On the Dynamical Height Zeta Functions

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Let $K$ be a global field (a number field or a function field over finite field) and let $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a polynomial map over $K$ whose degree is greater than one. By a polynomial map we mean a map with a totally ramified fixed point. Using the canonical height associated with $f$, Silverman defines a function associated with the dynamical system of a polynomial map on $\mathbb{P}^1$ [12]. We call such functions dynamical zeta functions which are commonly used in the theory of dynamical systems.

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Let $\phi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a morphism over $K$ with degree $d$ greater than one, the canonical height $\hat{H}_{\phi, E}$ associated with $\phi$ and a divisor class $E$ is given by the following formula [4, Theorem 1.1]

$$\hat{H}_{\phi, E}(P) = \lim_{n \to \infty} H_{E}(\phi^n(P))^{d^{-n}}$$

for all $P \in \mathbb{P}^1(K)$, where $H_{E}(P)$ is an ordinary Weil height in $\mathbb{P}^1$ associated to the divisor class of $E$. The canonical height $\hat{H}_{\phi, E}$ satisfies the following properties:

1. $\hat{H}_{\phi, E}$ is a Weil height associated with the divisor class $E$.
2. $\hat{H}_{\phi, E} \circ \phi = \hat{H}_{d\phi}$ where $d \geq 2$ is the degree of the morphism.

For more details, see [4].

We shall take $\phi$ to be a polynomial map and $E$ to be the divisor associated to the unique totally ramified fixed point of $\phi$ and fix an affine coordinate function $z$ whose polar divisor is $E$. In the following, the subscript $E$ will be omitted if there is no danger of confusion. The dynamical height zeta function that we will consider is

$$Z_{K}(\phi, s) = \sum_{\pi \in K} \hat{H}_{\pi}(z)^{-s} \quad \text{for} \quad \Re(s) > 2.$$
By results on the cardinality of rational points of bounded height in projective space ([10] for number fields and [11, 13] for function fields over finite field). The series $Z_K(\phi, s)$ converges absolutely and defines an analytic function for $\Re(s) > 2$. Silverman conjectured that $Z_K(\phi, s)$ has a meromorphic continuation to the whole complex plane with a simple pole at $s = 2$ and gave a conjectural formula of the residue at $s = 2$ of $Z_K(\phi, s)$.

In this note, we assume that the given morphism $\phi$ has mildly bad reduction which will be defined below in Section 3 and compute $Z_K(\phi, s)$ in the case of function field over finite fields. Our result confirms Silverman’s conjecture in our case. We now describe the contents of this paper.

In Section 1 we establish a counting lemma (Lemma 1.3) which gives the number of solutions in $K$ with given height to the diophantine inequalities

$$v(z - x_v) \geq n_v \quad \text{for each} \quad v \in S,$$

where $S$ is a finite set of primes of the function field $K$ and $x_v \in K_v$, $n_v \in \mathbb{Z}$ are prescribed data. With large height, one expects that the number of solutions of the inequalities is proportional to the size of each disc in question with center $x_v$ and radius $|\pi_v^n|$. In order to compute the dynamical height zeta function, one needs more precise information about the number of solutions and this needs extra work. The purpose of Section 1 is to attain this goal.

In Section 2, we define the partial height zeta function which is a series associated to the solutions of the diophantine inequalities studied in Section 1. Let

$$W(D_S, s) = \sum_{\pi} H(\pi)^{-s}$$

to be our partial height zeta function, where $\pi$ runs through all the solutions of the diophantine inequalities and $H(\cdot)$ is the ordinary Weil height function on $\mathbb{P}^1$. Instead of $W(D_S, s)$, we consider the generating function of the solutions

$$Z(D_S, t) = \sum f_{\text{deg}(\pi), v}.$$  

As an application of Lemma 1.3, $Z(D_S, t)$ is shown to be a rational function in $t$ (Theorem 2.1); moreover, it can be expressed in terms of the zeta function

$$Z_S(t) = \prod_{v \in S} (1 - t^{f_v})^{-1}$$

and an error term, where $f_v$ is the degree of the prime $v$. Letting $t = q^{-s}$ and substituting into $Z(D_S, t)$, $W(D_S, s)$ is a rational function in $q^{-s}$ and hence...
it has a meromorphic continuation to the whole complex plane. By considering $H(a)^{-s}$ as a continuous function on the adele ring $\mathbb{A}_K$ of $K$, $W(D_{uv}, s)$ has a representation as the integral of $H(a)^{-s}$ over a subset of the adele ring determined by the inequalities plus an error term which is regular at $s = 2$ (Theorem 2.2). The result is generalized in Corollary 2.3 which will be used in the computation of the dynamical height zeta functions.

Section 3 is devoted to the relationship between the reduction of morphism $\phi$ and the canonical local height associated to the morphism. First, we give the definition of mildly bad reduction: there exists a finite, projective model of $\mathbb{P}^1$ over $\mathcal{O}_K$ so that $\phi$ extends to an $\mathcal{O}_K$-morphism on the smooth part of the model. Note the distinction between mildly bad reduction and the notion of weak Néron model defined in [4] is that the $\mathcal{O}_K$-morphism is not required to be finite. For polynomial maps, the difference function between canonical local height and ordinary local height assumes only finitely many values on $K_v$ in the case of mildly bad reduction (Proposition 3.1). The same assertion in the case of good reduction follows directly from Call and Silverman’s result in [4]. The result in this section is the key to show that the dynamical height zeta functions have meromorphic continuation to the whole complex plane.

We prove our main result (Theorem 4.2) in Section 4. We show that the dynamical height zeta function associated to a polynomial map whose reduction is at worst mildly bad has a meromorphic continuation to the whole complex plane. The dynamical height zeta functions also have a representation in terms of the Dedekind zeta function $\zeta_K(s)$ and the integral of the canonical local heights $H_\phi, v(a_v)^{-s}$ (see Section 3) and the ordinary local heights $H_v(a_v)^{-s}$ over the local fields $K_v$ for $v$ the place where $\phi$ has mildly bad reduction. To be more precise, Let $S$ be a finite subset of the primes of $K$ at which $\phi$ has mildly bad reduction and let $r'_v$ be the Haar measure so that the ring of integers $\mathcal{O}_v$ of $K_v$ gets measure one. Denote the degree of $v$ by $f_v$, we have

**Theorem.** Let $K$ be a function field of genus $g$ over $\mathbb{F}_q$, and let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a polynomial map over $K$ with at worst mildly bad reduction. Then the dynamical height zeta function $Z_K(\phi, s)$ has a meromorphic continuation to the whole complex plane and

$$Z_K(\phi, s) = q^{1 - \frac{\zeta_K(s)}{\zeta_K(s)}} \prod_{v \in S} \int_{K_v} H_\phi, v(a_v)^{-s} d\tau'_v + R(s)$$

where $R(s)$ has the property that $\{\prod_{v \in S} (1 - q^{-n_v})\} \zeta_K(s) R(s)$ is a $\mathbb{Q}$-linear combination of $q^{-n}$ for finitely many rational numbers $n_v$.
As a result of the main theorem, the residue of the dynamical height zeta function is

$$\text{Res}_{s=2} \zeta_K(s) = \text{Res}_{s=2} \zeta_K(s-1) \left( \frac{q^{1-s}}{\zeta_K(2)} \prod_{v \in S} \int_{K_v} H_\phi(t^v)^{-2} \ dt^v \right).$$

For the quantity $\text{Res}_{s=2} \zeta_K(s-1)$ see [14, pp. 130 Theorem 4] and by direct computation $\int_{K_v} H_\phi(t^v)^{-2} \ dt^v = 1 + g^{-\deg v}$. An application of Silverman's conjecture on the dynamical height zeta function is to compute the cardinality of rational points with bounded canonical height. Namely, let $\omega_\phi(x)$ be the cardinality of the set

$$\{ x \in K \mid \hat{H}_\phi(x) \leq x \}.$$

If Silverman's conjecture is true for the polynomial map $\phi$, then $\omega_\phi(x) \sim (c_\phi/2) x^2$ for $x$ sufficiently large, where $c_\phi$ is the residue of $Z_K(\phi, s)$ at $s = 2$. Our result is the function field analogue of Silverman's conjecture about the dynamical height zeta function and provide positive evidences to support his conjecture.

Throughout this paper, the following data are fixed.

