there are $\xi \in \mathcal{F}((P', Q \cdot N_{P'}^{K}(Q))$, $\eta' \in \mathcal{F}'(P', R')$ and $v, w \in P$ such that, for any $u \in Q$, we have
\[ \xi(u)^w = \eta'(\varphi(\chi(u)))^v. \]

Consequently, since $\xi$, $\eta'$, $\varphi$ and $\chi$ are $\mathcal{F}'$-morphisms, the action by conjugation of $wv^{-1}$ on $v^{Q''}$ belongs to $\mathcal{F}'(v^{Q''})$, that is to say, we have a subgroup $U$ of $P'$ which is $\mathcal{F}'$-isomorphic to both $w^{Q''}$ and $v^{Q''}$, is fully normalized in $\mathcal{F}'$ and which admits $\mathcal{F}$-morphisms.
7.4.19 $\lambda : N_p^{(vQ''')} \to N_p^{(U)} \quad \text{and} \quad \lambda' : N_p^{(wQ''')} \to N_p^{(U)}$

and $\sigma \in \mathcal{F}'(U)$ fulfilling

7.4.20 $\lambda(vQ''') = U = \lambda'(wQ''') \quad \text{and} \quad \lambda'(wu''w^{-1}) = \sigma(\lambda(vu''v^{-1}))$

for any $u'' \in Q'''$; actually, according to 2.7 and Proposition 6.6, we can modify our choice of $U$, $\lambda$, $\lambda'$ and $\sigma$ in such a way that $U$ is fully normalized in $\mathcal{F}$ too; moreover, since $Q'''$ is fully $\mathcal{F}'(Q'')$-normalized in $\mathcal{F}$ and $P'$ is $\mathcal{F}$-stable, we get

7.4.21 $\lambda(N_p^{(vQ''')} = N_p^{(U)} = \lambda'(N_p^{(wQ'''}))$

and therefore $\sigma$ normalizes $\mathcal{F}_p'(U)$.

On the other hand, since $\mathcal{F}$ is a Frobenius $P$-category, $U$ is fully centralized in $\mathcal{F}$ (cf. Proposition 2.12) and it follows from statement 2.11.1 that the following restriction homomorphism is surjective

7.4.22 $\mathcal{F}(N_p^{(U)}) \to N_{\mathcal{F}(U)}(\mathcal{F}_p'(U))$

and therefore it maps $\mathcal{O}_p^p(\mathcal{F}(N_p^{(U)})_U)$ onto $\mathcal{O}_p^p(N_{\mathcal{F}(U)}(\mathcal{F}_p'(U)))$ which is contained in $N_{\mathcal{F}(U)}(\mathcal{F}_p'(U))$ and contains $\mathcal{O}_p^p(N_{\mathcal{F}(U)}(\mathcal{F}_p'(U)))$; since $\mathcal{F}_p'(U)$ is a Sylow $p$-subgroup of $\mathcal{F}'(U)$ and $\mathcal{O}_p^p(\mathcal{F}(N_p^{(U)}))_U$ is contained in $\mathcal{F}'(N_p^{(U)})_U$, it follows that the following restriction homomorphism is still surjective

7.4.23 $\mathcal{F}'(N_p^{(U)})_U \to N_{\mathcal{F}'(U)}(\mathcal{F}_p'(U))$

and therefore $\sigma$ can be lifted to $\theta \in \mathcal{F}'(N_p^{(U)})_U$.

At this point, from equalities 7.4.18 and 7.4.20, for any $u \in Q$ we get

7.4.24 $\lambda'(\xi(u)) = (\theta \circ \lambda)(\eta'(\varphi(\chi(u))))$,

so that, denoting by $\lambda^*$ and $\eta'^*$ the inverses of the corresponding isomorphisms and considering the composition $\beta$ of the $\mathcal{F}'$-morphisms

