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Quenched convergence of a sequence of superprocesses in $\mathbb{R}^d$ among Poissonian obstacles

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Abstract

We prove a convergence theorem for a sequence of super-Brownian motions moving among hard Poissonian obstacles, when the intensity of the obstacles grows to infinity but their diameters shrink to zero in an appropriate manner. The superprocesses are shown to converge in probability for the law **P** of the obstacles, and **P**-almost surely for a subsequence, towards a superprocess with underlying spatial motion given by Brownian motion and (inhomogeneous) branching mechanism $\psi(u, x)$ of the form $\psi(u, x) = u^2 + \kappa(x)u$, where $\kappa(x)$ depends on the density of the obstacles. This work draws on similar questions for a single Brownian motion. In the course of the proof, we establish precise estimates for integrals of functions over the Wiener sausage, which are of independent interest.

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Keywords: Super-Brownian motion; Random obstacles; Quenched convergence; Brownian motion; Wiener sausage

1. Introduction

1.1. Superprocesses in random media

The purpose of this article is to investigate the behaviour of super-Brownian motion among random obstacles, when the density of these obstacles grows to infinity but their diameter shrinks to zero in an appropriate manner. More precisely, let us fix $d \geq 2$ and a domain D of $\mathbb{R}^d$, and

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let $c : \mathbb{R}^d \to [0, \infty)$ be a bounded measurable function. For every $\varepsilon \in (0, \frac{1}{2})$, let us define an obstacle configuration by

$$\Gamma_\varepsilon = \bigcup_{x \in \mathcal{P}^\varepsilon} \overline{B}(x, \varepsilon),$$

where $\mathcal{P}^\varepsilon$ is a Poisson point process on $\mathbb{R}^d$ with intensity $\log(\varepsilon^{-1})c(x)\mathrm{d}x$ if $d = 2$ and $\varepsilon^{2-d}c(x)\mathrm{d}x$ if $d \geq 3$, and $\overline{B}(x, \varepsilon)$ denotes the closed ball of radius ε centered at x. This Poisson point process is defined on a probability space $(\boldsymbol{\Omega}, \mathcal{F}, \mathbf{P})$. On a different probability space, let us also consider a superprocess $\{X_t^\varepsilon,\ t \in [0, \infty)\}$ with critical branching mechanism $\psi(u) = u^2$ and underlying spatial motion given by Brownian motion killed when entering $D^c \cup \Gamma_\varepsilon$. Thus, for each ε, the superprocess X^ε can be seen as evolving in a random medium given by Γ_ε. A realization of $\{\Gamma_\varepsilon,\ \varepsilon \in (0, 1/2)\}$ will be called an *environment*.

We wish to understand the behaviour of X^ε when ε tends to zero. As in most works about random media, two points of view can be adopted: either we fix an environment (quenched approach), or we average over the possible realizations of $\bigcup_{\varepsilon > 0} \Gamma_\varepsilon$ (annealed approach). Although the results of this paper are set in the quenched framework, the main ingredients of their proofs are "annealed-type" calculations. Moreover, the latter approach is also useful in obtaining a better understanding of where the scaling comes from and of what the limiting process might be. To simplify the analysis, let us first assume that $D = \mathbb{R}^d$ and let us consider a single Brownian motion ξ, independent of the obstacles. Denote by P_x the probability measure under which ξ starts from x. Let us define the random time T_ε as the entrance time of ξ into the set Γ_ε, that is

$$T_\varepsilon := \inf\{t \geq 0 : \xi_t \in \Gamma_\varepsilon\}.$$

In addition, for all $0 \leq s \leq t$, we denote by $S_\varepsilon(s, t)$ the Wiener sausage of radius ε along the time interval $[s, t]$, defined as

$$S_\varepsilon(s, t) = \{y \in \mathbb{R}^d : \inf_{s \leq r \leq t} |\xi_r - y| \leq \varepsilon\} = \bigcup_{r \in [s,t]} \left(\xi_r + \overline{B}(0, \varepsilon)\right).$$

The probability that the Brownian motion ξ hits Γ_ε before time t is equal to the probability that the center of one of the obstacles lies in $S_\varepsilon(0, t)$. These centers are given by the Poisson point process $\mathcal{P}^\varepsilon$ and so, by averaging over the random obstacles and using Fubini's theorem, we obtain

$$\mathbf{E}\left[\mathrm{P}_0[T_\varepsilon > t]\right] = \mathrm{E}_0\left[\mathbf{P}[\mathcal{P}^\varepsilon \cap S_\varepsilon(0, t) = \emptyset]\right] = \mathrm{E}_0\left[\exp -s_d(\varepsilon) \int_{S_\varepsilon(0,t)} c(x)\mathrm{d}x\right], \tag{1}$$

where

$$s_d(\varepsilon) = \begin{cases} \log(\varepsilon^{-1}) & \text{if } d = 2, \\ \varepsilon^{2-d} & \text{if } d \geq 3. \end{cases}$$

In the case $c = \nu$, the integral in (1) is just ν times the volume $\lambda\left(S_\varepsilon(0, t)\right)$ of the Wiener sausage, whose asymptotics have been well studied owing to their connections with physical problems (see e.g. the introduction of [1,2] or [3]). Note that the large-t asymptotics of $\lambda\left(S_\varepsilon(0, t)\right)$ are essentially equivalent to its small-ε asymptotics thanks to the equality in law:

$$\lambda\left(S_1(0, t)\right) \stackrel{(d)}{=} t^{d/2} \lambda\left(S_{t^{-1/2}}(0, 1)\right).$$

A classical result of Kesten, Spitzer and Whitman (cf. [4], p.253) states that, if $d \geq 3$,

$$\lim_{\varepsilon \to 0} s_d(\varepsilon)\lambda\left(S_\varepsilon(0,t)\right) = k_d t \quad \text{a.s.}, \tag{2}$$

where $k_d = (d-2)\pi^{d/2}/\Gamma(d/2)$ ($k_3 = 2\pi$) is the Newtonian capacity of the unit ball. The Kesten–Spitzer–Whitman convergence result was in fact stated for the large-time asymptotics of $\lambda\left(S_\varepsilon(0,t)\right)$, but a scaling argument gives the previous statement, at least in the sense of convergence in probability. The convergence in (2) also holds if $d = 2$ (see [5]), with $k_2 = \pi$.

It is not hard to deduce from the preceding result that, at least when the function c is continuous,

$$\lim_{\varepsilon \to 0} s_d(\varepsilon) \int_{S_\varepsilon(0,t)} c(y)\mathrm{d}y = k_d \int_0^t c(\xi_s)\mathrm{d}s \quad a.s. \tag{3}$$

It then follows from (1) that

$$\lim_{\varepsilon \to 0} \mathbf{E}\left[\mathrm{P}_0[T_\varepsilon > t]\right] = \mathrm{E}_0\left[\exp -k_d \int_0^t c(\xi_s)\mathrm{d}s\right].$$

This argument, which is due to Kac [6], can be interpreted in the following way. When ε tends to zero, the obstacles become dense in $\mathbb{R}^d$ (at least if the function c is everywhere positive), and the Brownian motion ξ_t gets absorbed in the obstacles at rate $k_d c(\xi_t)$.

Going back to our initial problem about killed superprocesses, the result for a single Brownian particle suggests that the sequence X^ε should converge to the superprocess X^* with branching mechanism $\psi(u,x) = u^2 + k_d c(x)u$ and underlying spatial motion given by Brownian motion. We shall establish in this work that the distribution of X^* is, indeed, the limit of the distribution of X^ε as ε tends to 0, in **P**-probability. Here, the distribution of X^ε is a probability measure on the Skorokhod space $D_{\mathcal{M}_f(\mathbb{R}^d)}([0,\infty))$ of all càdlàg paths with values in $\mathcal{M}_f(\mathbb{R}^d)$ (the space of all finite measures on $\mathbb{R}^d$) and the preceding limit is in the sense of weak convergence. A stronger statement can be made, but only for subsequences: if the sequence ε_n decreases to 0 fast enough,

$$X^{\varepsilon_n} \stackrel{(d)}{\to} X^* \quad \text{as } n \to \infty, \ \mathbf{P}\text{-a.s.}$$

Here, $\stackrel{(d)}{\to}$ denotes convergence in distribution. Let us emphasize the meaning of this result: except for a set of zero **P**-measure, if we fix an environment, then the sequence of superprocesses X^{ε_n} evolving among these fixed obstacles converges in law to X^*. Theorem 1 and Corollary 1 are stated in a more general setting, allowing the superprocesses to reside only within a domain D of $\mathbb{R}^d$.

The question we address in this paper was motivated by analogous works on Brownian motion. An extensive literature is already available on this topic, reviewed for example in [7]. Owing to the well-known properties of Poisson point processes, they seem to be a natural way to encode traps and have been frequently exploited in investigations of the behaviour of Brownian motion moving among "hard" obstacles, where the particle is killed instantaneously when hitting an obstacle as described above, or among "soft" obstacles, within which the Brownian particle is killed at a certain rate. Our approach is close to ideas developed by Kac in [6], whose probabilistic method differs from the analytic method used by Papanicolaou and Varadhan [8] in a similar context. Both derive the convergence in the $L^2(\mathbf{P})$-norm of the semigroup of Brownian motion among random obstacles when the number of obstacles tends to infinity but their diameters tend to 0 (recall that **P** denotes the probability measure on the space where the obstacles are defined).