- $K$ function field of genus $g$ over the finite field $k = \mathbb{F}_q$,
- $M_K$ the set of normalized valuation of $K$,
- $K_v$ the $v$-adic completion of $K$ with respect to $v \in M_K$,
- $\mathcal{O}_v$ the ring of integers of $K_v$,
- $\pi_v$ a uniformizer of the maximal ideal of $\mathcal{O}_v$,
- $k_v$ the residue field $\mathcal{O}_v/\langle \pi_v \rangle$ at the place $v$,
- $f_v$ the degree of $v$, that is $\dim_k k_v$,
- $\mathcal{C}$ a smooth curve over $k$ with function field $K$,
- $\mathcal{K}_q$ a canonical divisor of the curve $\mathcal{C}$,
- $\text{Div}_k(\mathcal{C})$ the group of divisors of $\mathcal{C}$ defined over $\mathbb{F}_q$,
- $\text{Supp}(D)$ the support of divisor $D \in \text{Div}_k(\mathcal{C})$,
- $f_D$ the degree of divisor $D = \sum_{v \in M_K} \varepsilon(D) f_v$,
- $z$ an affine coordinate of $\mathbb{P}^1$.

We will let $v$ denote the corresponding prime divisor and $\varepsilon(D)$ denote the order of the divisor $D$ at the place $v$ in the following.

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1. PRELIMINARY LEMMAS

**Definition 1.** For any divisor $D \in \text{Div}_k(\mathbb{C})$, we let

$$L(D) = \{ x \in K^* : \text{div}(x) + D \geq 0 \} \cup \{ 0 \}.$$  

Given any positive divisor $E \in \text{Div}_k(\mathbb{C})$, $E = \sum_{v \in \text{Supp}(E)} n_v v$, where $n_v \geq 0$ and $n_v = 0$ for almost all $v$, if the support of $D$ is disjoint from that of $E$ then every element $x$ of $L(D)$ is $v$-integral for all $v \in \text{Supp}(E)$. We'll identify $L(D)$ as a subset of the ring $\prod_{v \in \text{Supp}(E)} \mathbb{C}_v$. The canonical map $\prod_{v \in \text{Supp}(E)} \mathbb{C}_v \rightarrow \mathbb{C}_E = \prod_{v \in \text{Supp}(E)} \mathbb{C}_v / \mathbb{C}_v^n$ gives a reduction of elements of $L(D)$. In the following, for any $a \in \prod_{v \in \text{Supp}(E)} \mathbb{C}_v$, we let $\bar{a}$ denote the residue of $a$ in $\mathbb{C}_E$.

For any divisor $D \in \text{Div}_k(\mathbb{C})$, we agree that $D = D_+ - D_-\mathbb{C}$ where $D_+, D_-$ are positive divisors disjoint support. We'll use the following notations:

$$\ell(D) \quad \text{dim}_K L(D),$$

$$L(D)_a \quad \{ x \in L(D) : \bar{x} = \bar{a} \},$$

$$\tilde{L}(D)_a \quad \{ x \in L(D)_a : (x)_a = D_+ \},$$

$N(D)$ the cardinality of $L(D)$,

$N(D)_a$ the cardinality of $L(D)_a$.

$N(D)_a$ the cardinality of $\tilde{L}(D)_a$.

The following lemma is concerned with the distribution of elements of $L(D)$ in the residue classes of $\mathbb{C}_E$.

**Lemma 1.1.** Let $D, E \in \text{Div}_k(\mathbb{C})$ be divisors such that $D$ and $E$ have disjoint support. Assume that $E$ is a positive divisor and $\deg(D - E) \geq 2g - 1$. Then the elements of $L(D)$ are equally distributed in each class of $\mathbb{C}_E$ and the number is equal to $q^{\ell(D)} - q^{\ell(D - E)}$.

**Proof.** Since $L(D)$ is a vector space, the number of elements of the same residue class in $\mathbb{C}_E$ is equal to the number of those of the zero class in $\mathbb{C}_E$. Thus, the number of elements in $L(D)$ which are of the same residue class in $\mathbb{C}_E$ is independent of the classes in $\mathbb{C}_E$. Since the total number of classes in $\mathbb{C}_E$ is $q^\ell$, it therefore suffices to prove the following equality

$$q^\ell N(D) = N(D - E).$$

Since $N(D) = q^{\ell(D)}$ and $N(D - E) = q^{\ell(D - E)}$, by the Riemann–Roch Theorem,

$$\ell(D) = \deg(D) + 1 - g + \ell(\mathbb{C}_E - D)$$

$$\ell(D - E) = \deg(D - E) + 1 - g + \ell(\mathbb{C}_E - D + E)$$

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The hypothesis on the degree of \( D \) gives
\[
\ell(K_{a} - D) = \ell(K_{a} - D + E) = 0.
\]
We conclude that \( q^{\deg N(D)} = N(D - E) \) and this proves the lemma.

**Remark.** Lemma 1.1 can be thought of as a variant of the Chinese Remainder Theorem. The lower bound \( 2g - 1 \) is best possible since in the case that \( \deg(D - E) = 2g - 2 \) and \( D - E \) is linearly equivalent to the canonical divisor \( K_{a} \), \( N(D - E) \neq s^{-\deg N(D)} \).

Let \( S \) denote a finite set of primes of \( K \). For each \( v \in S \), a fixed element \( x_{v} \in K_{v} \) and an integer \( n_{v} \) are given. Consider the following inequalities:
\[
v(z - x_{v}) \geq n_{v}, \quad v \in S, \quad (1)
\]
where \( z \in K \) and \( (z)_{x_{v}} = D \). In the computation of the partial height zeta functions, we need to know the number of the solutions to the inequalities above.

Before we compute the number of solutions, we define the following divisor determined by the solutions of inequalities (1).

**Definition 2.** Let \( (z)_{s} \) denote the components of \( \text{div}(z) \) whose supports are in \( S \), then define
\[
E_{0}^{v} - E_{0}^{v} = \inf((z)_{s}),
\]
where the function \( z \) ranging over the solutions of (1).

Since \( v(z - x_{v}) \geq n_{v} \) for all \( v \in S \) and \( v(z) \geq \min\{(v(x_{v}), v(z - x_{v}))\} \), both \( E^{0} \) and \( E^{v} \) exist. Moreover, we may change the centers \( x_{v} \) such that \( v(x_{v}) = v(E_{0}^{0} - E^{v}) \) for all \( v \in S \) since the valuations are non-archimedean.

**Lemma 1.2.** (A) \( v(E^{0} - E^{v}) \leq n_{v} \) for all \( v \in S \).

(B) If \( v(E^{0} - E^{v}) < n_{v} \), then \( v(z) = v(E^{0} - E^{v}) \) for all \( z \) satisfying (1).

**Proof.** Let \( m_{v} = v(E^{0} - E^{v}) \) for \( v \in S \).

(A) If \( v(x_{v}) < n_{v} \), then for all \( z \) satisfying (1), \( v(z) = v(x_{v}) < n_{v} \). Assume that \( v(x_{v}) \geq n_{v} \), we may choose \( x_{v} \) such that \( v(x_{v}) = n_{v} \). By the approximation theorem, there exists an element \( z \in K \) such that \( z \) satisfies (1) and \( v(z - x_{v}) > n_{v} \). It follows, \( v(z) = v(x_{v}) = n_{v} \). Therefore \( m_{v} \leq n_{v} \).

(B) Assume \( m_{v} < n_{v} \) for some \( v \in S \) and let \( z_{0} \) be a solution of (1) such that \( v(z_{0}) = m_{v} \). Then, for any \( z \) which is a solution of (1), \( v(z) \geq \min(v(z_{0}), v(z - z_{0})) \). But, \( v(z_{0}) = m_{v} < n_{v} \leq v(z - z_{0}) \), it follows \( v(z) = v(z_{0}) = m_{v} \).
The Möbius function can be defined on the monoid of non-negative divisors and given by the following formula:

\[
\mu(D) = \begin{cases} 
1 & \text{if } D = 0, \\
(-1)^i & \text{if } n_i = 1 \text{ for all } i, \\
0 & \text{if some } n_i > 0.
\end{cases}
\]

where \( D = \sum_{i=1}^{l} n_i v_i \) and \( n_i \geq 0 \) for all \( i = 1, \ldots, l \).

