On the Rate of Convergence of Fourier Series for Functions of HBMV*

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1. INTRODUCTION


If $f \in L_{2\pi}$ and there exists a constant $M$ so that for each $I = [a, b]$,

$$m_I(f) = |I|^{-1} \int_I |f(x) - f_I| \, dx < M,$$

(1.1)

where $|I| = b - a$, $f_I = |I|^{-1} \int_I f(x) \, dx$, then the function $f(x)$ is said to be a bounded mean oscillation function, written $f \in \text{BMO}$.

Let $A = \{\lambda_n\}$ be a nondecreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$ and suppose $I_n = [a_n, b_n]$ is a sequence of nonoverlapping subintervals of $[0, 2\pi]$. If there exists an $M$ so that, for each $I_n = [a_n, b_n]$,

$$\sum f(I_n)/\lambda_n < M,$$

where $f(I_n) = |f(b_n) - f(a_n)|$, then $f$ is said to be a $\lambda$-bounded variation function, denoted $f \in \text{ABV}$. In particular, if $\lambda_n = n$, we will let $\text{ABV} = \text{HBV}$ (harmonic bounded variation).

If for any real $a$ and each sequence $I_n = [a_n, b_n]$ of nonoverlapping intervals on $[a, a + 2\pi]$,

$$M_A(f) := \sup \sum_{n=1}^{\infty} m_{I_n}(f)/\lambda_n < +\infty,$$

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holds, then \( f \) is said to be \( A \)-bounded mean variation function, written \( f \in ABMV \).

X. L. Shi proved

(i) \( ABV \subseteq ABMV \subseteq BMO \).

(ii) If \( f \in HBMV \), then the partial sum of Fourier series of \( f \) \( S_n(x) \) converges to \( S(x) = \frac{1}{2}(f(x+0)+f(x-0)) \) at the points where \( S(x) \) is well defined and converges uniformly on closed intervals of continuity points of \( f \).

(iii) Let \( f \in ABMV \) and for \( I = [a, b] \) denote

\[
M_A(f, I) = \sup \sum_{l_n = I}^{x} m_{l_n}(f)/\lambda_n.
\]

If \( x_0 \in (a, b) \) is a continuity point of \( f \), then

\[
\lim_{y \to 0} M_A(f, [x_0 - y, x_0 + y]) = 0.
\]

On the other hand, an estimate on the rate of convergence of Fourier series for functions of bounded variation was first established by R. Bojanic [4]. Later R. Bojanic and D. Waterman (see [5]) considered the class between two extremes, \( BV \) and \( HBV \), by setting \( A = \{n^s \} \ (0 < s < 1) \). Recently, D. Waterman [6] proved that if \( \{\lambda_k/k\} \) is nonincreasing and \( f \in ABV \) and \( \pi/(n + 1) < a_n < a_{n+1} < \cdots < a_0 = \pi \), then

\[
\left| S_n(x) - \frac{1}{2}[f(x + 0) + f(x - 0)] \right| 
\leq (1 + 1/\pi) \frac{\lambda_{n+1}}{n+1} V(\pi) + \frac{\pi}{n+1} \sum_{i=0}^{n-1} V(a_i) [H(a_{i+1}) - H(a_i)] \tag{1.2}
\]

where \( V(t) = V_A(g, [0, t]) \), \( g_v(t) = f(x + t) + f(x - t) - f(x + 0) - f(x - 0) \) and \( H(t) \) is a continuous nonincreasing function on \( (0, \pi) \) such that

\[
H(t) = \lambda_k/t, \quad \text{for} \quad t = k\pi/(n+1), \ k = 1, \ldots, n + 1.
\]

The estimate (1.2) is not adequate for \( f \in HBV \). Hence, for the extreme \( HBV \), the question of estimate of the rate of convergence remains open to this day. In this paper we will solve the question for larger class \( HBMV \). As a special case, we obtain the rate for \( f \in HBV \). We define so-called pointwise modulus of continuity at \( x \) as follows:

\[
w(g; x, t) := \sup_{|y - x| \leq t, y} |g(y) - g(x)|.
\]
It is clear that if \( f \in \mathbb{ABV} \), then
\[
w(x, t) \leq V_A(g_x, [0, t]) = V(t).
\]

2. Results

Now we state our results.

