Journal of Complexity 27 (2011) 39–54

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

A unifying theorem for Newton's method on spaces with a convergence structure

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ARTICLE INFO

Article history:
Received 5 January 2010
Accepted 25 August 2010
Available online 17 September 2010

Keywords:
Newton's method
Banach space
Convergence structure
Fixed point
Kantorovich hypothesis

ABSTRACT

We present a semilocal convergence theorem for Newton's method (NM) on spaces with a convergence structure. Using our new idea of recurrent functions, we provide a tighter analysis, with weaker hypotheses than before and with the same computational cost as for Argyros (1996, 1997, 1997, 2007) [1–3,5], Meyer (1984, 1987, 1992) [13–15]. Numerical examples are provided for solving equations in cases not covered before.

1. Introduction

In this study we are concerned with the problem of approximating a zero of a nonlinear operator using Newton's method (NM).

A large number of problems in applied mathematics and in engineering are solved by finding the solutions of the nonlinear equation

$$F(x) = 0. \tag{1.1}$$

For example, dynamic systems are mathematically modeled by difference or differential equations and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving Eq. (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference,

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0885-064X/$ – see front matter
doi:10.1016/j.jco.2010.08.005

differential and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative — when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iterative methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

Our study is motivated by the elegant works by Meyer [15], Ezquerro and Hernández [11], and Proinov [17,18], and optimization considerations (see also [1–3,5,13,14]). Meyer provided a general structure for the convergence analysis of (NM) in [15], which enables us to consider, in the same framework, Kantorovich-type semilocal convergence results and theorems based on monotonicity considerations. Here, we use our new idea of recurrent functions to provide a tighter semilocal analysis of (NM) with weaker conditions and for the same computational cost as before (see [1–3,5,11,13,15,17,18]). Other approaches on the Kantorovich-type semilocal convergence of (NM) can also be found in [4–8,12,17,18].

The paper is organized as follows: Section 2 contains the terms and concepts needed in this study; Sections 3 and 4 contain two different semilocal convergence approaches for (NM). Finally, Section 5 develops special cases and numerical examples containing nonlinear integral equations of Chandrasekhar type and a two-point boundary value problem with Green's kernel.

2. Preliminaries

To make the study as self-contained as possible, we reintroduce concepts that can be found in [1–3,5,13,15].

Definition 2.1. A Banach space $\mathcal{X}$ with convergence structure is a triple $(\mathcal{X}, \mathcal{V}, \mathcal{W})$ satisfying:

(1) $(\mathcal{X}, \|\ .\ \|)$ is a real Banach space.
(2) $(\mathcal{V}, \mathcal{C}, \|\ .\ \|_{\mathcal{V}})$ is a real Banach space that is partially ordered by the closed convex cone $\mathcal{C}$; the norm $\|\ .\ \|_{\mathcal{V}}$ is assumed to be monotone on $\mathcal{C}$.
(3) $\mathcal{W}$ is a closed convex cone in $\mathcal{X} \times \mathcal{V}$ satisfying $\{0\} \times \mathcal{C} \subset \mathcal{W} \subset \mathcal{X} \times \mathcal{C}$.
(4) The following map $/./ : \mathcal{D} \longrightarrow \mathcal{C}$ is well defined:

$$/x/ := \inf\{p \in \mathcal{C} : (x, p) \in \mathcal{W}\}, \tag{2.1}$$

where

$$\mathcal{D} := \{x \in \mathcal{X} : \exists\, p \in \mathcal{C}, (x, p) \in \mathcal{W}\}.$$

(5) For every $x \in \mathcal{D}$, we have

$$\|x\| \leq \|/x/\|_{\mathcal{V}}. \tag{2.2}$$

We use standard properties of partial orderings on $\mathcal{V}$ and $\mathcal{X} \times \mathcal{V}$ (see [9,16]). The set $\mathcal{D}$ satisfies $\mathcal{D} + \mathcal{D} \subset \mathcal{D}$ and $s\mathcal{D} \subset \mathcal{D}$ for $s > 0$ whereas

$$U(a) := \{x \in \mathcal{X} : (x, a) \in \mathcal{W}\}$$

defines a generalized neighborhood of zero.

Definition 2.1 is motivated by the following examples (see [1–3,5,15]):

Example 2.2. Let $\mathcal{X} = \mathbb{R}^n$ equipped with the maximum norm.

(a) Let $\mathcal{V} = \mathbb{R}$, $\mathcal{W} = \{(x, p) \in \mathbb{R}^{n+1} : \|x\|_\infty \leq p\}$. This case is concerned in classical convergence analysis in a Banach space.
(b) Let $\mathcal{V} = \mathbb{R}^n$, $\mathcal{W} = \{(x, p) \in \mathbb{R}^{2n} : |x| \leq p\}$, i.e., if $x = (x_i)_{i=1}^n$ and $p = (p_i)_{i=1}^n$, then $|x| \leq p \iff x_i \leq p_i$, for all $1 \leq i \leq n$. This case is concerned in componentwise analysis and error estimates.
(c) Let $\mathcal{V} = \mathbb{R}^n$, $\mathcal{W} = \{(x, p) \in \mathbb{R}^{2n} : 0 \leq x \leq p\}$. This case is used in monotone convergence analysis.

Remark 2.3. The convergence analysis is based on monotonicity considerations in $\mathcal{X} \times \mathcal{V}$. Let $(x_n, p_n) \in \mathcal{W}^N$ be an increasing sequence; then

$$(x_n, p_n) \le (x_{n+m}, p_{n+m}) \Longrightarrow 0 \le (x_{n+m} - x_m, p_{n+m} - p_m).$$

If $p_n \longrightarrow p$, we obtain $0 \le (x_{m+n} - x_n, p - p_n)$. Using (2.2) of Definition 2.1, we have

$$\|x_{n+m} - x_n\| \le \|p - p_n\|_{\mathcal{V}} \xrightarrow[n\to\infty]{0} .$$

Then $\{x_n\}$ is a Cauchy sequence. When deriving error estimates we shall use the sequence $p_n = a_0 - a_n$ with a decreasing sequence $\{x_n\} \in \mathcal{C}$ to obtain the estimate

$$0 \le (x_{m+n} - x_n, a_n - a_{n+m}) \le (x_{m+n} - x_n, a_n).$$

If $x_n \longrightarrow x$ this implies the estimate $/x - x_n/ \le a_n$.