We set \( E = \sum_{v \in S} n_v v \) and \( E^\circ = \delta + \delta' \), where \( \delta, \delta' \) have disjoint support and

\[
v(E^\circ) = \begin{cases} 
-n_v & \text{if } v \in \text{Supp}(\delta'), \\
>-n_v & \text{if } v \in \text{Supp}(\delta).
\end{cases}
\]

**Lemma 1.3.** Let \( z \in K \) be a solution of the inequalities (1), then \((z)_w = D + \delta\), where \( D = D + E^\circ \) such that \( \text{Supp}(D) \cap S = \emptyset \) and \( 0 \leq E_0 \leq \delta' \). Let \( N(D)_w \) denote the number of solutions of (1) with \((z)_w = D + \delta\) and let \( \tilde{N}(D)_w \) denote the number of solutions of (1) with \((z)_w = D + \delta\). Then,

\[
\tilde{N}(D)_w = q^{\deg(D)} - q^{-\deg(D)} \sum_{0 < \delta' \leq D, \deg(D') \geq \eta_{E + \delta'}} \mu(D - D') q^{\deg(D')} + \sum_{0 < \delta' \leq D, \deg(D') < \eta_{E + \delta'}} \mu(D - D') N(D'_w),
\]

where \( \eta_{E + \delta'} = f_{E + \delta'} + 2g - 1 \).

**Proof.** By the approximation theorem, there exists an element \( c \in K \) such that \( v(z - c) \geq n_v \) for all \( v \in S \) and we may choose \( c \) so that \( v(c) = v(E^0 - E^\circ) \) for all \( v \in S \). Moreover, \( v(c) \leq n_v \) for all \( v \in S \) by Lemma 1.2(A). We replace (1) by the following:

\[
v(z - c) \geq n_v, \quad v \in S.
\]

Note that for any \( z \in K \) satisfies (1) if and only if it satisfies (2). Therefore, the number of solutions \( N(D)_w \) and \( \tilde{N}(D)_w \) are independent of the choice of the center \( c \).

If \( c = 0 \), then \( E^0 = E^\circ = 0 \). It follows \( n_v \)'s are positive for all \( v \in S \). The result follows from Lemma 1.1 and the inclusion-exclusion principle. We assume \( c \neq 0 \) and let

\[
(c)_0 = A_c + E^0 \quad (c)_w = B_c + E^\circ
\]
where $\text{Supp}(A_c + B_c)$ is disjoint from $S$. To solve the inequalities
\[ v(z - c) \geq v(E) \quad v \in S, \]
it is equivalent to solving
\[ v(x - 1) \geq v(E') \quad v \in S, \tag{3} \]
where $x = z/c$ and $E' = E - E^0 + E^\infty$. Since $v(c) \leq v(E)$ for all $v \in S$, $E'$ is a positive divisor. Thus, $x$ is $v$-integral for all $v \in S$ and hence the support of the polar divisor of $x$ is disjoint from that of $E'$. Put
\[ (x)_0 = D_0 + B \quad (x)_w = D + A, \]
where $\text{Supp}(D_0 + D) \cap \text{Supp}(A_c + B_c) = \emptyset$ and
\[ \text{Supp}(A) \subseteq \text{Supp}(A_c), \]
\[ \text{Supp}(B) \subseteq \text{Supp}(B_c). \]

The divisor of $z$ is
\[ \text{div}(z) = \text{div}(x) + \text{div}(c) = E^0 + (D_0 - E^\infty) + (A_c - A) + (B - B_c) - D. \]

Set
\[ \hat{A} = \sup (0, A - A_c), \]
\[ \hat{B} = \sup (0, B_c - B), \]
\[ \hat{E} = \sup (0, E^\infty - D_0). \]

By the definition of $\mathcal{E}$ and $\mathcal{E}'$,
\[ v(E') \begin{cases} > 0 & \text{if } v \in \text{Supp}(\mathcal{E}), \\ = 0 & \text{if } v \in \text{Supp}(\mathcal{E}'). \end{cases} \]

$\text{Supp}(D_0)$ is disjoint from $\text{Supp}(\mathcal{E})$ and therefore $\hat{E} = E_c + E_\infty$, $0 \leq E_\infty \leq \mathcal{E}'$. The polar divisor of $z$ is
\[ (z)_w = D + \hat{E} + \hat{A} + \hat{B}, \]
\[ = \mathcal{D} + \mathcal{E}. \]
to ease the notation, we let $D = D + E_x + A + B$. Since
\[ \hat{A} = -A = \text{sup} (-A, -A) \geq -A, \]
\[ \hat{B} = B = \text{sup} (B, B_x) \geq B_x, \]
\[ \hat{E} + D_0 = \text{sup} (D_0, E^\infty) \geq E^\infty, \]

\[ \text{div}(x) + \mathcal{D} + \mathcal{E} + A_x \geq B_x + E^\infty. \]

By definition, $z = xc$ and $E^\infty = \mathcal{E} + \mathcal{E}'$, for any $x \in L(\mathcal{D} + A_x - B_x - \mathcal{E}')$

\[ \text{div}(z) + \mathcal{D} + \mathcal{E} \geq E^0. \]

It follows that every $x \in L(\mathcal{D} + A_x - B_x - \mathcal{E}') \cap \mathcal{D}$ gives a $z$ satisfying (2) with

\[ \text{div}(z) = (z_0 - (\mathcal{D} + \mathcal{E})), \]

where $E^0 \leq (z_0)$ and $0 \leq \mathcal{D}' \leq \mathcal{D}$, and vice versa. We have

\[ N(D)_* = \sum_{0 \leq \mathcal{D}' \leq \mathcal{D}} \tilde{N}(\mathcal{D}')_* . \]

By the inclusion-exclusion principle and Lemma 1.1,

\[ \tilde{N}(\mathcal{D})_* = \sum_{0 \leq \mathcal{D}' \leq \mathcal{D}} \mu(\mathcal{D} - \mathcal{D}') N(\mathcal{D}')_* \]

\[ = \sum_{0 \leq \mathcal{D}' \leq \mathcal{D}} \mu(\mathcal{D} - \mathcal{D}') N(\mathcal{D}' + A_x - B_x - \mathcal{E}')_1 \]

\[ = \sum_{0 \leq \mathcal{D}' \leq \mathcal{D}} \mu(\mathcal{D} - \mathcal{D}') q^{(1 - r)q - \text{deg} \mathcal{E}'} q^{\text{deg} \mathcal{E}} \]

\[ + \sum_{0 \leq \mathcal{D}' \leq \mathcal{D}} \mu(\mathcal{D} - \mathcal{D}') N(\mathcal{D}')_* \]

\[ = q^{(1 - r)q - f_{E_x, E'}} \sum_{0 \leq \mathcal{D}' \leq \mathcal{D}} \mu(\mathcal{D} - \mathcal{D}') q^{\text{deg} \mathcal{E}} \]

\[ + \sum_{0 \leq \mathcal{D}' \leq \mathcal{D}} \mu(\mathcal{D} - \mathcal{D}') N(\mathcal{D}')_* \]

In the course of the computation we use the relation $\text{deg} (A_x + E^0 - B_x - E^\infty) = \text{deg} (\text{div}(x)) = 0$ and $E^\infty = \mathcal{E} + \mathcal{E}'$ to simplify the notations.
2. COMPUTATION OF PARTIAL HEIGHT ZETA FUNCTIONS

Let $S$ be a finite set of primes in $M_K$. For each $v \in S$, a fixed element $x_v \in \mathcal{O}_v$ and an integer $n_v \geq 0$ are given. We define a disc $D_v$ as follows:

$$D_v = \{ y \in K_v : |y - x_v|_v \leq |\pi^n_v|_v \}.$$ 

We identify $K$ with its image in $\prod_{v \in S} K_v$ arising from the canonical embedding $K \hookrightarrow K_v$ for each $v \in S$. Let

$$D_S = K \cap \prod_{v \in S} D_v.$$ 

Let $z$ be an affine coordinate on $\mathbb{P}^1$ and denote its pole by $\infty$ which is the point $[1, 0]$.

The ordinary height function on $\mathbb{P}^1$ is defined by

$$H(P) = \prod_{v \in M_K} H_v(P)$$

for any $P \in \mathbb{P}^1(\bar{K})$, where $H_v(\cdot)$ denotes a Weil local height associated with the divisor $\infty$ over $K_v$ and is defined by the well known formula

$$H_v(Q) = \max \{ 1, |z(Q)|_v \} \quad \text{for} \quad Q \in \mathbb{P}^1(\bar{K}) \setminus \{ \infty \}.$$ 

DEFINITION 3. Let $S, D_S$ be given as above, the partial height zeta function is the following series:

$$W(D_S, s) = \sum_{x \in D_S} H(x)^{-s} \quad \text{for} \quad \Re(s) > 2.$$ 

By results in [11, 13] on the cardinality of rational points with bounded height in $\mathbb{P}^n$ over function fields, the series in the definition converges absolutely for $\Re(s) > 2$.

