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# SURGERY FORMULAS FOR SPIN MANIFOLDS

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THE $s$-cobordism theorem and surgery together reduce the problem of classifying manifolds within one homotopy type $X$ to two main calculations:

(1) The group $[X, G/H]$; $H = O$, $PL$, or Top.
(2) The surgery obstruction $s$: $[X, G/H] \to \mathscr{L}_{\dim X}(\pi_1 X)$.

Calculation of (1) is made possible when $H = PL$ or Top by Sullivan's analysis of the homotopy type of $G/H$. When $H = O$, the problem is a mixture of $[X, G]$, which is hard to compute, and $\ker([X, BO] \to [X, BG])$, which is determined by Adams' work "on $J(X)$" plus the "Adams conjecture". Calculation of (2) depends first on the $\mathscr{L}$ groups, but even when $\pi_1 X = \{1\}$ may be unclear. However, if $X$ is a 1-connected manifold $M$, there are formulas which help to calculate $s$, see [4, 5]. When $\dim M = 4r$,

$$s(f) = f^*(u - 1) \cap [M]_\Delta$$

where

$$f: M \to G/\text{Top}.$$

$[M]_\Delta$ is Sullivan's $(KO)_{(\text{odd})}$ orientation for $M$; $u \in (KO)_{(\text{odd})} G/\text{Top}$ is the unit determined by fibre homotopy trivilization.

When $\dim M = 4r + 2$

$$s(f) = (V^2(\tau_M) \cup f^*k) \cap [M]_{H_{4r+2}}$$

where $\tau_M$ = tangent bundle, $V$ = total Wu class and $k$ is a universal class in $H^{4^*+2}(G/\text{Top}; Z_2)$. The main fact for calculating with $k$ is that when restricted to $G/O$, $k$ has components only in dimensions $2^i - 2$, see [3].

These two formulas play analogous roles in Sullivan's decomposition of $G/\text{Top}$. Their relation becomes closer in the very special case that $M$ is a smooth spin manifold (still 1-conn.), $r$ is even, and $f$ factors through $G/O$. Now $M$ is $KO$ oriented so $s_{8r}$ may be expressed integrally; and $s_{8r+2}$ simplifies to $V_{4r}{}^2 f^*(k_2)$ which can also be expressed in $KO$. The formulas now fit neatly into the calculations, via Adams, of (1).

First define a bilinear form $\langle\ ,\ \rangle$ on $KO\,(M^n_{\text{spin}})$ with values in $KO_n(pt)$, by $\langle a, b\rangle = (a \otimes b) \cap [M]_{\text{Dirac}}$. Here $[M]_{\text{Dirac}}$ is the $KO$ fundamental homology class, constructed with the $\Delta_+ - \Delta_-$ orientation of the normal bundle of $M$, see [2].

Next define a unit $u \in 1 + \widetilde{KO^\circ G/O}$ by comparing the (Dirac) orientation of the universal bundle to the orientation induced from the trivial bundle by fibre homotopy trivialization, see [1]. From $u$ form the class $\lambda^2 u/u^2$, which has filtration 2.

Finally we need characteristic classes for $M$. For Spin $(8r)$ bundles $E^{8r}$ there are classes $\rho^k E \in KO^\circ$ (base $(E)$) obtained from the action of $\psi^k$ on the (Dirac) orientation of $E$, see [1]. For $M^{8r}$, we will use $\rho^2\tau_M$. For Spin $(8r+2)$ bundles there is again a class $\rho^2 E^{8r+2}$ in $KO^\circ$, since the Spin $(8r+2)$ representation $\Delta = \prod_{i=1}^{4r+1} (Z_i^{1/2} + Z_i^{-1/2})$ is real. However for $M^{8r+2}$, it is not $\rho^2\tau_M$ we need, but rather "half of $\rho^2\tau_M$". Since $M$ is 1-connected, $M - pt$ has formal dimension $8r$ so $\tau_M$ splits as $2 + E^{8r}$ there. Also, $1 + E^{8r} = F^{8r+1}$ is uniquely determined by $\tau_M$ as a Spin $(8r+1)$ bundle. The real spin representation $\Delta_{8r+1}$ gives $\Delta_{8r+1}(F) = \Delta_{8r}(E) = \rho^2 E$ so $\rho^2 E$ is determined on $M - pt$ by $\tau_M$. Since the $KO$ sequence for $M - pt \subsetneq M \to S^{8r+2}$ is split by $\cap[M]$, we can extend $\rho^2 E$ over $M$. Let $\frac{1}{2}\rho^2\tau$ be any extension (there are two). $\frac{1}{2}\rho^2\tau$ is defined modulo an element of top filtration; and on $M - pt$ we have $2(\frac{1}{2}\rho^2\tau) = 2\rho^2 E = \rho^2\tau$.

