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On the Summation of Series Involving Bessel or Struve Functions

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The subject called "summation of series" can be viewed in two different ways. From one point of view, it means numerical summation which involves acceleration of convergence, and from the other, it represents a variety of summation formulas which are considered important, but which are often not useful. The aim of this paper is to produce a compromise between the two opposite approaches when series of Bessel or Struve functions are considered. The proposed method leads to series of Riemann zeta and related functions which converge much faster than the originals. In the most significant cases, closed-form formulas were obtained.

Key Words: Bessel functions; Struve functions; Riemann zeta and related functions.

1. INTRODUCTION

During the past few years, the advancement of a part of mathematical physics has been characterized by exhaustive utilization of series over special functions. For example, series over Bessel functions naturally appear in antenna and waveguide theory, plasma physics, etc.

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Apart from this, which is more than sufficient motivation for this work, is a large number of papers deal with some particular cases of the series

$$S_1(a,b,s,\alpha,F_\nu) = \sum_{n=1}^{\infty} \frac{(s)^{n-1} F_\nu((an-b)x)}{(an-b)^{\alpha}}, \tag{1}$$

$$S_2(a,b,s,\omega,m,\mu,F_\nu) = \sum_{n=1}^{\infty} \frac{(s)^{n-1}(an-b)^{\mu} F_\nu((an-b)x)}{(an-b)^{2m}\left((an-b)^2-\omega^2\right)}, \tag{2}$$

where $\alpha, \mu, \nu, \omega \in R$, $\alpha > 0$, $s = \pm 1$, $m \in N_0$, and F_ν represents Bessel, J_ν, or Struve, $\mathbf{H}_\nu$, functions of the first kind and of order ν, namely, $F_\nu = \{\begin{smallmatrix} J_\nu \\ \mathbf{H}_\nu \end{smallmatrix}\}$.

In the case of Bessel functions, the series (1) has already been treated. First, most of the particular cases have been known for a long time [3, 7, 12, 13, 18, 19]. Second, for $\alpha \in N$, $\nu \in N_0$, the short table of summable series was established in [16]. Some of these results are given in [11]. Third, the general formula was derived and represented in [17] and it involves some results from [2, 4]. It seems that this is the best published result until now, as far as the authors know. Finally, there is a possibility of extending this result for the case of Struve functions.

For the series (2), one can find in the literature only special cases. First, there are a lot of particular results cited in [13]. Second, there are a problem [5] proposed by Fettis and its solution [8] by Hansen. Independently, motivated by Fettis's "unsolved" problem [5], Lorch and Szego in [9] treated the special case of (2) ($a = 1$, $b = 0$), i.e., the series (in our notation)

$$S_2(1,0,s,\omega,m,-\nu-\delta,F_\nu) = \sum_{n=1}^{\infty} \frac{(s)^{n-1} F_\nu(nx)}{n^{2m+\delta+\nu}(n^2-\omega^2)},$$

$$F_\nu = \begin{Bmatrix} J_\nu \\ \mathbf{H}_\nu \end{Bmatrix}, \qquad \delta = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}. \tag{3}$$

This paper dealt with four types of (3), two of which involve Bessel and the other two Struve functions. In all cases, as they pointed out, the problem can be reduced to the summation of the series (3) for $\omega = 0$ which is, obviously, (1). In their paper this sum is obtained by induction. In [17], the sum is found independently for the Bessel function case in a general form.

Using some summations of trigonometric series from [17], in this paper we give a method for evaluating and representing the series (1), (2) as the series over Riemann zeta and related functions. The obtained results include the above mentioned particular cases.

Inspired by closed-form expressions of trigonometric and Bessel series whose general terms are reciprocal powers of integral variables [16, 17], in this paper special attention is paid to the series (1) and (2) which can be expressed in closed form under some appropriately imposed restrictions on a, b, α, and ν.

2. PRELIMINARIES

This section is aimed at shedding more light on the class of trigonometric series (4). The discussion presents both background material and recent developments.