Let

$$Z_K(t) = \sum_{0 \leq D} t^{\deg(D)},$$

$$Z_S(t) = \sum_{D \leq S, \text{Supp}(D) \cap S = \emptyset} t^{\deg(D)},$$

$$Z(D_S, t) = \sum_{x \in D_S} t^{\deg(x)},$$

$$\zeta_K(s) = Z_K(q^{-s}).$$

As the notations we used in Section 1, we’ll write $E = \sum_{v \in S} n_v v, E' = E + E'$ and $\text{div}(z) = (z)_0 - (\mathcal{D} + E)$, where $\mathcal{D} = D + E_v$ and $0 \leq E_v \leq E'$. 


Theorem 2.1. $Z(D_S, t)$ is a rational function in $t$. Moreover,

$$Z(D_S, t) = q^{1 - q^{f_{c,x} - \sigma}} \frac{Z_S(qt)}{Z_S(t)} t^{\deg d} \prod_{\nu \in \text{Supp}(d)} \left( 1 + \sum_{\ell = 0}^{n} \left( 1 - \frac{1}{q^{\ell}} \right) (q^{\ell})^t (t^{\ell})^t \right)$$

$$+ \frac{t^{\deg d}}{Z_S(t)} \prod_{\nu \in \text{Support}(d - E_1)} (1 - t^{\nu})$$

$$\times \sum_{\deg (D + E_1) < \deg D} \delta(D + E_1) t^{\deg D},$$

where $\delta(D) = N(D) - q^{1 - q^{f_{c,x} - \sigma}} q^{\deg D}$.

Proof. By definition,

$$Z(D_S, t) = \sum_{0 \leq D, (s) \in \sigma} \sum_{0 \leq D' \leq D, \deg (D' + d) < \deg D} \mu(D - D'), N(D') t^{\deg D'} t^{\deg (D')},$$

Put

$$(A) = q^{1 - q^{f_{c,x} - \sigma}} \sum_{0 \leq D} \sum_{0 \leq D' \leq D} \mu(D - D') q^{\deg (D')} q^{\deg (D)}$$

$$(B) = \sum_{0 \leq D} \sum_{0 \leq D' \leq D} \mu(D - D') N(D') t^{\deg D'} t^{\deg (D)}$$

$$- q^{1 - q^{f_{c,x} - \sigma}} \sum_{0 \leq D} \sum_{0 \leq D' \leq D} \mu(D - D') q^{\deg (D')} q^{\deg (D')},$$

By Lemma 1.3,

$$Z(D_S, t) = t^{\deg (D)} \{(A) + (B)\}.$$
Then,
\[
q^{-1 - \varepsilon - f x + x} (A) = \sum_{0 \leq D} \sum_{0 \leq E_{1} < E_{2}} \sum_{D_{1} + D_{2} = D_{1}} \mu(D_{2}) \mu(E_{2}) (qt)^{\deg(D_{1}) \cdot t^{\deg(E_{1})}} \cdot t^{\deg(E_{2})}
\]
\[
= \sum_{0 \leq D_{1}} (qt)^{\deg(D_{1})} \sum_{0 \leq D_{2}} \mu(D_{2}) t^{\deg(D_{2})}
\]
\[
\times \sum_{0 \leq E_{1} < E_{2}} \mu(E_{2}) q^{\deg(E_{1})} \cdot t^{\deg(E_{2})}
\]
\[
= \frac{Z_{S}(qt)}{Z_{S}(t)} \sum_{0 \leq E_{1} < E_{2}} \sum_{0 \leq E_{2}} \mu(E_{2}) q^{-\deg(E_{2})} (qt)^{\deg(E_{2})}
\]
\[
= \frac{Z_{S}(qt)}{Z_{S}(t)} \sum_{0 \leq E_{1}} \prod_{v \in \text{Supp}(E_{1})} \left( 1 + \sum_{i=1}^{n_{v}} \left( 1 - \frac{1}{q^{v}} \right) (q^{v})^{i} \cdot (t^{v})^{i} \right)
\]

Set \( \delta(\mathfrak{D}) = N(\mathfrak{D})_{0} - \frac{1}{q^{1-\varepsilon - f x + x}} q^{\deg(\mathfrak{D})} \), then

\[
(B) = \sum_{0 \leq D_{1} + D_{2} = \mathfrak{D}} \mu(D_{2}) \delta(\mathfrak{D}_{1}) t^{\deg(\mathfrak{D})}
\]
\[
= \sum_{\deg(\mathfrak{D}) \leq \eta_{x + x}} \sum_{0 \leq D_{1}} \mu(D_{2}) t^{\deg(D_{2})}
\]
\[
\times \sum_{0 \leq E_{1} < E_{2}} \mu(E_{2}) t^{\deg(E_{2})} \delta(\mathfrak{D}_{1}) t^{\deg(\mathfrak{D})}
\]
\[
= \frac{1}{Z_{S}(t)} \sum_{\deg(\mathfrak{D}) \leq \eta_{x + x}} \sum_{0 \leq E_{1} < E_{2}} \mu(E_{2}) t^{\deg(E_{2})} \delta(\mathfrak{D}_{1}) t^{\deg(\mathfrak{D})}
\]
\[
= \frac{1}{Z_{S}(t)} \sum_{\mathfrak{D}_{1}} \prod_{v \in \text{Supp}(\mathfrak{D}_{1})} (1 - t^{v})
\]
\[
\times \sum_{\deg(D_{1}) \leq \eta_{x + x}} \delta(D + E_{1}) t^{\deg(D_{1})}
\]

Let \( \mathbb{A}_{K} \) denote the adele ring of \( K \) and let \( \mathbb{D} \) denote a fundamental domain for \( \mathbb{A}_{K}/K \). For any adele vector \( a \), its \( x \)-component is denoted
by \( a \). We'll use \( A_{D_v} \) to denote the subset of \( A_K \) consisting of adele vectors \( a \) such that \( a_v \in D_v \) for all \( v \in S \). We choose a Haar measure \( \tau \) on \( A_K \) so that \( \tau(D) = 1 \). By [14, pp. 100, Cor. 1], if \( \tau' \) is another Haar measure on \( A_K \) for which \( \tau'\big(\prod_v \epsilon_v\big) = 1 \). Then, \( \tau' = q^{-s} \tau \).

The height function \( H(a) \) is a continuous function on \( A_K \) since the local height function \( H_v(x_v) \) is trivial on \( \mathcal{O}_v \). With these convention and putting \( t = q^{-s} \) back into \( Z(D_S, t) \), we have the following

**Theorem 2.2.** Let \( S \) be a finite set of primes of \( K \) and let \( D_v \) be closed discs whose radius is less than or equal to a prescribed number \( q^{-n_v} \), where \( n_v \in \mathbb{Z} \) for all \( v \in S \). Let \( D_S = K \cap \prod_v D_v \) and let \( W(D_S, s) \) be the partial height zeta function associated with \( D_S \) as defined in Definition 3. Then \( W(D_S, s) \) is a rational function in \( q^{-s} \) and can be expressed as follows.

\[
W(D_S, s) = \int_{A_{D_S}} H(\tau)^{-s} d\tau + R(s)
\]

where \( R(s) \) has the property that \( \prod_v (1 - q^{-n_v} \zeta_k(s)) R(s) \) is a polynomial in \( q^{-s} \).

**Proof.** By Theorem 2.1, \( Z(D_S, t) = A(t) + B(t) \), where

\[
A(t) = q^{1-s} q^{-\deg D} Z_S(t) t^{\deg D} \\
\times \prod_{v \in \text{Supp}(E)} \left\{ 1 + \sum_{i=0}^{m_v} \left( 1 - \frac{1}{q} \right)^i \left( \frac{q}{q_i} \right)^i (t_f)_i \right\}
\]

\[
B(t) = \frac{t^{\deg D}}{Z_S(t)} \sum_{E_1 \leq E} t^{\deg E_1} \prod_{v \in \text{Supp}(E - E_1)} (1 - t_f) \\
\times \delta(D + E_1) t^{\deg D}
\]

and \( W(D_S, s) = Z(D_S, q^{-s}) \). The rationality of \( W(D_S, s) \) in \( q^{-s} \) follows from the rationality of \( Z(D_S, q^{-s}) \) in \( t \). Observe that

\[
\int_{A_{D_v}} H_v(\tau'_v)^{-s} d\tau'_v = \frac{1 - q^{-s}}{1 - q^{-1 - s}} \frac{1}{n_v},
\]
where \( \tau' \) is the Haar measure on \( K_r \) such that \( \mathcal{O}_v \) gets measure one. By the fact that

\[
Z_S(q^{-r}) = \prod_{v \in S} (1 - q^{-\sigma_v})^{-1},
\]

\[
A(q^{-r}) = q^{1-\varepsilon} \left\{ \prod_{v \in S} \int_{K_v} H_v(\tau'_v)^{-\varepsilon} \, d\tau'_v \right\} q^{-f_{\sum} - r} q^{-s\varepsilon} \times \prod_{v \in \text{Supp}(\varepsilon')} \left\{ 1 + \sum_{i=1}^n \left( \frac{1}{q^{\varepsilon'}} \right)^i (q^{\varepsilon'})^i (q^{-s\varepsilon'})^i \right\}.
\]