Definition 2.4. We denote the space of multilinear, symmetric, bounded operators $A : \mathcal{X}^n \longrightarrow \mathcal{X}$ on a Banach space $\mathcal{X}$ by $\mathcal{L}(\mathcal{X}^n)$ and for an ordered Banach space $\mathcal{V}$, we let

$$\mathcal{L}_+(\mathcal{V}^n) = \{L \in \mathcal{L}(\mathcal{V}^n) : 0 \le x_i \,(1 \le i \le n) \Longrightarrow 0 \le L(x_1, \dots, x_n)\}.$$

A map $L \in \mathcal{C}^1(\mathcal{V}_L \longrightarrow V)$ on an open subset V_L of an ordered Banach space $\mathcal{V}$ is defined to be order convex on an interval $[a, b] \subset V_L$ if

$$c, d \in [a, b], \quad c \le d \Longrightarrow L'(d) - L'(c) \in \mathcal{L}_+(\mathcal{V}).$$

Definition 2.5. As the set of bounds for an operator $A \in \mathcal{L}(\mathcal{X}^n)$, we define

$$B(A) = \{L \in \mathcal{L}_+(\mathcal{V}^n) : (x_i, p_i) \in \mathcal{W} \Longrightarrow (A(x_1, \dots, x_n), L(p_1, \dots, p_n)) \in \mathcal{W}\}.$$

Lemma 2.6 ([13]). *Let $A : [0, 1] \longrightarrow \mathcal{L}(\mathcal{X}^n)$ and $L : [0, 1] \longrightarrow \mathcal{L}_+(\mathcal{V}^n)$ be continuous maps; then*

$$\forall t \in [0, 1] : L(t) \in B(A(t)) \Longrightarrow \int_0^1 L(t)\mathrm{d}t \in B\left(\int_0^1 A(t)\mathrm{d}t\right),$$

which will used for the remainder of Taylor's formula.

Finally the following conventions are needed: Let $T : \mathcal{Y} \longrightarrow \mathcal{Y}$ be a map on a subset $\mathcal{Y}$ of a normed space. Then $T^n(x)$ denotes the result of n-fold application of T and in the case of convergence, we write

$$T^\infty(x) = \lim_{n\longrightarrow\infty} T^n(x).$$

In particular, we define a right inverse through:

Definition 2.7. Let $A \in \mathcal{L}(\mathcal{X})$ and $y \in \mathcal{X}$ be given. Then we can write

$$A^\star y := z \Longleftrightarrow z \in T^\infty(0), \quad T(x) := (\mathcal{I} - A)x + y \Longleftrightarrow z = \sum_{j=0}^{\infty} (\mathcal{I} - A)^j y$$

provided this limit exists.

3. Semilocal convergence analysis of (NM)

We provide sufficient semilocal convergence conditions for (NM) to determine a zero $x^\star$ of operator G on Banach space. Our results are stated for the operator

$$F(x) = AG(x_0 + x), \tag{3.1}$$

where x_0 is the initial guess for (NM) and A is an approximation of $G'(x_0)^{-1}$. That is, the following result is affine invariant in the sense of [10].

Theorem 3.1. *Let $\mathcal{X}$ be a Banach space with a convergence structure $(\mathcal{X}, \mathcal{V}, \mathcal{W})$ with $\mathcal{V} = (\mathcal{V}, \mathcal{C}, \| \,.\, \|_{\mathcal{V}})$, an operator $F \in \mathcal{C}^1(\mathcal{X}_F \longrightarrow \mathcal{X})$ with $\mathcal{X}_F \subseteq \mathcal{X}$, an operator $L \in \mathcal{C}^1(\mathcal{V}_L \longrightarrow \mathcal{V})$ with $\mathcal{V}_L \subseteq \mathcal{V}$, an operator $L_0 \in \mathcal{C}^1(\mathcal{V}_{L_0} \longrightarrow \mathcal{V})$ with $\mathcal{V}_L \subseteq \mathcal{V}_{L_0}$ and a point $a \in \mathcal{C}$ such that the following hypotheses are satisfied:*

$$U(a) \subseteq \mathcal{X}_F \quad \text{and} \quad [0, a] \subseteq \mathcal{V}_L; \tag{3.2}$$

L is order convex on $[0, a]$, satisfying

$$L'(|x| + |y|) - L'(|x|) \in B(F'(x) - F'(x + y)) \tag{3.3}$$

for all $x, y \in U(a)$ with $|x| + |y| \leq a$;

L_0 is order convex on $[0, a]$, satisfying $L_0' \leq L'$ on $\mathcal{V}_L$ and

$$L_0'(|x|) - L_0'(0) \in B(F'(0) - F'(x)) \tag{3.4}$$

for all $x \in U(a)$ with $|x| \leq a$;

$$L_0'(0) \in B(I - F'(0)) \quad \text{and} \quad (-F(0), L(0)) \in \mathcal{W}; \tag{3.5}$$

$$L(a) \leq a; \tag{3.6}$$

and

$$L'(a)^n a \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.7}$$

Then sequence $\{x_n\}$ generated by (NM)

$$x_0 = 0, \quad x_{n+1} = x_n + F'(x_n)^\star(-F(x_n)) \tag{3.8}$$

is well defined, remains in $U(a)$ for all $n \geq 0$, and converges to the unique zero $x^\star$ of operator F in $U(a)$.

Moreover, the following estimates hold true for all $n \geq 0$:

$$/x_{n+1} - x_n/ \leq d_{n+1} - d_n, \tag{3.9}$$

$$/x_{n+1} - x^\star/ \leq b - d_n, \tag{3.10}$$

where

$$b = L^\infty(0) \tag{3.11}$$

is the minimal fixed point of operator L in $[0, a]$ and sequence $\{d_n\}$ is given by

$$d_0 = 0, \quad d_{n+1} = L(d_n) + L_0'(|x_n|)c_n, \quad c_n = /x_{n+1} - x_n/. \tag{3.12}$$

Furthermore, sequence $(x_n, d_n) \in (\mathcal{X} \times \mathcal{V})^N$ is well defined, remains in $\mathcal{W}^N$ and is monotone.

Remark 3.2. (a) If $L_0' = L'$ on $\mathcal{V}_L$, then hypotheses of Theorem 3.1 reduce to the ones in [15, Theorem 5]. However, note that in general

$$L_0' \leq L' \tag{3.13}$$

holds and $\frac{L'}{L_0'}$ can be arbitrarily large [4–8].

Condition (3.4) is not an additional hypothesis. In practice, computing operator L' also requires determining L_0'. Moreover, operator L_0' always exists (if L' exists). The benefits of introducing condition (3.4) are given in Remark 3.5.