Subsequently, Brownian motion among traps was studied in different settings, in particular by Sznitman, who devised the powerful method of enlargement of obstacles (see [7]).

The problem of super-Brownian motion or branching Brownian motion among random obstacles was addressed recently by Engländer in [9–11], the latter paper dealing with soft obstacles. However, Engländer considers the supercritical case (instead of critical super-Brownian motion as we do) and keeps the sizes of obstacles fixed. Within the obstacles, a particle does not die but branches at a slower rate. His interest is in the long-time asymptotics of the process and, in particular, the survival probability and the growth rate of the support. His techniques are mostly analytic, in contrast with the probabilistic tools of the present work.

1.2. Statement of the main result

Let us first introduce some notation and construct the sequence of superprocesses X^ε from the historical superprocess corresponding to a super-Brownian motion on $\mathbb{R}^d$, independent of the obstacles. We refer to [12] for more details on historical superprocesses and their applications. If E is a topological space, $\mathcal{M}_f(E)$ stands for the space of all finite Borel measures on E.

The (Brownian) historical superprocess can be defined as follows. Let $\mathcal{W}$ be the set of all finite continuous paths in $\mathbb{R}^d$, and note that $\mathbb{R}^d$ can be viewed as a subset of $\mathcal{W}$ by identifying x with the path of length zero and initial point x. Then, let $\tilde{\xi}$ be the continuous Markov process in $\mathcal{W}$ whose transition kernel is described as follows: If $\tilde{\xi}_0 = (w(r),\ 0 \leq r \leq s) \in \mathcal{W}$, the law of $\tilde{\xi}_t$ is the law under $\mathrm{P}_{w(s)}$ of the concatenation of the paths $(w(r),\ 0 \leq r \leq s)$ and $(\xi_r,\ 0 \leq r \leq t)$. The historical superprocess H is defined as the superprocess on $\mathcal{W}$ with branching mechanism $\psi(u) = u^2$ and underlying spatial motion given by $\tilde{\xi}$. Thus, H takes values in $\mathcal{M}_f(\mathcal{W})$. The super-Brownian motion X^0 on $\mathbb{R}^d$, starting at $\mu \in \mathcal{M}_f(\mathbb{R}^d)$, can then be recovered from the historical superprocess starting at μ (which is viewed as a finite measure on the paths of length zero) through the formula

$$\langle X_t^0, f\rangle = \int_{\mathcal{W}} H_t(\mathrm{d}w) f(w(t))$$

for all f bounded and measurable and all $t \geq 0$. Here, $\langle \nu, f\rangle$ denotes the integral of f against the measure ν.

We exploit this correspondence between the historical superprocess and super-Brownian motion further to construct the sequence of killed superprocesses which is of interest in this work. Let E be an open subset of $\mathbb{R}^d$, and recall the definition of the obstacle configuration Γ_ε. For every $\varepsilon > 0$, the superprocess $\{X_t^{\varepsilon,E}, t \in [0,\infty)\}$ is defined from the historical superprocess H via the formula

$$\langle X_t^{\varepsilon,E}, f\rangle = \int_{\mathcal{W}} H_t(\mathrm{d}w) f(w(t))\, \mathbb{I}_{\{\forall s\in[0,t],\ w(s)\in E\cap\Gamma_\varepsilon^c\}},$$

for all f bounded and measurable, and all $t \geq 0$. It is straightforward to verify that $X^{\varepsilon,E}$ is itself a super-Brownian motion with critical branching mechanism $\psi(u) = u^2$ and underlying spatial motion given by Brownian motion killed when entering $E^c \cup \Gamma_\varepsilon$. Furthermore, $X_0^{\varepsilon,E}$ is the restriction of μ to $E \cap \Gamma_\varepsilon^c$.

Recall that we defined $k_2 = \pi$ and $k_d = \frac{d-2}{\Gamma(d/2)}\pi^{d/2}$ for $d \geq 3$. We also introduce another superprocess $X^{*,E}$, with branching mechanism $\psi(u,x) = u^2 + k_d c(x) u$ and underlying spatial motion given by Brownian motion killed when it exits E.

In practice, E will be either D or a bounded open subset of D. When there is no ambiguity, we shall suppress the dependence on E in the notation. We choose a sequence ε_n such that $\sum_n |\log\varepsilon_n|^{-1} < \infty$ if $d = 2$, and $\sum_n \varepsilon_n|\log\varepsilon_n| < \infty$ if $d \geq 3$. For instance, we may fix $\alpha > 1$ and set $\varepsilon_n = \exp(-n^\alpha)$ if $d = 2$ and $\varepsilon_n = n^{-\alpha}$ if $d \geq 3$.

We will use the following notation.

- $\mathbb{P}_\mu$ is the (quenched) probability measure under which H starts at $\mu \in \mathcal{M}_f(\mathbb{R}^d) \subset \mathcal{M}_f(\mathcal{W})$. By the preceding correspondence, each superprocess $X^{\varepsilon,E}$ then starts under $\mathbb{P}_\mu$ from the restriction of μ to $E \cap \Gamma_\varepsilon^c$. It will be convenient to assume that $X^{*,E}$ is also defined under $\mathbb{P}_\mu$ and starts from the restriction of μ to E.
- To simplify notation, $X^{(n),E}$ will be a shorthand for the killed superprocess with parameter ε_n, and $\mathbb{P}_\mu^{(n),E}$ will be its law under $\mathbb{P}_\mu$. Likewise, $\mathbb{P}_\mu^{\varepsilon,E}$ (resp. $\mathbb{P}_\mu^{*,E}$) will be the law of $X^{\varepsilon,E}$ (resp. $X^{*,E}$) under $\mathbb{P}_\mu$.
- For all $t \geq 0$ and $x \in \mathbb{R}^d$, $\mathrm{P}_{t,x}$ will be a probability measure under which a Brownian motion ξ on $\mathbb{R}^d$, independent of the obstacles, starts from x at time t.
- $T^E := \inf\{t \geq 0 : \xi_t \in E^c\}$, $T_\varepsilon := \inf\{t \geq 0 : \xi_t \in \Gamma_\varepsilon\}$ and $T_{(n)} = T_{\varepsilon_n}$.

We can now state our main result.

Theorem 1. *For every $\mu \in \mathcal{M}_f(D)$, $\mathbf{P}$-a.s.*

$$\mathbb{P}_\mu^{(n),D} \Rightarrow \mathbb{P}_\mu^{*,D} \quad \text{as } n \to \infty,$$

where the symbol $\Rightarrow$ refers to the weak convergence of probability measures.

As an immediate corollary, we also have:

Corollary 1. *For every $\mu \in \mathcal{M}_f(D)$, the sequence $\mathbb{P}_\mu^{\varepsilon,D}$ converges in $\mathbf{P}$-probability to $\mathbb{P}_\mu^{*,D}$ as ε tends to zero. In other words, for every $\delta > 0$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$,*

$$\mathbf{P}\left[d\left(\mathbb{P}_\mu^{\varepsilon,D}, \mathbb{P}_\mu^{*,D}\right) > \delta\right] < \delta,$$

where d is the Prohorov metric on $\mathcal{M}_1(D_{\mathcal{M}_f(D)}[0,\infty))$ (here, $\mathcal{M}_1(D_{\mathcal{M}_f(D)}[0,\infty))$ is the space of all probability measures on $D_{\mathcal{M}_f(D)}[0,\infty)$).

The rest of the paper is devoted to the proofs of Theorem 1 and Corollary 1. In Section 2, we prove certain estimates for the rate of convergence in (3), which are of independent interest. These estimates are a key ingredient of the proof of Lemma 2 in Section 3. Then, we fix a bounded open subset B of D and prove the almost sure convergence of the distribution of $X^{(n),B}$ in two steps. First, we show in Section 3 that to each k-tuple $(t_1, \ldots, t_k)$, there corresponds a set of $\mathbf{P}$-measure zero outside which $(X_{t_1}^{(n),B}, \ldots, X_{t_k}^{(n),B})_{n\geq 1}$ converges in law to $(X_{t_1}^{*,B}, \ldots, X_{t_k}^{*,B})$. Second, we prove in Section 4 that, with $\mathbf{P}$-probability 1, the sequence of superprocesses $X^{(n),B}$ is tight in $D_{\mathcal{M}_f(D)}[0,\infty)$. In Section 5, we complete the proof for a general domain D. Starting with a bounded subset of D is required for technical reasons, to ensure the finiteness of certain integrals which appear in the proof.

2. Some estimates for the Wiener sausage

Let us define the set $\mathcal{B}_1$ as the set of all bounded Borel measurable functions c on $\mathbb{R}^d$ such that $\|c\| \leq 1$, where $\|c\|$ denotes the supremum norm of c. We have the following result (we write E_x for $E_{0,x}$ in the rest of the section).