We summarize here some of the results which have been or might be employed for Bessel, Struve, or other special function analysis.

As a matter of fact, the following trigonometric series [15–17] has already been quite widely used, in our earlier works, for a variety of problems in Bessel series,

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^{\alpha}} = \frac{c\pi}{2\Gamma(\alpha) f(\pi\alpha/2) x^{\alpha-1}} + \sum_{i=0}^{\infty} \frac{(-1)^i F(\alpha - 2i - \delta)}{(2i+\delta)!} x^{2i+\delta}, \quad (4)$$

where $f = \left\{ {\sin \atop \cos} \right\}$, $\delta = \left\{ {1 \atop 0} \right\}$, $\alpha \in R$, $\alpha > 0$, and where all relevant parameters are given in Table I, in which ζ, η, λ, and β represent Riemann zeta and related functions [1, 6].

TABLE I
Corresponding F and c

a	b	s	c	F	for
1	0	1	1	ζ	$0 < x < 2\pi$
		-1	0	η	$-\pi < x < \pi$
2	1	1	$\frac{1}{2}$	λ	$0 < x < \pi$
		-1	0	β	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

TABLE II
Closed-Form Cases

F	f	α
ζ, η, λ	sin cos	$2m+1$ $2m$
β	sin cos	$2m$ $2m+1$

Note that when $f(x) = \sin x$ and $\alpha \to 2m$ or $f(x) = \cos x$ and $\alpha \to 2m+1$, $m \in N_0$, the limiting value of the right-hand side of (4) should be taken into account [10, 14].

A special, but frequent case is when the right-hand side series truncate due to the vanishing of F functions (Table II),

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^{\alpha}} = (-1)^{(\alpha-\delta)/2} \frac{c\pi}{2(\alpha-1)!} x^{\alpha-1} + \sum_{i=0}^{M} \frac{(-1)^i F(\alpha-2i-\delta)}{(2i+\delta)!} x^{2i+\delta}, \quad (5)$$

where $\alpha \in N$, $M = (\alpha-1)/2$ for α odd, and $M = \alpha/2 - \delta$ for α even. This is the generalization of the results in [15]. Some particular cases of (5) can be found in [1, 7, 12, 19]. The importance of (5) is in the possibility of expressing the series (1), (2) in closed form.

According to (2), we next require the sum of the trigonometric series

$$S = \sum_{n=1}^{\infty} \frac{(s)^{n-1}(an-b)^{1-d} f((an-b)x)}{(an-b)^{2m}\left((an-b)^2-\omega^2\right)},$$

$$\omega \in R, \qquad \omega \neq an-b, \qquad m \in N_0, \qquad f = \begin{Bmatrix} \sin \\ \cos \end{Bmatrix}. \quad (6)$$

Series (6) for $m = 0$ will be denoted by S_0 and its representation, given in [7, 12, 19], is stated in general form,

$$S_0 = \frac{sd(1-b)}{2\omega^2} - \frac{s\pi \sin^{b-1}(\pi\omega/2)}{4\omega^d \cos(\pi\omega/2)} f\left(\omega x - \frac{\pi(s+1)(b+\omega)}{2a}\right), \quad (7)$$

where $\omega \in R$, $\omega \neq an - b$, and all relevant parameters are readable from Table III.

TABLE III

a	b	s	c	F	f	d	for
1	0	1	1	ζ	sin cos	0 1	$0 < x < 2\pi$
		-1	0	η	sin cos	0 1	$-\pi < x < \pi$
2	1	1	$\frac{1}{2}$	λ	sin cos	0 1	$0 < x < \pi$
		-1	0	β	sin cos	1 0	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

Using partial fractions decomposition,

$$\frac{1}{k^{2m}(k^2-\omega^2)} = \frac{1}{\omega^{2m}(k^2-\omega^2)} - \sum_{i=1}^{m} \frac{1}{k^{2i}\omega^{2m-2i+2}},$$

the series (6) can be written as

$$S = \frac{1}{\omega^{2m}} \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^{d-1}\left((an-b)^2-\omega^2\right)} - \sum_{i=1}^{m} \frac{1}{\omega^{2m-2i+2}} \sum_{n=1}^{\infty} \frac{(s)^{n-1 f((an-b)x)}}{(an-b)^{2i+d-1}}.$$

