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On the vanishing of Tor of the absolute integral closure

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Abstract

Let R be an excellent local domain of positive characteristic with residue field k and let R^+ be its absolute integral closure. If $\mathrm{Tor}_1^R(R^+,k)$ vanishes, then R is weakly F-regular. If R has at most an isolated singularity or has dimension at most two, then R is regular.
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1. Introduction

Recall that the *absolute integral closure* A^+ is defined for an arbitrary domain A as the integral closure of A inside an algebraic closure of the field of fractions of A. A key property of the absolute integral closure was discovered in [4]: for R an excellent local domain of positive characteristic, R^+ is a *balanced big Cohen–Macaulay algebra*, that is to say, any system of parameters on R is an R^+-regular sequence. It is well known that this implies that an excellent local domain R of positive characteristic is regular if, and only if, $R \to R^+$ is flat. Indeed, the direct implication follows since R^+ is a balanced big Cohen–Macaulay algebra of finite projective dimension (use, for instance, [8, Theorem IV.1]) and the converse follows since $R \to R^+$ and $R^{1/p} \to R^+$ are isomorphic whence both faithfully flat, implying that $R \to R^{1/p}$ is flat, and therefore, by Kunz's Theorem, that R is

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regular (here $R^{1/p}$ denotes the extension of R obtained by adding all pth roots of element of R; for more details, see [5, Theorem 9.1 and Exercise 8.8]).

Huneke [5, Exercise 8.8] points out that it is not known whether the weaker condition that all *Betti numbers* of R^+ vanish, that is to say, that all $\operatorname{Tor}_n^R(R^+, k)$ vanish for $n \geqslant 1$, already implies that R is regular. It is not hard to see, using that R^+ is a big Cohen–Macaulay algebra, that this is equivalent with requiring that only $\operatorname{Tor}_1^R(R^+, k)$ vanishes. The main result of this paper is then the following positive solution for isolated singularities.

Theorem 1.1. *Let $(R, \mathfrak{m})$ be an excellent local domain of positive characteristic with residue field k. Suppose R has either an isolated singularity or has dimension at most two. If $\operatorname{Tor}_1^R(R^+, k) = 0$, then R is regular.*

For arbitrary domains, we obtain at least the following theorem.

Theorem 1.2. *Let $(R, \mathfrak{m})$ be an excellent local domain of positive characteristic with residue field k. If $\operatorname{Tor}_1^R(R^+, k) = 0$, then R is weakly F-regular. In particular, R is normal, Cohen–Macaulay, pseudo-rational and any finite extension of R is split* (*i.e., R is a* splinter).

We have some more precise information on the vanishing of certain *Tor*'s in terms of the singular locus of R.

Theorem 1.3. *Let $(R, \mathfrak{m})$ be an excellent local domain of positive characteristic and let $\mathfrak{a}$ be an ideal defining the singular locus of R (e.g., $\mathfrak{a}$ is the Jacobian ideal of R). If $\operatorname{Tor}_1^R(R^+, k) = 0$, where k is the residue field of R, then $\operatorname{Tor}_n^R(R^+, M) = 0$ for all $n \geqslant 1$ and all finitely generated R-modules M for which $M/\mathfrak{a}M$ has finite length.*

The key observation in obtaining all these results, is that, in general, the vanishing of $\operatorname{Tor}_1^R(S, k)$ implies that $R \to S$ is *cyclically pure* (or *ideal-pure*), meaning that $IS \cap R = I$, for all ideals I of R. This is explained in Section 2. To prove Theorem 1.1, we need a result from [8]: if the first Betti number of a module over an isolated singularity vanishes, then the module has finite projective dimension. Now, the argument which proofs that $R \to R^+$ is flat when R is regular, yields the same conclusion under the weaker assumption that R^+ has finite projective dimension. This proves also the two-dimensional case, since we know already that R is normal.