**Theorem 2.1.** Let \( f \in \mathbb{ABMV} \); then
\[
\left| S_n(x) - \frac{1}{2} (f(x + 0) + f(x - 0)) \right| 
\leq 3w \left( \frac{7\pi}{(2n + 1)} \right) + 2\pi M_A(\pi) \left\{ \sum_{k=1}^{[n/2]} 1/\lambda_k \right. 
+ 2\pi \sum_{k=1}^{[n/2]} M_A((4k + 3) \pi/(2n + 1)) \left( \sum_{i=1}^{k} 1/\lambda_i \right),
\]
where \( w(t) = w(g_x, 0, t) \) and \( M_A(t) = M_A(g_x, [0, t]) \) and \( g_x(t) = f(x + t) + f(x - t) - f(x + 0) - f(x - 0). \)

**Theorem 2.2.** Let \( f \in \mathbb{BMV} \); then we have
\[
\left| S_n(x) - \frac{1}{2} (f(x + 0) + f(x - 0)) \right| 
\leq 3w \left( \frac{7\pi}{(2n + 1)} \right) + 2\pi M_H((4n \delta_n + 3) \pi/(2n + 1)) 
+ 2\pi \sum_{t=1}^{[n/2]} 1/\lambda_i 
+ 2\pi \sum_{k = \lceil n/2 \rceil + 1}^{[n/2] - 1} M_H((4k + 3) \pi/(2n + 1)) \left( \sum_{i=1}^{k} 1/\lambda_i \right),
\]
where \( \delta_n \downarrow 0 \) and \( \log \delta_n = o (\log n) \).
Set \( \delta_n = (\log n)^{-1} \); we have

**Corollary 2.3.** For \( f \in \mathbb{BMV} \),
\[
S_n(x) - \frac{1}{2} (f(x + 0) + f(x - 0)) 
= 0(w(7\pi/(2n + 1) + M_H(4\pi/\log n) + (\log \log n/\log n) M_H(\pi)).
\]
Noting (1.3), we obtain
Corollary 2.4. For $f \in \text{HBV}$,
\[
S_{n}(x) - \frac{1}{2}(f(x + 0) + f(x - 0)) = 0(V_{n}(4\pi/\log n) + (\log \log n/\log n)V_{n}(\pi)).
\]

3. Proofs

Proof of Theorem 2.1. We have
\[
\Delta_{n}(x) = S_{n}(x) - \frac{1}{2}(f(x + 0) + f(x - 0))
\]
\[
= \frac{1}{2\pi} \int_{0}^{\pi} g_{x}(t) \frac{\sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dt
\]
\[
= \frac{1}{2\pi} \int_{0}^{\pi/(2n + 1)} g_{x}(t) \frac{\sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dt
\]
\[
+ \frac{1}{2\pi} \sum_{k=1}^{2n} \int_{0}^{\pi/(2n + 1)} g_{x}(t + \frac{k\pi}{2n + 1}) \frac{\sin((n + \frac{1}{2})t + \frac{k\pi}{2n + 1})}{\sin \frac{1}{2}t + k\pi/(4n + 2)} \, dt
\]
\[
= \frac{1}{2\pi} \int_{0}^{\pi/(2n + 1)} g_{x}(t) \frac{\sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dt
\]
\[
+ \frac{1}{2\pi} \int_{0}^{\pi/(2n + 1)} \left\{ \sum_{k=1}^{n} g_{x}(t + \frac{(2k - 1)\pi}{2n + 1}) \frac{(\cos(n + \frac{1}{2})t)}{\sin \frac{1}{2}t + (2k - 1)\pi/(4n + 2)} \right\} \, dt
\]
\[
+ \frac{1}{2\pi} \int_{0}^{\pi/(2n + 1)} g_{x}(t + \frac{2k\pi}{2n + 1}) \frac{(-1)^k \sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t + 2k\pi/(4n + 2)} \, dt \}
\]
\[
:= R_{1} + R_{2}.
\]
(3.1)

It is obvious that
\[
R_{1} \leq \frac{1}{2}w(\pi/(2n + 1)).
\]
(3.2)