*Formulas.*

$$s_{8r}(f) = \tfrac{1}{4}\langle \rho^2\tau_M, f^*\lambda^2 u/u^2\rangle \varepsilon Z$$

$$s_{8r+2}(f) = \langle \tfrac{1}{2}\rho^2\tau_M, f^*\lambda^2 u/u^2\rangle \varepsilon Z_2 .$$

Notice that the second formula is independent of the choice of $\frac{1}{2}\rho^2\tau$ since Filtration $f^*(\lambda^2 u/u^2) > 0$.

The fact that $s_{8r+2}$ is a group homomorphism (Whitney sum on $G/O$) appears here as the *mod 2* identity $[\lambda^2(uv)/(uv)^2] = \lambda^2 u/u^2 + \lambda^2 v/v^2$. The fact that $s_{8r+2}$ vanishes for all $f$ if $M - pt$ has a 3-field comes from the divisibility of $\frac{1}{2}\rho^2 z$: if $\tau = 3 + D$ on $M - pt$ then $\rho^2 E = \Delta_{8r}(E) = 2\Delta_{8r-1}(D)$ so $\frac{1}{2}\rho^2\tau$ may be taken to be twice an extension of $\Delta_{8r-1}(D)$, e.g. this applies when $M$ is 2-connected.

Although the formulas are so similar, I do not know a unified proof for them. For $M^{8r+2}$, the fact that the Wu class $V$ is concentrated in dimensions $4j$ (using $M$ spin) gives $s_{8r+2}(f) = (V_{4r}{}^2 \cup f^*(k_2)) \cap [M]_{8r+2}$ as on p. 255 of [6]. Or, $s_{8r+2}(f) = W_{8r} \cup f^*(k_2)$. We can reduce the $KO$ formula to this. Returning to $\tau = 2 + E^{8r}$ on $M - pt$, we have $\rho^2 E = \Delta(E) = \Delta_+(E) + \Delta_-(E)$. If $x_\pm$ are extensions of $\Delta_\pm(E)$ over $M$, then $x_+ + x_-$ is a possible choice for $\frac{1}{2}\rho^2\tau$. Then in $KO_{8r+2}(pt) = Z_2$ we have

$$\langle \tfrac{1}{2}\rho^2\tau, f^*(\lambda^2 u/u^2)\rangle = \langle x_+ + x_-, f^*(\lambda^2 u/u^2)\rangle = \langle x_+ - x_-, f^*(\lambda^2 u/u^2)\rangle.$$

Now Filtration $(x_+ - x_-) \geq 8r$ since $\Delta_+ = \Delta_-$ on Spin $(8r-1)$ and $E$ splits to $1 + H^{8r-1}$ over the $8r-1$ skeleton; also filtration $f^*(\lambda^2 u/u^2) \geq 2$.

LEMMA 1. *Let $F^j = \ker(KO^\circ(M) \to KO^\circ(j-1$ skeleton of $M))$. There is a monomorphism $F^2/F^3 \subsetneq H^2(M, Z_2)$ given by the Stiefel–Whitney class $w_2$, and an epimorphism $H^{8r}(M, Z) \twoheadrightarrow F^{8r}/F^{8r+1}$. Given $x \in F^{8r}$, $y \in F^2$ suppose $w \to x$ and $y \to z$. Then $\langle x, y\rangle$ may be computed as $(w \pmod 2 \cup z) \cap [M]_{H_{8r+2}}$.*

*Proof.* This is a spectral sequence argument using

(a) $M$ is $KO$ oriented so all differentials into the last column are zero.

(b) On $F^{8r} \otimes F^2$, $\langle\ ,\ \rangle$ may be identified with $F^{8r} \otimes F^2 \to F^{8r+2}$.

(c) The pairing $KO^{-8r}(pt) \otimes KO^{-2}(pt) \to KO^{-(8r+2)}(pt)$ is the non trivial $Z \otimes Z_2 \to Z_2$.

