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Every strong digraph has a spanning strong subgraph with at most $n + 2\alpha - 2$ arcs

Stéphane Bessy and Stéphan Thomassé

Laboratoire LaPCS, U.F.R. de Mathématiques, Université Claude Bernard, Lyon 1, 50, avenue Tony Garnier, Bâtiment RECHERCHE [B], Domaine de Gerland Lyon 69367, Cedex 07, France

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Abstract

Answering a question of Adrian Bondy (Per. Comm.), we prove that every strong digraph has a spanning strong subgraph with at most $n + 2\alpha - 2$ arcs, where α is the size of a maximum stable set of D. Such a spanning subgraph can be found in polynomial time. An infinite family of oriented graphs for which this bound is sharp was given by Odile Favaron (Discrete Math. 146 (1995) 289). A direct corollary of our result is that there exists $2\alpha - 1$ directed cycles which span D. Tibor Gallai (Theory of Graphs and its Applications, Czech. Acad. Sci. Prague, 1964, p. 161) conjectured that α directed cycles would be enough.

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1. Introduction and known results

In this paper, cycles of length two are allowed. Since loops and multiple arcs play no role in this topic, we will simply assume that our digraphs are loopless and simple—when performing a contraction, we will implicitly delete the cycles of length one and reduce the multiple arcs to simple one. Let $D = (V, E)$ be a strong digraph. We are mainly concerned in this paper by the following problem: What is the minimum number of arcs of a strong spanning subgraph of D? This classical problem is known as the MSSS-problem, see for instance [1] for a survey on this topic, see [6,8] for its relationship with connectivity and [7,11] for some approximation algorithm. Let us say that a strong digraph $D = (V, E)$ is a *k-handle* if $k = |E| - |V| + 1$ (a 0-handle is simply a single vertex). We want to find the minimum k, for which there exists a k-handle which is a spanning subgraph of D. We

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E-mail address: bessy@univ-lyon1.fr (S. Bessy).

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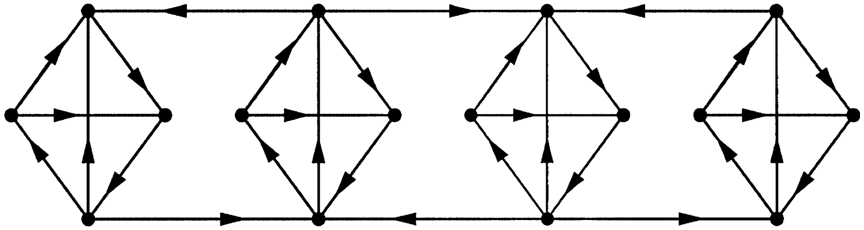
introduce now the key-definitions in this topic: a *handle* is a directed path $H := x_1, \dots, x_l$ in which we allow $x_1 = x_l$. We denote the restriction of H to $\{x_i, x_{i+1}, \dots, x_j\}$ by $H[x_i, x_j]$, and $\overset{\circ}{H} := H \setminus \{x_1, x_l\}$. The vertex x_1 is the *head* of H and x_l is the *tail* of H. If A and B are subgraphs of D, an (A, B)-*handle* of D is a handle with its head in $V(A)$, its tail in $V(B)$ and its internal vertices and arcs disjoint from $A \cup B$. We simply write (A)-handle instead of (A, A)-handle. A *handle basis* of D (or *ear decomposition*, see [1]) is a sequence $H_0, H_1, \dots, H_k$ of handles of D such that H_0 is a single vertex, H_i is a $(\cup\{H_j\colon j<i\})$-handle for all $i = 1, \dots, k$ and $D = \cup\{H_i\colon i = 0, \dots, k\}$. Clearly, a digraph has a handle basis $H_0, \dots, H_k$ if and only if D is a k-handle. Moreover, if D' is a minimum strong spanning subgraph of D, every H_i in any handle basis of D' has at least 2 arcs. It follows directly that D is spanned by a k-handle with $k \leqslant n-1$. Our goal in this paper is to prove the following theorem, where $\alpha(D)$ is the number of vertices of a maximum stable set of D, called the *stability* of D:

Theorem 1. *Every strong digraph D is spanned by a k-handle, with* $k \leqslant 2\alpha(D) - 1$.

