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On Stability for Symmetric Hyperbolic Systems, I

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1. INTRODUCTION

In a very wide spectrum of applications, the problem of stability plays a central role. There is a vast literature available in the field, but still the problem is solved only in special cases. The incompleteness of the literature is especially noticeable when the governing equations are partial differential equations. The normal procedure in a stability research for such equations is to *linearize* the equations of perturbation and confine the study to the simpler linear equations without any mathematical justification. Even though the linearization has been mathematically justified only in a few very special cases, it is a fact that usually the linear stability criteria are in excellent agreement with experimental observations.

In this paper we shall also confine our study to linear partial differential equations. The conventional approach to the problem of stability for such equations is the so-called *normal mode method*, which under certain restrictions transfers the problem to an eigenvalue problem for a partial differential operator with boundary conditions. In general this eigenvalue problem is very complicated. It is therefore customary to restrict the study to the special cases where the problem essentially is transferred to an eigenvalue problem for an ordinary differential operator. Even in these special cases the eigenvalue problem is not completely resolved. An alternative approach which has been given some attention in recent years is *Liapunov's direct method*, which in principle can be applied to all kinds of stability problems, including nonlinear problems. The results obtained by this method so far, however, are rather limited. The difficulties in applying the method consist of constructing functionals with specific properties.

A new approach to the problem of stability for linear hyperbolic equations was given by Eckhoff [1] on a somewhat speculative and conjectural basis. The approach is based on the generalized progressing wave expansion method described by Friedlander [2] and Ludwig [3], and it has been applied to problems of stability in fluid mechanics by Eckhoff and Storesletten [4, 5]. The purpose of this paper is to give a rigorous

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mathematical justification of this new approach to the problem of stability. The presentation we shall give will to a large extent be self-contained.

2. Assumptions and Formulation of the Problem

We shall consider a linear symmetric hyperbolic system of the form

$$Lu \overset{\text{def}}{=} \mathbf{u}_t + \sum_{\nu=1}^{n} \mathbf{A}^\nu \mathbf{u}_{x_\nu} + \mathbf{B}\mathbf{u} = 0, \tag{2.1}$$

where $\mathbf{u} = \{u_1,..., u_m\}$ are the dependent variables (i.e., the unknown functions), while $\mathbf{B}$ and $\mathbf{A}^\nu$ ($\nu = 1,..., n$) are given $m \times m$ matrices with real coefficients which may depend on the real independent variables $\mathbf{x} = \{x_1,..., x_n\}$ (space) and t (time). The matrices $\mathbf{A}^\nu$ are assumed to be symmetric.

We are primarily interested in real-valued solutions of (2.1). For our purpose, however, it is equivalent and more convenient to consider complex-valued solutions; we shall therefore do that in the following. We are particularily interested in solutions of the initial value problem for (2.1) with initial data

$$\lim_{t \to t_0^+} \mathbf{u}(\mathbf{x}, t) = \mathbf{g}(\mathbf{x}) \tag{2.2}$$

in a given metric space M_0 of m-dimensional vector-functions $\mathbf{g}(\mathbf{x})$ defined for $\mathbf{x} \in S$, where S is some fixed open set in R^n which may be bounded or unbounded.

We shall assume that the initial value problem (2.1), (2.2) for an arbitrarily given $\mathbf{g}(\mathbf{x}) \in M_0$ has a unique solution $\mathbf{u}(\mathbf{x}, t; t_0)$ in some metric space M for every $t \geqslant t_0 \geqslant 0$, where M also is a fixed space of vector-functions defined for $\mathbf{x} \in S$. More specifically we shall assume that the initial value problem (2.1), (2.2) is well posed in the sense that the family of mappings

$$\mathbf{f}_t\colon M_0 \times R_+^1 \to M \tag{2.3}$$

is continuous for every $t \geqslant t_0$, where $\mathbf{f}_t$ is defined by

$$\mathbf{f}_t(\mathbf{g}(\mathbf{x}), t_0) = \mathbf{u}(\mathbf{x}, t; t_0). \tag{2.4}$$

When $R^n \backslash S \neq \varnothing$, it follows from the well-established theory in the literature (see Courant and Hilbert [6] and the references quoted there) that the assumptions above usually imply that some boundary conditions must be built into the structure of M and M_0. In this case (2.1), (2.2) is therefore a mixed initial-boundary value problem. When $S = R^n$, (2.1), (2.2) is a global Cauchy problem.

In any case we let ρ_0 and ρ denote the metrics in the spaces M_0 and M, respectively. We may then define the concept of stability in the following way.

DEFINITION 1. The trivial solution $\mathbf{u} \equiv 0$ of (2.1) is said to be stable if for every $\varepsilon > 0$ and every $t_0 \geqslant 0$ it is possible to find a number $\delta = \delta(\varepsilon, t_0) > 0$ with the following property: If $g(\mathbf{x}) \in M_0$ is such that $\rho_0\{\mathbf{g}(\mathbf{x}), 0\} < \delta$, then $\rho\{\mathbf{u}(\mathbf{x}, t; t_0), 0\} < \varepsilon$ for every $t \geqslant t_0$. The trivial solution of (2.1) is said to be unstable if it is not stable.

This is essentially Liapunov's definition of stability as generalized by Movchan [7]. Most of the definitions of stability used in the literature can easily be shown to be equivalent to special cases of this definition.

The problem of stability for (2.1) is to determine the coefficient matrices $\mathbf{B}$ and $\mathbf{A}^\nu$ $(\nu = 1,..., n)$ for which the trivial solution of (2.1) is stable and unstable, respectively. In order to be able to study the problem of stability for (2.1), we have to introduce further assumptions. In particular, we have to be more specific about the metric spaces M_0 and M, since Definition 1 shows that the stability properties may depend strongly at least on the choice of metrics in these spaces. Several choices of these metrics ρ_0, ρ may a priori seem relevant for Definition 1 to be in agreement with the intuitive concept of stability appearing in applications. However, the speculations given in Eckhoff [1] seem to indicate that the most suitable metrics are those generated by the norms in the Sobolev spaces H_s. For simplicity, we shall therefore in this paper assume that the metrics ρ_0, ρ are both generated by the L^2 $(= H_0)$ norm in the straightforward way. Thus we shall assume that

$$\begin{aligned} \rho_0\{\mathbf{g}(\mathbf{x}), 0\} &= \left\{\int_S \mathbf{g}(\mathbf{x}) \cdot \mathbf{g}^*(\mathbf{x})\, dx_1 \cdots dx_n\right\}^{1/2} \\ \rho\{\mathbf{u}(\mathbf{x}, t; t_0), 0\} &= \left\{\int_S \mathbf{u}(\mathbf{x}, t; t_0) \cdot \mathbf{u}^*(\mathbf{x}, t; t_0)\, dx_1 \cdots dx_n\right\}^{1/2}, \end{aligned} \tag{2.5}$$

where the asterisk means complex conjugation. Furthermore, we assume that M is the subset of $L^2(S)$ which satisfies the boundary conditions imposed, and that these boundary conditions are of the type considered by Courant and Hilbert [6, p. 657] when $R^n \backslash S \neq \varnothing$. For simplicity we shall assume that M_0 consists of all smooth functions which have compact support in S (M_0 is then obviously a subset of $L^2(S)$ which satisfies the boundary conditions).

In order to keep the ideas as simple as possible, we shall assume that all the coefficients in (2.1) are smooth and bounded in $\overline{S \times R^1_+}$. Furthermore, we shall assume that all the first-order partial derivatives of the coefficients in the matrices $\mathbf{A}^\nu$ ($\nu = 1,..., n$) are bounded there. It is then well known that the Cauchy problem (2.1), (2.2) is *locally* well posed in L^2 (see Courant and Hilbert [6] or Mizohata [8]). If $S = R^n$ it is not difficult to show that the global Cauchy problem formulated above is well posed also, and that the solutions are smooth functions which have compact support in R^n for each $t \geqslant t_0$. If on the other hand $R^n \backslash S \neq \varnothing$, there seems to be no theory available in the literature which guarantees the assumed existence of solutions of (2.1), (2.2) unless additional assumptions are introduced (see, for instance, Rauch [9]). It is not necessary for us to introduce such further assumptions, since we have already assumed that the initial value problem (2.1), (2.2) is well posed.