Balanced big Cohen–Macaulay algebras in characteristic zero exist by the work of Hochster–Huneke, basically by a lifting procedure due to Hochster. However, the balanced big Cohen–Macaulay algebras obtained in [4] are not canonically defined. In [7], I give an alternative but canonical construction $\mathcal{B}(R)$ of a balanced big Cohen–Macaulay algebra for a $\mathbb{C}$-affine local domain R using ultraproducts and the absolute integral closure in positive characteristic. It follows from the present results that if $\operatorname{Tor}_1^R(\mathcal{B}(R), k) = 0$, where k is the residue field of R, then R is regular provided R has an isolated singularity or has dimension at most two (moreover,without these additional assumptions, R has at most rational singularities). This is the more interesting because it is not clear whether in general flatness of $R \to \mathcal{B}(R)$ implies regularity of R. For a further generalization to arbitrary excellent local domains, see the forthcoming [1].

2. Vanishing of Betti numbers and cyclic purity

We derive a simple criterion for a local ring homomorphism to be cyclically pure. We start with an easy lemma, the proof of which is included for sake of completeness.

Lemma 2.1. *Let A be a ring, $\mathfrak{a}$ an ideal in A and M and N two A-modules. If $\mathfrak{a}N = 0$ and $\operatorname{Tor}_1^A(M, N) = 0$, then $\operatorname{Tor}_1^{A/\mathfrak{a}}(M/\mathfrak{a}M, N) = 0$.*

Proof. One can derive this by aid of spectral sequences, but the following argument is more direct. Put $\overline{A} := A/\mathfrak{a}$. Since N is an $\overline{A}$-module, we can choose an exact sequence of $\overline{A}$-modules

$$0 \to \overline{H} \to \overline{F} \to N \to 0$$

with $\overline{F}$ a free $\overline{A}$-module. Tensoring with the $\overline{A}$-module $\overline{M} := M/\mathfrak{a}M$, we get an exact sequence

$$0 \to \operatorname{Tor}_1^{\overline{A}}(\overline{M}, N) \to \overline{M} \otimes_{\overline{A}} \overline{H} \to \overline{M} \otimes_{\overline{A}} \overline{F}.$$

Since the last two modules are equal to $M \otimes_A \overline{H}$ and $M \otimes_A \overline{F}$, respectively, and since $\operatorname{Tor}_1^A(M, N) = 0$, the last morphism in this exact sequence is injective. Therefore, $\operatorname{Tor}_1^{\overline{A}}(\overline{M}, N) = 0$, as required. □

Theorem 2.2. *Let $(R, \mathfrak{m})$ be a noetherian local ring with residue field k and let S be an arbitrary R-algebra. If $\operatorname{Tor}_1^R(S, k) = 0$ and $\mathfrak{m}S \neq S$, then $R \to S$ is cyclically pure. Moreover, if $\mathfrak{n}$ is an $\mathfrak{m}$-primary ideal, then*

$$(\mathfrak{n} :_R I)S = (\mathfrak{n}S :_S IS) \quad \textit{for every ideal } I \textit{ in } R.$$

Proof. Since $\operatorname{Tor}_1^R(S, k)$ vanishes, so does $\operatorname{Tor}_1^{R/\mathfrak{n}}(S/\mathfrak{n}S, k)$ by Lemma 2.1, for every $\mathfrak{m}$-primary ideal $\mathfrak{n}$. By the Local Flatness Criterion (see [6, Theorem 22.3]) applied to the artinian local ring $R/\mathfrak{n}$, the base change $R/\mathfrak{n} \to S/\mathfrak{n}S$ is flat, whence faithfully flat, since $\mathfrak{m}S \neq S$. In particular, this base change is injective, showing that $\mathfrak{n}S \cap R = \mathfrak{n}$. Since every ideal is the intersection of $\mathfrak{m}$-primary ideals by Krull's Intersection Theorem, the assertion follows.