Proposition 1. *For every $t \geq 0$, there exists a constant $C = C(t)$ such that for every $\varepsilon \in (0, \frac{1}{2}]$, if $d = 2$,*

$$\sup_{c\in\mathcal{B}_1}\sup_{x\in\mathbb{R}^2} \mathrm{E}_x\left[\left(|\log\varepsilon|\int_{S_\varepsilon(0,t)} c(y)\mathrm{d}y - \pi\int_0^t c(\xi_s)\mathrm{d}s\right)^2\right] \leq \frac{C}{|\log\varepsilon|^2},$$

and if $d \geq 3$,

$$\sup_{c\in\mathcal{B}_1}\sup_{x\in\mathbb{R}^d} \mathrm{E}_x\left[\left(\varepsilon^{2-d}\int_{S_\varepsilon(0,t)} c(y)\mathrm{d}y - k_d\int_0^t c(\xi_s)\mathrm{d}s\right)^2\right] \leq C\varepsilon^2|\log\varepsilon|^2.$$

Remark 1. In the case $c = 1$, the bounds of Proposition 1 follow from the known results for the fluctuations of the volume of the Wiener sausage [13]. However, it does not seem easy to derive Proposition 1 from the special case $c = 1$. Note that the latter case suggests that the bound $C\varepsilon^2|\log\varepsilon|^2$ could be replaced by $C\varepsilon^2|\log\varepsilon|$ if $d = 3$ and by $C\varepsilon^2$ if $d \geq 4$. These refinements will not be needed in our applications.

Proof of Proposition 1 (*For $d \geq 3$*). To simplify notation, we prove the desired bound only for $t = 1$. A scaling argument then gives the result for any $t \geq 0$. Let us set

$$h(\varepsilon) = \sup_{c\in\mathcal{B}_1}\sup_{x\in\mathbb{R}^d} \mathrm{E}_x\left[\left(\varepsilon^{2-d}\int_{S_\varepsilon(0,1)} c(y)\mathrm{d}y - k_d\int_0^1 c(\xi_s)\mathrm{d}s\right)^2\right].$$

As a first step, let us notice that

$$\int_{S_\varepsilon(0,1)} c(y)\mathrm{d}y = \int_{S_\varepsilon(0,1/2)} c(y)\mathrm{d}y + \int_{S_\varepsilon(1/2,1)} c(y)\mathrm{d}y - \int_{S_\varepsilon(0,1/2)\cap S_\varepsilon(1/2,1)} c(y)\mathrm{d}y.$$

Also,

$$\begin{aligned}&\varepsilon^{2-d}\int_{S_\varepsilon(0,1/2)} c(y)\mathrm{d}y - k_d\int_0^{1/2} c(\xi_s)\mathrm{d}s\\ &\quad = \varepsilon^{2-d}2^{-d/2}\int_{\tilde{S}_{\varepsilon\sqrt{2}}(0,1)} c\left(\frac{z}{\sqrt{2}}\right)\mathrm{d}z - \frac{k_d}{2}\int_0^1 c\left(\frac{\tilde{\xi}_s}{\sqrt{2}}\right)\mathrm{d}s,\end{aligned}$$

where $\tilde{\xi}_s = \sqrt{2}\,\xi_{s/2}$ for all $s \geq 0$ and $\tilde{S}_\varepsilon(a, b)$ is the Wiener sausage associated to $\tilde{\xi}$. Since the function $\tilde{c}(z) = c\left(\frac{z}{\sqrt{2}}\right)$ also belongs to $\mathcal{B}_1$, we obtain that

$$\mathrm{E}_x\left[\left(\varepsilon^{2-d}\int_{S_\varepsilon(0,1/2)} c(y)\mathrm{d}y - k_d\int_0^{1/2} c(\xi_s)\mathrm{d}s\right)^2\right] \leq \frac{1}{4}h(\varepsilon\sqrt{2}).$$

Likewise, using the Markov property at time $\frac{1}{2}$ and the preceding argument, we have

$$\mathrm{E}_x\left[\left(\varepsilon^{2-d}\int_{S_\varepsilon(1/2,1)} c(y)\mathrm{d}y - k_d\int_{1/2}^1 c(\xi_s)\mathrm{d}s\right)^2\right] \leq \frac{1}{4}h(\varepsilon\sqrt{2}).$$

On the other hand, we have $\lambda\left(S_\varepsilon(0, 1/2) \cap S_\varepsilon(1/2, 1)\right) = \lambda\left(S'_\varepsilon(0, 1/2) \cap S''_\varepsilon(0, 1/2)\right)$, where $\xi'_t = \xi_{1/2-t} - \xi_{1/2}$ and $\xi''_t = \xi_{1/2+t} - \xi_{1/2}$ for every $t \in [0, 1/2]$, and $S'_\varepsilon(0, 1/2)$, resp.

$S''_\varepsilon(0,1/2)$, denotes the Wiener sausage with radius ε associated to ξ', resp. ξ'', along the time interval $[0,1/2]$. Since ξ' and ξ'' are independent Brownian motions, we can use the following consequence of Corollary 3-2 in [5], and of [13], p.1012: There exists a constant $K_1(d)>0$ such that for every $\varepsilon\in(0,1/2]$

$$\mathrm{E}\left[\lambda\left(S_\varepsilon(0,1/2)\cap S_\varepsilon(1/2,1)\right)^2\right]\le\begin{cases}K_1\varepsilon^4, & d=3\\ K_1\varepsilon^8|\log\varepsilon|^2, & d=4\\ K_1\varepsilon^{2d}, & d\ge 5.\end{cases}$$

Coming back to the definition of $h(\varepsilon)$, and using the triangle inequality in L^2, the fact that

$$\mathrm{E}_x\left[\left(\int_{S_\varepsilon(0,1/2)\cap S_\varepsilon(1/2,1)}c(y)\mathrm{d}y\right)^2\right]\le\mathrm{E}\left[\lambda\left(S_\varepsilon(0,1/2)\cap S_\varepsilon(1/2,1)\right)^2\right]$$

and the preceding inequalities, we obtain

$$\begin{aligned}h(\varepsilon)\le\sup_{c\in\mathcal{B}_1}\sup_{x\in\mathbb{R}^d}\Bigg\{&\mathrm{E}_x\left[\left(\varepsilon^{2-d}\int_{S_\varepsilon(0,1/2)}c(y)\mathrm{d}y\right.\right.\\&\left.\left.+\varepsilon^{2-d}\int_{S_\varepsilon(1/2,1)}c(y)\mathrm{d}y-k_d\int_0^1c(\xi_s)\mathrm{d}s\right)^2\right]^{1/2}\\&+\mathrm{E}_x\left[\varepsilon^{4-2d}\left(\int_{S_\varepsilon(0,1/2)\cap S_\varepsilon(1/2,1)}c(y)\mathrm{d}y\right)^2\right]^{1/2}\Bigg\}^2\\ \le\Bigg\{&\left(\frac{1}{2}h(\varepsilon\sqrt{2})+2u(\varepsilon)\right)^{1/2}+K_1'\psi_d(\varepsilon)\Bigg\}^2,\end{aligned}\tag{4}$$

where $\psi_d(\varepsilon)=\varepsilon$ (resp. $\varepsilon^2|\log\varepsilon|$, resp. ε^2) if $d=3$ (resp. $d=4$, resp. $d\ge 5$) and

$$\begin{aligned}u(\varepsilon)=\sup_{c\in\mathcal{B}_1}\sup_{x\in\mathbb{R}^d}\Bigg|\mathrm{E}_x\Bigg[&\left(\varepsilon^{2-d}\int_{S_\varepsilon(0,1/2)}c(y)\mathrm{d}y-k_d\int_0^{1/2}c(\xi_s)\mathrm{d}s\right)\\&\times\left(\varepsilon^{2-d}\int_{S_\varepsilon(1/2,1)}c(y)\mathrm{d}y-k_d\int_{1/2}^1c(\xi_s)\mathrm{d}s\right)\Bigg]\Bigg|.\end{aligned}$$

Applying the Markov property at time $\frac{1}{2}$, we have

$$u(\varepsilon)=\sup_{c\in\mathcal{B}_1}\sup_{x\in\mathbb{R}^d}\left|\mathrm{E}_x\left[\left(\varepsilon^{2-d}\int_{S_\varepsilon(0,1/2)}c(y)\mathrm{d}y-k_d\int_0^{1/2}c(\xi_s)\mathrm{d}s\right)v(\varepsilon,\xi_{1/2})\right]\right|,$$

where

$$v(\varepsilon,z)=\mathrm{E}_z\left[\varepsilon^{2-d}\int_{S_\varepsilon(0,1/2)}c(y)\mathrm{d}y-k_d\int_0^{1/2}c(\xi_s)\mathrm{d}s\right].$$

We now use the following lemma.