The problem of stability is now defined in a very specific way. We should like to point out that the method we are going to describe in the following, essentially also works with considerably less restrictive hypotheses and for a number of other choices of the metric spaces M_0, M. This is partially discussed in Eckhoff [1]. Some remarks will also be given in Section 8.

3. An Energy Inequality

We shall in this section consider the inhomogeneous system

$$L\mathbf{w} = \mathbf{f}, \tag{3.1}$$

where $\mathbf{f}$ is a smooth complex-valued vector-function which for each $t \geqslant 0$ has compact support in S. In view of Duhamel's principle (see Courant and Hilbert [6]) it is easy to show that the assumptions introduced in the preceding section imply that the initial value problem for (3.1) is well posed in the same sense as for the homogeneous system (2.1).

We introduce a new dependent variable $\mathbf{W}$ in (3.1) by

$$\mathbf{w} = e^{\gamma t}\mathbf{W}, \tag{3.2}$$

where γ is a real constant which we shall choose later. Equation (3.1) then becomes

$$\mathbf{W}_t + \sum_{\nu=1}^{n} \mathbf{A}^\nu \mathbf{W}_{x_\nu} + (\mathbf{B} + \gamma \mathbf{I})\,\mathbf{W} = e^{-\gamma t}\mathbf{f}. \tag{3.3}$$

From this equation it easily follows that

$$\frac{\partial}{\partial t}(\mathbf{W} \cdot \mathbf{W}^*) + \sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu}(\mathbf{W} \cdot \mathbf{A}^\nu \mathbf{W}^*) + \mathbf{W} \cdot (\mathbf{H} + 2\gamma \mathbf{I})\,\mathbf{W}^*$$
$$= e^{-\gamma t}(\mathbf{W} \cdot \mathbf{f}^* + \mathbf{f} \cdot \mathbf{W}^*), \tag{3.4}$$

where

$$\mathbf{H} = \mathbf{B} + \mathbf{B}^T - \sum_{\nu=1}^{n} \mathbf{A}^{\nu}_{x_\nu} \tag{3.5}$$

is seen to be a symmetric matrix with smooth and bounded coefficients in $\overline{S \times R^1_+}$. Since we assume that the initial value of $\mathbf{W}$ has compact support in S, the support of $\mathbf{W}$ will be compact for every $t \geqslant 0$. Keeping $t \geqslant 0$ fixed, we may therefore integrate (3.4) over the domain S. Gauss' theorem (supplied by the boundary conditions when $R^n \backslash S \neq \varnothing$) then shows that the second term in (3.4) drops out. Thus we are left with

$$\begin{aligned} &\frac{d}{dt}\int_S \mathbf{W} \cdot \mathbf{W}^* \, dx_1 \cdots dx_n + \int_S \mathbf{W} \cdot (\mathbf{H} + 2\gamma \mathbf{I})\, \mathbf{W}^* \, dx_1 \cdots dx_n \\ &\quad = e^{-\gamma t} \int_S (\mathbf{W} \cdot \mathbf{f}^* + \mathbf{f} \cdot \mathbf{W}^*) \, dx_1 \cdots dx_n. \end{aligned} \tag{3.6}$$

From this equation we may deduce several useful results. First, by choosing γ sufficiently large we can always achieve that the matrix $\mathbf{H} + 2\gamma\mathbf{I}$ is positive definite at every point in $S \times R^1_+$, i.e., that

$$\mathbf{W} \cdot (\mathbf{H} + 2\gamma \mathbf{I})\, \mathbf{W}^* \geqslant 0 \tag{3.7}$$

everywhere for any $\mathbf{W}$. For solutions $\mathbf{u}(\mathbf{x}, t)$ of the *homogeneous* system (2.1), it then follows from (3.6) that

$$\frac{d}{dt}\int_S \mathbf{u} \cdot \mathbf{u}^* e^{-2\gamma t} \, dx_1 \cdots dx_n \leqslant 0, \tag{3.8}$$

which implies that for every $t \geqslant t_0$ we have

$$\begin{aligned} &\int_S \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}^*(\mathbf{x}, t) \, dx_1 \cdots dx_n \\ &\quad \leqslant e^{2\gamma(t - t_0)} \int_S \mathbf{u}(\mathbf{x}, t_0) \cdot \mathbf{u}^*(\mathbf{x}, t_0) \, dx_1 \cdots dx_n. \end{aligned} \tag{3.9}$$

Thus we have obtained an upper bound on the growth rate for the solutions of (2.1). In particular we see that the trivial solution of (2.1) is stable if (3.7) holds for $\gamma = 0$, or equivalently if the eigenvalues of the matrix $\mathbf{H}$ are all nonnegative at every point in $S \times R^1_+$. Thus the stability problem is solved for the hyperbolic systems (2.1) satisfying this condition. Unfortunately these so-called dissipative systems are very special, most of the hyperbolic systems appearing in applications do not belong to this class.

For any vectors $\mathbf{V}$, $\mathbf{W}$, we have

$$\mathbf{V} \cdot \mathbf{W}^* + \mathbf{W} \cdot \mathbf{V}^* \leqslant \mathbf{V} \cdot \mathbf{V}^* + \mathbf{W} \cdot \mathbf{W}^*. \tag{3.10}$$

If we take $\mathbf{V} = e^{-\gamma t}\mathbf{f}$, we obtain from (3.6) and (3.10) that

$$\frac{d}{dt}\int_S \mathbf{W} \cdot \mathbf{W}^* \, dx_1 \cdots dx_n + \int_S \mathbf{W} \cdot \{\mathbf{H} + (2\gamma - 1)\mathbf{I}\} \mathbf{W}^* \, dx_1 \cdots dx_n$$

$$\leqslant e^{-2\gamma t} \int_S \mathbf{f} \cdot \mathbf{f}^* \, dx_1 \cdots dx_n. \tag{3.11}$$

We now choose γ so large that

$$\mathbf{W} \cdot \{\mathbf{H} + (2\gamma - 1)\mathbf{I}\} \mathbf{W}^* \geqslant 0 \tag{3.12}$$

everywhere in $S \times R^1_+$ for any $\mathbf{W}$. For solutions $\mathbf{w}(\mathbf{x}, t)$ of the inhomogeneous system (3.1), it then follows from (3.11) that

$$\begin{aligned} &\frac{d}{dt}\int_S \mathbf{w} \cdot \mathbf{w}^* e^{-2\gamma t} \, dx_1 \cdots dx_n \\ &\leqslant e^{-2\gamma t} \int_S \mathbf{f} \cdot \mathbf{f}^* \, dx_1 \cdots dx_n \end{aligned} \tag{3.13}$$

which implies that for every $t \geqslant t_0 \geqslant 0$ we have

$$\begin{aligned} &\int_S \mathbf{w}(\mathbf{x}, t) \cdot \mathbf{w}^*(\mathbf{x}, t) \, dx_1 \cdots dx_n \\ &\leqslant e^{2\gamma(t - t_0)} \int_S \mathbf{w}(\mathbf{x}, t_0) \cdot \mathbf{w}^*(\mathbf{x}, t_0) \, dx_1 \cdots dx_n \\ &\quad + e^{2\gamma t} \int_{t_0}^t \left\{ e^{-2\gamma\tau} \int_S \mathbf{f}(\mathbf{x}, \tau) \cdot \mathbf{f}^*(\mathbf{x}, \tau) \, dx_1 \cdots dx_n \right\} d\tau. \end{aligned} \tag{3.14}$$

This inequality, which gives an upper bound for the solution of (3.1), is similar to the energy inequalities obtained, for instance, in Courant and Hilbert [6].