If $n$ is odd, set $n = 2m + 1$, then
\[
2\pi R_{2} = \int_{0}^{\pi/(2n + 1)} \left[ g_{x}(t + \frac{\pi}{2n + 1}) \frac{\cos(n + \frac{1}{2})t}{\sin \frac{1}{2}t + \pi/(4n + 2)}
\right.
\]
\[
- g_{x}(t + \frac{2\pi}{2n + 1}) \frac{\sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t + 2\pi/(4n + 2)} \right] \, dt
\]
HBMV FUNCTIONS

\[ + \int_{0}^{\pi/(2n+1)} \left\{ \cos \left( n + \frac{1}{2} \right) t \right. \]

\[ \times \sum_{k=1}^{m} \left[ g_x \left( t + \frac{4k + 1}{2n + 1} \pi \right) \sin^{-1} \frac{1}{2} \left( t + \frac{4k + 1}{2n + 1} \pi \right) \right. \]

\[ - g_x \left( t + \frac{4k - 1}{2n + 1} \pi \right) \sin^{-1} \frac{1}{2} \left( t + \frac{4k - 1}{2n + 1} \pi \right) \]

\[ + \sin \left( n + \frac{1}{2} \right) t \sum_{k=1}^{m} \left[ g_x \left( t + \frac{4k + 1}{2n + 1} \pi \right) \sin^{-1} \frac{1}{2} \left( t + \frac{4k + 1}{2n + 1} \pi \right) \right. \]

\[ - g_x \left( t + \frac{4k + 2}{2n + 1} \pi \right) \sin^{-1} \frac{1}{2} \left( t + \frac{4k + 2}{2n + 1} \pi \right) \left\} \right] \}

\[ dx \]

\[ := 2\pi (R_{21} + R_{22}). \]  

(3.3)

It is clear that

\[ |R_{21}| \leq \frac{3}{4} w(3\pi/(2n + 1)). \]  

(3.4)

Rewrite that

\[ 2\pi R_{22} = \int_{0}^{\pi/(2n+1)} \left\{ \cos \left( n + \frac{1}{2} \right) t \sum_{k=1}^{m} \left[ g_x \left( t + \frac{4k + 1}{2n + 1} \pi \right) \right. \right. \]

\[ - g_x \left( t + \frac{4k - 1}{2n + 1} \pi \right) \sin^{-1} \frac{1}{2} \left( t + \frac{4k - 1}{2n + 1} \pi \right) \]

\[ + \sin \left( n + \frac{1}{2} \right) t \sum_{k=1}^{m} \left[ g_x \left( t + \frac{4k + 1}{2n + 1} \pi \right) - g_x \left( t + \frac{4k + 2}{2n + 1} \pi \right) \right. \]

\[ \times \sin^{-1} \frac{1}{2} \left( t + \frac{4k + 2}{2n + 1} \pi \right) \left) \right] \right\} \}

\[ + \int_{0}^{\pi/(2n+1)} \left\{ \cos \left( n + \frac{1}{2} \right) t \sum_{k=1}^{m} g_x \left( t + \frac{4k + 1}{2n + 1} \pi \right) \right. \]

\[ \times \left[ \sin^{-1} \frac{1}{2} \left( t + \frac{4k + 1}{2n + 1} \right) - \sin^{-1} \frac{1}{2} \left( t + \frac{4k - 1}{2n + 1} \right) \right] \]

\[ + \sin \left( n + \frac{1}{2} \right) t \sum_{k=1}^{m} g_x \left( t + \frac{4k + 2}{2n + 1} \pi \right) \]

\[ \times \left[ \sin^{-1} \frac{1}{2} \left( t + \frac{4k + 2}{2n + 1} \pi \right) - \sin^{-1} \frac{1}{2} \left( t + \frac{4k + 2}{2n + 1} \pi \right) \right] \}

\[ \cdot dt \]

\[ := 2\pi (R'_{22} + R''_{22}). \]  

(3.5)
It is obvious that
\[ |R_{22}'| \leq \frac{2n + 1}{6\pi} \int_0^{\pi/(2n+1)} \sum_{k=1}^{m} \left( |g_k \left( t + \frac{4k - 1}{2n+1} \pi \right) - g_k \left( t + \frac{4k + 1}{2n+1} \pi \right)| \right) \, dt 
\]