7.4.25 $Q \cdot N_p^K(Q) \xrightarrow{\xi} N_p^{(wQ''')} \xrightarrow{\lambda'} N_p^{(U)} \xrightarrow{\theta^{-1}} N_p^{(U)} \xrightarrow{\lambda^*} N_p^{(vQ'''),}$

it is not difficult to check that

7.4.26 $\beta(Q \cdot N_p^K(Q)) \subset vQ'' \cdot N_p^K(vQ''') = \eta'(Q' \cdot N_p^K(Q'))$

and finally we have $\eta'^*(\beta(u)) = \varphi(\chi(u))$ for any $u \in Q$. □
7.5. With the notation of Theorem 7.4, $\mathcal{F}'$ is clearly unique and whenever $P' = H_{\hat{F}}$ we set $\mathcal{F}' = \mathcal{F}^b$ and call it the hyperfocal subcategory of $\mathcal{F}$; note that $(\mathcal{F}^b)^b = \mathcal{F}^b$. If $\hat{P}$ is a subgroup of $P$ and $\hat{F}$ is a Frobenius $\hat{P}$-subcategory of $\mathcal{F}$ then we have

$$H_{\hat{F}} \subset H_{\mathcal{F}} \cap \hat{P}$$

and we claim that $\hat{F}^b$ is contained in $\mathcal{F}^b$; indeed, for any subgroup $Q$ of $H_{\hat{F}}$, choose $\varphi \in \hat{F}(\hat{P}, Q)$ such that $Q' = \varphi(Q)$ is fully normalized in $\hat{F}$ (cf. 2.7); thus, we have

$$\psi \hat{F}^b(Q') = \psi \mathcal{F}_{H_{\hat{F}}}(Q') \cdot \psi \mathcal{O}^p(\hat{F}(Q')) \subset \mathcal{F}_{H_{\hat{F}}}(Q') \cdot \mathcal{O}^p(\mathcal{F}(Q')) \subset \mathcal{F}^b(Q')$$

and therefore we get $\hat{F}^b(Q) \subset \mathcal{F}^b(Q)$; now, our claim follows from Corollary 5.10. Moreover, if $\hat{F}$ is normal in $\mathcal{F}$ then it is easily checked that $\mathcal{F}(P)$ stabilizes $H_{\hat{F}}$ and it follows from Proposition 6.6 that $\hat{F}^b$ is normal in $\mathcal{F}$ too.

7.6. On the other hand, if $U$ is a $\mathcal{F}$-stable subgroup of $P$ then, setting $\tilde{P} = P/U$ and denoting by $\tilde{\mathcal{F}}$ the $U$-quotient of $\mathcal{F}$, it is easily checked that

$$H_{\tilde{F}} = (U \cdot H_{\mathcal{F}})/U$$

and we claim that $\tilde{\mathcal{F}}^b$ contains the image of $\mathcal{F}^b$ in $\tilde{\mathcal{F}}$; indeed, mutatis mutandis, for any subgroup $Q$ of $H_{\tilde{F}}$, choose $\varphi \in \mathcal{F}(P, Q)$ such that $Q' = \varphi(Q)$ is fully normalized in $\mathcal{F}$, so that we have

$$\mathcal{F}^b(Q') = \mathcal{F}_{H_{\tilde{F}}}(Q') \cdot \mathcal{O}^p(\mathcal{F}(Q'))$$

hence, the image of $\mathcal{F}^b(Q')$ in $\tilde{\mathcal{F}}(\tilde{Q}')$ is contained in $\tilde{\mathcal{F}}^b(\tilde{Q}')$ and therefore the image of $\mathcal{F}^b(Q)$ is contained in $\tilde{\mathcal{F}}^b(\tilde{Q})$; once again, our claim follows from Corollary 5.10. Our two last results are concerned by the hyperfocal subgroups of the centralizers in $\mathcal{F}$; we need them in [14].

**Proposition 7.7.** For any subgroup $Q$ of $P$ fully centralized in $\mathcal{F}$, the subgroup $Q \cdot H_{C_{\mathcal{F}}(Q)}$ is $\mathcal{F}$-nilcentralized and fully centralized in $\mathcal{F}$.

**Proof.** Actually, $Q$ is also fully $\mathcal{F}_Q(Q)$-normalized in $\mathcal{F}$ and thus there is a morphism $\eta: Q \cdot H_{C_{\mathcal{F}}(Q)} \to Q \cdot C_P(Q)$ in the Frobenius $Q \cdot C_P(Q)$-category $Q \cdot C_{\mathcal{F}}(Q)$ ($= N_{\mathcal{F}^0}(Q)(Q)$) such that $\eta(Q \cdot H_{C_{\mathcal{F}}(Q)})$ is fully centralized there; but, it follows from Lemma 2.17 that $\eta(Q \cdot H_{C_{\mathcal{F}}(Q)})$ is fully centralized in $\mathcal{F}$ and it is clear that $\eta(Q \cdot H_{C_{\mathcal{F}}(Q)}) = Q \cdot H_{C_{\mathcal{F}}(Q)}$. Then, for any subgroup $R$ in $C_P(Q \cdot H_{C_{\mathcal{F}}(Q)})$, any $p'$-subgroup $K$ of $\mathcal{F}(R \cdot Q \cdot H_{C_{\mathcal{F}}(Q)})$ which normalizes $R$ and centralizes $Q \cdot H_{C_{\mathcal{F}}(Q)}$ induces a $p'$-group of automorphisms of $R \cdot H_{C_{\mathcal{F}}(Q)}$ in the Frobenius $C_P(Q)$-category $C_{\mathcal{F}}(Q)$, so that