The final assertion follows from the flatness of $R/\mathfrak{n} \to S/\mathfrak{n}S$ (use, for instance, [6, Theorem 7.4]). □

Remark 2.3. Note that with notation from the theorem, we have that the induced map of affine schemes $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective, since the *fiber rings* $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$ are non-zero.

The following lemma shows that for a local Cohen–Macaulay ring, the vanishing of some Betti number of a big Cohen–Macaulay algebra is equivalent with the vanishing of all of its Betti numbers.

Lemma 2.4. *If $(R, \mathfrak{m})$ is a local Cohen–Macaulay ring with residue field k and if S is a big Cohen–Macaulay R-algebra, such that* $\mathrm{Tor}_j^R(S,k) = 0$ *for some $j \geqslant 1$, then* $\mathrm{Tor}_n^R(S,k) = 0$, *for all $n \geqslant 1$.*

Proof. Let $\boldsymbol{x}$ be a maximal R-regular sequence which is also S-regular. Put $I := \boldsymbol{x}R$. Since $\mathrm{Tor}_j^R(S,k)$ vanishes, so does $\mathrm{Tor}_j^{R/I}(S/IS,k)$ by [6, Lemma 2, p. 140], so that S/IS has finite flat dimension over R/I by the Local Flatness Criterion. However, since the finitistic weak dimension is at most the dimension of a ring by [2, Theorem 2.4], it follows that S/IS is flat over R/I. Therefore, $0 = \mathrm{Tor}_n^{R/I}(S/IS,k) = \mathrm{Tor}_n^R(S,k)$, for all $n \geqslant 1$. □

Therefore, below, we may replace everywhere the condition that $\mathrm{Tor}_1^R(S,k) = 0$ by the weaker condition that some $\mathrm{Tor}_j^R(S,k) = 0$, provided we also assume that R is Cohen–Macaulay. In fact, if j is either 1 or 2, we do not need to assume that R is Cohen–Macaulay, since this then holds automatically.

Proposition 2.5. *If $(R, \mathfrak{m})$ is a noetherian local ring with residue field k and if S is a big Cohen–Macaulay R-algebra, such that either* $\mathrm{Tor}_1^R(S,k)$ *or* $\mathrm{Tor}_2^R(S,k)$ *vanishes, then R is Cohen–Macaulay.*

Proof. I claim that $IS \cap R = I$, for some parameter ideal I of R. By a standard argument, it then follows that R is Cohen–Macaulay (see, for instance, the argument in [7, Theorem 4.2]). For $j = 1$, we can use Lemma 2.1 to conclude that $\mathrm{Tor}_1^{R/I}(S/IS,k) = 0$, so that by the argument above, $R/I \to S/IS$ is faithfully flat. For $j = 2$, we reason as follows. Let

$$0 \to M \to F \to S \to 0$$

be a short exact sequence with F free. It follows that $\mathrm{Tor}_1^R(M,k)$ is equal to $\mathrm{Tor}_2^R(S,k)$, whence is zero. Therefore, by the same argument as before, M/IM is flat over R/I. On the other hand, since we may choose I so that it is generated by an S-regular sequence, we get that $\mathrm{Tor}_1^R(S, R/I) = 0$ (indeed, the canonical morphism $I \otimes S \to IS$ is easily seen to be injective). Hence we get an exact sequence

$$0 \to M/IM \to F/IF \to S/IS \to 0$$

showing that S/IS has finite flat dimension, whence is flat, since R/I is artinian. □

Is there a counterexample in which some $\mathrm{Tor}_j^R(S,k)$ vanishes for some big Cohen–Macaulay algebra S and some $j > 2$, without R being Cohen–Macaulay?

3. Proofs

Recall that an excellent local ring of positive characteristic is called *F-rational*, if some ideal generated by a system of parameters is tightly closed, and *weakly F-regular*, if every ideal is tightly closed. It is well known that for excellent local rings, weakly F-regular implies splinter, and F-rational implies Cohen–Macaulay and normal [5, Theorem 4.2]. By [9, Theorem 3.1], an F-rational ring is pseudo-rational.