4. Special Families of Solutions

We shall now construct families of solutions of (2.1) depending on a real *frequency parameter* ω in the following way (with $i = \sqrt{-1}$)

$$\mathbf{u}_\omega(\mathbf{x}, t) = \mathbf{a}_0(\mathbf{x}, t) \exp\{i\omega\varphi(\mathbf{x}, t)\} + \frac{1}{i\omega}\mathbf{v}(\mathbf{x}, t; \omega). \tag{4.1}$$

Here we want to determine *the phase function* φ and *the amplitude* $\mathbf{a}_0$ such that *the remainder* $\mathbf{v}$ can be shown to be bounded as $\omega \to \infty$. By substituting (4.1) into (2.1) we get

$$L\mathbf{u}_\omega = \left\{ i\omega \left(\varphi_t \mathbf{I} + \sum_{\nu=1}^{n} \varphi_{x_\nu} \mathbf{A}^\nu \right) \mathbf{a}_0 + L\mathbf{a}_0 \right\} \exp(i\omega\varphi) + \frac{1}{i\omega} L\mathbf{v}$$
$$= 0. \tag{4.2}$$

In order that (4.1) shall have the required properties, it is natural to assume that φ and $\mathbf{a}_0$ satisfy

$$\left(\varphi_t \mathbf{I} + \sum_{\nu=1}^{n} \varphi_{x_\nu} \mathbf{A}^\nu \right) \mathbf{a}_0 = 0. \tag{4.3}$$

Since we assume that $\mathbf{a}_0 \neq 0$, this equation can only be satisfied when the phase function φ satisfies the characteristic equation

$$\det \left\{ \varphi_t \mathbf{I} + \sum_{\nu=1}^{n} \varphi_{x_\nu} \mathbf{A}^\nu \right\} = 0, \tag{4.4}$$

which is a partial differential equation of order 1 and degree m. On introducing the notation

$$\lambda = -\varphi_t, \qquad \xi^\nu = \varphi_{x_\nu} \quad (\nu = 1, \dots, n), \qquad \mathbf{E} = \sum_{\nu=1}^{n} \xi^\nu \mathbf{A}^\nu, \tag{4.5}$$

(4.4) shows that λ must be an eigenvalue of the symmetric matrix $\mathbf{E}$. If

$$\lambda = \Omega(\mathbf{x}, t, \xi^1, \dots, \xi^n) \tag{4.6}$$

is an eigenvalue of $\mathbf{E}$, we see that (4.4) is satisfied when

$$\varphi_t + \Omega(\mathbf{x}, t, \varphi_{x_1}, \dots, \varphi_{x_n}) = 0, \tag{4.7}$$

which is a partial differential equation of order 1 and degree 1. The eigenvalues of the matrix $\mathbf{E}$ are called *the characteristic roots* associated with the symmetric hyperbolic system (2.1). To the different characteristic roots there correspond different families of phase functions which again will correspond to different families of solutions of the form (4.1).

Now let $\varphi(\mathbf{x}, t)$ be a real-valued solution of (4.7) which is such that $\nabla\varphi = \{\varphi_{x_1}, \dots, \varphi_{x_n}\} \neq 0$. Suppose that Ω for this solution φ is an eigenvalue of fixed multiplicity μ, say, in the considered domain. Equation (4.3) then shows that

$$\mathbf{a}_0 = \sum_{l=1}^{\mu} \sigma_l \boldsymbol{r}_l, \tag{4.8}$$

where $\mathbf{r}_1,\ldots,\mathbf{r}_\mu$ are orthonormal eigenvectors associated with the eigenvalue Ω, and $\sigma_1,\ldots,\sigma_\mu$ are scalar functions to be determined. In order to do that, we shall write the remainder in the following way

$$\mathbf{v}(\mathbf{x}, t; \omega) = \mathbf{a}_1(\mathbf{x}, t) \exp\{i\omega\varphi(\mathbf{x}, t)\} + \mathbf{w}(\mathbf{x}, t; \omega). \tag{4.9}$$

In view of (4.3), (4.2) then becomes

$$\left\{L\mathbf{a}_0 + \left(\varphi_t \mathbf{I} + \sum_{\nu=1}^{n} \varphi_{x_\nu} \mathbf{A}^\nu\right) \mathbf{a}_1 + \frac{1}{i\omega} L\mathbf{a}_1\right\} \exp(i\omega\varphi) + \frac{1}{i\omega} L\mathbf{w} = 0 \tag{4.10}$$

which is satisfied if the following equations hold

$$L\mathbf{a}_0 + \left(\varphi_t \mathbf{I} + \sum_{\nu=1}^{n} \varphi_{x_\nu} \mathbf{A}^\nu\right) \mathbf{a}_1 = 0, \tag{4.11}$$

$$(L\mathbf{a}_1) \exp(i\omega\varphi) + L\mathbf{w} = 0. \tag{4.12}$$

Equation (4.11) may be considered as a system of linear algebraic equations for $\mathbf{a}_1$. When φ has the assumed properties, (4.11) therefore has a solution if and only if

$$\mathbf{r}_l \cdot L\mathbf{a}_0 = 0 \qquad (l = 1,\ldots,\mu). \tag{4.13}$$

Substituting (4.8) into (4.13) yields the following system of partial differential equations for the functions σ_l $(l = 1,\ldots,\mu)$:

$$(\sigma_l)_t + \sum_{\nu=1}^{n} \sum_{k=1}^{\mu} (\mathbf{r}_l \cdot \mathbf{A}^\nu \mathbf{r}_k)(\sigma_k)_{x_\nu} + \sum_{k=1}^{\mu} (\mathbf{r}_l \cdot L\mathbf{r}_k)\, \sigma_k = 0. \tag{4.14}$$

This obviously is a symmetric hyperbolic system. As we shall see in the next section, (4.14) is often considerably simpler to study than the original system (2.1).

Let us now assume that we have found a phase function φ and an amplitude $\mathbf{a}_0$ satisfying (4.7), (4.8) and (4.14) such that $\mathbf{a}_0 \exp(i\omega\varphi)$ is smooth in $S \times R^1_+$ and has compact support in S for each $t \geqslant 0$. The remainder can then be determined from (4.9) and (4.11), (4.12). As for $\mathbf{a}_1$, we pick any smooth solution of (4.11) which for each $t \geqslant 0$ has support contained in the support of $\mathbf{a}_0$. The final term $\mathbf{w}$ is then determined from the inhomogeneous hyperbolic system (4.12) together with the initial value of $\mathbf{w}$ at $t = t_0$ and the same boundary conditions as were imposed for (2.1). Thus (4.12) satisfies the conditions assumed for (3.1) in the preceding section, $\mathbf{w}$

therefore exists and satisfies inequality (3.14) with $\mathbf{f} = -(L\mathbf{a}_1)\exp(i\omega\varphi)$. If we take $t_0 = 0$ and $\mathbf{w}(\mathbf{x}, 0; \omega) \equiv 0$ this inequality becomes for $t \geqslant 0$

$$\int_S \mathbf{w}(\mathbf{x}, t; \omega) \cdot \mathbf{w}^*(\mathbf{x}, t; \omega)\, dx_1 \cdots dx_n \leqslant e^{2\gamma t} \int_0^t \left\{ e^{-2\gamma\tau} \int_S L\mathbf{a}_1(\mathbf{x}, \tau) \cdot L\mathbf{a}_1^*(\mathbf{x}, \tau)\, dx_1 \cdots dx_n \right\} d\tau. \tag{4.15}$$

For the remainder (4.9), the triangle inequality gives

$$\left\{ \int_S \mathbf{v} \cdot \mathbf{v}^*\, dx_1 \cdots dx_n \right\}^{1/2} \leqslant \left\{ \int_S \mathbf{a}_1 \cdot \mathbf{a}_1^*\, dx_1 \cdots dx_n \right\}^{1/2} + \left\{ \int_S \mathbf{w} \cdot \mathbf{w}^*\, dx_1 \cdots dx_n \right\}^{1/2}. \tag{4.16}$$

Since $\mathbf{a}_1$ is smooth and has compact support in S for each $t \geqslant 0$, it follows from (4.15), (4.16) that for any $T \geqslant 0$ there exists a finite number $M(T)$ which is independent of ω and such that

$$\left\{ \int_S \mathbf{v}(\mathbf{x}, t; \omega) \cdot \mathbf{v}^*(\mathbf{x}, t; \omega)\, dx_1 \cdots dx_n \right\}^{1/2} \leqslant M(T) \tag{4.17}$$

for every $t \in [0, T]$ and for every $\omega > 0$. This is the desired boundedness of the remainder.