LEMMA 2.

$$w_2\left(\frac{\lambda^2 u}{u^2}\right) = k_2 \text{ on } G/O.$$

*Proof.* $H^2(G/O; Z_2) = Z_2$. The generator of $\pi_2 G/O = Z_2$ gives a map $S^2 \to G/O$ whose associated normal map is $\tilde{T} \xrightarrow{\phi} S^2$; here $\tilde{T}$ is the torus with exotic framing. Since $f^*k_2 = s_2(f) = \text{Arf}(\tilde{T}) \neq 0, k_2 \neq 0$ on $G/O$. Since $f^*u \cap [S^2]_{\text{Dirac}} = 1 \cap [\tilde{T}] \neq 0$, we must have $f^*u = 1 + g$ where $g$ generates $\widetilde{KO}^\circ S^2$, so $w_2 f^* \dfrac{\lambda^2 u}{u^2} = w_2 g \neq 0$; i.e. $w_2 \dfrac{\lambda^2 u}{u^2}$ and $k_2$ are non-zero elements of $H^2 = Z_2$.

Returning to $\langle \frac{1}{2}\rho^2\tau, f^*(\lambda^2 u/u^2)\rangle$ we must choose $w \to x_+ - x_-$ to apply Lemma 1. Since $M - pt \subsetneq M$ induces isomorphisms of $H^{8r}(-; Z)$ and $F^{8r}/F^{8r+1}$ we can choose $w$ on $M - pt$. Here $x_+ - x_-$ is the $KO$ Euler class, $\Delta_+ - \Delta_-$, for $E$, so we choose $w$ universally to be the $H^*(-; Z)$ Euler class using $H^{8r}(M\text{ Spin }(8r); Z) \twoheadrightarrow F^{8r}/F^{8r+1}$ ($M$ Spin $(8r)$). Thus $w(\text{mod } 2) = w_{8r}(E) = w_{8r}(\tau)$ and Lemma 1 reads $\langle x_+ - x_-, f^*(\lambda^2 u/u^2)\rangle = w_{8r}(\tau) \cup w_2 f^*(\lambda^2 u/u^2)$. Lemma 2 then completes the reduction of $\langle \frac{1}{2}\rho^2\tau, f^*(\lambda^2 u/u^2)\rangle$ to $w_{8r}(\tau) \cup f^*(k_2)$.

For $M^{8r}$, we first rewrite Hirzebruch's signature formula as Sign $M^{8r}_{\text{spin}} = \langle \rho^2\tau, 1\rangle$.

This may be proved, e.g. by the rational calculation

$$\begin{aligned}\rho^2\tau \cap [M] &= (\text{ph}(\rho^2\tau) \cup \hat{A}(\tau)) \cap [M]_{H_{8r}} \\ &= \left(\prod_{i=1}^{4r} (e^{x_i/2} + e^{-x_i/2}) \cup \prod_{i=1}^{4r} \left(\frac{x_i}{e^{x_i/2} - e^{-x_i/2}}\right)\right) \cap [M]_{H_{8r}} = L(\tau) \cap [M]_{H_{8r}}.\end{aligned}$$

Given $M^{8r} \to G/O$, form the associated normal map $N^{8r} \xrightarrow{\phi} M^{8r}$. $\nu(N) = \phi^*(\nu(M) + f^*\xi)$ where $\xi$ is the universal "bundle with fibre homotopy trivialization". It follows that $\rho^2\tau_N = \phi^*\left(\rho^2\tau_M \otimes f^* \dfrac{\psi^2 u}{u}\right)$ and $\forall x$, $\phi^* x \cap [N] = (x \otimes f^*(u^{-1})) \cap [M]$. Therefore

$$\begin{aligned}s_{8r}(f) = \tfrac{1}{8}[\text{Sign } M - \text{Sign } N] &= \tfrac{1}{8}[\langle \rho^2\tau_M, 1\rangle_M - \langle \rho^2\tau_N, 1\rangle_N] \\ &= \frac{1}{8}\left\langle \rho^2\tau_M, f^*\left(1 - \frac{\psi^2 u}{u^2}\right)\right\rangle_M = \frac{1}{4}\left\langle \rho^2\tau_M, f^*\frac{\lambda^2 u}{u^2}\right\rangle_M\end{aligned}$$

as claimed.

The signature formula may be written in the spirit of $V^2$ if we localize away from 2. On $(KO^\circ)_{(\text{odd})}$, the bilinear from $\langle\ ,\ \rangle$ is non degenerate into $(Z)_{(\text{odd})}$; this follows from Bott periodicity as in [4]. Let $\Sigma \in (KO^\circ)_{(\text{odd})} M$ be the Wu class characterized by $\langle \Sigma, x\rangle = \langle 1, \psi^2 x\rangle$. The Wu relation here is $\psi^2\Sigma = \rho^2\tau$, so we have the formula

$$\text{Sign } M^{8r}_{\text{spin}} = \langle \Sigma, \Sigma\rangle.$$

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