Applying the Abel transformation to \( R_{22}' \), we obtain
\[ |R_{22}''| = \frac{1}{\pi} \sin \frac{\pi}{4n+2} \int_0^{\pi/(2n+1)} \left[ -\cos \left( n + \frac{1}{2} \right) \tan \left( t + \frac{5\pi}{2n+1} \right) \right] \, dt 
\]

\[ \times \sum_{k=1}^{m} \frac{\cos \left( \frac{1}{2}(t + 4k\pi/(2n+1)) \right)}{\sin \left( 1/2(t + 4(k-1)\pi/(2n+1)) \right) \sin \left( 1/2(t + 4(k+1)\pi/(2n+1)) \right)} \right] \, dt 
\]

\[ + \frac{1}{\pi} \sin \frac{\pi}{4n+2} \int_0^{\pi/(2n+1)} \left( -\cos \left( n + \frac{1}{2} \right) \right) t 
\]

\[ \times \sum_{k=2}^{m} \left[ g_k \left( t + \frac{4k + 1}{2n+1} \pi \right) - g_k \left( t + \frac{4k - 3}{2n+1} \pi \right) \right] \left( t + \frac{4k - 2}{2n+1} \pi \right) \right] \, dt 
\]

\[ \times \sum_{i=k}^{m} \frac{\cos \left( \frac{1}{2}(t + 4i\pi/(2n+1)) \right)}{\sin \left( 1/2(t + 4(i-1)\pi/(2n+1)) \right) \sin \left( 1/2(t + 4(i+1)\pi/(2n+1)) \right)} \right] \, dt 
\]

\[ \leq w(7\pi/(2n+1)) + \int_0^{\pi/(2n+1)} \sum_{k=2}^{m} \left| g_k \left( t + \frac{4k + 1}{2n+1} \pi \right) - g_k \left( t + \frac{4k - 3}{2n+1} \pi \right) \right| \, dt 
\]

\[ - \sum_{i=k}^{m} \frac{2n + 1}{(4i - 1)(4i + 1)} \, dt 
\]

\[ \times \int_0^{\pi/(2n+1)} \sum_{k=2}^{m} \left| g_k \left( t + \frac{4k + 2}{2n+1} \pi \right) - g_k \left( t + \frac{4k - 2}{2n+1} \pi \right) \right| \, dt 
\]

\[ \times \sum_{i=k}^{m} \frac{2n + 1}{(4i)(4i + 2)} \, dt 
\]
\[ \leq w\left(\frac{7\pi}{(2n+1)}\right) + \frac{2n+1}{6} \]

\[ \times \int_0^{\pi/(2n+1)} \left\{ \sum_{k=2}^m k^{-1} \left| g_x\left(t + \frac{4k+1}{2n+1}\pi\right) - g_x\left(t + \frac{4k-3}{2n+1}\pi\right)\right| \right. \]

\[ + \left. \sum_{k=2}^m k^{-1} \left| g_x\left(t + \frac{4k+2}{2n+1}\pi\right) - g_x\left(t + \frac{4k-2}{2n+1}\pi\right)\right| \right\} dt \]

\[ \leq w\left(\frac{7\pi}{(2n+1)}\right) + \frac{2n+1}{6} \]

\[ \times \int_0^{\pi/(2n+1)} \left\{ \sum_{k=2}^m k^{-1} \left| g_x\left(t + \frac{4k+1}{2n+1}\pi\right) - g_x\left(t + \frac{4k-1}{2n+1}\pi\right)\right| \right. \]

\[ + \left. \sum_{k=2}^m k^{-1} \left| g_x\left(t + \frac{4k+2}{2n+1}\pi\right) - g_x\left(t + \frac{4k}{2n+1}\pi\right)\right| \right\} dt \]

\[ + \frac{2n+1}{6} \int_0^{\pi/(2n+1)} \left\{ \sum_{k=2}^m k^{-1} \left| g_x\left(t + \frac{4k-1}{2n+1}\pi\right) - g_x\left(t + \frac{4k-3}{2n+1}\pi\right)\right| \right. \]

\[ + \left. \sum_{k=2}^m k^{-1} \left| g_x\left(t + \frac{4k}{2n+1}\pi\right) - g_x\left(t + \frac{4k-2}{2n+1}\pi\right)\right| \right| dt. \] (3.7)