Proof of Theorem 1.2. Suppose R is as in the statement of the theorem, so that in particular $\operatorname{Tor}_1^R(R^+, k)$ vanishes. By Theorem 2.2, the embedding $R \to R^+$ is cyclically pure. In order to show that R is weakly F-regular, it suffices to show by [5, Theorem 1.5] that every $\mathfrak{m}$-primary ideal is tightly closed. Towards a contradiction, suppose that $\mathfrak{n}$ is an $\mathfrak{m}$-primary ideal which is not tightly closed. Therefore, we can find a u in the tight closure of $\mathfrak{n}$ such that $(\mathfrak{n} :_R u) = \mathfrak{m}$. By Theorem 2.2, we have

$$(\mathfrak{n}R^+ :_{R^+} u) = \mathfrak{m}R^+. \tag{1}$$

By definition, there is a $c \in R$ not contained in any minimal prime of R such that $cu^q \in \mathfrak{n}^{[q]}$, for all powers $q = p^e$ (as usual, $I^{[q]}$ denotes the ideal generated by the qth powers of elements in an ideal I). Since therefore $c^{1/q}u \in \mathfrak{n}R^+$, we get from (1) that $c^{1/q} \in \mathfrak{m}R^+$ whence $c \in \mathfrak{m}^q R^+$. By cyclical purity, $c \in \mathfrak{m}^q$ for all q, contradiction.

In particular, R is F-rational whence pseudo-rational, normal and Cohen–Macaulay (in fact, R is Cohen–Macaulay, by Proposition 2.5, and normal, by the cyclic purity of $R \to R^+$). Since R is normal, it follows from [3] that $R \to R^+$ is pure. Let us give a direct argument for showing that R is a splinter. Let $R \subset S$ be a finite extension. In order to show that this is split, we may factor out a minimal prime of S and hence assume that S is a domain. So $R \subset S$ extends to the pure map $R \to R^+$ and hence is itself pure. Since a pure map with finitely generated cokernel is split [6, Theorem 7.14], we showed that any finite extension splits (as a module). □

Proof of Theorem 1.1. The vanishing of $\operatorname{Tor}_1^R(R^+, k)$ implies that R is Cohen–Macaulay by Theorem 1.2. Since R^+ is a balanced big Cohen–Macaulay algebra and since R has an isolated singularity, we get from [8, Theorem IV.1] that $R \to R^+$ is flat. As already observed, this implies that R is regular. If R has dimension at most 2, then by Theorem 1.2, it is normal and therefore has an isolated singularity, so that the previous argument applies. □

Recall that by the argument at the end of the previous section, the vanishing of a single $\operatorname{Tor}_j^R(R^+, k)$ implies already that R is regular, if apart from being an isolated singularity, we also assume that R is Cohen–Macaulay, when $j \geqslant 3$. In order to derive a regularity criterion from Theorem 1.1, we need a lemma on flatness over artinian local Gorenstein rings of embedding dimension one.

Lemma 3.1. *Let $(A, \mathfrak{m})$ be an artinian local ring of embedding dimension one and let M be an arbitrary A-module. Then M is A-flat if, and only if,* $\operatorname{Ann}_M(I) = \mathfrak{m}M$, *where I denotes the socle of A, that is to say,* $I = \operatorname{Ann}_A(\mathfrak{m})$.

Proof. By assumption $\mathfrak{m} = xA$, for some $x \in A$. It follows that the socle I of A is equal to $x^{e-1}A$, where e is the smallest integer for which $x^e = 0$. I claim that $\mathrm{Ann}_M(x^{e-i}) = x^i M$, for all i. We will induct on i, where the case $i = 1$ is just our assumption. For $i > 1$, let $\mu \in M$ be such that $x^{e-i}\mu = 0$. Therefore, $x^{e-i+1}\mu = 0$, so that by our induction hypothesis, $\mu \in x^{i-1}M$, say, $\mu = x^{i-1}\nu$. Since $0 = x^{e-i}\mu = x^{e-1}\nu$, we get $\nu \in xM$ whence $\mu \in x^i M$, as required.