To motivate this result, we invite the reader to check that the bound is sharp when D is chosen in the following family of examples due to O. Favaron and drawn for the illustrative case $\alpha = 4$.



Theorem 1 is one of the corollary of the following conjecture of Chen and Manalastas, which is explicitly stated in [1,3].

Conjecture 1. *Every strong digraph with stability* α *is spanned by the disjoint union of some* k_i*-handles, where* $k_i > 0$ *for all i and the sum of the* k_i *being at most* α.

To see that Conjecture 1 implies Theorem 1, observe that such a disjoint union has exactly $n + k - c$ arcs where c is the number of components and k is the sum of the k_i. Consequently, making this disjoint union strong requires at most $2c - 2$ new arcs, and thus D is spanned by a strong digraph with at most $n + k + c - 2 \leqslant n + 2\alpha - 2$ arcs, since $\alpha \geqslant c$. Conjecture 1 also implies the following result [9], once conjectured by Las-Vergnas:

Theorem 2. *Every strong digraph with stability* $\alpha > 1$ *is spanned by the disjoint union of* $\alpha - 1$ *paths.*

But the real motivation of Conjecture 1 is to prove the following long-standing conjecture of Gallai [5]: every strong digraph is spanned by the union of α cycles. For all these reasons, Conjecture 1 seems to be the very challenge of this topic. It is verified for $\alpha = 1$, this is the well-known result of Camion: *every strong tournament has a hamilton cycle*. The case $\alpha = 2$ is the following theorem of Chen and Manalastas [4]: *Every strong digraph with stability* 2 *is spanned by two cycles, intersecting one another on a* (*possibly empty*) *path*. The case $\alpha = 3$ can be found in [10]. The link between the MSSS-problem and the stability number is the classical Gallai–Milgram's theorem: *every digraph D is spanned by the disjoint union of* $\alpha(D)$ *directed paths*. It suggests that the involved number of handles in a handle basis should be related to α. In [3], Bondy proposed the following refinement of Gallai–Milgram's theorem. The proof is by induction on k, and can also be found in [1,2].

Theorem 3. *Let D be a digraph and $\{P_i\colon\ 1 \leqslant i \leqslant k\}$ be a spanning set of disjoint directed paths of D. If $k > \alpha(D)$, there exists a spanning set of disjoint paths $\{P'_i\colon\ 1 \leqslant i \leqslant k-1\}$ of D such that every head* (*resp. tail*) *of a P'_i is the head* (*resp. the tail*) *of a P_j.*

This theorem provides the key-operation of this paper—the main snag being that strong connectivity is certainly not preserved under such a path exchange. The difficult part of the proof is to find some structures (the so-called tree-handle systems) on which we can perform Theorem 3.

2. Completion

An *out-arborescence* is an oriented tree in which every vertex has indegree at most 1. The one vertex with indegree 0 is called the *root*. The vertices with outdegree 0 are the *leaves*. The dual definitions hold for *in-arborescence*. A *bi-arborescence A* is a tree obtained by identifying the root of an in-arborescence A_- and the root of an out-arborescence A_+. The vertices of A_- (resp. A_+) are the *in-vertices* (resp. the *out-vertices*) of A. The vertices of A with indegree 0 (resp. outdegree 0) are the *in-leaves* (resp. the *out-leaves*) of A. The common root is both an in-vertex and an out-vertex of A, we call it the *centre* of A. Observe that the *centre* of A can also be a leaf of A, when A_- or A_+ is a single vertex. The vertices of A which are not leaves are the *internal* vertices of A. A bi-arborescence is *plain* if it has at least two in-leaves and two out-leaves. Let $D = (V, E)$ be a strong digraph and S a subset of V. We denote by $D[S]$ the induced restriction of D on S. If A and B are two nonempty subsets of V, an (A, B)-*completion* is a set $\{A_1, \dots, A_r\}$ of bi-arborescences such that:

(i) Every A_i is a subgraph of D.
(ii) The internal vertices of A_i do not belong to $A \cup B$.
(iii) For all $i \neq j$, $V(A_i) \cap V(A_j) \subseteq A \cup B$.
(iv) In the graph $D[A \cup B] \cup A_1 \cup \cdots \cup A_r$ (called *completed graph*), every vertex $a \in A$ is the head of an (a, B)-path and every vertex $b \in B$ is the tail of an (A, b)-path.