By comparing with Friedlander [2], Ludwig [3] or Eckhoff [1], we see that (4.1) consists of the leading term in *a generalized progressing wave solution* of (2.1) and the remainder. With the imposed assumptions, (4.17) shows that on any finite time interval the leading term in (4.1) is an approximation of a family of exact solutions of (2.1) where the error can be made arbitrarily small by choosing ω sufficiently large. As a first step towards the significance of this in the study of the problem of stability for (2.1), we now establish the following result:

PROPOSITION. *The trivial solution* $\mathbf{u} = 0$ *of* (2.1) *is unstable if there exists a number* $\varepsilon_0 > 0$ *with the following property: For any* $\delta > 0$ *it is possible to find a family of solutions* (4.1) *which satisfies the assumptions imposed above and which is such that for some numbers* $t_1 \geqslant t_0 \geqslant 0$ *we have*

$$\rho_0\{\mathbf{a}_0(\mathbf{x}, t_0), 0\} < \delta \qquad \textit{and} \qquad \rho\{\mathbf{a}_0(\mathbf{x}, t_1), 0\} > \varepsilon_0. \tag{4.18}$$

Proof. Suppose that such a number $\varepsilon_0 > 0$ exists. If we choose

$$\omega > \frac{M(t_0)}{\delta - \rho_0\{\mathbf{a}_0(\mathbf{x}, t_0), 0\}} + \frac{M(t_1)}{\rho\{\mathbf{a}_0(\mathbf{x}, t_1), 0\} - \varepsilon_0}, \tag{4.19}$$

the triangle inequality and (2.5), (4.1), (4.17) implies that

$$\rho_0\{\mathbf{u}_\omega(\mathbf{x}, t_0), 0\} < \delta \qquad \text{and} \qquad \rho\{\mathbf{u}_\omega(\mathbf{x}, t_1), 0\} > \varepsilon_0. \tag{4.20}$$

Since $\varepsilon_0 > 0$ is fixed while $\delta > 0$ is arbitrary, this shows that the trivial solution of (2.1) is unstable. Q.E.D.

The essence of the above proposition is that the trivial solution $\mathbf{u} = 0$ of (2.1) is unstable if the trivial solution $\boldsymbol{\sigma} = \{\sigma_1, ..., \sigma_\mu\} = 0$ of (4.14) is unstable (when the imposed assumptions are satisfied). This could also be expressed in the following way: In order that the trivial solution of (2.1) shall be stable, it is necessary that the trivial solution of (4.14) is stable. In the following sections we shall exploit this further.

5. The Transport Equations

In order that the considerations in the preceding section shall be of any value in the study of the stability problem for (2.1), it is necessary that there exist families of solutions (4.1) satisfying all the assumptions imposed: The real-valued phase function φ satisfies (4.7) and is such that Ω is an eigenvalue of fixed multiplicity in the considered domain, the amplitude $\mathbf{a}_0$ satisfies (4.8) and (4.14) such that the leading term $\mathbf{a}_0 \exp(i\omega\varphi)$ is smooth in $S \times R^1_+$ and has compact support in S for each $t \geqslant 0$. In this section we shall study how these assumptions can be satisfied.

In addition to the assumption that Ω is an eigenvalue of fixed multiplicity in the considered domain, we shall also assume that Ω as well as the associated eigenvectors $\mathbf{r}_1, ..., \mathbf{r}_\mu$ are smooth with respect to all the variables $\mathbf{x}$, t and $\boldsymbol{\xi} = \{\xi^1, ..., \xi^n\}$ in the considered domain. These assumptions are satisfied if the hyperbolic system (2.1) has characteristics with constant multiplicity (see Courant and Hilbert [6, p. 626]). If (2.1) has characteristics with nonuniform multiplicity, the assumptions imply that we have to limit the considered class of initial data for (4.7)

$$\varphi(\mathbf{x}, 0) = \varphi_0(\mathbf{x}). \tag{5.1}$$

In some cases only minor restrictions on φ_0 are necessary, while in others it may be impossible to find suitable restrictions. Thus if the hyperbolic system (2.1) has characteristics with nonuniform multiplicity, the assumptions imposed on Ω and $\mathbf{r}_1, ..., \mathbf{r}_\mu$ may in some cases put serious limitations on the applicability of the theory in this paper. On this basis, we shall refer to these assumptions as *the multiplicity assumptions*.

The Cauchy problem (4.7), (5.1) may at least locally be uniquely solved by the method of characteristics, i.e., by solving the following system of ordinary differential equations (see Courant and Hilbert [6, Chap. 2])

$$\frac{dx_\nu}{dt}=\frac{\partial\Omega}{\partial\xi^\nu},\qquad \frac{d\xi^\nu}{dt}=-\frac{\partial\Omega}{\partial x_\nu}\qquad (\nu=1,\dots,n)\qquad \text{(5.2a, b)}$$

called the ray (or bicharacteristic) equations associated with (2.1), together with the initial data

$$x_\nu(0)=x_{\nu 0},\qquad \xi^\nu(0)=\xi_0^\nu\qquad (\nu=1,\dots,n),\qquad \text{(5.3a, b)}$$

where $\mathbf{x}_0=\{x_{10},\dots,x_{n0}\}\in S$ and $\boldsymbol{\xi}_0=\{\xi_0^1,\dots,\xi_0^n\}=\nabla\varphi_0(\mathbf{x}_0)$. Unfortunately, the solution $\varphi(\mathbf{x},t)$ of (4.7), (5.1) thus obtained, will not in general exist as a smooth function in the whole region $S\times R_+^1$. First, the existence of so-called *caustics* may have this implication. Second, the *boundaries* may cause difficulties when $R^n\backslash S\neq\varnothing$. We shall consider these problems below.

From the definition of Ω and $\mathbf{r}_1,\dots,\mathbf{r}_\mu$ we have for $k,l=1,\dots,\mu$

$$\{-\Omega\mathbf{I}+\mathbf{E}\}\cdot\mathbf{r}_k=\mathbf{r}_l\cdot\{-\Omega\mathbf{I}+\mathbf{E}\}\equiv 0.\qquad (5.4)$$

Differentiation with respect to ξ^j leads to

$$\left\{-\frac{\partial\Omega}{\partial\xi^j}\mathbf{I}+\mathbf{A}^j\right\}\cdot\mathbf{r}_k+\{-\Omega\mathbf{I}+\mathbf{E}\}\cdot\frac{\partial\mathbf{r}_k}{\partial\xi^j}\equiv 0.\qquad (5.5)$$

Using (5.4), this implies that

$$\begin{aligned}\mathbf{r}_l\cdot\mathbf{A}^j\mathbf{r}_k&=0 &&\text{when } l\neq k,\\ &=\frac{\partial\Omega}{\partial\xi^j} &&\text{when } l=k.\end{aligned}\qquad (5.6)$$

The system of equations (4.14) for the amplitude $\boldsymbol{\sigma}$ may therefore be interpreted as a system of ordinary differential equations

$$\frac{d}{dt}\sigma_l=-\sum_{k=1}^{\mu}(\mathbf{r}_l\cdot L\mathbf{r}_k)\,\sigma_k\qquad (l=1,\dots,\mu)\qquad (5.7)$$

along the rays determined by (5.2). Equations (5.7) are the so-called *transport equations* for the hyperbolic system (2.1).