(b) Assume conditions (3.2)–(3.4) of Theorem 3.1 hold and

$$\exists t \in (0, 1) : L(a) \leq ta.$$

Then there exists $a' \in [0, ta]$ satisfying all conditions of Theorem 3.1. The zero $z \in U(a')$ is unique in $U(a)$ [15].

Let $L \in \mathcal{C}^1(\mathcal{V}_L \to \mathcal{V})$ be a map satisfying the conditions of Theorem 3.1. Then L is monotone. Therefore, sequences $b_n = L^n(0)$ and $c_n = L^n(a)$ are monotone with

$$0 \leq b_n \leq b_{n+1} \leq c_{n+1} \leq c_n \leq a.$$

In view of (3.7), we conclude that the sequence $\{c_n - b_n\}$ converges to zero [13]. That is, we obtain that $b = L^{\infty}(0)$ is well defined and is the smallest solution of $L(p) \leq p$ in $[0, a]$.

In the case of Remark 3.2(b), we obtain

$$0 \leq c_n - b_n \leq t^n a.$$

That is, $b = L^{\infty}(0)$ is well defined and the following inequalities hold:

$$L'(b)(a-b) \leq L(a) - L(b) \leq t(a-b)$$

$$b = L(b) \leq L(a) \leq ta \Longrightarrow b \leq \frac{t}{1-t}(a-b),$$

so

$$L'(b)^n b \leq \frac{t}{1-t} L'(b)^n (a-b) \leq \frac{t^{n+1}}{1-t}(a-b) \longrightarrow 0.$$

Therefore, $a' = b$ satisfies the additional hypothesis of Remark 3.2(b).

As a special case, we obtain the following result for affine maps:

Corollary 3.3 ([15]). *Let $L \in L_+(\mathcal{V})$ and $a, p \in \mathcal{C}$ be given such that*

$$Lp + a \leq p \quad \text{and} \quad L^n p \longrightarrow 0.$$

Then the map

$$(I - L)^{\star} : [0, a] \longrightarrow [0, a]$$

is well defined and continuous.

As substitute for the Banach lemma we use:

Lemma 3.4 ([15]). *Let $A \in L(\mathcal{X})$, $L \in B(A)$, $y \in \mathcal{D}$ and $p \in \mathcal{C}$ be given as in Theorem 3.1 such that*

$$Lp + /y/ \leq p \quad \text{and} \quad L^n p \longrightarrow 0.$$

Then $x = (I - A)^{\star} y$ is well defined, $x \in \mathcal{D}$ and

$$/x/ \leq (I - L)^{\star} /y/ \leq p.$$

The proof of Lemma 3.4 is easy. Simply note that the sequence $\{b_n\}$ defined by

$$b_0 = 0, \quad b_{n+1} = Lb_n + /y/ \leq p$$

is well defined and converges to

$$b = (I - L)^{\star} /y/ \leq p.$$

If we consider $x_{n+1} = Ax_n + y$, $x_0 = 0$, then the sequence (x_n, b_n) is monotone in $\mathcal{X} \times \mathcal{V}$. Hence, the statement follows from the general principles in Section 2.

Proof of Theorem 3.1. Conditions of Theorem 3.1 are satisfied for b replacing a. We shall show that for each n, there exists x_{n+1} solving

$$p = (I - F'(x_n))p + (-F(x_n)). \tag{3.14}$$

Let $n = 1$ and $p = b$. Using (3.3)–(3.6) we get

$$\begin{aligned} |I - F'(0)|b + |-F(0)| &\leq L_0'(0)b + |-F(0)| \\ &\leq L'(0)b + |-F(0)| \\ &\leq L(b) - L(0) + L(0) = L(b) = b. \end{aligned} \tag{3.15}$$

Hence, x_1 is well defined and $(x_1, b) \in \mathcal{W}$.

We also have

$$x_1 = (I - F'(0))x_1 + (-F(0)), \tag{3.16}$$

so

$$|x_1| \le L_0'(0)|x_1| + L(0) = d_1, \tag{3.17}$$

and consequently

$$d_1 = L_0'(0)|x_1| + L(0) \le L'(0)b + L(0) \le L(b) - L(0) + L(0) = L(b) = b. \tag{3.18}$$

Let us assume that (x_k, d_k) is well defined and monotone for all $k \le n$, with

$$0 \le (x_{k-1}, d_{k-1}) \le (x_k, d_k), \quad d_k \le b. \tag{3.19}$$

Using (3.4) and (3.5), we have

$$\begin{aligned} |I - F'(x_k)| &\le |(I - F'(0)) + (F'(0) - F'(x_k))| \\ &\le |I - F'(0)| + |F'(0) - F'(x_k)| \\ &\le L_0'(0) + L_0'(|x_k|) - L_0'(0) = L_0'(|x_k|), \end{aligned} \tag{3.20}$$

so

$$L_0'(|x_k|) \in B(I - F'(x_k)). \tag{3.21}$$

In view of Lemma 3.4, we must solve for p:

$$L_0'(|x_k|)p + |-F(x_k)| \le p. \tag{3.22}$$

We need an estimate on $|-F(x_k)|$. By Taylor's theorem, (3.3) and Lemma 3.4, we obtain in turn

$$\begin{aligned} |-F(x_k)| &= |-F(x_k) + F(x_{k-1}) + F'(x_{k-1})(x_k - x_{k-1})| \\ &\le \int_0^1 [L'(|x_{k-1}| + tc_{k-1}) - L'(|x_{k-1}|)]c_{k-1} \mathrm{d}t \\ &= L(|x_{k-1}| + c_{k-1}) - L(|x_{k-1}|) - L'(|x_{k-1}|)c_{k-1} \\ &\le L(d_{k-1} + d_k - d_{k-1}) - L(d_{k-1}) - L'(|x_{k-1}|)c_{k-1} \\ &\le L(d_k) - L(d_{k-1}) - L_0'(|x_{k-1}|)c_{k-1} \\ &= L(d_k) - d_k. \end{aligned} \tag{3.23}$$

Let $p = b - d_k$. Then, we have by (3.20) and (3.23)

$$\begin{aligned} L_0'(|x_k|)p + |-F(x_k)| + d_k &\le L'(d_k)(b - d_k) + L(d_k) \\ &\le L(b) - L(d_k) + L(d_k) = L(b) = b. \end{aligned} \tag{3.24}$$

Hence, x_{k+1} is well defined by Lemma 3.4, and $c_k \le b - d_k$. Therefore, d_{k+1} is well defined too and we obtain

$$\begin{aligned} d_{k+1} &\le L(d_k) + L_0'(d_k)(b - d_k) \\ &\le L(d_k) + L'(d_k)(b - d_k) \\ &\le L(d_k) + L(b) - L(d_k) = L(b) = b. \end{aligned} \tag{3.25}$$

In view of the estimate

$$\begin{aligned} c_k + d_k &\le L_0'(|x_k|)c_k + |-F(x_k)| + d_k \\ &\le L_0'(|x_k|)c_k + L(d_k) = d_{k+1}, \end{aligned} \tag{3.26}$$

we deduce the monotonicity

$$(x_k, d_k) \le (x_{k+1}, d_{k+1}), \tag{3.27}$$

which also implies (3.9).