An (A,B)-completion C is *minimum* if $\sum_1^r |V(A_i)|$ is minimum. Observe that if C is minimum, every leaf f of A_i belongs to $A \cup B$. Indeed, if $f \notin A \cup B$, the vertex f only belongs to one bi-arborescence A_i and thus $(C \setminus \{A_i\}) \cup \{A_i \setminus f\}$ is still a completion, a contradiction to the minimality of C. It follows from this observation that every vertex $v \notin A \cup B$ in the completed graph of a minimum (A,B)-completion is the head of a (v,B)-path and the tail of an (A,v)-path: indeed the vertex v is certainly in a unique bi-arborescence A_i, therefore there exists a path from v to an out-leaf l of T. If $l \in B$ we are done, and if $l \in A$, by (iv), there is a path from l to B in the completed graph. Similarly, an (A,v)-path exists.

Lemma 1. *Let C be a minimum (A,B)-completion of a strong digraph $D=(V,E)$. If a bi-arborescence T of C has more than one out-leaf, all the out-leaves of T belong to $B \setminus A$. Similarly, if T has more than one in-leaf, all the in-leaves of T belong to $A \setminus B$.*

Proof. Suppose that T has one out-leaf $f \in A$ and another out-leaf $g \in B$. We consider $C' = (C \setminus \{T\}) \cup \{T \setminus f\}$. Since $T \setminus f$ has an out-leaf in B, the completed graph of C' still satisfies the first part of (iv). Also, since f belongs to A, deleting f from T does not affect the second part of (iv). Now we suppose that all the out-leaves of T belong to $A \setminus B$. In the completed graph, consider a path P from the centre c of T to a vertex b of B. We call f the last out-leaf of T on the path P. We claim that $C' = (C \setminus \{T\}) \cup \{T'\}$, where $T' := T_- \cup T[c,f]$, is an (A,B)-completion. Indeed, in the completed graph of C', every in-leaf of T' is still the head of a path with tail in B. The second part of (iv) is still satisfied since we deleted out-leaves of T which belong to A. We again contradicts the minimality of C, therefore the leaves of T form a subset of $B \setminus A$. The proof for the in-leaves follows by directional duality. □

Lemma 2. *If $D=(V,E)$ is a strong digraph and A,B are two nonempty subsets of V, D admits an (A,B)-completion.*

Proof. We proceed by induction on $|A|$. If $A=\{a\}$, there exists a spanning out-arborescence T rooted at a. Now we consider the set of sub-arborescences A_i of T which have no internal vertices in $A \cup B$ and are maximal with respect to inclusion for this property. This set is clearly an (A,B)-completion since it satisfies (i)–(iii) by construction, and its completed graph contains T as a subgraph, therefore it satisfies (iv). Now, we suppose that $|A|>1$, and, for some $a \in A$, we apply the induction hypothesis in order to find an $(A \setminus \{a\}, B)$-completion $\{A_1, \dots, A_n\}$. We assume, without loss of generality, that this completion is minimum and denote its completed graph by D_c. If the vertex a is a vertex of D_c, we are done since every vertex x of D_c is the head of an (x,B)-path. Otherwise, we consider a shortest directed (a, D_c)-path P in D. We denote by t the tail of P. If $t \in A \cup B$, the set $\{A_1, \dots, A_n, P\}$ is an (A,B)-completion. If t is an in-vertex of A_i, the set $\{A_1, \dots, A_{i-1}, A_i \cup P, A_{i+1}, \dots, A_n\}$ is an (A,B)-completion. If t is an out-vertex of A_i and $A_i \cup P$ is a bi-arborescence (with new centre t, this can only happen when A_i is the union of an in-arborescence and a path which contains t as an internal vertex), $\{A_1, \dots, A_{i-1}, A_i \cup P, A_{i+1}, \dots, A_n\}$ is an (A,B)-completion. If t is an out-vertex of A_i and $A_i \cup P$ is not a bi-arborescence, A_i

has more than one out-leaf, and thus, by Lemma 1 all its out-leaves belong to B. We denote by $(A_i)_t$ the sub-out-arborescence of A_i with root t, and by A'_i the bi-arborescence $A_i \setminus (A_i)_t$. Finally, $\{A_1, \dots, A_{i-1}, A'_i, (A_i)_t \cup P, A_{i+1}, \dots, A_n\}$ is an (A, B)-completion. $\square$