By the definition of \( E^\varepsilon = \varepsilon + \varepsilon' \) and Lemma 1.2, \( H_v(z) \) is either 1 or \( q^{\varepsilon(\varepsilon')} \) for any \( z \) satisfying (1) and for all \( v \in S \setminus \text{Supp} \varepsilon' \). The measure of \( D_v \) is \( q^{-\sigma_v} \), therefore,

\[
q^{-f_{\sum} - r} q^{-s\varepsilon} = \prod_{v \in S \setminus \text{Supp}(\varepsilon')} \int_{D_v} H_v(\tau'_v)^{-\varepsilon} \, d\tau'_v.
\]

One last thing to notice is the equality

\[
1 + \sum_{i=1}^n \left( \frac{1}{q^{\varepsilon'}} \right)^i (q^{\varepsilon'})^i (q^{-s\varepsilon'})^i = \int_{D_v} H_v(\tau'_v)^{-\varepsilon} \, d\tau'_v \quad v \in \text{Supp}(\varepsilon').
\]

Combine these identities and the remark mentioned above that \( \tau = q^{1-\varepsilon} \prod \tau'_v \), we conclude that

\[
A(q^{-r}) = \int_{\mathcal{A}_K} H(\tau)^{-\varepsilon} \, d\tau.
\]

Let \( R(s) = B(q^{-s}) \). Use the fact \( \prod_{v \in S} (1 - q^{-\sigma_v}) \zeta_K(s) = Z_S(q^{-s}) \), the assertion about \( R(s) \) follows trivially from the same assertion about \( B(t) \).  

Theorem 2.2 is easily seen to be extended to a more general situation. Namely, we define \( D_v \) to be the set of a closed disc minus a disjoint union of finitely many smaller discs or the complement of a disjoint union of finitely many discs in \( K_r \) and call \( D_v \) a domain in \( K_r \). \( D_v \) is a bounded domain in the former case and is called unbounded domain in the latter case. We still denote \( D_S = K \cap \prod_{v \in S} D_v \) and \( \mathcal{A}_K \), the subset of \( \mathcal{A}_K \) consisting of adele vectors \( a \) such that \( a_v \in D_v \) for all \( v \in S \) Then,
Corollary 2.3. Given $D_v$ a domain in $K_v$ for $v \in S$, the partial height zeta function associated to $D_S$

$$W(D_S, s) = \sum_{x \in D_S} H(x)^{-s}$$

is a rational function in $q^{-s}$. Moreover,

$$W(D_S, s) = \int_{\mathfrak{A}_{D_S}} H(\tau)^{-s} d\tau + R(s).$$

where $R(s)$ has the property that $\left[ \prod_{v \in S} (1 - q^{-s}) \right] \zeta_{K}(s) R(s)$ is a polynomial in $q^{-s}$.

Proof. For any $v \in S$, assume $D_v = D_{v,0} \bigcup_{i=1}^{l_v} D_{v,i}$, where the union is a disjoint union and $D_{v,i}$, $i = 1, \ldots, l_v$, are bounded closed discs. If $D_v$ is a bounded domain, then $D_{v,0}$ is also a closed disc, otherwise, $D_{v,0} = K_v$. Then,

$$\prod_{v \in S} D_v = \prod_{v \in S} \left( D_{v,0} \bigcup_{i=1}^{l_v} D_{v,i} \right) = \prod_{v \in S} D_{v,0} \bigcup \prod_{v \in S} D_{v,i}$$

where the unions are disjoint unions and $0 \leq j \leq l_v$ with $j \neq 0$ for some $j$. Let $D_{S,0} = K \cap \prod_{v \in S} D_{v,0}$. Then,

$${\mathfrak{A}}_{D_S} = {\mathfrak{A}}_{D_{S,0}} \bigcup {\mathfrak{A}}_{D_{S,\mu}}$$

where $\mu = \{ v, j \}_{v \in S}$ and $D_{v,j}$ is the subset of adele ring consisting of adele vectors $a_v \in D_{v,j}$ for $v \in S$. Since

$$D_S = D_{S,0} \bigcup D_{S,\mu},$$

$$W(D_S, s) = W(D_{S,0}, s) - \sum_{\mu} W(D_{S,\mu}, s).$$

Note that the union of $D_{S,\mu}$ is a disjoint union and the sum of $W(D_{S,\mu}, s)$ is a finite sum. Therefore, it suffices to show the assertion for each $W(D_{S,\mu}, s)$ where $\mu = \{ v, j \}_{v \in S}$ or $\mu = 0$. For $\mu = \{ v, j \}_{v \in S}$, $D_{S,\mu} = K \cap \prod_{v \in S} D_{v,j}$. If all the $D_{v,j}$ are bounded closed discs, the assertion is Theorem 2.2.

Assume $D_{v,0} = K_v$ for some $v^* \in S$ and let $S' = S \setminus \{ v^* \}$. Then, $D_{S,\mu} = D_{S',\mu'} = K \cap \prod_{v \in S'} D_{v,j}$ and $\mathfrak{A}_{D_{S,\mu}} = \mathfrak{A}_{D_{S',\mu'}}$ where $\mu' = \{ v, j \}_{v \in S'}$. By
induction on the number of unbounded $D_{x,0}$'s and the fact that if $S = \emptyset$ then

$$W(D_{S,\mu}, s) = q^{-s} \zeta_K(s-1) \zeta_K(s)$$

$$= \int_{\Delta_K} H(t)^{-s} \, dt.$$ 

We conclude that the assertion is true for any $D_{S,\mu}$ and so is the corollary.

3. REDUCTION OF MORPHISMS AND CANONICAL LOCAL HEIGHTS

Given a morphism $\phi: \mathbb{P}^1 \to \mathbb{P}^1$ over $K_v$, write $\phi(z) = f(z)/g(z) f(z), \ g(z) \in O_K[z]$ as a quotient of two polynomials. We would like to know if $\phi(z)$ is still a morphism by reducing modulo $\pi_v$ for $v \in M_K$. In the following, we will say that $\phi$ has good reduction at $v \in M_K$ if the resultant of polynomials $f$ and $g$ is a unit in $O_v$ and $\phi$ has bad reduction otherwise. In the case of bad reduction, if there exists a projective scheme $X$ of finite type over $\mathcal{C}_v$ and an $O_v$-morphism $X \to \mathbb{P}^1$ which is an isomorphism on the generic fiber such that $\phi$ extends to an $\mathcal{C}_v$-morphism which maps the smooth part of $X$ to itself, we say that $\phi$ has mildly bad reduction at $v$. If no such $X$ exists, we say $\phi$ has chaotic reduction at $v$. Note that the number of places where $\phi$ has bad reduction is finite.

Remark. In [4], Call and Silverman have introduced a notion of weak Néron model. A pair $(\mathbb{P}^1/K_v, \phi)$ is said to have a weak Néron model if there exists a smooth, finite model $X/\mathcal{C}_v$ of $\mathbb{P}^1$ with the property that every point of $\mathbb{P}^1(K_v)$ extends to a section in $X(\mathcal{C}_v)$ and a finite morphism $\Phi$ from $X$ to $X$ over $\mathcal{C}_v$ which extends $\phi$. In our case, $\Phi$ is not required to be a finite morphism.

For the convenience of our discussion, we shall use logarithmic height instead of multiplicative height in this section. The ordinary local height is denoted by $\lambda_v = \log_q H_v$. In the following, we will restrict ourselves to the case of polynomial maps with degrees greater than one. Recall that by a polynomial map we mean a map $\phi$ with a totally ramified point, denoted by $\infty$. In the case of polynomial map the canonical height $\lambda_{\phi,v}$ is computed by Tate's averaging procedure. We have the following formula:

$$\lambda_{\phi,v}(P) = \lim_{n \to \infty} \frac{(\lambda_v - \phi^n)(P)}{d^n} \quad \text{for} \quad P \in \mathbb{P}^1(K_v) \setminus \{\infty\}$$

(cf. [4, Theorem 2.1]), where $d$ is the degree of the polynomial $\phi$. 