Lemma 1. *There exists a constant* $K_2 > 0$ *such that for all* $z \in \mathbb{R}^d$, $\varepsilon \in (0, \frac{1}{2}]$ *and* $c \in \mathcal{B}_1$

$$|v(\varepsilon, z)| \le K_2\, \varepsilon.$$

We postpone the proof of Lemma 1 and complete the case $d \ge 3$ of the Proposition. By Lemma 1, we have

$$\begin{aligned}|u(\varepsilon)| &\le K_2\, \varepsilon \sup_{c\in\mathcal{B}_1} \sup_{x\in\mathbb{R}^d} \mathrm{E}_x\left[\left(\varepsilon^{2-d}\int_{S_\varepsilon(0,1/2)} c(y)\mathrm{d}y - k_d\int_0^{1/2} c(\xi_s)\mathrm{d}s\right)^2\right]^{1/2}\\ &\le \frac{K_2}{2}\,\varepsilon h(\varepsilon\sqrt{2})^{1/2}.\end{aligned}$$

From (4), we obtain for every $\varepsilon \in (0, \frac{1}{2}]$

$$h(\varepsilon) \le \left(\left(\frac{1}{2}h(\varepsilon\sqrt{2}) + K_2\varepsilon h(\varepsilon\sqrt{2})^{1/2}\right)^{1/2} + K_1'\psi_d(\varepsilon)\right)^2.$$

Let us set $g(\varepsilon) = \varepsilon^{-1}h(\varepsilon)^{1/2}$. We thus have for $\varepsilon \in (0, \frac{1}{2}]$:

$$g(\varepsilon) \le \left(g(\varepsilon\sqrt{2})^2 + \sqrt{2}K_2 g(\varepsilon\sqrt{2})\right)^{1/2} + K_1'\varepsilon^{-1}\psi_d(\varepsilon). \tag{5}$$

Fix $r \in (1/4, 1/2]$ and set $u_n = g(r2^{-n/2})$ for every integer $n \ge 0$. Rewriting (5) in terms of u_n and noting that $\varepsilon^{-1}\psi_d(\varepsilon) = 1$ if $d = 3$ and $\varepsilon^{-1}\psi_d(\varepsilon) = o(1)$ as $\varepsilon \to 0$ if $d \ge 4$, we obtain for a constant $K_1'' > 0$ (independent of n)

$$u_{n+1} \le (u_n^2 + \sqrt{2}K_2u_n)^{1/2} + K_1'' = u_n\left(1 + \frac{\sqrt{2}K_2}{u_n}\right)^{1/2} + K_1'' \le u_n + \frac{\sqrt{2}K_2}{2} + K_1''.$$

It follows that $u_n \le u_0 + n\left(K_2 2^{-1/2} + K_1''\right)$ for every $n \ge 0$, from which we can conclude that there exists a constant K_3 such that for all $\varepsilon \in (0, 1/2]$,

$$g(\varepsilon) \le K_3|\log\varepsilon|$$

and thus

$$h(\varepsilon) \le K_3^2\varepsilon^2|\log\varepsilon|^2. \quad \square$$

Proof of Lemma 1. We may assume that $z = 0$, and we fix $c \in \mathcal{B}_1$ (the constant K_2 will not depend on c). First, we have

$$\mathrm{E}_0\left[k_d\int_0^{1/2} c(\xi_s)\mathrm{d}s\right] = k_d\int_{\mathbb{R}^d}\mathrm{d}y c(y)\int_0^{1/2}\frac{\mathrm{d}s}{(2\pi s)^{d/2}}\exp\left(-\frac{|y|^2}{2s}\right). \tag{6}$$

Let us define the random times $\tau_\varepsilon(y)$ and $L_\varepsilon(y)$ for all $\varepsilon > 0$ and $y \in \mathbb{R}^d$ by

$$\begin{aligned}\tau_\varepsilon(y) &= \inf\{t \ge 0 : |\xi_t - y| \le \varepsilon\},\\ L_\varepsilon(y) &= \sup\{t \ge 0 : |\xi_t - y| \le \varepsilon\},\end{aligned}$$

with the conventions that $\inf\emptyset = +\infty$ and $\sup\emptyset = 0$. We thus have

$$\mathrm{E}_0\left[\int_{S_\varepsilon(0,\frac{1}{2})} c(y)\mathrm{d}y\right] = \int_{\mathbb{R}^d} \mathrm{d}y c(y) \mathrm{P}_0\left[\tau_\varepsilon(y) \leq \frac{1}{2}\right] = \int_{\mathbb{R}^d} \mathrm{d}y c(y) \mathrm{P}_0\left[0 < L_\varepsilon(y) \leq \frac{1}{2}\right] + \int_{\mathbb{R}^d} \mathrm{d}y c(y) \mathrm{P}_0\left[\tau_\varepsilon(y) \leq \frac{1}{2} < L_\varepsilon(y)\right]. \tag{7}$$

On the one hand,

$$\left|\int \mathrm{d}y c(y) \mathrm{P}_0\left[\tau_\varepsilon(y) \leq \frac{1}{2} < L_\varepsilon(y)\right]\right| \leq \int \mathrm{d}y \mathrm{P}_0\left[\tau_\varepsilon(y) \leq \frac{1}{2} \leq L_\varepsilon(y)\right].$$

We have

$$\int \mathrm{d}y \mathrm{P}_0\left[\tau_\varepsilon(y) \leq \frac{1}{2} \leq L_\varepsilon(y)\right] = \mathrm{E}_0\left[\lambda\left(S_\varepsilon(0,1/2) \cap S_\varepsilon(1/2,\infty)\right)\right] = \mathrm{E}_0\left[\lambda\left(S_\varepsilon(0,1/2) \cap S'_\varepsilon(0,\infty)\right)\right],$$

where S'_ε denotes the Wiener sausage associated to a Brownian motion ξ' independent of ξ and also started from 0 under P_0. If $d = 3$, it is easily checked that

$$\mathrm{E}_0\left[\lambda\left(S_\varepsilon(0,1/2) \cap S'_\varepsilon(0,\infty)\right)\right] = O(\varepsilon^2) \tag{8}$$

(use the fact that $\mathrm{P}_0\left[y \in S'_\varepsilon(0,\infty)\right] = \frac{\varepsilon}{|y|} \wedge 1$, together with the bound (3.d) in [13]). If $d \geq 4$,

$$\mathrm{E}_0\left[\lambda\left(S_\varepsilon(0,1/2) \cap S'_\varepsilon(0,\infty)\right)\right] = \varepsilon^d \mathrm{E}_0\left[\lambda\left(S_1\left(0,\varepsilon^{-2}/2\right) \cap S'_1(0,\infty)\right)\right] = \begin{cases} O(\varepsilon^4|\log\varepsilon|), & \text{if } d = 4,\\ O(\varepsilon^d), & \text{if } d \geq 5 \end{cases} \tag{9}$$

by [13], p. 1010.

Let $\nu_{\varepsilon,y}(\mathrm{d}z)$ denote the equilibrium measure of the ball $\overline{B}(y,\varepsilon)$, that is the unique finite measure on the sphere $\partial B(y,\varepsilon)$ such that for every x with $|x-y| > \varepsilon$,

$$\mathrm{P}_x\left[\tau_\varepsilon(y) < \infty\right] = \int \nu_{\varepsilon,y}(\mathrm{d}z) G(z-x),$$

where $G(z) = \int_0^\infty (2\pi s)^{-d/2} \exp(-|z|^2/2s)\mathrm{d}s = c_d|y|^{2-d}$ is the Green function of d-dimensional Brownian motion (c_d is a constant depending only on d). By a classical formula of probabilistic potential theory (see [14], p. 61–62) we have

$$\mathrm{P}_0\left[0 < L_\varepsilon(y) \leq \frac{1}{2}\right] = \int_0^{1/2} \mathrm{d}s \int \nu_{\varepsilon,y}(\mathrm{d}z) \frac{1}{(2\pi s)^{d/2}} \exp\left(-\frac{|z|^2}{2s}\right).$$

It is well known that $\nu_{\varepsilon,y} = k_d \varepsilon^{d-2} \pi_{\varepsilon,y}$, where $\pi_{\varepsilon,y}$ denotes the uniform distribution on the sphere of radius ε centered at y. Recalling (6)–(9), we can write

$$|v(\varepsilon,z)| = \left|k_d \int_{\mathbb{R}^d} \mathrm{d}y c(y) \int_0^{1/2} \frac{\mathrm{d}s}{(2\pi s)^{d/2}} \left\{\int \pi_{\varepsilon,y}(\mathrm{d}z)\left(\mathrm{e}^{-|z|^2/2s} - \mathrm{e}^{-|y|^2/2s}\right)\right\}\right| + O(\phi_d(\varepsilon))$$

$$\leq k_d \int_{\mathbb{R}^d} \mathrm{d}y \int_0^{1/2} \frac{\mathrm{d}s}{(2\pi s)^{d/2}} \left\{ \int \pi_{\varepsilon,y}(\mathrm{d}z) \left| \mathrm{e}^{-|z|^2/2s} - \mathrm{e}^{-|y|^2/2s} \right| \right\} + O(\phi_d(\varepsilon))$$

where $\phi_d(\varepsilon) = \varepsilon$ (resp. $\varepsilon^2 |\log \varepsilon|$, resp. ε^2) if $d = 3$ (resp. $d = 4$, resp. $d \geq 5$). It follows that

$$\int_{|y|\leq 10\varepsilon} \mathrm{d}y \int_0^{1/2} \frac{\mathrm{d}s}{(2\pi s)^{d/2}} \int \pi_{\varepsilon,y}(\mathrm{d}z) \left| \mathrm{e}^{-|z|^2/2s} - \mathrm{e}^{-|y|^2/2s} \right|$$
$$\leq 2 \int_{|z|\leq 11\varepsilon} \mathrm{d}z \int_0^{1/2} \frac{\mathrm{d}s}{(2\pi s)^{d/2}} \mathrm{e}^{-|z|^2/2s} \leq 2 \int_{|z|\leq 11\varepsilon} \mathrm{d}z\, G(z) = O(\varepsilon^2).$$

On the other hand, we can find constants C and C' such that if $|y| > 10\varepsilon$ and $|z - y| = \varepsilon$,

$$\left| \exp\left(-\frac{|z|^2}{2s}\right) - \exp\left(-\frac{|y|^2}{2s}\right) \right| \leq C \left| \frac{|z|^2 - |y|^2}{s} \right| \exp\left(-\frac{|y|^2}{4s}\right)$$
$$\leq C' \varepsilon \frac{|y|}{s} \exp\left(-\frac{|y|^2}{4s}\right).$$