It follows inductively from (3.12) that

$$L^k(0) \le d_k \le b, \tag{3.28}$$

which together with $L^k(0) \longrightarrow b$, implies that $d_k \longrightarrow b$ as $k \longrightarrow \infty$.

By Section 2, sequence $\{x_n\}$ converges to some $x^\star \in U(b)$. By setting $k \longrightarrow \infty$ in (3.23), we deduce that $x^\star$ is a zero of operator F.

Estimate (3.10) now follows from (3.9) by using standard majorization techniques (see [5,6,11]).

The uniqueness statement is given in [13], where the modified Newton method

$$x_{n+1} = x_n - F(x_n) \tag{3.29}$$

is considered. Clearly, sequence $(x_n, L^n(0))$ is monotone in $\mathcal{X} \times \mathcal{V}$.

Moreover, if there exists a zero $y^\star \in U(a)$ of F, then it was shown in [13] that

$$|y^\star - x_n| \le L^n(a) - L^n(0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.30}$$

which implies $\lim_{n\to\infty} x_n = y^\star$. But, we showed that $\lim_{n\to\infty} x_n = x^\star$. Hence, we deduce

$$x^\star = y^\star. \tag{3.31}$$

That completes the proof of Theorem 3.1.

Remark 3.5. If equality holds in (3.13) then our Theorem 3.1 reduces to [15, Theorem 5]. Otherwise (i.e., if $L_0' < L'$), the former theorem improves the latter. Indeed, the majorizing sequence $\{q_n\}$ used in [15] is given by

$$q_0 = 0, \quad q_{n+1} = L(q_n) + L'(|x_n|)c_n. \tag{3.32}$$

In view of (3.12) and (3.32), a simple inductive argument shows

$$d_n < q_n, \quad (n \ge 1) \tag{3.33}$$

and

$$d_{n+1} - d_n < q_{n+1} - q_n, \quad (n \ge 1). \tag{3.34}$$

Hence, sequence $\{d_n\}$ is a tighter majorizing sequence for $\{x_n\}$ than $\{q_n\}$. As already noted in Remark 3.2, these advantages are obtained under the same hypotheses and for the same computational cost as in [15].

By simply replacing L' by L_0' in the definitions of the operators involved, we can also improve the a posteriori estimates given in [15] under the hypotheses of Theorem 3.1. More, precisely in order for us to obtain a posteriori estimates, we define

$$R_n(p) = (I - L_0'(|x_n|))^\star S_n(p) + c_n \tag{3.35}$$

where

$$S_n(p) = L(|x_n| + p) - L(|x_n|) - L_0'(|x_n|)p. \tag{3.36}$$

Operator S_n is monotone on the interval $I_n = [0, a - |x_n|]$; moreover, if there exists $p_n \in \mathcal{C}$ such that $|x_n| + p_n \le a$, and

$$S_n(p_n) + L_0'(|x_n|)(p_n - c_n) \le p_n - c_n, \tag{3.37}$$

then operator $R_n : [0, p_n] \longrightarrow [0, p_n]$ is well defined by Corollary 3.3, and monotone. We then have

$$\begin{aligned} d_n + c_n \le d_{n+1} &\Rightarrow L(a) - L(d_n) - L_0'(|x_n|)c_n \le a - d_n - c_n \\ &\Rightarrow S_n(a - d_n) + L_0'(|x_n|)(a - d_n - c_n) \le a - d_n - c_n, \end{aligned} \tag{3.38}$$

which implies that $a - d_n$ is a suitable choice for p_n.

Other ways for choosing suitable p_n are given by the following:

Proposition 3.6. *Assume that*

$$R_n(p) \le p \quad \text{for some } p \in I_n. \tag{3.39}$$

Then

$$c_n \le R_n(p) = \overline{p} \le p \tag{3.40}$$

and

$$R_{n+1}(\overline{p} - c_n) \le \overline{p} - c_n. \tag{3.41}$$

Proof. Using (3.23), the order convexity of L_0, L and the estimate

$$S_n(p) + L_0'\left(|x_n|\right)\left(\overline{p} - c_n\right) = \overline{p} - c_n, \tag{3.42}$$

we get

$$S_{n+1}\left(\overline{p} - c_n\right) + |-F(x_{n+1})| + L_0'\left(|x_{n+1}|\right)\left(\overline{p} - c_n\right) \le \overline{p} - c_n. \tag{3.43}$$

That completes the proof of Proposition 3.6. □

Proposition 3.7. *Assume that the conditions of Theorem 3.1 hold and there exists a solution $p_n \in I_n$ satisfying*

$$R_n(p) \le p. \tag{3.44}$$

Define a sequence

$$a_n = p_n, \quad a_{m+1} = R_m(a_m) - c_m \quad (m \ge n). \tag{3.45}$$

Then the following a posteriori estimate holds:

$$/x^\star - x_m/ \le a_m. \tag{3.46}$$

Proof. An induction argument shows

$$R_m(a_m) \le a_m, \tag{3.47}$$

so

$$a_{m+1} + c_m \le a_m, \tag{3.48}$$

and consequently we deduce the monotonicity of $(x_m, a_n - a_m)$ in $\mathcal{X} \times \mathcal{V}$.