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The global (logarithmic) ordinary height is denoted by \( h = \log_q H \) and the global (logarithmic) canonical height is denoted by \( \hat{h}_\phi = \log_q \hat{H}_\phi \). It has been shown in [4] that the canonical height \( \hat{h}_\phi \) has a decomposition as a sum of canonical local heights \( \hat{h}_{\phi,v} \). The canonical local heights \( \hat{h}_{\phi,v} \) are characterized by the following properties:

1. \( \hat{h}_{\phi,v} : \mathbb{P}^1_{K_v} \setminus \{ \infty \} \to \mathbb{R}^+ \) is a Weil local height associated with the divisor \( \{ \infty \} \).

2. \( \hat{h}_{\phi,v}(\phi P) = d \hat{h}_{\phi,v}(P) \) for all \( P \in \mathbb{P}^1_{K_v} \setminus \{ \infty \} \), where \( d > 1 \) is the degree of \( \phi \).

Set \( \hat{\gamma}_v(P) = \hat{h}_{\phi,v}(P) - \hat{\gamma}(P) \). We describe the behavior of \( \hat{\gamma}_v \) with respect to the reduction of \( \phi \) in the following.

**Proposition 3.1.** Let \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) be a polynomial map over \( K_v \). Assume that \( \phi \) has good reduction or mildly bad reduction, then \( \hat{\gamma}_v \) assumes only finitely many values on \( \mathbb{P}^1(K_v) \). Moreover, if \( \phi \) has good reduction, then \( \hat{\gamma}_v \equiv 0 \) on \( \mathbb{P}^1(K_v) \).

Since \( \phi \) has at worst mildly bad reduction, we let \( X \) denote a finite, projective model of \( \mathbb{P}^1 \) over \( \mathcal{O}_v \) so that \( \phi \) extends to an \( \mathcal{O}_v \)-morphism on the smooth part \( X^{\text{sm}} \) of \( X \). It follows that each \( K_v \)-rational point of \( \mathbb{P}^1(K_v) \) extends uniquely to a \( \mathcal{O}_v \)-point and factors through \( X^{\text{sm}} \). Let \( E_j \) be the subset of \( \mathbb{P}^1(K_v) \) such that for each point \( P \in E_j \), its closure \( \overline{P} \) in \( X^{\text{sm}} \) intersects with \( X_j^{\text{sm}} \). Then, \( E_j \)'s are domains defined in Section 2 (cf. [3]) and \( \mathbb{P}^1(K_v) \) is the disjoint union of \( E_j \) for \( j = 1, \ldots, l \). Let \( E_i \) be the one that contains the point \( \infty \), and \( E_i \) is an unbounded domain. Since \( \phi \) is a morphism on \( X^{\text{sm}} \), \( \phi \) takes each irreducible component of \( X^{\text{sm}} \) into another irreducible component. Therefore, \( \phi \) maps each domain \( E_i \) into another domain \( E_i \) for some \( i \).

By [4, Theorem 6.1], the canonical local height \( \hat{h}_{\phi,v} \) can be computed by a certain intersection index on \( X^{\text{sm}} \) provided \( (X^{\text{sm}}/\mathcal{O}_v, \phi) \) is a weak Néron model for \( (\mathbb{P}^1/K_v, \phi) \). Since we'll quote this result in the proof of Proposition 3.1, we state it in the following. For the proof, see [4].

**Theorem 3.2.** Suppose \( \mathcal{O}_v \) is a weak Néron model for \((V/K_v, \phi)\) over \( \mathcal{O}_v \). Let \( \hat{\gamma}_{D,\phi} \) be a canonical local height associated to the divisor \( D \) and morphism \( \phi \). Then, there exist rational numbers \( \gamma_1, \ldots, \gamma_n \) so that for all \( P \in V(K_v) \setminus \text{Supp}(D) \),

\[
\hat{\gamma}_{D,\phi}(P) = B \left( D + \sum_{j=1}^{n} \gamma_j \mathcal{O}_v \right)
\]

(5)
where $\bar{D}$ denotes the closure of $D$ in $V$ and $V_{\bar{s}}$ denotes the irreducible components of the special fiber of $V$.

**Lemma 3.3.** Let $E$ be a subset of $\mathbb{P}^1(K_v)$. Assume that $\phi: E \to E$. If there is some constant $c$ such that $\lambda_\phi)(z)) = d\lambda_\phi(z) + c$ for all $z \in E$, then

$$\lambda_{\phi, s}(z) = \lambda_s(z) + \frac{c}{d-1}$$

for all $z \in E$.

**Proof.** This is an easy consequence of formula (4). Let

$$\lambda_{\phi, n} = \frac{\lambda_{\phi}(d^n)}{d^n}.$$ 

By assumption, $\lambda_{\phi, n} = \lambda_{\phi, n-1} + (c/d^n)$. Therefore,

$$\lambda_{\phi, n} = \lambda_n + \sum_{i=1}^{n} \frac{c}{d^i}.$$ 

Apply formula (4) and use the fact that $d > 1$, we get the formula for $\lambda_{\phi, s}$.

**Proof of Proposition 3.1.** In the case of good reduction, $\phi$ extends to a finite morphism on $\mathbb{P}^1_{\overline{\mathcal{O}}_v}$. Then, $(\mathbb{P}^1/K_v, \phi)$ has a weak Néron model $(\mathbb{P}^1/\mathcal{O}_v, \phi/\mathcal{O}_v)$. Our conclusion follows from Theorem 3.2. In fact, we may choose $\lambda = \lambda_{s}$ in this case.

Assume that $\phi$ has mildly bad reduction, then $\phi(E_i) \subseteq (E_i)$ for some $i$. Since there are only finitely many domains, the orbit of any $E_i$ must be stationary after a finite iterate of $\phi$. If all the domains of the orbit of $E_i$ are finite domains, then $\lambda_{\phi, s}(E_i) = 0$ since $\lambda_s$ is bounded on each domain of the orbit. On the other hand, $X$ is a finite model for $\mathbb{P}^1$ over $\mathcal{O}_v$, $X$ can be obtained by blowing up some closed points on $\mathbb{P}^1_{\overline{\mathcal{O}}_v}$. It follows that $\lambda_s$ is constant on $E_i$ and hence $\lambda_{\phi, s}$ is constant on $E_i$.

It remains to show that $\phi^m(E_i) \subseteq E_i$ for some positive integer $m$. First, we contend that $\gamma_{\phi, s}$ is constant on $E_i$.

Since $X_{\phi}$ is of dimension one over the residue field $k(v)$ and $\Phi((X_{\phi})_{\phi}) \subseteq (X_{\phi})_{\phi}$, $\Phi$ is either finite or constant on $(X_{\phi})_{\phi}$. In the former case, it can be reduced to the case of Theorem 3.2. It follows that the logarithmic canonical local height is an intersection index on $X_{\phi}$ and $\gamma_{\phi, s}$ is constant on $E_i$. If $\Phi$ is constant on $(X_{\phi})_{\phi}$, we can write $z \cdot \phi = \phi_{j}(u) = (g_{j}(u))/\pi^{n}$ on $E_i$ for some nonnegative integer $e$, where $u = \pi^{e}z$ is some local coordinate on $E_i$ with integer $n$ nonnegative and possibly $|u - e_{i}| \geq 1$ for
some finitely many $c_i \in \mathbb{C}$. The numerator $g_i(u) = \sum_{i=0}^d a_i u^i$ is a polynomial with coefficients in $\mathbb{C}$, and $v(a_i) = 0$. Since $\Phi$ is a well defined morphism on $(X^m)_v$, $v(g(u)) = 0$ for the subset of $E_i$ with $v(u) \geq 0$ if $c \neq 0$. It follows

$$\lambda_i(\phi P) = d \lambda_i(P) + dn - c$$

for all $P \in E_i$. Apply Lemma 3.3, $\lambda_{\phi,v} = \lambda_v + c/(d-1)$ where $c = dn - c$. This proves our contention.

Suppose $\phi(E_{\mu}) \subseteq E_i$. Since $E_{\mu}$ is a bounded domain, it’s enough to show that $\lambda_{\phi,v}$ is constant on $E_{\mu}$. We must have either $v(\phi(P)) \geq 0$ for all $P \in E_{\mu}$ or $v \circ \phi = \text{constant} \cdot \text{constant}$ on $E_{\mu}$ if $v(\phi(P)) < 0$ for some $P \in E_{\mu}$, otherwise $\Phi$ will not be a morphism on $(X^m)_v$. Therefore, $\lambda_i(\phi P)$ is constant for all $P \in E_{\mu}$. It follows $\lambda_{\phi,v}(\phi P)$ is constant for all $P \in E_{\mu}$. By the functorial property of $\lambda_{\phi,v}$, $\lambda_{\phi,v}(P) = (\lambda_{\phi,v}(\phi P))/d$. We conclude that $\lambda_{\phi,v}$ is constant on $E_{\mu}$.