3. Spanning a neighbourhood

In this part, we show that, given a vertex w of a strong digraph D, there exists a k-handle which spans w and the neighbours of w, where k is at most the stability of the neighbourhood of w. This result is the core of our proof, we introduce for this the notion of tree-handle system. Given a vertex v in a digraph D, we denote by $N^+_D(v)$ the set of out-neighbours of v in D, and by $N^-_D(v)$ the set of in-neighbours of v in D. We write also the $d^+_D(v) := |N^+_D(v)|$ and $d^-_D(v) := |N^-_D(v)|$.

Theorem 4. *If $D = (V, E)$ is a strong digraph and w is a vertex of D, the set $\{w\} \cup N^+_D(w) \cup N^-_D(w)$ is contained in a p-handle D', where D' is a subgraph of D and $p \leqslant \alpha(D[\{w\} \cup N^+_D(w) \cup N^-_D(w)])$.*

Proof. We will simply denote $N^+_D(w)$ by w^+, $N^-_D(w)$ by w^- and $\alpha(D[\{w\} \cup w^+ \cup w^-])$ by α. We proceed by induction on E. To simplify a bit, we first treat the case $w^+ \cap w^- \neq \emptyset$. Assume for this that a vertex v belongs to $w^+ \cap w^-$, and that strong connectivity is lost when the arc vw or the arc wv is deleted (otherwise we simply remove the arc—the stability is unchanged). In this case, D consists of the union of two strong digraphs D_1 and D_2, such that $w \in D_1$ and $v \in D_2$ and the unique arcs between D_1 and D_2 are vw and wv. By the induction hypothesis, $\{w\} \cup N^+_{D_1}(w) \cup N^-_{D_1}(w)$ is spanned by at most an $(\alpha - 1)$-handle, to which we add the handle wvw.

From now on, we suppose that w^+ and w^- are disjoint sets. Let $C := \{B_1, \dots, B_r\}$ be a minimum $(w^+ \cup \{w\}, w^- \cup \{w\})$-completion. Observe that the completed graph is strong, therefore, we may suppose that the completed graph is exactly D, otherwise we apply the induction hypothesis. If one of the B_i is an out-arborescence, say with root r and set of leaves L. We construct a digraph D^* by removing from D the internal vertices of B_i and adding the set S consisting of all (r, L)-arcs. Observe that B_i has at least one internal vertex, otherwise $\{B_1, \dots, B_r\} \setminus \{B_i\}$ would be a $(w^+ \cup \{w\}, w^- \cup \{w\})$-completion since the leaves of B_i belong to $\{w\} \cup w^+ \cup w^-$. Thus, we can apply the induction hypothesis to D^* and span $\{w\} \cup w^+ \cup w^-$ by a k-handle H, where $k \leqslant \alpha(D^*[\{w\} \cup w^+ \cup w^-]) \leqslant \alpha$. Since H is strong and C is minimum, the set of arcs S is included in the arc set of H, thus $H' := (H \setminus S) \cup B_i$ is a k-handle and satisfies the conclusion of Theorem 4. We can now assume that every B_i is a plain bi-arborescence. If for some i, B_i has at least two internal vertices, we can consider instead of D the digraph D^* in which all the internal vertices of B_i are contracted to a single vertex. Again, we apply the induction hypothesis to D^* to conclude. From now on, we assume that every B_i is plain and has a single internal vertex b_i. From Lemma 1, it follows that the out-leaves of B_i belong to w^- and the in-leaves of B_i belong to w^+.