That completes the proof of Proposition 3.7. □

The properties of R_n imply the existence of $R_n^\infty(0)$, which is a suitable choice for p_n in Proposition 3.7. Hence we arrive at:

Corollary 3.8. *Assume that the conditions of Theorem 3.1 hold and there exists $p \in I_n$ satisfying*

$$R_n(p) \le p.$$

Then the following a posteriori estimates hold:

$$|x^* - x_n| \le R_n^\infty(0) \le p. \tag{3.49}$$

As already noted in [15], in view of the estimate

$$S_n(p) + /-F(x_n)/ + L_0'(|x_n|)p \le p \Longrightarrow R_n(p) \le p, \tag{3.50}$$

one may consider further majorization:

$$Q_n(p) = L(|x_{n-1}| + c_{n-1} + p) - L(|x_{n-1}|) - L_0'(|x_{n-1}|)c_{n-1}. \tag{3.51}$$

Note that an application to a two-point boundary value problem was given in [15] in the case $L_0' = L_0$. Further discussion on applications and practical aspects can be found in [3,5,13,14,19].

Remark 3.9. If $L'_0 = L'$, then our a posteriori estimates reduce to the ones in [15]. However, if $L'_0 < L'$, then our a posteriori estimates are tighter. So far, we have shown how to improve on the error estimates given in [15]. We are now wondering whether we can also weaken the sufficient convergence conditions (3.6) and (3.7).

It turns out that this can be done (see Section 4), using our new idea of recurrent functions [5–7].

4. (NM) and recurrent functions

We need to define some operator sequences.

Definition 4.1. Let $\eta \in \mathcal{C}$. Define operators

$$f_n, h_n, \beta_n : [0, 1) \longrightarrow \mathcal{X}$$

and

$$\delta : I_\delta = \left[1, \frac{1}{1-\gamma}\right] \times [0, 1)^4 \longrightarrow \mathcal{X}, \quad \gamma \in [0, 1)$$

by

$$f_n(\gamma) = \left\{\int_0^1 \left(L'\left(\left(\frac{1-\gamma^{n-1}}{1-\gamma} + t\gamma^{n-1}\right)\eta\right) - L'\left(\frac{1-\gamma^{n-1}}{1-\gamma}\eta\right)\right) \mathrm{d}t + \gamma L'_0\left(\frac{1-\gamma^n}{1-\gamma}\eta\right)\right\} - \gamma, \tag{4.1}$$

$$\begin{aligned} h_n(\gamma) = {} & \left\{\int_0^1 \left(L'\left(\left(\frac{1-\gamma^n}{1-\gamma} + t\gamma^n\right)\eta\right) - L'\left(\left(\frac{1-\gamma^{n-1}}{1-\gamma} + t\gamma^{n-1}\right)\eta\right)\right) \mathrm{d}t \right. \\ & \left. + \left(L'\left(\frac{1-\gamma^{n-1}}{1-\gamma}\eta\right) - L'\left(\frac{1-\gamma^n}{1-\gamma}\eta\right)\right) + \gamma\left(L'_0\left(\frac{1-\gamma^{n+1}}{1-\gamma}\eta\right) - L'_0\left(\frac{1-\gamma^n}{1-\gamma}\eta\right)\right)\right\}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \overline{\beta_n}(\gamma) = {} & \int_0^1 \left(L'\left(\left(\frac{1-\gamma^{n+1}}{1-\gamma} + t\gamma^{n+1}\right)\eta\right) + L'\left(\left(\frac{1-\gamma^{n-1}}{1-\gamma} + t\gamma^{n-1}\right)\eta\right)\right. \\ & \left. - 2L'\left(\left(\frac{1-\gamma^n}{1-\gamma} + t\gamma^n\right)\eta\right)\right) \mathrm{d}t \\ & + \left(2L'\left(\frac{1-\gamma^n}{1-\gamma}\eta\right) - L'\left(\frac{1-\gamma^{n-1}}{1-\gamma}\eta\right) - L'\left(\frac{1-\gamma^{n+1}}{1-\gamma}\eta\right)\right) \\ & + \gamma\left(L'_0\left(\frac{1-\gamma^{n+2}}{1-\gamma}\eta\right) + L'_0\left(\frac{1-\gamma^n}{1-\gamma}\eta\right) - 2L'_0\left(\frac{1-\gamma^{n+1}}{1-\gamma}\eta\right)\right), \end{aligned}$$

$$\beta_n(\gamma) = \overline{\beta_n}(\gamma), \tag{4.3}$$

$$\begin{aligned} & \overline{\delta}(v_1, v_2, v_3, v_4, \gamma) \\ & \quad = \int_0^1 \left(L'((v_1 + v_2 + v_3 + tv_4)\eta) + L'((v_1 + tv_2)\eta) - 2L'((v_1 + v_2 + tv_3)\eta)\right) \mathrm{d}t \\ & \quad + \left(2L'((v_1 + v_2)\eta) - L'(v_1\eta) - L'((v_1 + v_2 + v_3)\eta)\right) \\ & \quad + \gamma\left(L'_0((v_1 + v_2 + v_3 + v_4)\eta) + L'_0((v_1 + v_2)\eta) - 2L'_0((v_1 + v_2 + v_3)\eta)\right), \end{aligned}$$

$$\delta(v_1, v_2, v_3, v_4, \gamma) = \overline{\delta}(v_1, v_2, v_3, v_4, \gamma). \tag{4.4}$$

Moreover, define function $f_\infty : [0, 1) \longrightarrow \mathcal{X}$ by

$$f_\infty(\gamma) = \lim_{n \longrightarrow \infty} f_n(\gamma). \tag{4.5}$$

It then follows from (4.1) and (4.5) that

$$f_\infty(\gamma) = b\left(L'\left(\frac{\eta}{1-\gamma}\right) + \gamma L'_0\left(\frac{\eta}{1-\gamma}\right)\right) - \gamma. \tag{4.6}$$

It can also easily be seen from (4.1)–(4.4) that the following identities hold:

$$f_{n+1}(\gamma) = f_n(\gamma) + h_n(\gamma), \tag{4.7}$$

$$h_{n+1}(\gamma) = h_n(\gamma) + \beta_n(\gamma), \tag{4.8}$$

and for

$$v_1 = \sum_{i=0}^{n-2} \gamma^i, \quad v_2 = \gamma^{n-1}, \quad v_3 = \gamma^n, \quad v_4 = \gamma^{n+1}, \tag{4.9}$$

we have

$$\delta(v_1, v_2, v_3, v_4, \gamma) = \beta_n(\gamma). \tag{4.10}$$

Finally, let us define sequence $\{t_n\}$ by

$$t_0 = 0, \quad t_1 = L_0(0) + L'_0(0)\eta,$$
$$t_{n+1} = t_n + L'_0(t_n)(t_n - t_{n-1}) + \int_0^1 (L'(t_{n-1} + t(t_n - t_{n-1})) - L'(t_{n-1}))\mathrm{d}t(t_n - t_{n-1}). \tag{4.11}$$

We need the following result on majorizing sequences for (NM).