It follows $\lambda_{\phi,v}(\phi P)$ is constant for all $P \in E_{\mu}$. By the functorial property of $\lambda_{\phi,v}$, $\lambda_{\phi,v}(P) = (\lambda_{\phi,v}(\phi P))/d$. We conclude that $\lambda_{\phi,v}$ is constant on $E_{\mu}$. Since $\phi^m(E_i) \subseteq E_i$, by induction on $m$, $\lambda_{\phi,v}$ is a constant on $E_i$. This completes the proof of the proposition.

4. THE DYNAMICAL HEIGHT ZETA FUNCTIONS

In the following proposition we compute the dynamical height zeta function of a morphism over the rational function field with everywhere good reduction. Here we use a slightly general definition for good reduction: a morphism $\phi: \mathbb{P}^1 \to \mathbb{P}^1$ over $K_v$ is said to have good reduction at $v$ if there exist a model $X$ which is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ over $\mathbb{C}$, so that $\phi$ extends to an $\mathbb{C}$-morphism from $X$ to itself. This definition amounts to saying that there exists a linear map $L_v$ over $K_v$ such that $L_v^{-1} \circ \phi \circ L_v$ has good reduction in the sense defined in Section 3. (To make a distinction in the proof of the following proposition, we’ll call the latter case strictly good reduction.)

**Proposition 4.1.** Let $\phi: \mathbb{P}^1 \to \mathbb{P}^1$ be a polynomial map over the rational function field $K = \mathbb{F}_q(t)$ which has everywhere good reduction.

(a) After a change of variables, $\phi$ can be defined over the constant field $\mathbb{F}_q$.

(b) The dynamical height zeta function is given by

$$Z_K(\phi, s) = \int_{\mathbb{A}_K} \mathcal{H}_s^{-1}(\tau) \, d\tau$$

$$= q^{-s} \frac{\zeta_K(s-1)}{\zeta_K(s)} \quad \text{for} \quad \Re(s) > 2,$$

$$= q \frac{1 - q^{-s}}{1 - q^{-2}}.$$
Proof. (a) By definition, there exist a linear maps $L_v, v \in M_K,$ such that $L_v^{-1} \cdot \phi \cdot L_v$ has strictly good reduction at $v.$ Since $\phi$ is a polynomial map, $L_v$ is of the form $x_v z + \beta_v$ where $x_v, \beta_v \in K_v.$ The key point is to show there exists a linear map $L(z) = az + \beta$ with $a, \beta \in K$ so that $L_v^{-1} \cdot \phi \cdot L$ is of strictly good reduction at every prime $v.$

Let $\phi(z) = \sum_{n \geq 0} a_n z^n, a_n \in F_q(t).$ Then $L_v^{-1} \cdot \phi \cdot L_v(z) = a_n x_v^{n-1} z^n + \text{(lower terms)}.$ It follows $v(a_n x_v^{n-1}) = 0$ by the definition of strictly good reduction (Section 3). Therefore, $\sum_{v \in M_{F_q(t)}} v(a_n) f_v = 1/(n-1) \sum_{v \in M_{F_q(t)}} v(a_n) f_v = 0.$ Consider the equations:

$$|x - \beta_v|_v \leq |x|_v, \quad v \in M_{F_q(t)}.$$

Since $\prod_{v \in M_{F_q(t)}} |x_v|_v = 1$ there exists a solution $\beta \in F_q(t).$ Set $\beta = \beta_v + \mu, \forall v,$ $v(\mu_v) \geq 0$ and $L_v'(z) = x_v z + \beta = x_v(z + \mu_v) + \beta_v = L_v \cdot T_v$ where $T_v(z) = z + \mu_v$, then $L_v^{-1} \cdot \phi \cdot L = (L_v^{-1} \cdot \phi \cdot L_v) \cdot T_v.$ Because $L_v^{-1} \cdot \phi \cdot L_v$ is of strictly good reduction at $v$ by assumption and $v(\mu_v) \geq 0,$ it follows that $L_v^{-1} \cdot \phi \cdot L_v$ is of strictly good reduction at $v.$ Since $\sum_{v \in M_{F_q(t)}} v(x_v) f_v = 0,$ there exists an element $x$ of $F_q(t)$ such that $v(x) = v(x_v)$ for all $v \in M_{F_q(t)}.$ Now it's an easy task to verify that the linear map $L(z) = az + \beta$ is the desired one. That is $L_v^{-1} \cdot \phi \cdot L$ has strictly good reduction at $v$ for all $v \in M_{F_q(t)}.$

Let $L^{-1} \cdot \phi \cdot L = \sum_{n \geq 0} b_n z^n, b_n \in F_q(t).$ Then $v(b_n) \geq 0$ for all $v \in M_{F_q(t)}.$ Therefore, $b_n \in F_q[t].$

(b) By Proposition 3.1, the canonical local height $\hat{H}_v$ equals the ordinary local height $H_v.$ Therefore, $\hat{H}_v = H.$ The result can be obtained by direct computation or one can proceed as follows.

We may choose a fundamental domain $D$ for $\mathcal{A}_{F_q(t)/F_q(t)}$ so that for any adele vector $a \in D,$ $a_v \in C_v$ for every $v \in M_{F_q(t)}.$ Then,

$$Z_K(\phi, s) = \sum_{x \in \mathcal{A}_{F_q(t)}} H(x)^{-s}; \quad \Re(s) > 2$$

$$= \sum_{x \in \mathcal{A}_{F_q(t)}} \int_\Omega H(x)^{-s} \, dx; \quad \tau(\Omega) = 1$$

$$= \sum_{x \in \mathcal{A}_{F_q(t)}} \int_\Omega H(\tau + x)^{-s} \, dx + \sum_{x \in \mathcal{A}_{F_q(t)}} \int_\Omega \{H(x)^{-s} - H(\tau + x)^{-s}\} \, dx$$

$$= \int_{\mathcal{A}_{F_q(t)}} H(\tau)^{-s} \, dx + R(s)$$

To compute the error term $R(s),$ observe that $H_v(a_v + x) = H_v(x)$ for all $a_v \in C_v$ and all $v \in M_{F_q(t)}.$ It follows that $H(a + x) = H(x)$ for all $a \in D.$
Therefore, $R(s) = 0$. The last equality of the proposition follows from the local integration

$$\int_{\kappa_v} H_\phi(\tau'_v)^{-r} d\tau'_v = \frac{1 - q^{-s'}}{1 - q^{1 - r} p} \quad \text{where} \quad \tau'_v(\kappa_v) = 1$$

and the definition of $\zeta_{\phi, v}(s)$:

$$\zeta_{\phi, v}(s) = \prod_{v \in M_{\phi}(v)} (1 - q^{-s})^{-1} = \frac{1}{(1 - q^{-s})(1 - q^{1 - s})}.$$  

**Theorem 4.2.** Let $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a polynomial map over $K$ and let $S$ be the set of all primes where $\phi$ has bad reduction. Assume that $\phi$ has mildly bad reduction at all $v \in S$. Then the dynamical height zeta function $Z_K(\phi, s)$ has a meromorphic continuation to the whole complex plane and $Z_K(\phi, s) = \int_{A_K} \hat{H}_\phi(\tau)^{-r} dt + R(s),$

where $R(s)$ has the property that $\{ \prod_{v \in S} (1 - q^{-s_v}) \} \zeta_K(s) R(s)$ is a $\mathbb{Q}$-linear combination of $q^{n_s}$ for finitely many rational numbers $n_s$.

**Proof.** By assumption, $\phi$ has mildly bad reduction at $v$ for every $v \in S$. We let $X$ denote a projective model of $\mathbb{P}^1$ over $\mathbb{C}$ so that $\phi$ extends to a morphism $\Phi$ from $X^\text{sm}$ to itself over $\mathbb{C}$. Let $X_v$ denote the local model of $\mathbb{P}^1$ over $\mathcal{O}_v$ and let $\Phi_v$ denote the morphism extending $\phi$ on $X_v$. Assume the special fiber $X_{v, 0}$ of $X_v$ has a decomposition $X_{v, 0} = \bigcup_{n=1}^{l_v} X_{v, 0}$ into the union of irreducible components. Let $D_{v, i}$ be the subset of $K_v$ as defined in Section 3 and let $D_{v, i}$ be the unbounded domain. We have $K_v = \bigcup_{n=1}^{l_v} D_{v, i}$ of $K_v$ into a disjoint union of domains.