The above considerations show that the leading term $\mathbf{a}_0\exp(i\omega\varphi)$ may be calculated by solving ordinary differential equations only, i.e., (5.2) and (5.7). From (5.7) it follows that $\mathbf{a}_0\exp(i\omega\varphi)$ is different from 0 only along the family of rays which starts at the points in the support of $\mathbf{a}_0$ at $t=0$. The support of $\mathbf{a}_0\exp(i\omega\varphi)$ will therefore remain in S for each $t\geqslant 0$ if and only if no member of this family of rays hits the boundary of S within a finite interval of time. Furthermore, we see that $\mathbf{a}_0 \exp(i\omega\varphi)$ will be smooth in $S \times R^1_+$ if the $\mathbf{x}$-components of this family of rays do not cross each other or have envelopes for $t \geqslant 0$ (this is the nonexistence of caustics). Finally, we see that the support of $\mathbf{a}_0 \exp(i\omega\varphi)$ is compact for every $t \geqslant 0$ since we assume that the support of $\mathbf{a}_0$ at $t=0$ is compact.

In conclusion, we may say that the assumptions imposed for the leading term $\mathbf{a}_0 \exp(i\omega\varphi)$ in (4.1), lead to very restrictive assumptions for the family of rays starting at the points in the support of $\mathbf{a}_0$ at $t=0$. It is very fortunate, however, that these assumptions may be shown to holds in important applications. The easiest case to handle in this context, is that where the characteristic root in question is a *linear* function with respect to $\boldsymbol{\xi}$, i.e.,

$$\Omega = \sum_{\nu=1}^{n} e_\nu(\mathbf{x}, t)\, \xi^\nu, \tag{5.8}$$

where $e_1,..., e_n$ are smooth and bounded in $\overline{S \times R^1_+}$. In this case the ray equations (5.2) decouple and (5.2a) becomes

$$\frac{dx_\nu}{dt} = e_\nu(\mathbf{x}, t) \qquad (\nu = 1,..., n). \tag{5.9}$$

The standard theory for such systems of ordinary differential equations immediately implies that caustics cannot exist. Thus the leading term $\mathbf{a}_0 \exp(i\omega\varphi)$ in (4.1) will in this case always satisfy the assumptions imposed if no ray (i.e, no solution of (5.9)) which starts at a point in S hits the boundary of S within a finite interval of time. It is very fortunate that this attractive case seems to be the most important one in a stability research in fluid mechanics, for instance (see Eckhoff and Storesletten [4, 5]).

When the considered characteristic root Ω is a *nonlinear* function with respect to $\boldsymbol{\xi}$, the $\mathbf{x}$-components of the family of rays starting at the points in the support of $\mathbf{a}_0$ at $t=0$ will no longer be independent of the initial function φ_0 given in (5.1) and (5.3). In this case caustics will necessarily appear for a large variety of smooth functions φ_0 (some considerations on this are given in Eckhoff [1]). At least in special cases, however, it may be possible to overcome this difficulty by a careful restriction of the considered functions φ_0. It is obvious, for instance, that caustics cannot appear if it is possible to restrict φ_0 such that any pair of rays are either parallel or diverging for every $t \geqslant 0$. It is probably not possible in general to satisfy this condition, but in a large number of special cases it is at least in principle possible to design restrictions on φ_0 which imply that this condition is satisfied. The easiest case to handle in this context, is that where the matrices $\mathbf{A}^\nu$ $(\nu = 1,..., n)$ in (2.1) are independent of $\mathbf{x}$, t since the rays are then straight lines. If the considered functions φ_0 are restricted to be linear with respect to $\mathbf{x}$, any pair of rays will be parallel in this case. The considerably larger class of initial functions φ_0 for which any pair of associated rays are either parallel or diverging for every $t \geqslant 0$, may of course also be used in this case. As a conclusion we may say that when the considered characteristic root Ω is nonlinear with respect to $\boldsymbol{\xi}$, it seems possible in many cases to show that our assumptions hold if we put suitable restrictions on φ_0 and if $S = R^n$. When $R^n \backslash S \neq \varnothing$, on the other hand, the prospects seem very limited. Usually it will not be possible to restrict φ_0 such that no ray reaches the boundary of S within a finite interval of time. Some further remarks on the case where Ω is nonlinear with respect to $\boldsymbol{\xi}$ will be given in Section 8.

We shall close this section by elaborating the right-hand side in the transport equations (5.7).

We record that

$$\begin{aligned} L\mathbf{r}_k = {} & \frac{\partial \mathbf{r}_k}{\partial t} + \sum_{\nu=1}^{n} \mathbf{A}^\nu \frac{\partial \mathbf{r}_k}{\partial x_\nu} + \mathbf{B}\mathbf{r}_k \\ & + \sum_{j=1}^{n} \left\{ \varphi_{x_j t} \mathbf{I} + \sum_{\nu=1}^{n} \varphi_{x_j x_\nu} \mathbf{A}^\nu \right\} \frac{\partial \mathbf{r}_k}{\partial \xi^j}. \end{aligned} \tag{5.10}$$

Differentiation with respect to x_j in (4.7) leads to

$$\varphi_{x_j t} + \sum_{\nu=1}^{n} \frac{\partial \Omega}{\partial \xi^\nu} \varphi_{x_j x_\nu} + \frac{\partial \Omega}{\partial x_j} = 0. \tag{5.11}$$

If we differentiate (5.5) with respect to ξ^ν, we obtain

$$\begin{aligned} & -\frac{\partial^2 \Omega}{\partial \xi^j \partial \xi^\nu} \mathbf{r}_k + \left\{ -\frac{\partial \Omega}{\partial \xi^j} \mathbf{I} + \mathbf{A}^j \right\} \frac{\partial \mathbf{r}_k}{\partial \xi^\nu} \\ & + \left\{ -\frac{\partial \Omega}{\partial \xi^\nu} \mathbf{I} + \mathbf{A}^\nu \right\} \frac{\partial \mathbf{r}_k}{\partial \xi^j} + \{ -\Omega \mathbf{I} + \mathbf{E} \} \frac{\partial^2 \mathbf{r}_k}{\partial \xi^j \partial \xi^\nu} \equiv 0. \end{aligned} \tag{5.12}$$

By combining (5.4), (5.10), (5.11), (5.12), we may write the transport equations (5.7) in the following way

$$\begin{aligned} \frac{d}{dt} \sigma_l = {} & - \sum_{k=1}^{\mu} r_l \cdot \left\{ \frac{\partial \mathbf{r}_k}{\partial t} + \sum_{\nu=1}^{n} \mathbf{A}^\nu \frac{\partial \mathbf{r}_k}{\partial x_\nu} + \mathbf{B}\mathbf{r}_k \right\} \sigma_k \\ & + \sum_{k=1}^{\mu} \sum_{j=1}^{n} \frac{\partial \Omega}{\partial x_j} \mathbf{r}_l \cdot \frac{\partial \mathbf{r}_k}{\partial \xi^j} \sigma_k \\ & - \frac{1}{2} \sum_{j=1}^{n} \sum_{\nu=1}^{n} \varphi_{x_j x_\nu} \frac{\partial^2 \Omega}{\partial \xi^j \partial \xi^\nu} \sigma_l \qquad (l = 1, \ldots, \mu). \end{aligned} \tag{5.13}$$

In the attractive case where Ω is linear with respect to ξ, i.e., given by (5.8), we see that the last sum of terms in (5.13) drops out. In this case therefore, (5.2) and (5.13) constitute a closed system of ordinary differential equations which in principle can be solved without explicit knowledge of $\varphi(\mathbf{x}, t)$.

When the considered characteristic root Ω is nonlinear with respect to ξ, (5.2) and (5.13) do not by themselves constitute a closed system since (5.13) depends on the quantities $\varphi_{x_j x_\nu}$ along the rays. It is not difficult to close the system (5.2), (5.13) by adding transport equations for these quantities $\varphi_{x_j x_\nu}$ along the rays (this is done in Eckhoff [1]), but this will enlarge the number of equations by $\frac{1}{2}n(n+1)$. Again it is clear that it is considerably more difficult in general to study families of solutions (4.1) which correspond to a nonlinear characteristic root Ω, than those corresponding to a linear one.