$$[K, R \cdot H_{C_{\mathcal{F}}(Q)}] \subset H_{C_{\mathcal{F}}(Q)};$$
hence, since $K$ centralizes $H_{C_F(Q)}$, $K$ centralizes $R \cdot H_{C_F(Q)}$ (cf. [6, Chapter 5, Theorem 3.2]) and therefore we get $K = \text{id}_{R \cdot Q \cdot H_{C_F(Q)}}$.

**Proposition 7.8.** There is a contravariant functor $h \setminus c_F$ from $\mathcal{F}$ to the exterior quotient $\tilde{\mathcal{G}}_\mathcal{T}$ of the category of groups $\mathcal{G}_\mathcal{T}$, unique up to natural isomorphisms, mapping any subgroup $Q$ of $P$ fully centralized in $F$ on the quotient $C_P(Q)/H_{C_F(Q)}$ and any $F$-morphism $\varphi : R \to Q$ between subgroups of $P$ fully centralized in $F$, on the class of the group homomorphism

$$C_P(Q)/H_{C_F(Q)} \to C_P(R)/H_{C_F(R)}$$

induced by a morphism $\zeta \in \mathcal{F}(R \cdot C_P(R), \varphi(R) \cdot C_P(Q))$ fulfilling $\zeta(\varphi(v)) = v$ for any $v \in R$.

**Proof.** Since $\mathcal{F}$ is equivalent to the full subcategory over the set of subgroups of $P$ fully centralized in $F$, in order to define a contravariant functor $\mathcal{F} \to \tilde{\mathcal{G}}_\mathcal{T}$ up to natural isomorphisms, it suffices to define it over this full subcategory.

With the notation above, the existence of $\zeta$ follows from statement 2.11.1 applied to the inverse of the isomorphism $R \cong \varphi(R)$ induced by $\varphi$; then, it is clear that $\zeta(C_P(Q)) \subset C_P(R)$ and the restriction $\xi : C_P(Q) \to C_P(R)$ of $\zeta$ is $(C_F(Q), C_F(R))$-functorial since, for any subgroup $U$ of $C_P(Q)$ and any morphism $\theta \in \mathcal{F}(Q \cdot C_P(Q), Q \cdot U)$ fulfilling $\theta(U) \subset C_P(Q)$ and $\theta(u) = u$ for any $u \in Q$, the group homomorphism $\eta : R \cdot \xi(U) \to R \cdot \xi(C_P(Q))$ mapping $v\xi(u)$ on $\xi(\varphi(v)\theta(u))$, for any $v \in R$ and any $u \in U$, clearly induces a $C_F(R)$-morphism from $\xi(U)$ to $\xi(C_P(Q))$. Consequently, we have

$$\zeta(H_{C_F(Q)}) \subset H_{C_F(R)}$$

and therefore $\zeta$ induces the announced homomorphism 7.8.2.

Moreover, for another choice $\zeta'$ of $\zeta$, it is quite clear that the isomorphism $\zeta(C_P(Q)) \cong \zeta'(C_P(Q))$ induced by them is a $C_F(R)$-isomorphism and therefore, according to Lemma 7.2, there is $z \in C_P(R)$ such that, for any $u \in C_P(Q)$, we have

$$\zeta'(u) \equiv \zeta(u)z \mod H_{C_F(R)},$$

so that the group homomorphisms induced by $\zeta$ and by $\zeta'$ determine the same morphism in $\tilde{\mathcal{G}}_\mathcal{T}$.

$$h \setminus c_F(\varphi) : C_P(Q)/H_{C_F(Q)} \to C_P(R)/H_{C_F(R)}.$$  

Finally, for a third subgroup $T$ of $P$ fully centralized in $F$ and for any $\mathcal{F}$-morphism $\psi : T \to R$, it is easily checked that

$$h \setminus c_F(\varphi \circ \psi) = h \setminus c_F(\psi) \circ h \setminus c_F(\varphi).$$
References