Set

$$D_{S, \mu} = K \cap \prod_{v \in S} D_{v, i}, \quad \text{where} \quad \mu = (\{ v, i_v \})_{v \in S}.$$  

It follows therefore $K = \bigcup_{\mu} D_{S, \mu}$, where $\mu$ runs through all the indices $(\{ v, i_v \})_{v \in S}$, $0 \leq i_v \leq l_v$, and the union is a disjoint union. We also have the decomposition of the adele ring into disjoint union of subsets $A_{D_{S, \mu}}$. Let $\hat{\beta} = \hat{H}_\phi/H$ and $\hat{\beta}_v = \hat{H}_\phi_v/H_v$, then $\hat{\beta} = \prod_{v \in M_K} \hat{\beta}_v$. By Proposition 3.1, $\hat{\beta}_v \equiv 1$ if $v \notin S$ and $\hat{\beta}_v$ is constant on $D_{v, i}$ if $v \in S$. Thus, $\hat{\beta} = \prod_{v \in S} \hat{\beta}_v$ and $\hat{\beta}$ is a constant $\hat{\beta}_{S, \mu}$ on $A_{D_{S, \mu}}$, and $\hat{\beta}$ is viewed as a continuous function on $A_K$. Hence,
Z_K(\phi, s) = \sum_{x \in K} \hat{H}_\phi(x)^{-s} \\
= \sum_{\mu} \sum_{x \in D_{S,\mu}} \hat{H}_\phi(x)^{-s} \\
= \sum_{\mu} \sum_{x \in D_{S,\mu}} \hat{\rho}(x)^{-s} \hat{H}(x)^{-s} \\
= \sum_{\mu} \hat{\beta}_{S,\mu} W(D_{S,\mu}, s) \\
= \sum_{\mu} \hat{\beta}_{S,\mu} \left( \int \hat{H}(\tau)^{-s} \, d\tau + R_{S,\mu}(s) \right) \text{ by Cor. 2.3} \\
= \sum_{\mu} \int \hat{H}(\tau)^{-s} \, d\tau + \sum_{\mu} \hat{\beta}_{S,\mu} R_{S,\mu}(s) \\
= \int \hat{H}(\tau)^{-s} \, d\tau + R(s).

Since each partial height zeta function $W(D_{S,\mu}, s)$ is a rational function in $q^{-s}$, it has a meromorphic continuation to the whole complex plane. As the computation above shows, the meromorphic continuation of $Z_K(\phi, s)$ follows directly from that of $W(D_{S,\mu}, s)$ and the property of $R(s)$ follows from that of $R_{S,\mu}$.

We give two examples of polynomial maps which have at worst mildly bad reduction.

**Example 1.** Let $K = \mathbb{F}_q(t)$ be the rational function field with indeterminate $t$ and let $\phi(z) = z^2 + 1/f$ where $f(t) = g(t)^m \in \mathbb{F}_q[t]$ with $g$ squarefree and $m$ odd. For simplicity, we consider the case that $m = 1$. It follows immediately from the definition that $\phi$ has good reduction at $v$ if and only if $v(1/f) \geq 0$. Thus $\phi$ has good reduction at $v$ if and only if $v = 1/t$ or $v \mid f$. For any $v$ such that $v \mid f$, we have $v(f) = 1$ since $f$ is squarefree.

Consider $\phi$ as a morphism over $K_v$ and write $\phi(z) = z^2 + u_v/\pi_v$. Then $\phi$ extends to an $\ell_v$-morphism on $X_v^{sm}$ where $X_v$ is obtained by blowing up the point $z = \infty$ of the special fiber of $\mathbb{P}^1_{\ell_v}$.

**Example 2.** Let $K = \mathbb{F}_q(t)$ and let $p$ be the characteristic of $K$. Let $\phi(z) = (z/f)^p + 1$ where $f \in \mathbb{F}_q[t]\setminus\mathbb{F}_q$ be a polynomial in $t$. $\phi$ has good reduction at $v$ if and only if $v(f) = 0$. To simplify the argument, for those primes $v$ such that $v(f) = m_v \neq 0$ we assume that $m_v < p$. The corresponding model $X_v$ on which $\phi$ extends to an $\ell_v$-morphism is obtained as follows:
(i) \( v = l/t \). By choosing appropriate uniformizer we may write \( \phi(z) = (\pi_m^\circ z)^r + 1 \) over \( K_v \). Then by a consecutive dilation of the point \( z = \infty \) (the point where the section of the point \( \infty \) intersects with the special fiber) \( m_v + 1 \) times (see \([8]\)), \( \phi \) extends to an \( \mathcal{O}_v \)-morphism on the smooth part of the final model.

(ii) \( v \mid f \). In this case, we may write \( \phi(z) = (z/\pi_m^\circ)^r + 1 \) for an appropriate uniformizer at \( v \). Then by a consecutive dilation of the point \( z = 0 \) (the point where the section of the point \( z = 0 \) intersects with the special fiber) \( m_v + 1 \) times, \( \phi \) extends to an \( \mathcal{O}_v \)-morphism on the smooth part of the final model.

In terms of \( \zeta_k(s) \), we write \( Z_K(\phi, s) \) as follows.

\[
Z_K(\phi, s) = \int_{\mathcal{O}_K} \hat{H}_\phi(\tau)^{-s} \, d\tau + R(s)
\]

\[
= q^{1-s} \prod_{v \notin S} \int_{\mathcal{O}_K} H_v(\tau_v)^{-s} \, d\tau_v \prod_{v \in S} \int_{\mathcal{O}_K} \hat{H}_{\phi_v}(\tau_v)^{-s} \, d\tau_v + R(s)
\]

\[
= q^{1-s} \zeta_k(s-1) \zeta_k(s) \prod_{v \in S} \int_{\mathcal{O}_K} H_v(\tau_v)^{-s} \, d\tau_v + R(s)
\]

The expression in the last equality is the function field analogue of Silverman’s conjectural formula for \( Z_K(\phi, s) \) over a number field \([12]\). Since by Theorem 4.2, \( R(s) \) is regular at \( s = 2 \). It follows that the residue of \( Z_K(\phi, s) \) is

\[
\text{Res}_{s=2} Z_K(\phi, s) = \text{Res}_{s=2} \zeta_k(s-1) \left\{ \frac{q^{1-s}}{\zeta_k(2)} \prod_{v \in S} \int_{\mathcal{O}_K} H_v(\tau_v)^{-2} \, d\tau_v \right\}.
\]

An immediate consequence of Theorem 4.2 is to estimate the cardinality of the set of rational points with bounded canonical height. That is, the cardinality of the following set

\[
\{ x \in K \mid \hat{H}_\phi(x) \leq x \}
\]

Let \( \omega_\phi(x) \) denote the cardinality of this set.

**Corollary 4.3.** Let \( \phi: \mathbb{P}^1 \to \mathbb{P}^1 \) to a polynomial map over \( K \). Suppose that Theorem 4.2 holds for the dynamical height zeta function \( Z_K(\phi, s) \) associated with \( \phi \). Then, for positive real number \( x \in (\hat{H}_\phi(P) \mid P \in \mathbb{P}^1(K)) \),

\[
\omega_\phi(x) = \frac{c_\phi}{2} x^2 + O(x^{1/2+s}),
\]
where \( c_\phi = \text{Res}_{s=2} Z_k(\phi, s) \) and the constant in \( O \) depends only on \( \phi \) and \( \epsilon > 0 \).

**Proof.** Since \( \omega_k(x) \) is a monotonically increasing function and \( Z_k(\phi, s) \) converges absolutely for \( \Re(s) > 2 \). The assertion of the corollary is an exercise using Perron Sum Formula [1, Sec. 9.9] and complex contour integration. Namely,

\[
\omega_k(x) = \frac{1}{2\pi i} \int_{\Gamma} \frac{Z_k(\phi, s) x^s}{s} ds,
\]

for any \( x \) such that \( x/\cap_k(P) \notin \mathbb{Z} \) where \( \Gamma \) is any line to the right of \( s = 2 \) and parallel to the \( y \)-axis.

By formula (8), \( Z_k(\phi, s) \) has simple poles at \( s = 2 \) and zeros of \( \zeta_k(s) \theta \). By the Riemann hypothesis for function fields, \( \Re(\theta) = \frac{1}{2} \) and \( Z_k(\phi, s) x^s \) is bounded as \( \Re(s) \to \infty \), we move the contour to the vertical line \( \Gamma' \) with \( \Re(s) = \frac{1}{2} + \epsilon, 0 < \epsilon < 1 \). The residue of the integrand gives the main term of \( \omega_k(x) \) and the integrand along \( \Gamma' \) gives the error term. \( \square \)

**REFERENCES**