6. The Stability Equations

Now let $\varphi(\mathbf{x}, t)$ denote a real-valued solution of (4.7), and let $\mathbf{x}(t; \mathbf{x}_0)$ denote the corresponding rays, i.e., the solutions of (5.2a), (5.3a) when $\xi = \nabla\varphi$ has been substituted. Suppose that the family of these rays which start in some bounded open set $Q \subset S$ at $t = 0$, remains in S for every $t \geqslant 0$, is not in conflict with the multiplicity assumptions, and is free from caustics. For families of solutions (4.1) of (2.1) with $\operatorname{supp} \mathbf{a}_0(\mathbf{x}, 0) \subset Q$, the proposition of Section 4 is then concerned with the evolution of the quantity

$$\begin{aligned} \rho\{\mathbf{a}_0(\mathbf{x}, t), 0\} &= \left\{\int_S \sum_{l=1}^{\mu} |\sigma_l(\mathbf{x}, t)|^2 \, dx_1 \cdots dx_n\right\}^{1/2} \\ &= \left\{\sum_{l=1}^{\mu} \int_Q |P_l(t; \mathbf{x}_0)|^2 \, dx_{10} \cdots dx_{n0}\right\}^{1/2}, \end{aligned} \tag{6.1}$$

where

$$P_l(t; \mathbf{x}_0) = |J|^{1/2} \sigma_l(\mathbf{x}(t; \mathbf{x}_0), t) \qquad (l = 1, \dots, \mu), \tag{6.2}$$

$$J = \frac{\partial(x_1, \dots, x_n)}{\partial(x_{10}, \dots, x_{n0})}. \tag{6.3}$$

For fixed $\mathbf{x}_0 \in Q$, we obtain from (5.7) and (6.2)

$$\begin{aligned} \frac{d}{dt} P_l &= |J|^{1/2} \frac{d}{dt} \sigma_l + \frac{1}{2} |J|^{-1/2} \frac{d}{dt} |J| \, \sigma_l \\ &= -\sum_{k=1}^{\mu} (\mathbf{r}_l \cdot L\mathbf{r}_k) P_k + \frac{1}{2} |J|^{-1} \frac{d}{dt} |J| \, P_l \qquad (l = 1, \dots, \mu). \end{aligned} \tag{6.4}$$

This system of ordinary differential equations (6.4), which determines $\mathbf{P} = \{P_1, ..., P_\mu\}$ along the rays, we shall call *the stability equations* for the hyperbolic system (2.1).

DEFINITION 2. The trivial solution $\mathbf{P} = 0$ of the stability equations (6.4) is said to be uniformly unstable in the compact set $R \subset S$ if there exist a number $\varepsilon > 0$ and a bounded open set $Q \subset S$ with the following properties:

(a) $R \subset Q$;

(b) There exists a real-valued solution $\varphi(\mathbf{x}, t)$ of (4.7) which is such that the corresponding family of rays starting in Q at $t = 0$ remains in S for every $t \geqslant 0$, is not in conflict with the multiplicity assumptions, and is free from caustics;

(c) For any $\delta_1 > 0$ it is possible to find a smooth solution $\mathbf{P}(t; \mathbf{x}_0)$ of (6.4) along these rays with supp $\mathbf{P}(0; \mathbf{x}_0) \subset Q$, and numbers $t_1 \geqslant t_0 \geqslant 0$ which are such that

$$\left\{ \sum_{l=1}^{\mu} |P_l(t_0; \mathbf{x}_0)|^2 \right\}^{1/2} < \delta_1 \qquad \text{for every } \mathbf{x}_0 \in Q, \tag{6.5}$$

$$\left\{ \sum_{l=1}^{\mu} |P_l(t_1; \mathbf{x}_0)|^2 \right\}^{1/2} > \varepsilon \qquad \text{for every } \mathbf{x}_0 \in R. \tag{6.6}$$

THEOREM. *The trivial solution* $\mathbf{u} = 0$ *of* (2.1) *is unstable if the trivial solution* $\mathbf{P} = 0$ *of the stability equations* (6.4) *is uniformly unstable in some compact set* $R \subset S$ *of positive measure.*

Proof. Suppose that the assumptions in the theorem are fulfilled and let $\delta > 0$ be arbitrarily given. Choose $\delta_1 = \{\text{Volume of } Q\}^{-1/2}\,\delta$ and let $\mathbf{P}(t; \mathbf{x}_0)$ be the corresponding solution of (6.4) specified in Definition 2(c). From (6.1) and (6.5), (6.6) it then follows that the corresponding leading term in the family of solutions (4.1) is such that

$$\rho_0\{\mathbf{a}_0(\mathbf{x}, t_0), 0\} < \delta_1\{\text{Volume of } Q\}^{1/2} = \delta, \tag{6.7}$$

$$\rho\{\mathbf{a}_0(\mathbf{x}, t_1), 0\} > \varepsilon\{\text{Volume of } R\}^{1/2}. \tag{6.8}$$

Thus all the requirements in the proposition of Section 4 are fulfilled if we choose $\varepsilon_0 = \varepsilon\{\text{Volume of } R\}^{1/2}$. Q.E.D.

If we have been able to fulfill the requirements in Definition 2(b) (the problems in this connection were discussed in the preceding section), the essence of the above theorem is that we may obtain results on the problem of stability for the hyperbolic system (2.1) by a study of the stability problem for the linear system of ordinary differential equations (6.4). In fact, the requirements in Definition 2(c) is that for every $\mathbf{x}_0 \in R$, the trivial solution $\mathbf{P} = 0$ of (6.4) is unstable in the usual sense for a linear system of ordinary differential equations (see for instance Cesari [10]). In Definition 2(c) it is also required that this instability must be uniform with respect to $\mathbf{x}_0 \in R$, but this uniformity may usually be shown to hold by simple arguments since (6.4) is a linear system with smooth coefficients.

In order to elaborate the right-hand side in (6.4), we note that the difference between the stability equations (6.4) and the transport equations (5.7) is due to the last term in (6.4). In order to elaborate that term, we record that (5.2a) implies

$$x_q(t + \Delta t; \mathbf{x}_0) = x_q(t; \mathbf{x}_0) + \Delta t \left(\frac{\partial \Omega}{\partial \xi^q}\right)_0 + O(\Delta t^2) \qquad (q = 1, \dots, n), \tag{6.9}$$

where $(\cdots)_0$ means that $\mathbf{x} = \mathbf{x}(t; \mathbf{x}_0)$ and $\boldsymbol{\xi} = \nabla \varphi(\mathbf{x}(t; \mathbf{x}_0), t)$ are substituted. From the chain rule it follows that

$$J(t + \Delta t, \mathbf{x}_0) = \det\{j_1, \dots, j_n\}\, J(t, \mathbf{x}_0), \tag{6.10}$$

where j_p, $p = 1, \dots, n$ are the n-dimensional column vectors with elements

$$\frac{\partial\{x_q(t + \Delta t, \mathbf{x}_0)\}}{\partial\{x_p(t, \mathbf{x}_0)\}} \qquad (q = 1, \dots, n). \tag{6.11}$$