Substituting (7) in the first sum, and (5) for $\alpha = 2i + d - 1$ in the second, we finally obtain the formula

$$S = \frac{sd(1-b)}{2\omega^{2m+2}} - \frac{s\pi \sin^{b-1}(\pi\omega/2)}{4\omega^{2m+d}\cos(\pi\omega/2)} f\left(\omega x - \frac{\pi(s+1)(b+\omega)}{2a}\right) + \frac{c\pi}{2} \sum_{i=1}^{m} \frac{(-1)^{i+d} x^{2i+d-2}}{\omega^{2m-2i+2}(2i+d-2)!} - \sum_{i=1}^{m} \sum_{k=0}^{i-\delta^d} \frac{(-1)^k F(2i-2k+d-1-\delta)}{\omega^{2m-2i+2}(2k+\delta)!} x^{2k+\delta}, \tag{8}$$

where $\omega \neq an - b$, $f = \left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}$, $\delta = \left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right\}$, and all relevant parameters are given in Table III.

We need some additional explanation. We have to determine the value of $i - \delta^d$, which is the second upper bound in the double sum in (7), and that value is actually the upper bound M of the sum in (1). We do it in the following way. When d and δ take the value 0, $i - \delta^d$ is normally an indefinite expression. In this case, $\alpha = 2i - 1$ (because $d = 0$); that is, α is odd, which implies $M = i - 1$, and this value should be taken as an upper bound. Accordingly, we refer to Table III in all cases other than $d = 0$, $\delta = 0$.

Here we note in passing that it is always possible to find limiting values of (8) when $\omega \to 0$. In that case (8) is in full agreement with (5).

3. OUTLINE OF THE BASIC PROCEDURE

The object of this section is to utilize the results from the preceding one. First, we will use them in the summation of the series (1).

3.1. We will start with the well-known integral representation of Bessel/Struve functions [1],

$$F_\nu(z) = \frac{2(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu + 1/2)} \int_0^{\pi/2} \sin^{2\nu}\theta f(z\cos\theta)\, d\theta, \tag{9}$$

where $\operatorname{Re}\nu > -\frac{1}{2}$, $F_\nu = \left\{ {J_\nu \atop \mathbf{H}_\nu} \right\}$, $f = \left\{ {\cos \atop \sin} \right\}$. Substituting (9) into (1) and interchanging the order of summation and integration, the series (1) can be rewritten in the form of

$$S_1 = \frac{2(x/2)^\nu}{\Gamma(1/2)\Gamma(\nu + 1/2)} \int_0^{\pi/2} \sin^{2\nu}\theta \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x\cos\theta)}{(an-b)^{\alpha-\nu}}\, d\theta,$$

$$\alpha - \nu > 0.$$

Obviously, the part of the integrand is of the type (4), and that fact yields the integral [12],

$$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x\, dx = \frac{1}{2} B\left(\frac{\mu}{2}, \frac{\nu}{2}\right), \qquad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0. \tag{10}$$

We will omit the details and merely state the general formula (12).

3.2. Bearing in mind that, as a consequence of $\alpha - \nu > 0$, (12) is not of the most general character, we recall the integral representation of Bessel/Struve functions, but of integral order:

$$F_n(x) = \frac{1}{\pi}\int_0^\pi f(x\sin\theta - n\theta)\,d\theta, \quad n \in N_0, \quad F_n = \begin{Bmatrix} J_n \\ \mathbf{H}_n \end{Bmatrix}, \quad f = \begin{Bmatrix} \cos \\ \sin \end{Bmatrix}. \tag{11}$$

The same procedure as above leads to the integrals of the type (see [12])

$$\int_0^\pi \sin^\mu x f(\nu x)\,dx = \frac{\pi}{2^\mu} f\left(\frac{\nu\pi}{2}\right) \frac{\Gamma(\mu+1)}{\Gamma((\mu+\nu)/2+1)\Gamma((\mu-\nu)/2+1)},$$
$$f = \begin{Bmatrix} \sin \\ \cos \end{Bmatrix}, \qquad \operatorname{Re}\mu > -1,$$

and to the final result (13).