Then from (3.5)-(3.7),

\[ |R_{22}| \leq w\left(\frac{7\pi}{(2n+1)}\right) + \frac{2}{9} (2n+1) \]

\[ \times \int_0^{\pi/(2n+1)} \left\{ \sum_{k=2}^m k^{-1} \left| g_x\left(t + \frac{4k-1}{2n+1}\pi\right) - g_x\left(t + \frac{4k+1}{2n+1}\pi\right)\right| \right. \]

\[ + \left. \left| g_x\left(t + \frac{4k}{2n+1}\pi\right) - g_x\left(t + \frac{4k+2}{2n+1}\pi\right)\right| \right\} dt + \frac{2n+1}{6} \]

\[ \times \int_0^{\pi/(2n+1)} \left\{ \sum_{k=2}^m k^{-1} \left| g_x\left(t + \frac{4k-1}{2n+1}\pi\right) - g_x\left(t + \frac{4k-3}{2n+1}\pi\right)\right| \right. \]

\[ + \left. \left| g_x\left(t + \frac{4k}{2n+1}\pi\right) - g_x\left(t + \frac{4k-2}{2n+1}\pi\right)\right| \right| dt \] (3.8)

\[ = w\left(\frac{7\pi}{(2n+1)}\right) + \frac{2(2n+1)}{9m} \]

\[ \times \int_0^{\pi/(2n+1)} \left\{ \sum_{k=2}^m k^{-1} \left| g_x\left(t + \frac{4k-1}{2n+1}\pi\right) - g_x\left(t + \frac{4k+1}{2n+1}\pi\right)\right| \right. \]

\[ + \left. \left| g_x\left(t + \frac{4k}{2n+1}\pi\right) - g_x\left(t + \frac{4k+2}{2n+1}\pi\right)\right| \right\} dt + \frac{2n+1}{6m} \]
\begin{align*}
&\times \int_0^{\pi/(2n+1)} \sum_{k=2}^{m} \left\{ \left| g_x \left( t + \frac{4k-1}{2n+1} \pi \right) - g_x \left( t + \frac{4k-3}{2n+1} \pi \right) \right| 
+ \left| g_x \left( t + \frac{4k}{2n+1} \pi \right) - g_x \left( t + \frac{4k-2}{2n+1} \pi \right) \right| \right\} dt \\
&+ \frac{2(2n+1)}{9} \int_0^{\pi/(2n+1)} \sum_{k=2}^{m} \sum_{i=2}^{k-2} \left\{ \left| g_x \left( t + \frac{4i-1}{2n+1} \pi \right) - g_x \left( t + \frac{4i-3}{2n+1} \pi \right) \right| 
+ \left| g_x \left( t + \frac{4i}{2n+1} \pi \right) - g_x \left( t + \frac{4i-2}{2n+1} \pi \right) \right| \right\} dt.
\end{align*}

Denote

\begin{align*}
I_k &= \left[ \frac{4k-1}{2n+1} \pi, \frac{4k+3}{2n+1} \pi \right], \quad k = 1, 2, \ldots, m \\
I'_k &= \left[ \frac{4k-3}{2n+1} \pi, \frac{4k+1}{2n+1} \pi \right], \quad k = 2, \ldots, m.
\end{align*}

We have

\begin{align*}
|R_{22}| &\leq w(7\pi/(2n+1)) + \frac{2(2n+1)}{9m} \sum_{k=1}^{m} \int_{(4k-3)/(2n+1)\pi}^{(4k+1)/(2n+1)\pi} \left| g_x(u) - g_{I_k} \right| du \\
&+ \frac{2n+1}{6m} \sum_{k=2}^{m} \int_{(4k-3)/(2n+1)\pi}^{(4k+1)/(2n+1)\pi} \left| g_x(u) - g_{I'_k} \right| du \\
&+ \frac{2(2n+1)}{9} \sum_{k=2}^{m} \sum_{i=2}^{k-2} \int_{(4i+1)/(2n+1)\pi}^{(4i+3)/(2n+1)\pi} \left| g_x(u) - g_{I_i} \right| du \\
&+ \frac{2n+1}{6} \sum_{k=2}^{m} \sum_{i=2}^{k-1} \int_{(4i+3)/(2n+1)\pi}^{(4i+1)/(2n+1)\pi} \left| g_x(u) - g_{I_i} \right| du \\
&\leq w(7\pi/(2n+1)) + \frac{\pi}{m} \sum_{k=1}^{m} m_{I_k}(g_x) + \frac{2\pi}{3m} \sum_{k=2}^{m} m_{I'_k}(g_x) \\
&+ \pi \sum_{k=1}^{m-1} \sum_{i=1}^{k-2} m_{I_i}(g_x) + \frac{2\pi}{3} \sum_{k=2}^{m-1} \sum_{i=2}^{k-2} m_{I_i}(g_x).
\end{align*}
Arranging \( \{m_k(g_x)\}, \{m_k^*(g_x)\}, \{m_k^*(g_x)\}, \{m_k(g_x)\} \) in nonincreasing order and applying Chebyshev inequality [7], we have