We introduce the notion of *tree-handle system* of D. We define it as a set

$$TH = \{W, A_1, A_2, \dots, A_k | P_1, P_2, \dots, P_l\},$$

where W and $A_i, 1 \leqslant i \leqslant k$, are some bi-arborescences whose centres are, respectively, w and a_i, and $P_j, 1 \leqslant j \leqslant l$, are some handles (possibly arcs) with the additional conditions:

(i) The sets $V(W), V(A_1), \dots, V(A_k), \ V(\mathring{P}_1), \dots, V(\mathring{P}_l)$ are pairwise disjoint.
(ii) The digraph $\bigcup\{W, A_1, A_2, \dots, A_k, P_1, P_2, \dots, P_l\}$ is a spanning subgraph of D. We call it the *realization* of TH, and we denote it by R.
(iii) The head (resp. the tail) of $P_j, 1 \leqslant j \leqslant l$, is an out-vertex (resp. an in-vertex) of an A_i or W.
(iv) Every vertex x of D, except possibly w, verifies $d_R^+(x) \geqslant 1$ and $d_R^-(x) \geqslant 1$.
(v) For all $i, 1 \leqslant i \leqslant k$, the out-neighbours (resp. the in-neighbours) of a_i in R are in-neighbours (resp. out-neighbours) of w in D.

We call l and k the *handle index* and the *tree index* of TH, respectively. Observe that in R, every vertex x different from w is the tail of an (w^+, x)-path and the head of an (x, w^-)-path. Thus, by the minimality of the completion C, every arc of $B_i, 1 \leqslant i \leqslant r$, must be an arc of R. In particular, every centre of B_i is also the centre of an A_j. We will call *special* such a bi-arborescence A_j (to say it differently, A_j is special if its centre does not belong to $\{w\} \cup w^+ \cup w^-$, and, conversely, if a vertex is not in $\{w\} \cup w^+ \cup w^-$, it is the centre of a special bi-arborescence). Keep in mind that a special bi-arborescence is necessarily plain. Let us prove now that D admits a tree-handle system:

An *out-fork* is an out-arborescence with height exactly 1 (i.e. consists of one root and a non-empty set of leaves), an *in-fork* is defined analogously. We denote by X the subset of vertices of w^+ which have an out-neighbour in w^- and by Y the subset of vertices of w^- which have an in-neighbour in w^+. In particular, every vertex of X has an out-neighbour in Y, and every vertex of Y has an in-neighbour in X. It is routine to check that $X \cup Y$ is spanned by a disjoint union of out-forks with root in X and leaves in Y and in-forks with root in Y and leaves in X. We denote by F this union of forks. Since D is strong, for every vertex $y \in w^-$, there exists an (u, y)-path in $D[w^-]$ with $u \in Y$ or u is an out-leaf of some B_i. Equivalently, there exists a disjoint union O of out-arborescences with set of roots $Y \cup \{b_i\colon\ i = 1, \dots, r\}$ and set of vertices $w^- \cup \{b_i\colon\ i = 1, \dots, r\}$. By a similar argument, there exists a disjoint union I of in-arborescences with set of roots $X \cup \{b_i\colon\ i = 1, \dots, r\}$ and set of vertices $w^+ \cup \{b_i\colon\ i = 1, \dots, r\}$. Now $F \cup O \cup I$ is a disjoint union of bi-arborescences whose centres are the roots of the forks of F and the $\{b_i\colon\ i = 1, \dots, r\}$. We denote by $A_1, \dots, A_k$ the subset of these bi-arborescences whose centre is not a leaf. We denote by $A'_1, \dots, A'_p$ the bi-arborescences whose centre is a leaf. Every in-leaf of A_i or A'_j is in w^+, and every out-leaf of A_i or A'_j is in w^-. We denote by IL the set of wl arcs where l is an in-leaf of A_i and by OL the set of lw arcs where l is an out-leaf of A_i, for all $i = 1, \dots, k$. We also denote by IL' the set of wl arcs where l is an in-leaf but not the centre of A'_j and by OL' the set of lw arcs where l is an out-leaf but not the centre of A'_j, for all $j = 1, \dots, p$. Finally, we denote by CI the set of wl arcs where l is an in-leaf and the centre of A'_j and CO the set of lw arcs where l is an out-leaf and the centre of A'_j. Observe that $W := \{w\} \cup A'_1 \cup \cdots \cup A'_p \cup CI \cup CO$ is a bi-arborescence with centre w. We have the tree-handle system $TH = \{W, A_1, \dots, A_k | IL \cup OL \cup IL' \cup OL'\}$, all the handles of which are arcs.