The preceding discussion was intended not only to demonstrate the procedure for obtaining the closed-form formulas for series over Bessel/Struve functions but also to give the generalized concept for summation of series and acceleration of their convergence. The results are series over Riemann zeta functions and other known sums of reciprocal powers, and they converge more rapidly than the original ones or have, in certain cases, closed-form expressions.

Many other problems can be reduced to simpler ones using the proposed concept.

3.3. We successfully applied the described technique the series (2); using (9), we obtained

$$S_2 = \frac{2(x/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_0^{\pi/2} \sin^{2\nu}\theta \times \sum_{n=1}^\infty \frac{(s)^{n-1}(an-b)^{\mu+\nu}}{(an-b)^{2m}\left((an-b)^2-\omega^2\right)} f((an-b)x\cos\theta)\,d\theta.$$

By the application of (8) for $\mu + \nu = 1 - d$, in a similar way we derived the final result (14). On the other hand, the representation (11) yields to the formula (15).

4. MAIN RESULTS

The following expressions for the sums (1) are obtained by using the procedure given in Section 3.1,

$$S_1(a, b, s, \alpha, F_\nu) = \frac{c\pi(x/2)^{\alpha-1}}{2\Gamma((\alpha-\nu+1)/2)\Gamma((\alpha+\nu+1)/2)f(\pi[(\alpha-\nu)/2])} + \sum_{i=0}^{\infty}\frac{(-1)^i F_{i,\nu}(x/2)^{\nu+2i+\delta}}{G_{i,\nu}}, \tag{12}$$

$$F_{i,\nu} = F(\alpha-\nu-2i-\delta), \qquad G_{i,\nu} = \Gamma\left(i+1+\frac{\delta}{2}\right)\Gamma\left(\nu+i+1+\frac{\delta}{2}\right),$$

where $\alpha, \nu \in R$, $\alpha > 0$, $\nu > -\frac{1}{2}$, $\alpha > \nu$, $F_\nu = \{{J_\nu \atop \mathbf{H}_\nu}\}$, $f = \{{\cos \atop \sin}\}$, $\delta = \{{0 \atop 1}\}$, and s, a, b, c, F are given in Table I. When $\alpha - \nu = \{{2k+1 \atop 2k}\}$, $k \in N_0$, one should work with limiting values or with principal values of gamma functions. Even in the case $\alpha - \nu = 1$, formula (12) is the correct one. Truncation of the second series in (12) due to the vanishing of F functions gives all closed-form cases (see Table II for $\alpha - \nu$ instead of α).

If $\nu = m \in N_0$, the procedure given in Section 3.1 yields

$$S_1(a, b, s, \alpha, F_m) = \frac{c(-1)^{r\delta}(-1)^{(m-r)/2}\pi(x/2)^{\alpha-1}}{2\Gamma((\alpha-m+1)/2)\Gamma((\alpha+m+1)/2)g(\pi(\alpha/2))} + \sum_{i=0}^{\infty}\frac{(-1)^i F_{i,m}(x/2)^{m+2i+\delta}}{G_{i,m}}, \tag{13}$$

$$F_{i,m} = F(\alpha-m-2i-\delta), \quad G_{i,m} = \Gamma\left(i+1+\frac{\delta}{2}\right)\Gamma\left(m+i+1+\frac{\delta}{2}\right),$$

where $\alpha \in R$, $\alpha > 0$, $m \in N_0$, $F_m = \{{J_m \atop \mathbf{H}_m}\}$, $f = \{{\sin \atop \cos}\}$, $\bar{f} = \{{\sin \atop \cos}\}$, $\delta = \{{0 \atop 1}\}$, s, a, b, c, F are readable from Table I. Independent of that, $m = \{{2k \atop 2k+1}\}$, $k \in N_0$, $g = \{{f \atop \bar{f}}\}$, $r = \{{0 \atop 1}\}$.