\[
|R_{22}| \leq w\left(\frac{7\pi}{(2n+1)} + \pi M_A(\pi)\right)\left(\sum_{i=1}^{m}\frac{1}{\lambda_i}\right)
+ \frac{2}{3} \pi M_A(\pi)\left(\sum_{i=1}^{m}\frac{1}{\lambda_i}\right)
+ \pi \sum_{k=1}^{m-1} M_A((4k+3)\frac{\pi}{(2n+1)})\left(\sum_{i=1}^{k}\frac{1}{\lambda_i}\right)
+ \frac{2}{3} \pi \sum_{k=2}^{m-1} M_A((4k+1)\frac{\pi}{(2n+1)})\left(\sum_{i=1}^{k}\frac{1}{\lambda_i}\right)
+ \frac{2}{3} \pi \sum_{k=1}^{m-1} M_A((4k+3)\frac{\pi}{(2n+1)})\left(\sum_{i=1}^{k}\frac{1}{\lambda_i}\right)
\]

Finally, combining (3.1)-(3.4) and (3.9), (2.1) follows for odd \( n \). For even \( n \) (2.1) can be similarly proved. The proof is completed.

**Proof of Theorem 2.2.** Let \( n \) be odd. From (3.8) we have

\[
|R_{22}| \leq w\left(\frac{7\pi}{(2n+1)}\right)
+ \left\{\frac{2}{9} \left(\frac{2n+1}{6}\right)\right\} \sum_{k=1}^{\left\lfloor \frac{m\pi}{(2n+1)} \right\rfloor} k^{-1} \left[ g_x\left(t + \frac{4k-1}{2n+1} \pi\right) \right. \\
- g_x\left(t + \frac{4k+1}{2n+1} \pi\right) + \left| g_x\left(t + \frac{4k}{2n+1} \pi\right) \right| g_x\left(t + \frac{4k+2}{2n+1} \pi\right) \right| dt
\]

(3.9)
\[ w(\frac{\pi}{2n + 1}) \leq \sum_{k = 1}^{m} \left( \int_{-\pi/(2n + 1)}^{\pi/(2n + 1)} g_k(t + 4k - 1) \right) + \int_{-\pi/(2n + 1)}^{\pi/(2n + 1)} \left| g_x(t + \frac{4k - 3}{2n + 1}) - g_x(t + \frac{4k - 2}{2n + 1}) \right| dt \]

\[ = w(7\pi/(2n + 1)) + r'_{22} + r''_{22}. \]  

(3.10)

For \( r'_{22} \), we have

\[ r'_{22} \leq \frac{5\pi}{3} M_{H}(4n[\delta_n] + 3) \pi/(2n + 1). \]  

(3.11)

For \( r''_{22} \) applying the Abel transformation and the method similar to that in proof of Theorem 2.1, we obtain

\[ r''_{22} \leq \frac{5\pi}{3} M_{H}(\pi) \left( \sum_{i=1}^{[n/2]} 1/\lambda_i \right) \]

\[ + \frac{14}{9} \pi \sum_{k = [n\delta_n] + 1}^{[n/2] - 1} M_{H}(4k + 3) \pi/(2n + 1) \left( k \sum_{i=1}^{k} 1/\lambda_i \right). \]  

(3.12)

From (3.10)–(3.12) it follows that

\[ |R_{22}| \leq w(7\pi/(2n + 1)) + \frac{5}{3} \pi M_{H}(4n[\delta_n] + 3) \pi/(2n + 1) \]

\[ + \frac{14}{9} \pi \sum_{k = [n\delta_n] + 1}^{[n/2] - 1} M_{H}(4k + 3) \pi/(2n + 1) \left( k \sum_{i=1}^{k} 1/\lambda_i \right). \]  

(3.13)

Combining (3.1)–(3.4) and (3.13), we obtain (2.2). For even \( n \) (2.2) holds still. The proof of Theorem 2.2 is completed.

REFERENCES