Our next goal is to prove that there exists a tree-handle system with handle index at most α. We consider for this a tree-handle system $TH = \{W, A_1, A_2, \dots, A_k | P_1, P_2, \dots, P_l\}$ which satisfies the following conditions:

(a) l is minimum.
(b) Subject to (a), k is minimum.
(c) Subject to (a) and (b), $|V(P_1)| + |V(P_2)| + \cdots + |V(P_l)|$ is maximum.

Let us prove that TH is *complete*, that is, it verifies the property:

(vi) Except in the case $x = w$, every out-leaf x of A_i or W satisfies $d_R^+(x) \geqslant 2$ and every in-leaf x of A_i or W satisfies $d_R^-(x) \geqslant 2$. In other words, x is the head or the tail of at least two handles of TH.

Consider for this an out-leaf x of a bi-arborescence of TH and assume that x is the head of a unique handle P of TH. Without loss of generality, we can suppose that $P = P_1$. We consider several cases:

1. Assume that x belongs to the bi-arborescence W. If $x = w$, condition (vi) holds vacuously. If $x \neq w$, x has an in-neighbour x' in W which is an out-vertex of W. In this case, we extend P with x', i.e. we consider $TH' = \{W \setminus \{x\}, A_1, A_2, \dots, A_k | x'x \cup P_1, P_2, \dots, P_l\}$. Note that the realization of TH is exactly $\bigcup TH'$, thus TH' is still a tree-handle system. However, the total length of the handles of TH' has increased, a contradiction to condition (c).
2. Now, assume that x belongs to a bi-arborescence A_i for some i, say A_1. If $x \neq a_1$, we conclude as previously in the case $x \neq w$. If $x = a_1$, we denote by t the tail of P_1:
    - The simplest case arises when t does not belong to A_1, say $t \in A_2$. Note that the bi-arborescence A_1 has no out-vertex except a_1 and that a_1 is the head of the unique handle P_1. Consider $TH' = \{W, A_1 \cup P_1 \cup A_2, A_3, \dots, A_k | P_2, \dots, P_l\}$ and observe that TH and TH' have the same realization which implies, as previously, that TH' is a tree-handle system. However, the handle index has strictly decreased and this contradicts condition (a). The same argument holds if $t \in A_i, 3 \leqslant i \leqslant k$ or if $t \in W$.
    - Suppose now that t belongs to A_1, in particular t is an in-vertex of A_1. Again, we modify TH in order to find a contradiction. There exists a path Q from t

to a_1 in A_1. The union $Q \cup P_1$ forms a cycle C in R which contains a_1. Note that, by property (v) in the definition of tree-handle systems, the in-neighbour and the out-neighbour of a_1 in C, respectively, belong to w^+ and to w^-. Since $w^+ \cap w^- = \emptyset$, it follows that C has at least three vertices, all of these apart possibly a_1 being neighbours of w. Indeed, the vertices of $D \setminus (\{w\} \cup w^+ \cup w^-)$ are the centres of special bi-arborescences, so $C \setminus \{a_1\} \subseteq w^- \cup w^+$. Thus, we can exhibit two vertices y and z in $C \setminus \{a_1\}$ such that yz is an arc of $C, y \in w^-$ and $z \in w^+$. To conclude, observe that $A = (A_1 \cup P_1 \cup yw) \setminus yz$ is an in-arborescence rooted at w and that wz forms a handle from W to A. Now, we consider $TH' = \{W \cup A, A_2, \dots, A_k | wz, P_2, \dots, P_k\}$ and check that TH' is a tree-handle system. Conditions (i) and (ii) clearly hold. The unique added handle is the arc wz, since w is an out-vertex of $W \cup A$ and z is an in-vertex of $W \cup A$, property (iii) is satisfied. To check property (iv), observe that z is the unique leaf possibly created by our modifications, and in this case z is an in-leaf of $W \cup A$ and the tail of the handle wz. Finally, we have not created new arborescence, which implies that property (v) still holds. Consequently, TH' is a tree-handle system with the same handle index than TH but with lower tree index, a contradiction to condition (b).