If we disregard $O(\Delta t^2)$ terms, (6.9) gives that

$$j_p = \left\{ \begin{matrix} \Delta t \left\{ \left(\dfrac{\partial^2 \Omega}{\partial x_p\, \partial \xi^1}\right)_0 + \sum_{\nu=1}^{n} \left(\dfrac{\partial^2 \Omega}{\partial \xi^\nu\, \partial \xi^1}\right)_0 (\varphi_{x_\nu x_p})_0 \right\} \\ \cdots \\ \Delta t \left\{ \left(\dfrac{\partial^2 \Omega}{\partial x_p\, \partial \xi^{p-1}}\right)_0 + \sum_{\nu=1}^{n} \left(\dfrac{\partial^2 \Omega}{\partial \xi^\nu\, \partial \xi^{p-1}}\right)_0 (\varphi_{x_\nu x_p})_0 \right\} \\ 1 + \Delta t \left\{ \left(\dfrac{\partial^2 \Omega}{\partial x_p\, \partial \xi^p}\right)_0 + \sum_{\nu=1}^{n} \left(\dfrac{\partial^2 \Omega}{\partial \xi^\nu\, \partial \xi^p}\right)_0 (\varphi_{x_\nu x_p})_0 \right\} \\ \Delta t \left\{ \left(\dfrac{\partial^2 \Omega}{\partial x_p\, \partial \xi^{p+1}}\right)_0 + \sum_{\nu=1}^{n} \left(\dfrac{\partial^2 \Omega}{\partial \xi^\nu\, \partial \xi^{p+1}}\right)_0 (\varphi_{x_\nu x_p})_0 \right\} \\ \cdots \\ \Delta t \left\{ \left(\dfrac{\partial^2 \Omega}{\partial x_p\, \partial \xi^n}\right)_0 + \sum_{\nu=1}^{n} \left(\dfrac{\partial^2 \Omega}{\partial \xi^\nu\, \partial \xi^n}\right)_0 (\varphi_{x_\nu x_p})_0 \right\} \end{matrix} \right\} \tag{6.12}$$

Equation (6.12) shows that all the off-diagonal elements in the matrix $\{j_1, \dots, j_n\}$ are $O(\Delta t)$ terms. If we disregard $O(\Delta t^2)$ terms, it therefore suffices to consider the product of the main diagonal elements in $\det\{j,\dots,j_n\}$. Thus we obtain from (6.10) and (6.12)

$$J(t+\Delta t, \mathbf{x}_0) = J(t, \mathbf{x}_0)\left[1 + \Delta t\left\{\sum_{p=1}^{n}\left(\frac{\partial^2 \Omega}{\partial x_p\,\partial \xi^p}\right)_0 + \sum_{p=1}^{n}\sum_{\nu=1}^{n}\left(\frac{\partial^2 \Omega}{\partial \xi^\nu\,\partial \xi^p}\right)_0 (\varphi_{x_\nu x_p})_0\right\} + O(\Delta t^2)\right], \tag{6.13}$$

which implies that in (6.4) we may set

$$\frac{1}{2}|J|^{-1}\frac{d}{dt}|J| = \frac{1}{2}\sum_{p=1}^{n}\frac{\partial^2 \Omega}{\partial x_p\,\partial \xi^p} + \frac{1}{2}\sum_{p=1}^{n}\sum_{\nu=1}^{n}\frac{\partial^2 \Omega}{\partial \xi^\nu\,\partial \xi^p}\varphi_{x_\nu x_p}. \tag{6.14}$$

By combining (5.13) and (6.14), we may write the stability equations (6.4) in the following way

$$\frac{d}{dt}P_l = -\sum_{k=1}^{\mu}\mathbf{r}_l\cdot\left\{\frac{\partial \mathbf{r}_k}{\partial t} + \sum_{\nu=1}^{n}\mathbf{A}^\nu\frac{\partial \mathbf{r}_k}{\partial x_\nu} + \mathbf{B}\mathbf{r}_k\right\}P_k + \sum_{k=1}^{\mu}\sum_{j=1}^{n}\frac{\partial \Omega}{\partial x_j}\mathbf{r}_l\cdot\frac{\partial \mathbf{r}_k}{\partial \xi^j}P_k + \frac{1}{2}\sum_{p=1}^{n}\frac{\partial^2 \Omega}{\partial x_p\,\partial \xi^p}P_l \qquad (l = 1,\dots,\mu). \tag{6.15}$$

The most striking difference between the transport equations (5.13) and the stability equations (6.15) is that the latter together with the ray equations (5.2) *always* constitute a closed system of ordinary differential equations. The terms which prevented that for (5.13) were

$$-\frac{1}{2}\sum_{j=1}^{n}\sum_{\nu=1}^{n}\varphi_{x_j x_\nu}\frac{\partial^2 \Omega}{\partial \xi^j\,\partial \xi^\nu}\sigma_l. \tag{6.16}$$

It is well known that the leading term in (4.1) will blow up at a caustic. In (5.13) this blowing up is caused by the terms (6.16) which we therefore shall call *the focussing terms* in the transport equations (5.13) (see Eckhoff [1]). Since (5.2) and (6.15), on the other hand, constitute a closed system, (6.15) may be solved along each ray without any knowledge of neighbouring rays. Thus the presence of caustics will have no effect at all on the solutions of the stability equations (6.15).

As we recorded in the preceding section, the focussing terms (6.16) drop out of (5.13) when the characteristic root Ω is linear with respect to $\boldsymbol{\xi}$, i.e., when Ω is given by (5.8). The difference between the transport equations

(5.13) and the stability equations (6.15) is in this case due to the following terms in (6.15)

$$\frac{1}{2} \sum_{p=1}^{n} \frac{\partial^2 \Omega}{\partial x_p \, \partial \xi^p} P_l. \tag{6.17}$$

When Ω is given by (5.8), we see that

$$\sum_{p=1}^{n} \frac{\partial^2 \Omega}{\partial x_p \, \partial \xi^p} = \sum_{p=1}^{n} \frac{\partial e_p}{\partial x_p} = \nabla \cdot \mathbf{e}, \tag{6.18}$$

where $\mathbf{e} = \{e_1, ..., e_n\}$ is the bicharacteristic vector field which by (5.9) determines the rays (or bicharacteristics). From the analogy in fluid mechanics, it is clear that the quantity $\nabla \cdot \mathbf{e}$ may be interpreted as an expression for the "compressibility" of the rays. We shall therefore call (6.17) *the compression terms* in (6.15).

The compression terms can be shown to vanish in several cases of interest. The most obvious such cases are those where the matrices $\mathbf{A}^\nu$ ($\nu = 1, ..., n$) in (2.1) are independent of $\mathbf{x}$. Also the cases considered in Eckhoff and Storesletten [4, 5] as well as cases in magnetohydrodynamics belong to this category. In these cases with vanishing compression terms, the transport equations (5.13) and the stability equations (6.15) are therefore identical when Ω is a linear function with respect to $\boldsymbol{\xi}$.

7. Applications in the Most Attractive Cases

Let us first consider a hyperbolic system (2.1) which has characteristics with constant multiplicity. Suppose that associated with (2.1) there is a characteristic root Ω which is a linear function with respect to $\boldsymbol{\xi}$, i.e., Ω is given by (5.8). If no ray starting at a point in S hits the boundary of S within a finite interval of time, the requirements in Definition 2(b) are then always satisfied. Thus we may consider any bounded open set $Q \subset S$ and any initial function φ_0 in (5.1). In particular, we may for instance take $\varphi_0(\mathbf{x})$ as a linear function, which in view of (5.3b) may be written as

$$\varphi_0(\mathbf{x}) = \boldsymbol{\xi}_0 \cdot \mathbf{x}, \tag{7.1}$$

where $\boldsymbol{\xi}_0 \neq 0$ is an arbitrarily given real n-dimensional vector. At least in principle we may then solve the initial value problem (5.2), (5.3) for every $\mathbf{x}_0 \in S$ and every $t \geqslant 0$. By substituting the solutions of (5.2), (5.3)

$$\mathbf{x} = \mathbf{x}(t; \mathbf{x}_0), \qquad \boldsymbol{\xi} = \boldsymbol{\xi}(t; \mathbf{x}_0, \boldsymbol{\xi}_0) \tag{7.2}$$

into the stability equations (6.15), we obtain a closed linear system of ordinary differential equations

$$\frac{d}{dt}\mathbf{P} = \mathbf{A}(t; \mathbf{x}_0, \boldsymbol{\xi}_0)\,\mathbf{P}. \tag{7.3}$$

According to the theorem in the preceding section, the trivial solution $\mathbf{u} = 0$ of (2.1) is unstable if we are able to find a vector $\boldsymbol{\xi}_0 \neq 0$ which is such that the trivial solution $\mathbf{P} = 0$ of (7.3) is unstable for every $\mathbf{x}_0 \in Q$, where $Q \subset S$ is some bounded open set. In fact, it is then easy to show that there exists a compact set $R \subset Q$ of positiv measure in which $\mathbf{P} = 0$ is uniformly unstable. So, in order that the trivial solution $\mathbf{u} = 0$ of (2.1) shall be stable, it is necessary that no vector $\boldsymbol{\xi}_0$ with the above properties exists. By using the well-established theory of stability for linear ordinary differential equations (see Cesari [10]), we may in this way obtain stability criteria on the coefficients in the hyperbolic system (2.1). For certain models in fluid mechanics, this has been done by Eckhoff and Storesletten [4, 5].