Thus, with a constant K which may vary from line to line, we have

$$\int_{|y|>10\varepsilon} \mathrm{d}y \int_0^{1/2} \frac{\mathrm{d}s}{(2\pi s)^{d/2}} \int \pi_{\varepsilon,y}(\mathrm{d}z) \left| \exp\left(-\frac{|z|^2}{2s}\right) - \exp\left(-\frac{|y|^2}{2s}\right) \right|$$
$$\leq K\varepsilon \int_{|y|>10\varepsilon} \mathrm{d}y |y| \int_0^{1/2} \mathrm{d}s\, s^{-d/2-1} \exp\left(-\frac{|y|^2}{4s}\right)$$
$$= K\varepsilon \int_{|y|>10\varepsilon} \mathrm{d}y |y|^{1-d} \int_0^{1/(2|y|^2)} \mathrm{d}s' s'^{-d/2-1} \mathrm{e}^{-1/4s'}$$
$$\leq K\varepsilon.$$

Combining the above, the proof of Lemma 1 is complete. □

Proof of Proposition 1 (*For* $d = 2$). Let us define

$$h(\varepsilon) = \sup_{c\in\mathcal{B}_1} \sup_{x\in\mathbb{R}^d} \mathrm{E}\left[\left(|\log \varepsilon| \int_{S_\varepsilon(0,1)} c(y)\mathrm{d}y - \pi \int_0^1 c(\xi_s)\mathrm{d}s \right)^2 \right].$$

By Corollary 3-2 in [5], we have

$$\mathrm{E}\left[\lambda \left(S_\varepsilon\left(0, \frac{1}{2}\right) \cap S_\varepsilon\left(\frac{1}{2}, 1\right) \right)^2 \right] \leq \frac{K_1}{|\log \varepsilon|^4}.$$

The same technique as in the previous case yields

$$h(\varepsilon) \leq \left(\left(\frac{1}{2} h(\varepsilon\sqrt{2}) + h(\varepsilon\sqrt{2})^{1/2} \sup_{c\in\mathcal{B}_1} \sup_{x\in\mathbb{R}^2} |v(\varepsilon, z)| \right)^{1/2} + \frac{\sqrt{K_1}}{|\log \varepsilon|} \right)^2, \qquad (10)$$

where

$$v(\varepsilon, z) = \mathrm{E}_z\left[|\log \varepsilon| \int_{S_\varepsilon(0,1/2)} c(y)\mathrm{d}y - \pi \int_0^{1/2} c(\xi_s)\mathrm{d}s \right].$$

We now use the following result, whose rather technical proof is deferred to the Appendix: There exists a constant K_2 such that, for $\varepsilon \in (0, 1/2]$,

$$\sup_{c \in \mathcal{B}_1} \sup_{z \in \mathbb{R}^d} |v(\varepsilon, z)| \leq \frac{K_2}{|\log \varepsilon|}. \tag{11}$$

Hence, if $g(\varepsilon) = |\log \varepsilon| h(\varepsilon)^{1/2}$, we have for $\varepsilon \in (0, 1/2]$,

$$g(\varepsilon) \leq \left(\frac{1}{2} \frac{(\log \varepsilon)^2}{(\log \varepsilon \sqrt{2})^2}\, g(\varepsilon\sqrt{2})^2 + K_2 \frac{|\log \varepsilon|}{|\log \varepsilon\sqrt{2}|} g(\varepsilon\sqrt{2}) \right)^{1/2} + \sqrt{K_1}. \tag{12}$$

From (12), we can use arguments similar to the case $d \geq 3$ to infer that the function $g(\varepsilon)$ is bounded over $(0, 1/2]$. Thus, there exists a constant K_3 such that for all $\varepsilon \in (0, 1/2]$,

$$h(\varepsilon) \leq \frac{K_3}{|\log \varepsilon|^2}. \quad \square$$

3. Almost sure convergence of the finite-dimensional distributions of $X^{(n),B}$

In the following, we fix a bounded open subset B of D and consider only the superprocesses killed outside B. We therefore suppress the dependence on B in the notation. In particular, $T = T^B$.

The following proposition is the first step in the proof of Theorem 1.

Proposition 2 (*Convergence of the Finite-dimensional Distributions*). *Let $\mu \in \mathcal{M}_f(B)$, $p \in \mathbb{N}$ and $t_1 < \cdots < t_p \in [0, \infty)$. Then, under $\mathbb{P}_\mu$*

$$(X^{(n)}_{t_1}, \ldots, X^{(n)}_{t_p}) \overset{(d)}{\to} (X^*_{t_1}, \ldots, X^*_{t_p})$$

as $n \to \infty$, on a set of **P**-*probability* 1.

Proof of Proposition 2. We fix an environment. Let $p \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_p$ and $f_1, \ldots, f_p \in \mathcal{B}_{b+}(\mathbb{R}^d)$ be measurable, nonnegative and bounded functions. In the following, we shall denote $(t_1, \ldots, t_p)$ by $\mathbf{t}$ and $(f_1, \ldots, f_p)$ by $\mathbf{f}$.

Let $\mu \in \mathcal{M}_f(B)$. Following the notation in [15], we have:

$$\mathbb{E}_\mu \left[\exp - \sum_{i=1}^{p} \langle X^\varepsilon_{t_i}, f_i \rangle \right] = \exp - \langle \mu, w^\varepsilon_0 \rangle,$$
$$\mathbb{E}_\mu \left[\exp - \sum_{i=1}^{p} \langle X^*_{t_i}, f_i \rangle \right] = \exp - \langle \mu, w^*_0 \rangle,$$

where $w^\varepsilon = (w^\varepsilon_t(x);\ t \geq 0,\ x \in B)$ and $w^* = (w^*_t(x); t \geq 0,\ x \in B)$ are the unique nonnegative solutions to the following integral equations: for all $x \in B$ and $t \geq 0$,

$$w^\varepsilon_t(x) + \mathrm{E}_{t,x} \left[\int_t^\infty \mathrm{d}s\, w^\varepsilon_s(\xi_s)^2 \mathbb{I}_{\{s < T \wedge T_\varepsilon\}} \right] = \sum_{i=1}^{p} \mathrm{E}_{t,x} \left[f_i(\xi_{t_i}) \mathbb{I}_{\{t_i < T \wedge T_\varepsilon\}} \right], \tag{13}$$

$$w^*_t(x) + \mathrm{E}_{t,x} \left[\int_t^\infty \mathrm{d}s \left(w^*_s(\xi_s)^2 + k_d c(\xi_s) w^*_s(\xi_s) \right) \mathbb{I}_{\{s < T\}} \right] = \sum_{i=1}^{p} \mathrm{E}_{t,x} \left[f_i(\xi_{t_i}) \mathbb{I}_{\{t_i < T\}} \right], \tag{14}$$

where by convention $\mathrm{E}_{t,x}[f(\xi_s)] = 0$ if $s < t$. By the standard argument of the proof of the Feynman–Kac formula, the integral equation (14) for w^* is equivalent to

$$w_t^*(x) + \mathrm{E}_{t,x}\left[\int_t^\infty \mathrm{d}s\, w_s^*(\xi_s)^2 \mathrm{e}^{-k_d\int_t^s c(\xi_u)\mathrm{d}u}\, \mathbb{I}_{\{s<T\}}\right] = \sum_{i=1}^p \mathrm{E}_{t,x}\left[f_i(\xi_{t_i})\mathrm{e}^{-k_d\int_t^{t_i} c(\xi_u)\mathrm{d}u}\mathbb{I}_{\{t_i<T\}}\right]. \quad (15)$$

The equivalence of the two integral equations (14) and (15) corresponds to the well-known fact that super-Brownian motion with branching mechanism $\psi(u,x) = u^2 + k_d c(x)u$ can also be constructed as the superprocess with branching mechanism $\psi(u) = u^2$ and underlying spatial motion given by Brownian motion killed at rate $k_d c(x)$.

Remark 2. Since w^ε and w^* are nonnegative, (13) and (14) imply that w_t^* and w_t^ε are equal to zero whenever $t > t_p$ (recall that by convention, the right-hand side of (13) or (14) is zero when $t > t_p$). Likewise, $w_t^\varepsilon(x) = 0$ if $x \in \Gamma_\varepsilon \cap B$, for every $t \geq 0$.