We proceed similarly if x is an in-leaf of a bi-arborescence. Since all these cases give a contradiction, $TH = \{W, A_1, A_2, \dots, A_k | P_1, P_2, \dots, P_l\}$ is a complete tree-handle system. Now we want to achieve our bound, that is we want to prove that l, the handle index of TH, is at most α. First observe that if one of the handles P_j is an arc, we can simply remove it from TH and still have a tree-handle system: the reason for this is simply that removing P_j cannot harm condition (iv) in the definition of tree-handle system since TH is complete. By minimality of l, all the handles have length at least 2. Suppose for contradiction that $l > \alpha$. Since $\mathscr{P}$ is a subset of $w^+ \cup w^-$, its stability is at most α, therefore we can apply Theorem 3 to the set of disjoint paths $\mathscr{P} := \{\mathring{P}_1, \mathring{P}_2, \dots, \mathring{P}_l\}$, in order to get a set of disjoint paths $\mathscr{P}' := \{P'_1, P'_2, \dots, P'_{l-1}\}$. Since the head of P'_i is the head of some $\mathring{P}_a$ and the tail of P'_i is the tail of some $\mathring{P}_b$, the path P'_i extends naturally to a handle $H_i := hP'_it$ where h is the head of P_a and t is the tail of P_b. Let us show that $TH' := \{W, A_1, A_2, \dots, A_k | H_1, H_2, \dots, H_{l-1}\}$ is a tree-handle system. Conditions (i) and (ii) are still satisfied. Since the sets of heads and tails of $\mathscr{P}'$ are subsets of the sets of heads and tails of $\mathscr{P}$, condition (iii) holds for TH'. Since TH is complete and exactly one head and one tail of $\mathscr{P}$ are lost, the condition (iv) holds. Finally, no new out or in-neighbours of any a_i is created in TH', so condition (v) still holds. Thus TH' has handle index $l - 1$, a contradiction to condition (a). Consequently, the handle index l of TH is at most α. Our last step is to span D with an l-handle. By minimality, we recall that every handle of TH is non trivial (i.e. has length at least 2).

Consider for this a subgraph D' of D with vertex set V' and arc set E', which is maximal with respect to $|V'|$ and verifies the following conditions:

(I) For some $p \in \{0, \dots, l\}$, D' is a p-handle and contains the vertices of at least p handles of TH.

(II) For all $j = 1, \dots, l$ either $V(\overset{\circ}{P_j}) \cap V' = \emptyset$ or $V(P_j) \subseteq V'$.

(III) For all $i = 1, \dots, k$, either $V(A_i) \cap V' = \emptyset$ or $D'[V(A_i) \cap V']$ is a sub-bi-arborescence of A_i which contains a_i. Morever $D'[V(W) \cap V']$ is a sub-bi-arborescence of W which contains w.

Since the singleton digraph $\{w\}$ satisfies (I)–(III), such a D' exists. We prove that D' necessarily spans D, and thus achieve our goal:

- Let us assume that there exists a (V')-handle H in R which is not an arc. We denote its head by h and its tail by t. By condition (III), H is not contained in a bi-arborescence A_i or W. Therefore H contains an internal vertex of some handle P of TH. By condition (II), it follows that P is contained in H. Thus, $D' \cup H$ contains at least one handle of TH which is not in D', in particular $D' \cup H$ satisfies condition (I). If $\overset{\circ}{H}$ contains an internal vertex v of some handle P_v, the whole handle P_v is contained in $D' \cup H$, so condition (II) is also satisfied. To check that condition (III) is still satisfied, suppose that $\overset{\circ}{H}$ contains a vertex of some bi-arborescence A_i which is disjoint from D'. Denote by a the first vertex of $\overset{\circ}{H} \cap A_i$ along H and by b the last one. Since H is included in R, a is an in-vertex of A_i, b is an out-vertex of A_i, and $H[a, b]$ is included in A_i, and thus forms a sub-bi-arborescence which contains a_i. Now if $\overset{\circ}{H}$ contains a vertex v of some bi-arborescence A_i (resp. W) which meets D', assume without loss of generality that v is an in-vertex of A_i (resp. W). Since H is included in R and $a_i \in D'$ (resp. $w \in D'$), the path $H[v, t]$ is included in the set of in-vertices of A_i (resp. W). In both of these two cases, $D' \cup H$ satisfies condition (III).
- If there is no (V')-handle in R and $V \neq V'$, since both the in and the out-degree in R of any vertex different from w are greater than one, R contains a cycle C which is disjoint from D'. The cycle C contains the centre a_i of a bi-arborescence A_i of TH. Let us denote by a_j the next centre of an A_j on C, that is, no internal vertex of $C[a_i, a_j]$ is the centre of some bi-arborescence of TH. If a_i is the unique centre which is contained in C, we simply choose $a_j := a_i$. Since $C[a_i, a_j]$ contains a handle of TH, it has length at least two. By property (v) of a tree-handle system, the out-neighbour a_i^+ of a_i in C belongs to w^-, and the in-neighbour a_j^- of a_j in C belongs to w^+. Since w^+ and w^- are disjoint sets, $a_i^+ \neq a_j^-$. In particular, we can find in $C[a_i^+, a_j^-]$ two consecutive vertices x and y such that $x \in w^-$ and $y \in w^+$. The subgraph $D' \cup wy \cup C[y, x] \cup xw$ of D contains at