Lemma 4.2. *Assume that there exist η, $a \in \mathcal{C}$ and $\alpha \in (0, 1)$ such that*

$$\frac{\eta}{1-\alpha} \in [0, a]; \tag{4.12}$$

$$0 \le L'_0(t_1) + \int_0^1 (L'(tt_1) - L'(0))\mathrm{d}t \le \alpha I; \tag{4.13}$$

$$\delta(v_1, v_2, v_3, v_4, \gamma) \ge 0 \quad \text{on } I_\delta, \tag{4.14}$$

$$h_1(\alpha) \ge 0, \tag{4.15}$$

and

$$f_\infty(\alpha) \le 0, \tag{4.16}$$

where 0 *and I are the zero endomorphism and the identity operator on $\mathcal{X}$, respectively. Then the iteration $\{t_n\}$ ($n \ge 0$) given by* (4.10) *is non-decreasing, bounded from above by*

$$t^{\star\star} = \frac{\eta}{1-\alpha}, \tag{4.17}$$

and converges to its unique least upper bound $t^\star$ satisfying

$$t^\star \in \langle 0, t^{\star\star}\rangle. \tag{4.18}$$

Moreover, the following error bounds hold for all $n \ge 0$:

$$0 \le t_{n+1} - t_n \le \alpha(t_n - t_{n-1}) \le \alpha^n \eta, \tag{4.19}$$

and

$$t^\star - t_n \le \frac{\eta}{1-\alpha}\alpha^n. \tag{4.20}$$

Proof. Estimate (4.19) holds if

$$0 \le L_0'(t_n) + \int_0^1 (L'(t_{n-1} + t(t_n - t_{n-1})) - L(t_{n-1}))\mathrm{d}t \le \alpha I \tag{4.21}$$

holds for all $n \ge 1$.

In view of (4.13) and (4.17), estimate (4.21) holds for $n = 1$. We also have by (4.10) and (4.21) that

$$0 \le t_2 - t_1 \le \alpha(t_1 - t_0).$$

Let us assume that (4.19) and (4.21) hold for all $k \le n$. Then, we have

$$t_n \le \frac{1-\alpha^n}{1-\alpha}\eta. \tag{4.22}$$

Moreover, (4.19) and (4.21) will hold if

$$\left\{\int_0^1 \left(L'\left(\left(\frac{1-\alpha^{n-1}}{1-\alpha} + t\alpha^{n-1}\right)\eta\right) - L'\left(\frac{1-\alpha^{n-1}}{1-\alpha}\eta\right)\right)\mathrm{d}t + \alpha L_0'\left(\frac{1-\alpha^n}{1-\alpha}\eta\right)\right\} - \alpha \le 0. \tag{4.23}$$

Estimate (4.23) motivates us to define functions f_n (for $\gamma = \alpha$) and show instead

$$f_n(\alpha) \le 0. \tag{4.24}$$

We have by (4.7)–(4.10), (4.14) and (4.15) that

$$f_{n+1}(\alpha) \ge f_n(\alpha). \tag{4.25}$$

In view of (4.5) and (4.25), estimate (4.24) will hold if (4.16) holds true. The induction is completed.

It follows that iteration $\{t_n\}$ is non-decreasing, bounded from above by $t^{\star\star}$ (given by (4.17)) and hence converges to $t^\star$ satisfying (4.18).

Finally, estimate (4.20) follows from (4.19) by using standard majorizing techniques [5,6]. That completes the proof of Lemma 4.2. □

We can show the following semilocal convergence result for (NM).

Theorem 4.3. *Let $\mathcal{X}$ be a Banach space with a convergence structure $(\mathcal{X}, \mathcal{V}, \mathcal{W})$ with $\mathcal{V} = (\mathcal{V}, \mathcal{C}, \|\,.\,\|_{\mathcal{V}})$, an operator $F \in \mathcal{C}^1(\mathcal{X}_F \longrightarrow \mathcal{X})$ with $\mathcal{X}_F \subseteq \mathcal{X}$, an operator $L \in \mathcal{C}^1(\mathcal{V}_L \longrightarrow \mathcal{V})$ with $\mathcal{V}_L \subseteq \mathcal{V}$, an operator $L_0 \in \mathcal{C}^1(\mathcal{V}_{L_0} \longrightarrow \mathcal{V})$ with $\mathcal{V}_L \subseteq \mathcal{V}_{L_0}$ and a point $a \in \mathcal{C}$ such that the following hypotheses are satisfied for (3.2)–(3.5):*

$$L(\eta) \le \eta, \quad \eta = L_0^\infty(0), \quad \eta \le a; \tag{4.26}$$

$$L'(\eta)^n\eta \longrightarrow 0 \quad \text{as } n \longrightarrow \infty; \tag{4.27}$$

and the hypotheses of Lemma 4.2 hold for $|x_1| \le \eta$ where x_1 solves

$$p = (I - F'(0))p + (-F(0)). \tag{4.28}$$

Then sequence $\{x_n\}$ generated by (NM) is well defined, remains in $U(t^\star)$ for all $n \ge 0$, and converges to a zero $x^\star$ of operator F in $U(t^\star)$.

Moreover, the following estimates hold true for all $n \ge 0$:

$$/x_{n+1} - x_n/ \le t_{n+1} - t_n, \tag{4.29}$$

$$/x_{n+1} - x^\star/ \le t^\star - t_n. \tag{4.30}$$

Furthermore, sequence $(x_n, t_n) \in (\mathcal{X} \times \mathcal{V})^N$ is well defined, remains in $\mathcal{W}^N$ and is monotone.