By integrating over B the difference between (13) and (15), we obtain:

$$\begin{aligned}\int_B \mathrm{d}x|w_t^\varepsilon(x) - w_t^*(x)| \leq{}& \int_B \mathrm{d}x\left|\sum_{i=1}^p \mathrm{E}_{t,x}\left[f_i(\xi_{t_i})\mathbb{I}_{\{t_i<T\}}\left(\mathbb{I}_{\{t_i<T_\varepsilon\}} - \mathrm{e}^{-k_d\int_t^{t_i} c(\xi_u)\mathrm{d}u}\right)\right]\right| \\ &+ \int_B \mathrm{d}x\left|\mathrm{E}_{t,x}\left[\int_t^\infty \mathrm{d}s\mathbb{I}_{\{s<T\}}\left(\mathbb{I}_{\{s<T_\varepsilon\}} - \mathrm{e}^{-k_d\int_t^s c(\xi_u)\mathrm{d}u}\right) w_s^\varepsilon(\xi_s)^2\right]\right| \\ &+ \int_B \mathrm{d}x\left|\mathrm{E}_{t,x}\left[\int_t^\infty \mathrm{d}s\mathrm{e}^{-k_d\int_t^s c(\xi_u)\mathrm{d}u}\mathbb{I}_{\{s<T\}}(w_s^*(\xi_s)^2 - w_s^\varepsilon(\xi_s)^2)\right]\right|. \quad (16)\end{aligned}$$

Let us start with the third term in the right-hand side of (16). The functions w^ε and w^* are bounded by $C_{\mathbf{f}} := \sum_{i=1}^p \|f_i\|$, hence bounding $\mathbb{I}_{\{s<T\}}$ by $\mathbb{I}_B(\xi_s)$ and $\mathrm{e}^{-k_d\int_t^s c(\xi_u)\mathrm{d}u}$ by 1 yields

$$\begin{aligned}&\int_B \mathrm{d}x\left|\mathrm{E}_{t,x}\left[\int_t^\infty \mathrm{d}s\mathrm{e}^{-k_d\int_t^s c(\xi_u)\mathrm{d}u}\mathbb{I}_{\{s<T\}}(w_s^*(\xi_s)^2 - w_s^\varepsilon(\xi_s)^2)\right]\right| \\ &\leq 2C_{\mathbf{f}}\int_B \mathrm{d}x\mathrm{E}_{t,x}\left[\int_t^\infty \mathrm{d}s\mathbb{I}_B(\xi_s)|w_s^*(\xi_s) - w_s^\varepsilon(\xi_s)|\right] \\ &= 2C_{\mathbf{f}}\int_t^\infty \mathrm{d}s\int_{B\times B}\mathrm{d}x\mathrm{d}z\,|w_s^*(z) - w_s^\varepsilon(z)|p_{s-t}(x,z) \\ &\leq 2C_{\mathbf{f}}\int_t^\infty \mathrm{d}s\int_B \mathrm{d}z|w_s^*(z) - w_s^\varepsilon(z)|. \quad (17)\end{aligned}$$

In the preceding estimates, $p_r(\cdot,\cdot)$ denotes the transition density at time r of d-dimensional Brownian motion. The last inequality stems from the observation that $\int_B p_{s-t}(x,z)\mathrm{d}x = \int_B p_{s-t}(z,x)\mathrm{d}x \leq 1$.

We next show that the first two terms of (16) converge towards 0 **P**-a.s. The key ingredient is the following result.

Lemma 2. *Let $t_1 \in [0,\infty)$ and let $f \in \mathcal{B}_{b+}(\mathbb{R}^d)$ be a bounded nonnegative measurable function. Then, there exists a constant $K = K(c, t_1, d)$ such that, for every $t \in [0,\infty)$, $x \in B$*

and $\varepsilon \in (0, 1/2)$, *if* $d = 2$

$$\mathbf{E}\left[\mathrm{E}_{t,x}\left[f(\xi_{t_1})\mathbb{I}_{\{t_1<T\}}\left(\mathbb{I}_{\{t_1<T_\varepsilon\}} - \mathrm{e}^{-\pi\int_t^{t_1} c(\xi_u)\mathrm{d}u}\right)\right]^2\right] \le K\|f\|^2 \frac{1}{|\log\varepsilon|},$$

and if $d \ge 3$,

$$\mathbf{E}\left[\mathrm{E}_{t,x}\left[f(\xi_{t_1})\mathbb{I}_{\{t_1<T\}}\left(\mathbb{I}_{\{t_1<T_\varepsilon\}} - \mathrm{e}^{-k_d\int_t^{t_1} c(\xi_u)\mathrm{d}u}\right)\right]^2\right] \le K\ \|f\|^2\ \varepsilon|\log\varepsilon|.$$

The proof of Lemma 2 is postponed until the end of the section. Let us temporarily fix $t \in [0, t_p]$. Applying the lemma with $\varepsilon = \varepsilon_n$, we obtain for every $\delta > 0$ and every $i \in \{1, \dots, p\}$

$$\begin{aligned}
&\mathbf{P}\left[\int_B \left|\mathrm{E}_{t,x}\left[f_i(\xi_{t_i})\mathbb{I}_{\{t_i<T\}}(\mathbb{I}_{\{t_i<T_{(n)}\}} - \mathrm{e}^{-k_d\int_t^{t_i} c(\xi_u)\mathrm{d}u})\right]\right|\mathrm{d}x > \delta\right] \\
&\le \frac{1}{\delta^2}\,\mathbf{E}\left[\left(\int_B \left|\mathrm{E}_{t,x}\left[f_i(\xi_{t_i})\mathbb{I}_{\{t_i<T\}}(\mathbb{I}_{\{t_i<T_{(n)}\}} - \mathrm{e}^{-k_d\int_t^{t_i} c(\xi_u)\mathrm{d}u})\right]\right|\ \mathrm{d}x\right)^2\right] \\
&\le \frac{\lambda(B)}{\delta^2}\int_B \mathbf{E}\left[\mathrm{E}_{t,x}\left[f_i(\xi_{t_i})\mathbb{I}_{\{t_i<T\}}(\mathbb{I}_{\{t_i<T_{(n)}\}} - \mathrm{e}^{-k_d\int_t^{t_i} c(\xi_u)\mathrm{d}u})\right]^2\right]\mathrm{d}x \\
&\le \begin{cases} \lambda(B)^2 K\ \|f_i\|^2\delta^{-2}\ |\log\varepsilon_n|^{-1} & \text{if } d = 2, \\ \lambda(B)^2 K\ \|f_i\|^2\delta^{-2}\ \varepsilon_n|\log\varepsilon_n| & \text{if } d \ge 3, \end{cases}
\end{aligned}$$

which is summable by our assumptions on $(\varepsilon_n)_{n\ge1}$. Hence, by the Borel–Cantelli lemma,

$$\mathbf{P}\text{-a.s.},\ \int_B \left|\mathrm{E}_{t,x}\left[f_i(\xi_{t_i})\mathbb{I}_{\{t_i<T\}}(\mathbb{I}_{\{t_i<T_{(n)}\}} - \mathrm{e}^{-k_d\int_t^{t_i} c(\xi_u)\mathrm{d}u})\right]\right|\ \mathrm{d}x \to 0$$

as n tends to infinity. The first term of (16) is bounded above by a finite sum of such terms, therefore it converges to 0 **P**-a.s, for each fixed $t \in [0, t_p]$.

Let us set

$$A_{\mathbf{f},\mathbf{t}} := \left\{(\omega, t) \in \Omega \times [0, t_p] : \int_B \mathrm{d}x \left|\mathrm{E}_{t,x}\left[\sum_{i=1}^p f_i(\xi_{t_i})\mathbb{I}_{\{t_i<T\}}(\mathbb{I}_{\{t_i<T_{(n)}\}} - \mathrm{e}^{-k_d\int_t^{t_i} c(\xi_u)\mathrm{d}u})\right]\right| \to 0\right\}.$$

If λ_1 denotes the Lebesgue measure on $\mathbb{R}$, we have by Fubini's theorem $\mathbf{P}\otimes\lambda_1(A^c_{\mathbf{f},\mathbf{t}}) = \int_0^{t_p}\mathrm{d}t\mathbf{P}\left(\{\omega : (\omega, t) \in A^c_{\mathbf{f},\mathbf{t}}\}\right) = 0$, which gives (i) in the following lemma:

Lemma 3. (i) *There exists a measurable subset* $\tilde{\Omega}_{\mathbf{f},\mathbf{t}}$ *of* $\boldsymbol{\Omega}$, *with* $\mathbf{P}(\tilde{\Omega}_{\mathbf{f},\mathbf{t}}) = 0$, *such that for every* $\omega \in \boldsymbol{\Omega}\setminus\tilde{\Omega}_{\mathbf{f},\mathbf{t}}$,

$$\int_B \left|\mathrm{E}_{t,x}\left[\sum_{i=1}^p f_i(\xi_{t_i})\mathbb{I}_{\{t_i<T\}}\left(\mathbb{I}_{\{t_i<T_{(n)}\}} - \mathrm{e}^{-k_d\int_t^{t_i} c(\xi_u)\mathrm{d}u}\right)\right]\right|\mathrm{d}x \to 0 \quad \textit{as } n\to\infty$$

for all $t \ge 0$, *except for* t *belonging to a Lebesgue null subset* $\tilde{T}_{\mathbf{f},\mathbf{t},\omega}$ *of* $\mathbb{R}_+$.

(ii) *There exists also a measurable subset* $\hat{\Omega}_{\mathbf{f},\mathbf{t}}$ *of* $\boldsymbol{\Omega}$, *with* $\mathbf{P}(\hat{\Omega}_{\mathbf{f},\mathbf{t}}) = 0$, *such that for every* $\omega \in \boldsymbol{\Omega}\setminus\hat{\Omega}_{\mathbf{f},\mathbf{t}}$,

$$\int_0^\infty \mathrm{d}s \int_B \mathrm{d}x \left| \mathrm{E}_{0,x} \left[\mathbb{I}_{\{s<T\}} \left(\mathbb{I}_{\{s<T_{(n)}\}} - \mathrm{e}^{-k_d \int_0^s c(\xi_u)\mathrm{d}u} \right) w_{s+t}^{(n)}(\xi_s)^2 \right] \right| \to 0 \quad \textit{as } n \to \infty$$

for all $t \geq 0$, except on a Lebesgue null subset $\hat{T}_{\mathbf{f},\mathbf{t},\omega}$ of $\mathbb{R}_+$. Here, $w^{(n)} = w^{\varepsilon_n}$ is the function given by (13) corresponding to the superprocess $X^{(n)}$.