least one handle of TH which is not in D' (indeed $C[a_i, a_j]$ contains exactly one handle of TH), in particular it satisfies condition (I). Conditions (II) and (III) are also easily verified.

So, $V = V'$ and hence, D' is a p-handle which spans D, with $p \leqslant l \leqslant \alpha$. □

4. The main theorem. The algorithmic aspect

Finally, we prove Theorem 1, which is an easy corollary of the previous result.

Proof of Theorem 1. Let us fix a vertex w_0 of D. According to Theorem 4, we can cover $\{w_0\} \cup N_D^+(w_0) \cup N_D^-(w_0)$ by a k_1-handle H_1 with $k_1 \leqslant \alpha(D[N_D^+(w_0) \cup N_D^-(w_0)])$. We contract this k_1-handle to form a digraph D_1 and call w_1 the contracted vertex. We again apply Theorem 4, and cover $\{w_1\} \cup N_{D_1}^+(w_1) \cup N_{D_1}^-(w_1)$ by a k_2-handle H_2 with $k_2 \leqslant \alpha(D_1[N_{D_1}^+(w_1) \cup N_{D_1}^-(w_1)])$.

Perform these contractions until only one vertex w_p remains. For $l = 1, \dots, p$, we denote by V_l the set of vertices of D contracted to w_l and which were not contracted to w_{l-1}, observe that the stability of $D[V_l]$, denoted by α_l, is greater or equal to k_l. Moreover, if an arc of D has its endvertices in V_i and V_j, we clearly have $|i-j| \leqslant 1$. Consequently, $1 + \alpha_2 + \alpha_4 + \cdots \leqslant \alpha(D)$ and $\alpha_1 + \alpha_3 + \alpha_5 + \cdots \leqslant \alpha(D)$. Now, let $D'_p := \{w_p\}$ and, starting with $j := p$, inductively replace w_j in D'_j by the k_j-handle H_j to form the digraph D'_{j-1}. The spanning subgraph D'_0 of D is a k-handle where k is the sum of the k_j. Moreover, $k \leqslant \alpha_1 + \alpha_2 + \cdots + \alpha_p \leqslant 2\alpha - 1$. □

To conclude this paper, we invite the reader to check that an algorithm can easily be derived from our proof. The calculation of a completion in which every arc is necessary can be done in polynomial time. The reduction of a tree-handle system can be performed in $O(|V|)$, and the path-exchange of Theorem 3 can be calculated in $O(|E|)$. From this, the calculation of a $(2\alpha - 1)$-handle which spans D can be done in polynomial time. Although the calculation of the minimum k for which a strong digraph D admits a spanning k-handle cannot be approximated up to any fixed factor (we leave this as an exercise for the reader), the best known bound (see [11]) is the following: there exists an algorithm which calculates a spanning k-handle of a digraph D where $(n+k-1)/(n+l-1) \leqslant 3/2$, where l is the minimum value for an l-handle spanning D. Our approach gives a better bound for dense graphs, that is when $\alpha < n/4$.

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