By applying the procedure from Section 3.3. to the formula (2), we get the result

$$S_2 = \frac{(x/2)^{\nu}}{\omega^{2m+2}} \left[\frac{s(1-\mu-\nu)(1-b)}{2\Gamma(\nu+1)} + \frac{c\sqrt{\pi}}{2} \sum_{i=1}^{m} \frac{(-1)^{i+1-\mu-\nu}\Gamma(i+(\mu+\nu)/2)\Gamma(i-(\mu+\nu)/2)\,\omega^{2i}}{(2i-\mu-\nu-1)!\Gamma(i+(\nu-\mu+1)/2)} \times x^{2i-\mu-\nu-1} - \frac{1}{\sqrt{\pi}} \sum_{i=1}^{m} \sum_{k=0}^{i-\delta^{1-\mu-\nu}} \times \frac{(-1)^k F(2i-2k-\mu-\nu-\delta)\Gamma(k+(\delta+1)/2)\,\omega^{2i}}{(2k+\delta)!\Gamma(\nu+k+1+\delta/2)} x^{2k+\delta} \right]$$

$$- \frac{s\pi\omega^{\mu-2m-1}\sin^{b-1}(\pi\omega/2)}{4\cos \pi\omega/2} \left[(-1)^{\delta} f\left(\frac{\pi(s+1)(b+\omega)}{2a} \right) J_{\nu}(\omega x) + \bar{f}\left(\frac{\pi(s+1)(b+\omega)}{2a} \right) \mathbf{H}_{\nu}(\omega x) \right], \quad (14)$$

where $\operatorname{Re} \nu > -\frac{1}{2}$, $F_{\nu} = \{ {J_{\nu} \atop \mathbf{H}_{\nu}} \}$, $f = \{ {\cos \atop \sin} \}$, $\delta = \{ {0 \atop 1} \}$. The rest of the parameters are given in Table III, where $d = 1 - \mu - \nu$. Here we give a similar explanation for the sum in (8). When now $\mu + \nu$ and δ take 1 and 0, respectively, $i - \delta^{1-\mu-\nu}$ is again an indefinite expression, and because $d = 1 - \mu - \nu = 0$, there follows $\alpha = 2i - 1$, which implies $M = i - 1$, and this value should be taken as an upper bound. So we refer to Table III in all cases other than $\mu + \nu = 1$, $\delta = 0$.

For $\mu, \nu \in N_0$, we have

$$S_2 = r(-1)^{\delta} \frac{s(1-t)(1-b)}{\pi\omega^{2m+2}(2p-1)} + (-1)^{\delta r} \times \left[\frac{c\pi}{2} \sum_{i=1}^{m} \frac{(-1)^{i+1-t+p-r}(x/2)^{2i-t-1}}{\omega^{2m-2i+2}\Gamma(i+p-(t+r-1)/2)\Gamma(i-p-(t-r-1)/2)} - \sum_{i=1}^{m} \sum_{k=0}^{i-\delta^{1-t}} \times \frac{(-1)^{k+p-r}F(2i-2k-t-\delta)(x/2)^{2k+\delta}}{\omega^{2m-2i+2}\Gamma(k+p+1+(\delta-r)/2)\Gamma(k-p+1+(\delta+r)/2)} \right]$$

$$- \frac{s\pi\omega^{t-2m-1}}{4\sin^{1-b}(\pi\omega/2)\cos(\pi\omega/2)}$$
$$\times\left[(-1)^{\delta} f\left(\frac{\pi(s+1)(b+\omega)}{2a}\right) J_{2p-r}(\omega x) + \bar{f}\left(\frac{\pi(s+1)(b+\omega)}{2a}\right)\mathbf{H}_{2p-r}(\omega x)\right], \quad (15)$$