(iii) *Finally, for all $x \in B$ there exists a negligible set $\Omega_{\mathbf{f},\mathbf{t},0}(x)$ outside which*

$$\left| \mathrm{E}_{0,x} \left[\sum_{i=1}^p f_i(\xi_{t_i}) \mathbb{I}_{\{t_i<T\}} \left(\mathbb{I}_{\{t_i<T_{(n)}\}} - \mathrm{e}^{-k_d \int_0^{t_i} c(\xi_u)\mathrm{d}u} \right) \right] \right|$$

and

$$\left| \mathrm{E}_{0,x} \left[\int_0^\infty \mathrm{d}s \mathbb{I}_{\{s<T\}} \left(\mathbb{I}_{\{s<T_{(n)}\}} - \mathrm{e}^{-k_d \int_0^s c(\xi_u)\mathrm{d}u} \right) w_s^{(n)}(\xi_s)^2 \right] \right|$$

converge to 0 as $n \to \infty$.

Both (ii) and (iii) can be obtained from Lemma 2 in a way similar to the derivation of (i). Note that in (ii), we may replace the integral over $[0, \infty)$ by the integral over $[0, t_p]$ (since $w_r^{(n)} \equiv 0$ if $r \geq t_p$) and that the functions $w_r^{(n)}$ are uniformly bounded by $C_{\mathbf{f}}$.

The first term of the right-hand side of (16), with $\varepsilon = \varepsilon_n$, converges to 0 as $n \to \infty$ provided that $\omega \notin \tilde{\Omega}_{\mathbf{f},\mathbf{t}}$ and $t \notin \tilde{T}_{\mathbf{f},\mathbf{t},\omega}$, by Lemma 3 (i). For the second term, we have

$$\begin{aligned} &\int_B \mathrm{d}x \left| \mathrm{E}_{t,x} \left[\int_t^\infty \mathrm{d}s \mathbb{I}_{\{s<T\}} \left(\mathbb{I}_{\{s<T_{(n)}\}} - \mathrm{e}^{-k_d \int_t^s c(\xi_u)\mathrm{d}u} \right) w_s^{(n)}(\xi_s)^2 \right] \right| \\ &\quad = \int_B \mathrm{d}x \left| \mathrm{E}_{0,x} \left[\int_0^\infty \mathrm{d}s \mathbb{I}_{\{s<T\}} \left(\mathbb{I}_{\{s<T_{(n)}\}} - \mathrm{e}^{-k_d \int_0^s c(\xi_u)\mathrm{d}u} \right) w_{s+t}^{(n)}(\xi_s)^2 \right] \right| \\ &\quad \leq \int_0^\infty \mathrm{d}s \int_B \mathrm{d}x \left| \mathrm{E}_{0,x} \left[\mathbb{I}_{\{s<T\}} \left(\mathbb{I}_{\{s<T_{(n)}\}} - \mathrm{e}^{-k_d \int_0^s c(\xi_u)\mathrm{d}u} \right) w_{s+t}^{(n)}(\xi_s)^2 \right] \right|, \end{aligned} \tag{18}$$

which converges to 0 as $n \to \infty$ by Lemma 3 (ii), if $\omega \notin \hat{\Omega}_{\mathbf{f},\mathbf{t}}$ and $t \notin \hat{T}_{\mathbf{f},\mathbf{t},\omega}$.

Finally, for $\omega \in \left(\tilde{\Omega}_{\mathbf{f},\mathbf{t}} \cup \hat{\Omega}_{\mathbf{f},\mathbf{t}}\right)^c$ and $t \in \left(\tilde{T}_{\mathbf{f},\mathbf{t},\omega} \cup \hat{T}_{\mathbf{f},\mathbf{t},\omega}\right)^c$, the first two terms of the right-hand side of (16) converge to 0 as $n \to \infty$. Recalling (17), we obtain

$$\int_B \mathrm{d}x |w_t^{(n)}(x) - w_t^*(x)| \leq b_n(t) + 2C_{\mathbf{f}} \int_t^{t_p} \mathrm{d}s \int_B \mathrm{d}z\, |w_t^{(n)}(z) - w_t^*(z)|,$$

where $b_n(t) \to 0$ as $n \to \infty$ provided ω and t are as above. Besides, for every t,

$$|b_n(t)| \leq 2\lambda(B)(1 + t_p)(C_{\mathbf{f}} + C_{\mathbf{f}}^2). \tag{19}$$

Set for every $t \in [0, t_p]$,

$$G_n(t) := \int_B \mathrm{d}x |w_{t_p-t}^{(n)}(x) - w_{t_p-t}^*(x)|.$$

Then,

$$G_n(t) \leq b_n(t_p - t) + K_{\mathbf{f}} \int_0^t \mathrm{d}s\, G_n(s),$$

where $K_{\mathbf{f}} := 2C_{\mathbf{f}}$. By iterating this inequality as in the proof of Gronwall's lemma, we obtain for all $k \geq 1$, $n \geq 1$ and $t \in [0, t_p]$

$$G_n(t) \le b_n(t_p - t) + \sum_{i=0}^{k-2} K_{\mathbf{f}}^{i+1} \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_i} \mathrm{d}s_{i+1}\, b_n(t_p - s_{i+1})$$
$$+ K_{\mathbf{f}}^k \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_{k-1}} \mathrm{d}s_k G_n(s_k)$$
$$\le b_n(t_p - t) + \sum_{i=0}^{k-2} K_{\mathbf{f}}^{i+1} \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_i} \mathrm{d}s_{i+1}\, b_n(t_p - s_{i+1})$$
$$+ \lambda(B) K_{\mathbf{f}}^{k+1} \frac{t_p^k}{k!}.$$

Fix $\varepsilon > 0$ and let $k \ge 2$ be such that $\lambda(B) K_{\mathbf{f}}^{k+1} \frac{t_p^k}{k!} \le \frac{\varepsilon}{2}$. For $\omega \notin \left(\tilde{\Omega}_{\mathbf{f},\mathbf{t}} \cup \hat{\Omega}_{\mathbf{f},\mathbf{t}}\right)$, $b_n(r)$ converges to 0 as $n \to \infty$ except on a Lebesgue null set of values of r, and thus by dominated convergence

$$\int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_i} \mathrm{d}s_{i+1}\, b_n(t_p - s_{i+1}) \to 0$$

for every $t \in [0, t_p]$ and $i \in \{0, \ldots, k-2\}$. In particular, for such ω and for every $t \in [0, t_p]$, we have

$$K_{\mathbf{f}}^{i+1} \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_i} \mathrm{d}s_{i+1}\, b_n(t_p - s_{i+1}) \le \frac{\varepsilon}{2k}$$

for all n sufficiently large. If moreover t is such that $t_p - t \in \left(\tilde{T}_{\mathbf{f},\mathbf{t},\omega} \cup \hat{T}_{\mathbf{f},\mathbf{t},\omega}\right)^c$, then we have also $b_n(t_p - t) \le \frac{\varepsilon}{2k}$ if n is large. Hence, we have $G_n(t) \le \varepsilon$ when n is large. Since ε was arbitrary, we can conclude that for all ω and t as specified above, $G_n(t)$ converges to 0. Equivalently, for all $\omega \in \left(\tilde{\Omega}_{\mathbf{f},\mathbf{t}} \cup \hat{\Omega}_{\mathbf{f},\mathbf{t}}\right)^c$ and $t \in \left(\tilde{T}_{\mathbf{f},\mathbf{t},\omega} \cup \hat{T}_{\mathbf{f},\mathbf{t},\omega}\right)^c$,

$$\lim_{n\to\infty} \int_B \mathrm{d}x |w_t^{(n)}(x) - w_t^*(x)| = 0. \tag{20}$$

We next consider the asymptotic behaviour of $|w_0^{(n)}(x) - w_0^*(x)|$. In the same way as in (16) but now without integrating over B, we have for every $x \in B$

$$|w_0^{(n)}(x) - w_0^*(x)| \le \left| \mathrm{E}_{0,x}\left[\sum_{i=1}^p f_i(\xi_{t_i}) \mathbb{I}_{\{t_i < T\}} \left(\mathbb{I}_{\{t_i < T_{(n)}\}} - \mathrm{e}^{-k_d \int_0^{t_i} c(\xi_u)\mathrm{d}u} \right) \right] \right|$$
$$+ \left| \mathrm{E}_{0,x}\left[\int_0^\infty \mathrm{d}s \mathbb{I}_{\{s<T\}} \left(\mathbb{I}_{\{s<T_{(n)}\}} - \mathrm{e}^{-k_d \int_0^s c(\xi_u)\mathrm{d}u} \right) w_s^{(n)}(\xi_s)^2 \right] \right|$$
$$+ 2C_{\mathbf{f}} \mathrm{E}_{0,x}\left[\int_0^\infty \mathrm{d}s \mathbb{I}_B(\xi_s) |w_s^*(\xi_s) - w_s^{(n)}(\xi_s)| \right]. \tag{21}$$