Flatness now follows by the Local Flatness Criterion [6, Theorem 22.3]. Indeed, it suffices to show that $A/xA \to M/xM$ is flat and $xA \otimes M \cong xM$. The first assertion is immediate since A/xA is a field. For the second assertion, observe that $xA \cong A/x^{e-1}A$ and by what we just proved $xM \cong M/\mathrm{Ann}_M(x) \cong M/x^{e-1}M$. It follows that $xA \otimes M$ is isomorphic with xM, as required. □

Corollary 3.2. *Let $(R, \mathfrak{m})$ be a d-dimensional excellent local Cohen–Macaulay domain of positive characteristic. Suppose that there exists an ideal I in R generated by a regular sequence such that $\mathfrak{m}/I$ is a cyclic module. Suppose also that R has either an isolated singularity or that $d \leqslant 2$. If for each finite extension domain $R \subset S$, we can find a finite extension $S \subset T$, such that*

$$\big(IS :_S (I :_R \mathfrak{m})S\big) \subset \mathfrak{m}T, \tag{2}$$

then R is regular.

Proof. Let $(x_1, \dots, x_i)$ be the regular sequence generating I and write $\mathfrak{m} = I + xR$. If $i < d$ then necessary $i = d - 1$ and $\mathfrak{m}$ is generated by d elements, so R is regular. Hence assume $i = d$, that is to say, I is $\mathfrak{m}$-primary. It follows that $\overline{R} := R/I$ is an artinian local ring with maximal ideal $x\overline{R}$. Let e be the smallest integer for which $x^e \in I$. Hence the socle of $\overline{R}$ is $x^{e-1}\overline{R}$. Let $\overline{R^+} := R^+/IR^+$. I claim that

$$\mathrm{Ann}_{\overline{R^+}}\big(x^{e-1}\big) = x\overline{R^+}.$$

Assuming the claim, Lemma 3.1 yields that $\overline{R^+}$ is $\overline{R}$-flat. Therefore, if k is the residue field of R, then $\mathrm{Tor}_1^{\overline{R}}(\overline{R^+}, k) = 0$. But $(x_1, \dots, x_d)$ is both R-regular and R^+-regular, so that $\mathrm{Tor}_1^R(R^+, k) = 0$. Regularity of R then follows from Theorem 1.1.

To prove the claim, one inclusion is clear, so assume that $a \in R^+$ is such that $ax^{e-1} \in IR^+$. Choose a finite extension $R \subset S \subset R^+$ containing a and such that we already have a relation $ax^{e-1} \in IS$. By assumption, we can find a finite extension T of S, such that $(IS : x^{e-1}) \subset \mathfrak{m}T$. Hence $a \in \mathfrak{m}T$. Since T maps to R^+, we get $a \in \mathfrak{m}R^+$, and hence $a \in x\overline{R^+}$, as we wanted to show. □

The condition that $\mathfrak{m}$ is cyclic modulo a regular sequence is in this case equivalent with R being Cohen–Macaulay with regularity defect at most one (recall that the *regularity defect* of R is by definition the difference between its embedding dimension and its Krull dimension). If R is regular, then (2) is true for any $\mathfrak{m}$-primary ideal I of R (use the fact that $R \to R^+$ is flat).