where $F_{2p-r} = \{{J_{2p-r} \atop \mathbf{H}_{2p-r}}\}$, $f = \{{\cos \atop \sin}\}$, $\delta = \{{0 \atop 1}\}$, $p \in N$. When $t = \{{1-d \atop d}\}$, then $r = \{{0 \atop 1}\}$. All other relevant parameters are in Table III. Regarding the upper bound $i - \delta^{1-t}$, the same explanation as for the previous cases, where we have an indefinite expression, is valid. Thus, Table III should be referred to in all cases other than $\delta = 0$, $t = 1$. For $f = \{{\cos \atop \sin}\}$, we denote $\bar{f} = \{{\sin \atop \cos}\}$.

5. DISCUSSION AND CONCLUSION

Many particular cases of our formula (12) are cited in the literature. However, they include only those cases with $\alpha = \nu + m$, $m \in N$, while (12) holds true for $\alpha > \nu > -\frac{1}{2}$, $\alpha > 0$. For instance, formula (12) for $a = 0$, $b = 0$, $s = \pm 1$, $\alpha = \nu + 2k$, $F = J$, gives the sums 13, 14 from [13, p. 678],

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k+\nu}} J_\nu(nx) = \frac{(-1)^{k+1} x^{2k+\nu}}{(2k)!2^{\nu+1}\sqrt{\pi}} \sum_{n=0}^{2k} \frac{\Gamma(k-(n-1)/2)}{\Gamma(k+\nu+1-n/2)} \binom{2k}{n} \left(\frac{2\pi}{x}\right)^n B_n,$$
$$k = 1,2,3,\ldots; \qquad \operatorname{Re}\nu > -2k - \tfrac{1}{2};\ 0 < x < 2\pi,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k+\nu}} J_\nu(nx) = \frac{(-1)^{k+1} x^{2k+\nu}}{(2k)!2^{\nu+1}\sqrt{\pi}} \sum_{n=0}^{2k} \binom{2k}{n} B_n \sum_{i=0}^{2k-n} \binom{2k-n}{i} \frac{2^n \pi^{n+i}}{x^{n+i}} G_{ik},$$

$$\operatorname{Re}\nu > -2k - \tfrac{1}{2};\ -\pi < x < \pi, \qquad G_{ik} = \frac{\Gamma(k-(n+i-1)/2)}{\Gamma(k+\nu+1-(n+i)/2)}.$$

These two particular cases were proved by induction in [9] and then by a different method independently in [17]. Notice that the general formula (12) for $F_\nu = J_\nu$ is the formula (8) [17, p. 384]. For $F_\nu = \mathbf{H}_\nu$ in [9], two particular cases of (12), were proved by induction for $\alpha = \nu + 2k$, $k \in N$, $a = 1$, $b = 0$, $s = \pm 1$. Furthermore, special cases of (13), but only for $F_\nu = J_\nu$, can be found in [13].

In [13], there is only one particular case of (14), i.e., [13, p. 679, formula 24],

$$\sum_{k=1}^{\infty} (\pm 1)^k \frac{k^{-\nu}}{k^2 - a^2} J_\nu(kx) = \frac{2^{-\nu-1}x^\nu}{a^2\Gamma(\nu+1)} - \frac{\pi}{2}a^{-\nu-1}\left[J_\nu(ax)\begin{Bmatrix}\cot a\pi \\ \operatorname{cosec} a\pi\end{Bmatrix} + \mathbf{H}_\nu(ax)\begin{Bmatrix}1 \\ 0\end{Bmatrix}\right],$$

where $\left\{\begin{smallmatrix}0 < x < 2\pi \\ 0 < x \le \pi\end{smallmatrix}\right\}$; $\operatorname{Re}\nu > -\frac{5}{2}$. Note that this is $S_2(1, 0, \pm 1, a, 0, -\nu, J_\nu)$ in our notation, and there are no cases for $m \neq 0$. In [8] is given the sum $S_2(1, 0, -1, a, m, -\nu, J_\nu)$ and in [9] are given four particular cases, i.e., $S_2(1, 0, \pm 1, c, m, -\nu, F_\nu)$.