Proof. As in Theorem 3.1, we have that b_0 is the smallest fixed point of operator L_0 in $[0, a]$ guaranteed to exist by (3.5), (4.26), (4.27), and Lemma 3.4 since

$$\begin{aligned} L_0'(0)\eta + |-F(0)| &\leq L_0(\eta) - L_0(0) + L_0(0) \\ &= L_0(\eta) = \eta. \end{aligned} \tag{4.31}$$

Eq. (4.28) is satisfied for $p = \eta$. Therefore, x_1 is well defined and $(x_1, \eta) \in \mathcal{W}$. We also have

$$|x_1| \leq L_0'(0)|x_1| + L(0) = L_0'(0)\eta + L_0(0) = t_1,$$

and

$$t_1 \leq L_0(\eta) - L_0(0) + L_0(0) = L_0(\eta) \leq \eta \leq t^\star.$$

We also have

$$\begin{aligned} L_0'(|x_1|)(t_1 - t_0) + |-F(x_1)| &\leq L_0'(t_1)(t_1 - t_0) + \int_0^1 (L'(|x_0| + tc_0) - L'(|x_0|))c_0 \mathrm{d}t \\ &= L_0'(t_1)(t_1 - t_0) + \int_0^1 (L'(tc_0) - L'(0))c_0 \mathrm{d}t \\ &= t_2 - t_1 \leq \alpha(t_1 - t_0), \end{aligned}$$

which together with Lemma 3.4 implies that x_2 is well defined and (4.29) holds for $n = 1$. Let us assume that (x_k, t_k) is well defined and monotone for all $k \leq n$, i.e.,

$$0 \leq (x_{k-1}, t_{k-1}) \leq (x_k, t_k), \quad \text{and} \quad t_k \leq t^\star, \quad k = 1, \ldots, n.$$

In view of (3.4), (3.6), (3.20), (3.23), and the definition of sequence $\{t_n\}$, we have in turn, for $p = t_k - t_{k-1}$,

$$\begin{aligned} &L_0'(|x_k|)(t_k - t_{k-1}) + |-F(x_k)| \\ &\quad\leq L_0'(t_k)(t_k - t_{k-1}) + \int_0^1 [L'(|x_{k-1}| + tc_{k-1}) - L'(|x_{k-1}|)]c_{k-1} \mathrm{d}t \\ &\quad\leq L_0'(t_k)(t_k - t_{k-1}) + \int_0^1 [L'(|t_{k-1}| + t(t_k - t_{k-1})) - L'(|t_{k-1}|)](t_k - t_{k-1}) \mathrm{d}t \\ &\quad= t_{k+1} - t_k \leq \alpha(t_k - t_{k-1}). \end{aligned} \tag{4.32}$$

It follows from (4.32) and Lemma 3.4 that x_{k+1} is well defined, and (4.29) holds for all n. Sequence t_{k+1} is well defined too and bounded above by $t^\star$. According to Section 2, $\{x_n\}$ converges to some $x^\star \in U(t^\star)$. By setting $k \longrightarrow \infty$ in the upper bound of $|-F(x_k)|$ given in (4.32), we obtain that $x^\star$ is a zero of operator F.

Estimate (4.30) follows from (4.29) as in Theorem 3.1. That completes the proof of Theorem 4.3. □

Remark 4.4. (a) Note that $t^{\star\star}$, given in closed form by (4.17), can replace $t^\star$ in Theorem 3.1.

(b) In view of the proof of Theorem 4.3, it follows that sequence $\{s_n\}$ given by

$$\begin{aligned} &s_0 = 0, \quad s_1 = L_0(0) + L_0'(0)|x_1|, \\ &s_{n+1} = s_n + L_0'(|x_n|)c_n + \int_0^1 (L'(|x_{n-1}| + tc_{n-1}) - L'(|x_{n-1}|))c_{n-1} \mathrm{d}t \end{aligned}$$

is also a finer majorizing sequence for $\{x_n\}$ than t_n.

(c) **The monotone case.** This is a particular case of Theorem 3.1 (or Theorem 4.3) but is omitted here, since it follows along the lines of Theorem 13 in [15], where $\mathcal{X}$ is itself partially ordered and satisfies the conditions for $\mathcal{V}$ in Definition 2.1. We set $\mathcal{X} = \mathcal{V}$, $\mathcal{D} = \mathcal{C}^2$ and $/./ = I$ (see Case 3 in Example 2.2). Then (NM) is given by

$$\begin{aligned} &u_0 = u, \quad u_{n+1} = u_n + (AG'(u_n))^\star(-AG(u_n)), \\ &G \in \mathcal{C}^1(\mathcal{V}_G \longrightarrow \mathcal{Y}), \quad A \in L(\mathcal{Y} \longrightarrow \mathcal{X}), \\ &u, v \in \mathcal{V}, \quad L_0(p) = p - F(p), \quad [u, v] \subseteq \mathcal{V}_G, \quad \text{and} \quad a = v - u. \end{aligned}$$

5. Application and special cases

Application 5.1. *Let $\mathcal{X}$ be a Banach space with real norm $\|.\|$. We shall check the conditions of Theorem 3.1, and [15, Theorem 5]. Let us assume for simplicity that $F'(0) = I$ and that there exists a monotone operator $E : [0, a] \longrightarrow \mathbb{R}$ such that*

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \le E(\|x - y\|)\|x - y\| \tag{5.1}$$

for all $x, y \in U(a)$.

Define L by

$$L(p) = \eta + \int_0^p \mathrm{d}s \int_0^s \mathrm{d}tE(t), \quad \eta \ge \|F'(x_0)^{-1}F(x_0)\|. \tag{5.2}$$

We have to solve (3.6) for $E(t) \le E(a) = \ell$, i.e.,

$$\eta + \frac{1}{2}\ell a^2 \le a, \tag{5.3}$$

which is possible if

$$h_K = \ell\eta \le \frac{1}{2}. \tag{5.4}$$

Condition (5.4) is the—famous for its simplicity and clarity—Kantorovich sufficient convergence hypothesis for (NM) [12, Chapter 12], [4–6,16].

In view of (5.1), there exists a monotone operator $E_0 : [0, a] \longrightarrow \mathbb{R}$ such that

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le E_0(\|x - x_0\|)\|x - x_0\| \tag{5.5}$$

for all $x \in U(a)$.

Define operator L_0 by

$$L_0(p) = \eta + \int_0^p \mathrm{d}s \int_0^s \mathrm{d}tE_0(t). \tag{5.6}$$

Set

$$E_0(a) = \ell_0.$$

Then it can easily be seen that the hypotheses of Lemma 4.2 and Theorem 4.3 hold if

$$h_{AH} = \overline{\ell}\eta \le \frac{1}{2}, \tag{5.7}$$

where

$$\overline{\ell} = \frac{1}{8}\left(\ell + 4\,\ell_0 + \sqrt{\ell^2 + 8\ell_0\ell}\right), \tag{5.8}$$

and

$$\alpha = \frac{2\ell}{\ell + \sqrt{\ell^2 + 8\ell_0\ell}}. \tag{5.9}$$

In view of (5.4) and (5.7), we have

$$h_K \le \frac{1}{2} \Longrightarrow h_{AH} \le \frac{1}{2}, \tag{5.10}$$

but not necessarily vice versa, unless $\ell_0 = \ell$.