Proof of Theorem 1.3. Let $(R, \mathfrak{m})$ be as in the statement of Theorem 1.3. In particular, R is Cohen–Macaulay by Theorem 1.2. Let M be a finitely generated R-module such that $M/\mathfrak{a}M$ has finite length. Let I be the annihilator of M. By Nakayama's Lemma, $M/\mathfrak{a}M$ having finite length implies that $I + \mathfrak{a}$ is $\mathfrak{m}$-primary. We will induct on the dimension e of M. If $e = 0$, so that M has finite length, the vanishing of $\mathrm{Tor}_n^R(R^+, M)$ follows from Lemma 2.4 and a well-known inductive argument on the length of M (see, for instance, [8, Corollary II.6]). Hence assume $e > 0$ and let H be the largest submodule of finite length in M. The Tor long exact sequence obtained from

$$0 \to H \to M \to M/H \to 0$$

shows that it suffices to prove the result for M/H instead of M. Therefore, after modding out H, me may assume that M has positive depth. By prime avoidance and since $I + \mathfrak{a}$ is $\mathfrak{m}$-primary, we can find an M-regular element $x \in \mathfrak{a}$. The short exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

gives rise to a long exact sequence

$$\mathrm{Tor}_{n+1}^R\big(R^+, M/xM\big) \to \mathrm{Tor}_n^R\big(R^+, M\big) \xrightarrow{x} \mathrm{Tor}_n^R\big(R^+, M\big),$$

for all $n \geqslant 1$. Since the left most module is zero by induction on e, multiplication with x on $\mathrm{Tor}_n^R(R^+, M)$ is injective, for all $n \geqslant 1$. In particular, we have for each n an embedding

$$\mathrm{Tor}_n^R\big(R^+, M\big) \subset \big(\mathrm{Tor}_n^R\big(R^+, M\big)\big)_x = \mathrm{Tor}_n^{R_x}\big(\big(R^+\big)_x, M_x\big). \tag{3}$$

Since $x \in \mathfrak{a}$, the localization R_x is regular. Therefore $R_x \to (R_x)^+$ is flat. An easy calculation shows that $(R_x)^+ = (R^+)_x$ (see [4, Lemma 6.5]). In particular, $\mathrm{Tor}_n^{R_x}((R^+)_x, M_x) = 0$, and hence $\mathrm{Tor}_n^R(R^+, M) = 0$ by (3). □

If R has dimension three, then $\mathrm{Tor}_n^R(R^+, R/\mathfrak{p})$ vanishes for every $n \geqslant 1$ and every prime ideal $\mathfrak{p}$ of R not in the singular locus of R, since R is normal by Theorem 1.2 and hence $\mathfrak{a}$ has height at least two. On the other hand, we have the following non-vanishing result.

Corollary 3.3. *Let $(R, \mathfrak{m})$ be an excellent local domain of positive characteristic. If $\mathfrak{p}$ is a prime ideal defining an irreducible component of the singular locus of R, then $\mathrm{Tor}_1^R(R^+, R/\mathfrak{p})$ is non-zero.*

Proof. Assume $\mathrm{Tor}_1^R(R^+, R/\mathfrak{p})$ vanishes. Hence so does $\mathrm{Tor}_1^{R_\mathfrak{p}}((R^+)_\mathfrak{p}, k(\mathfrak{p}))$, where $k(\mathfrak{p})$ is the residue field of $\mathfrak{p}$. Since $(R^+)_\mathfrak{p}$ is equal to $(R_\mathfrak{p})^+$ by [4, Lemma 6.5] and since $R_\mathfrak{p}$ has an isolated singularity, it follows from Theorem 1.1 that $R_\mathfrak{p}$ is regular, contradicting the choice of $\mathfrak{p}$. □

In view of Lemma 2.4 we can generalize this even further: if R is Cohen–Macaulay, then each $\mathrm{Tor}_n^R(R^+, R/\mathfrak{p})$ is non-zero for $n \geqslant 1$ and for $\mathfrak{p}$ defining an irreducible component of the singular locus of R.

Note added in proof

I. Aberbach has recently announced a proof of Theorem 1.1 without the isolated singularities condition on R.

Acknowledgments

The original proof of Theorem 1.2 only established F-rationality; the argument yielding weak F-regularity is due to I. Aberbach, whom I thank for letting me reproduce it here. I also thank the anonymous referee, for providing a simplified argument for Theorem 1.3.

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