In order to avoid the restriction $\mu + \nu = 0$ or 1, we have derived the formula (15) for $\nu \in N_0$ using the integral representation (11) for Bessel/Struve functions. Particular cases of (15) can be found in [13], but only for $m = 0$. For example, [13, p. 679, formula 22] reads

$$\sum_{k=1}^{\infty} \frac{k}{k^2 - a^2} J_{2n+1}(kx) = \frac{\pi}{2}\left[-\mathbf{H}_{2n+1}(ax) - \cot a\pi J_{2n+1}(ax)\right], \qquad 0 < x < 2\pi.$$

In [9], there is no formula which includes such cases. The sums of the series (15) for $m \neq 0$ are not in the literature.

REFERENCES

1. M. Abramowitz and A. Stegun, "Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables," Dover Publications, New York, 1972.
2. R. A. Askey, "Theory and Application of Special Functions," Academic Press, New York, 1975.
3. B. Berkesh, Einige Formeln über unendlichen Reihen Besselsher Funktionen, *Glasnik Mat. Fiz. Astr.* **10** (1955), 161–170.
4. B. C. Berndt, The evaluation of character series by contour integration, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **386** (1972), 25–29.

5. H. E. Fettis, A summation of Bessel functions—problem 84-18, *SIAM Rev*. **26** (1984), 430–431.
6. M. L. Glasser, The evaluation of lattice sums. I. Analytic procedure, *J. Math. Phys*. **14** (1973), 409–413.
7. I. S. Gradshtein and I. M. Ryzhik, "Tablicy Integralov, Summ, Ryadov i Proizvedenii," Nauka, Moscow, 1971.
8. E. R. Hansen, Solution of the problem 84-18, *SIAM Rev*. **27** (1985), 455–457.
9. L. Lorch and P. Szego, Closed expressions for some infinite series of Bessel and Struve functions, *J. Math. Anal. Appl*. **122** (1987), 47–57.
10. D. S. Mitrinović and D. Adamović, "Nizovi i redovi," Naučna knjiga, Beograd, 1980.
11. D. M. Petković, Infinite sums of Bessel functions, problem 85-14, solution by M. L. Glasser, *SIAM Rev*. **28** (1986), 402–403.
12. A. P. Prudnikov, Y. A. Brychov, and O. I. Marichev, "Integraly i Ryady. Elementarnye Funktsii," Nauka, Moscow, 1981.
13. A. P. Prudnikov, Y. A. Brychov, and O. I. Marichev, "Integraly i Ryady. Spetsialnye Funktsii," Nauka, Moscow, 1983.
14. D. V. Slavić, On summation of trigonometric series, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz*. **263** (1969), 103–114.
15. M. S. Stanković and D. M. Petković, O sumiranju nekih redova pomoću Riemannovih zeta funkcija, *Informatica* '81, *Ljubljana* **3** (1981), 106–108.
16. M. S. Stanković, D. M. Petković, and M. V. Djurić, Short table of summable series of Bessel functions, *in* "V Conference on Applied Mathematics" (Z. Bohte, ed.), 147–152, Univ. Ljubljana, Ljubljana, 1986.
17. M. S. Stanković, D. M. Petković, and M. V. Djurić, Closed form expressions for some series involving Bessel functions of the first kind, *in* "Numerical Methods and Approximation Theory III" (G. V. Milovanović, ed.), 379–389, Univ. Niš, Niš, 1988.
18. G. N. Watson, "A Treatise on the Theory of Bessel Functions," Cambridge Univ. Press, Cambridge, UK, 1958.
19. A. M. Zaezdnyi, "Garmonycheskii sintez v radiotekhnike i elektrosyazi," Energiya, Leningrad, 1972.