Hence, in this special case, our Theorem 4.3 is weaker than [15, Theorem 5].

In the rest of the study, we provide examples, where (5.4) is violated but (5.7) is satisfied and $\ell_0 < \ell$. More applications can be found in [1–7,13–15].

Example 5.2. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$ equipped with the max-norm and

$$x_0 = (1, 1)^T, \quad U_0 = \{x : \|x - x_0\| \le 1 - \varpi\}, \quad \varpi \in \left[0, \frac{1}{2}\right).$$

Define function F on U_0 by

$$F(x) = (\xi_1^3 - \varpi, \xi_2^3 - \varpi), \quad x = (\xi_1, \xi_2)^T. \tag{5.11}$$

Using hypotheses of Theorem 3.1, we get

$$\eta = \frac{1}{3}(1 - \varpi), \quad \ell_0 = 3 - \varpi, \quad \text{and} \quad \ell = 2(2 - \varpi).$$

The condition (5.4) is violated, since

$$\frac{4}{3}(1 - \varpi)(2 - \varpi) > 1 \quad \text{for all } \varpi \in \left[0, \frac{1}{2}\right).$$

Hence, there is no guarantee that (NM) converges to $x^\star = (\sqrt[3]{\varpi}, \sqrt[3]{\varpi})^T$, starting at x_0.

However, our condition (5.7) is true for all $\varpi \in I = [.450339002, \frac{1}{2})$. Hence, the conclusions of our Theorem 3.1 can apply for solving Eq. (5.11) for all $\varpi \in I$.

Example 5.3. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$ be the space of real-valued continuous functions defined on the interval $[0, 1]$ with norm

$$\|x\| = \max_{0 \le s \le 1} |x(s)|.$$

Let $\theta \in [0, 1]$ be a given parameter. Consider the "cubic" integral equation

$$u(s) = u^3(s) + \lambda u(s) \int_0^1 q(s, t) u(t) \mathrm{d}t + y(s) - \theta. \tag{5.12}$$

Here the kernel $q(s, t)$ is a continuous function of two variables defined on $[0, 1] \times [0, 1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0, 1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0, 1]$. Equations of the form (5.12) arise in the kinetic theory of gases [5]. For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s, t) = \frac{s}{s+t}$, for all $s \in [0, 1]$ and $t \in [0, 1]$, with $s + t \neq 0$. If we let $\mathcal{D} = U(u_0, 1 - \theta)$, and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) - x(s) + \lambda x(s) \int_0^1 q(s, t) x(t) \mathrm{d}t + y(s) - \theta, \tag{5.13}$$

for all $s \in [0, 1]$, then every zero of F satisfies Eq. (5.12). We have the estimates

$$\max_{0 \le s \le 1} \left| \int \frac{s}{s + t} \mathrm{d}t \right| = \ln 2.$$

Therefore, if we set $\xi = \|F'(u_0)^{-1}\|$, then it follows from hypotheses of Theorem 3.1 that

$$\eta = \xi(|\lambda| \ln 2 + 1 - \theta),$$
$$\ell = 2\xi(|\lambda| \ln 2 + 3(2 - \theta)) \quad \text{and} \quad \ell_0 = \xi(2|\lambda| \ln 2 + 3(3 - \theta)).$$

It follows from Theorem 3.1 that if condition (5.7) holds, then problem (5.12) has a unique solution near u_0. This assumption is weaker than the one given before using the Newton–Kantorovich hypothesis (5.4).

Note also that $\ell_0 < \ell$ for all $\theta \in [0, 1]$.

Example 5.4. Consider the following nonlinear boundary value problem [5]:

$$\begin{cases} u'' = -u^3 - \gamma u^2 \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 Q(s,t)(u^3(t) + \gamma u^2(t))\mathrm{d}t \tag{5.14}$$

where Q is the Green function

$$Q(s,t) = \begin{cases} t(1-s), & t \le s \\ s(1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0 \le s \le 1} \int_0^1 |Q(s,t)| = \frac{1}{8}.$$

Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$, with norm

$$\|x\| = \max_{0 \le s \le 1} |x(s)|.$$

Then problem (5.14) is in the form (1.1), where $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)](s) = x(s) - s - \int_0^1 Q(s,t)(x^3(t) + \gamma x^2(t))\mathrm{d}t,$$

and

$$G(x)(s) = 0.$$

It is easy to verify that the Fréchet derivative of F is defined in the form

$$[F'(x)v](s) = v(s) - \int_0^1 Q(s,t)(3x^2(t) + 2\gamma x(t))v(t)\mathrm{d}t.$$

If we set $u_0(s) = s$ and $\mathcal{D} = U(u_0, R)$, then since $\|u_0\| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. It follows that $2\gamma < 5$; then

$$\|I - F'(u_0)\| \le \frac{3\|u_0\|^2 + 2\gamma\|u_0\|}{8} = \frac{3 + 2\gamma}{8},$$
$$\|F'(u_0)^{-1}\| \le \frac{1}{1 - \frac{3+2\gamma}{8}} = \frac{8}{5 - 2\gamma},$$
$$\|F(u_0)\| \le \frac{\|u_0\|^3 + \gamma\|u_0\|^2}{8} = \frac{1+\gamma}{8},$$
$$\|F(u_0)^{-1}F(u_0)\| \le \frac{1+\gamma}{5 - 2\gamma}.$$

On the other hand, for $x, y \in \mathcal{D}$, we have

$$[(F'(x) - F'(y))v](s) = -\int_0^1 Q(s,t)(3x^2(t) - 3y^2(t) + 2\gamma(x(t) - y(t)))v(t)\mathrm{d}t.$$

Consequently (see [5]),

$$\|F'(x) - F'(y)\| \le \frac{\gamma + 6R + 3}{4}\|x - y\|,$$
$$\|F'(x) - F'(u_0)\| \le \frac{2\gamma + 3R + 6}{8}\|x - u_0\|.$$

Therefore, the conditions of Theorem 3.1 hold with

$$\eta = \frac{1+\gamma}{5-2\gamma}, \quad \ell = \frac{\gamma + 6R + 3}{4}, \quad \ell_0 = \frac{2\gamma + 3R + 6}{8}.$$

Note also that $\ell_0 < \ell$.

Finally note that the results obtained here compare favorably to the ones given in [11,17,18] for the special case where $\mathcal{X}$ is a Banach space with real norm $\| \, . \, \|$.

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