The above procedure may also be applied in many cases where the hyperbolic system (2.1) has characteristics with nonuniform multiplicity. The modification of the procedure to these cases will consist of a restriction of the considered values of the vector $\boldsymbol{\xi}_0$ and possibly also a restriction of the considered set Q. These restrictions must be designed such that the multiplicity assumptions are satisfied along the rays (7.2) for the considered vectors $\boldsymbol{\xi}_0$ and for $\mathbf{x}_0 \in Q$. The easiest case to handle in this context is that where the matrices $\mathbf{A}^\nu$ ($\nu = 1,..., n$) in (2.1) are independent of $\mathbf{x}$, $\mathbf{t}$. Since Ω is independent of $\mathbf{x}$, $\mathbf{t}$ in this case, and $\boldsymbol{\xi} \equiv \boldsymbol{\xi}_0$ along the rays by (5.2b), it suffices to restrict $\boldsymbol{\xi}_0$ in such a way that we keep off the branch points for all the characteristic roots associated with the hyperbolic system (2.1). It is not necessary to restrict the set Q in this case.

In the case where the matrices $\mathbf{A}^\nu$ ($\nu = 1,..., n$) in (2.1) are independent of $\mathbf{x}$, t and $S = R^n$, the above procedure also works when the considered characteristic root Ω is a nonlinear function with respect to $\boldsymbol{\xi}$. If φ_0 is given by (7.1), the solution of (4.7), (5.1) is in this case

$$\varphi(\mathbf{x}, t) = \boldsymbol{\xi}_0 \cdot \mathbf{x} - t\Omega(\boldsymbol{\xi}_0) \tag{7.4}$$

and the rays are parallel straight lines

$$x_\nu(t) = x_{\nu 0} + t\,\frac{\partial \Omega}{\partial \xi^\nu}(\boldsymbol{\xi}_0) \qquad (\nu = 1,..., n). \tag{7.5}$$

The stability equations (6.15) simplify in this case to

$$\frac{dP_l}{dt} = -\sum_{k=1}^{\mu} \{\mathbf{r}_l \cdot \mathbf{B}\mathbf{r}_k\}\, P_k \qquad (l = 1,..., \mu), \tag{7.6}$$

where (7.5) is substituted if $\mathbf{B}$ depends on $\mathbf{x}$. We note that the focussing terms vanish for the phase function (7.4), the stability equations (7.6) and the transport equations (5.13) are therefore identical. According to the theorem in the preceding section, the trivial solution $\mathbf{u} = 0$ of (2.1) is unstable if we are able to find a vector $\boldsymbol{\xi}_0$ which is not in conflict with the multiplicity assumptions and which is such that the trivial solution $P = 0$ of (7.6) is unstable for every $\mathbf{x}_0 \in Q$, where $Q \subset R^n$ is some bounded open set.

If the matrix $\mathbf{B}$ as well as the matrices $\mathbf{A}^\nu$ $(\nu = 1,..., n)$ in (2.1) are independent of $\mathbf{x}$, t, (7.6) will for each $\boldsymbol{\xi}_0$ which is not in conflict with the multiplicity assumptions be an autonomous linear system of ordinary differential equations. The stability properties for the trivial solution $\mathbf{P} = 0$ of (7.6) are therefore determined by calculating the eigenvalues of the $\mu \times \mu$ matrix

$$\{\mathbf{r}_l \cdot \mathbf{B}\mathbf{r}_k\} \qquad (l, k = 1,..., \mu) \tag{7.7}$$

(see, for instance, Cesari [10]). Thus stability criteria for the hyperbolic system (2.1) may in this case be obtained by solving algebraic equations only. In view of (7.4), these criteria must necessarily be special cases of the criteria which can be obtained by the normal mode method (plane waves). It should be noted, however, that the normal mode method leads to an algebraic equation which often is considerably more difficult to solve than the algebraic equations appearing in our approach.

8. Some Remarks

The most striking advantage of our approach to the stability problem for the hyperbolic system (2.1), is probably that it essentially is a local analysis involving ordinary differential equations and algebraic equations only. Thus the geometry of the problem may in many cases represent a considerably less serious obstacle in our approach than in the normal mode approach, since the latter leads to a global eigenvalue problem for a partial differential operator. Furthermore, our approach leads to regular problems of stability for linear systems of ordinary differential equations (usually nonautonomous); the singularities which lead to the continuous spectrum and similar problems in the normal mode approach, will therefore not cause any special difficulties in our approach.

The assumptions necessary for applying our approach to the problem of stability are of a completely different nature than the assumptions necessary for applying the conventional approaches. Dependence on t in the coefficients of (2.1), for instance, does not cause any special difficulties in our approach. So far we have not studied the stability of nonstationary phenomena, however, but this seems to be a promising area for future research. As we recorded in the preceding sections, the assumptions necessary for applying our approach seem very restrictive, but they are satisfied in important applications. Furthermore, the assumptions may in many cases be considerably less restrictive than their appearance at first glance might indicate. Caustics, for instance, which initially seemed to be a serious obstacle in our approach, probably do not represent any obstacle at all. In fact, since the presence of caustics has no effect at all on the solutions of the stability equations, it is reasonable to conjecture that the conclusions drawn in Section 7 hold also in the cases where caustics appear in the solution of (4.7), (5.1), (7.1). The discussion in Ludwig [11, 12] support this conjecture very nicely. When $R^n \backslash S \neq \varnothing$, the boundaries also seemed to be a serious obstacle especially in connection with characteristic roots which are nonlinear functions with respect to ξ. At least in special cases, however, it seems possible to overcome the difficulties by a careful study of what happens when the rays hit the boundary. In the simplest cases only a *reflection* will occur, and it may be possible to take this reflection into account in the family of solutions (4.1). In general, however, problems with creeping waves and the like may appear (see Ludwig [12]). Finally, it may be worth mentioning that the multiplicity assumptions also may be relaxed substantially in special cases (some remarks on this is given in Eckhoff [1]).

As we noted in Section 2, our approach to the problem of stability for (2.1) will essentially also work for a number of other choices of the metrics ρ_0 and ρ. It is not difficult to see that the instabilities detected by application of the theorem in Section 6 are genuine instabilities in the sense that they will appear in instabilities for almost any reasonable choice of the metrics ρ_0, ρ. For other choices of ρ_0, ρ, however, there may be additional instabilities. In many cases it will be possible to detect such additional instabilities by suitable modifications of our approach (some remarks on this is given in Eckhoff [1]).

Summarizing, we may say that neither the assumptions necessary for applying our approach, nor our choice of metric spaces M_0, M do usually represent obstacles which seriously limit the applicability of our approach to the problem of stability for (2.1). The value of our approach is therefore essentially limited only by the fact that it leads to conditions for stability in (2.1) which only are necessary (i.e., sufficient conditions for instability). It is easy to construct examples where the trivial solution of (2.1) is unstable, but where the instability cannot be detected by our approach (consider for instance systems (2.1) with constant coefficients and compare with the results obtained by the normal mode method). On the other hand, the necessary conditions for stability obtained by our approach can in many cases be shown to be fairly close to sufficient conditions for stability obtained by other means (see for instance Eckhoff and Storesletten [4, 5]).

In a subsequent paper we shall improve our approach such that further necessary conditions for stability may be obtained. This will be done by considering not only the leading-order terms in the family of solutions (4.1), but also taking into account higher order terms.

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