Let us fix $x \in B$. By Lemma 3 (iii), there exists a $\mathbf{P}$-negligible set $\Omega_{\mathbf{f},\mathbf{t},0}(x)$ outside which the first two terms in the right-hand side of (21) converge to 0. Besides, for any $\delta > 0$,

$$\mathrm{E}_{0,x}\left[\int_0^\infty \mathrm{d}s \mathbb{I}_B(\xi_s) |w_s^*(\xi_s) - w_s^{(n)}(\xi_s)| \right] = \int_0^\infty \mathrm{d}s \int_B \mathrm{d}z\ p_s(x,z) |w_s^*(z) - w_s^{(n)}(z)|$$
$$\le 2C_{\mathbf{f}}\delta + \frac{1}{(2\pi\delta)^{d/2}} \int_\delta^\infty \mathrm{d}s \int_B \mathrm{d}z\ |w_s^*(z) - w_s^{(n)}(z)|,$$

using the bound $p_s(x,z) \leq (2\pi\delta)^{-d/2}$ if $s \geq \delta$. If in addition $\omega \in \left(\tilde{\Omega}_{\mathbf{f},\mathbf{t}} \cup \hat{\Omega}_{\mathbf{f},\mathbf{t}}\right)^c$, then by (20) and dominated convergence (recall that w_s^* and $w_s^{(n)}$ vanish for $s > t_p$),

$$\int_\delta^\infty \mathrm{d}s \int_B \mathrm{d}z\, |w_s^*(z) - w_s^{(n)}(z)| \to 0$$

and so $\limsup |w_0^{(n)}(x) - w_0^*(x)| \leq 2C_{\mathbf{f}}\delta$. Since δ was arbitrary, it follows that $\lim |w_0^{(n)}(x) - w_0^*(x)| = 0$.

To summarize, for all $x \in B$ and $\omega \in \left(\tilde{\Omega}_{\mathbf{f},\mathbf{t}} \cup \hat{\Omega}_{\mathbf{f},\mathbf{t}} \cup \Omega_{\mathbf{f},\mathbf{t},0}(x)\right)^c$ (of $\mathbf{P}$-probability 1),

$$\lim_{n\to\infty} |w_0^{(n)}(x) - w_0^*(x)| = 0.$$

From the latter result, we can obtain the convergence of the finite-dimensional distributions of $X^{(n)}$ towards the corresponding ones for X^*. For all $x \in B$, $\mathbf{P}\left[w_0^{(n)}(x) \to w_0^*(x)\right] = 1$, so by applying once again Fubini's theorem, we have

$$\mathbf{P}\text{-a.s.,} \quad \mu\text{-a.e.,} \quad w_0^{(n)}(x) \to w_0^*(x) \quad \text{as } n \to \infty. \tag{22}$$

Since the $w^{(n)}$ are bounded by $C_{\mathbf{f}}$, dominated convergence and (22) give

$$\exp -\langle \mu, w_0^{(n)}(\cdot)\rangle \to \exp -\langle \mu, w_0^*(\cdot)\rangle.$$

Our construction from the historical superprocess makes it obvious that $X^{(n)}$ is stochastically bounded by X^0. It follows that the sequence of the distributions of $\{(X_{t_1}^{(n)}, \ldots, X_{t_p}^{(n)}), n \in \mathbb{N}\}$ is relatively compact. Therefore, if we choose a countable set of p-tuples $(f_1, \ldots, f_p)$ such that the corresponding family of maps $(\mu_1, \ldots, \mu_p) \mapsto \exp -\sum_{i=1}^p \langle \mu_i, f_i\rangle$ is convergence determining, we obtain that $(X_{t_1}^{(n)}, \ldots, X_{t_p}^{(n)})$ converges in distribution to $(X_{t_1}^*, \ldots, X_{t_p}^*)$ on a set of $\mathbf{P}$-probability 1 (which a priori depends on $(t_1, \ldots, t_p)$). This completes the proof of Proposition 2. □

Proof of Lemma 2. The quantity of interest vanishes if $t > t_1$, and so we need only consider the case $t \leq t_1$. In that case,

$$\begin{aligned}
&\mathbf{E}\left[\left(\mathrm{E}_{t,x}\left[f(\xi_{t_1})\mathbb{I}_{\{t_1<T\}}\left(\mathbb{I}_{\{t_1<T_\varepsilon\}} - \mathrm{e}^{-k_d\int_t^{t_1} c(\xi_u)\mathrm{d}u}\right)\right]\right)^2\right] \\
&\quad = \mathbf{E}\Big[\mathrm{E}_{t,x}\Big[\mathbb{I}_{\{t_1<T\}}\mathbb{I}_{\{t_1<T'\}} f(\xi_{t_1}) f(\xi'_{t_1})\Big(\mathbb{I}_{\{t_1<T_\varepsilon\}}\mathbb{I}_{\{t_1<T'_\varepsilon\}} - \mathbb{I}_{\{t_1<T_\varepsilon\}}\mathrm{e}^{-k_d\int_t^{t_1} c(\xi'_u)\mathrm{d}u} \\
&\qquad - \mathbb{I}_{\{t_1<T'_\varepsilon\}}\mathrm{e}^{-k_d\int_t^{t_1} c(\xi_u)\mathrm{d}u} + \mathrm{e}^{-k_d\int_t^{t_1}(c(\xi_u)+c(\xi'_u))\mathrm{d}u}\Big)\Big]\Big],
\end{aligned} \tag{23}$$

where ξ' is another Brownian motion, independent of ξ, T' and T'_ε are defined in an obvious way and we have kept the notation $\mathrm{P}_{t,x}$ for the probability measure under which the two Brownian motions start from x at time t. Recall that $S_\varepsilon(s,t)$ denotes the Wiener sausage of radius ε along the time interval $[s,t]$ associated to ξ, and define $S'_\varepsilon(s,t)$ in a similar way. Then,

$$\begin{aligned}
\mathbf{E}\left[\mathbb{I}_{\{t_1<T_\varepsilon\}}\mathbb{I}_{\{t_1<T'_\varepsilon\}}\right] &= \mathbf{P}\left[\mathcal{P}^\varepsilon \cap (S_\varepsilon(t,t_1) \cup S'_\varepsilon(t,t_1)) = \emptyset\right] \\
&= \exp\left\{-s_d(\varepsilon)\int_{S_\varepsilon(t,t_1)\cup S'_\varepsilon(t,t_1)} c(y)\mathrm{d}y\right\}
\end{aligned}$$

and similarly

$$\mathbf{E}\left[\mathbb{I}_{\{t_1<T_\varepsilon\}}\right] = \exp\left\{-s_d(\varepsilon)\int_{S_\varepsilon(t,t_1)} c(y)\mathrm{d}y\right\}.$$

Set $\tilde{t}_1 = t_1 - t$. By Fubini's theorem and a simple symmetry argument, the quantity in (23) is equal to

$$\begin{aligned}
&\mathrm{E}_{t,x}\Big[\mathbb{I}_{\{t_1<T\}}\mathbb{I}_{\{t_1<T'\}} f(\xi_{t_1}) f(\xi'_{t_1})\Big\{2\mathrm{e}^{-k_d\int_t^{t_1} c(\xi'_u)\mathrm{d}u}\Big(\mathrm{e}^{-k_d\int_t^{t_1} c(\xi'_u)\mathrm{d}u} - \mathrm{e}^{-s_d(\varepsilon)\int_{S_\varepsilon(t,t_1)} c(y)\mathrm{d}y}\Big) \\
&\quad + \mathrm{e}^{-s_d(\varepsilon)\int_{S_\varepsilon(t,t_1)\cup S'_\varepsilon(t,t_1)} c(y)\mathrm{d}y} - \mathrm{e}^{-k_d\int_t^{t_1}(c(\xi_u)+c(\xi'_u))\mathrm{d}u}\Big\}\Big] \\
&\le 2\|f\|^2\mathrm{E}_{t,x}\left[\left|\mathrm{e}^{-k_d\int_t^{t_1} c(\xi'_u)\mathrm{d}u} - \mathrm{e}^{-s_d(\varepsilon)\int_{S_\varepsilon(t,t_1)} c(y)\mathrm{d}y}\right|\right] \\
&\quad + \|f\|^2\mathrm{E}_{t,x}\left[\left|\mathrm{e}^{-s_d(\varepsilon)\int_{S_\varepsilon(t,t_1)\cup S'_\varepsilon(t,t_1)} c(y)\mathrm{d}y} - \mathrm{e}^{-k_d\int_t^{t_1}(c(\xi_u)+c(\xi'_u))\mathrm{d}u}\right|\right] \\
&\le 2\|f\|^2\mathrm{E}_{0,x}\left[\left|k_d\int_0^{\tilde{t}_1} c(\xi_u)\mathrm{d}u - s_d(\varepsilon)\int_{S_\varepsilon(0,\tilde{t}_1)} c(y)\mathrm{d}y\right|\right] \\
&\quad + \|f\|^2\mathrm{E}_{0,x}\Bigg[\Bigg|s_d(\varepsilon)\int_{S_\varepsilon(0,\tilde{t}_1)} c(y)\mathrm{d}y + s_d(\varepsilon)\int_{S'_\varepsilon(0,\tilde{t}_1)} c(y)\mathrm{d}y \\
&\qquad - s_d(\varepsilon)\int_{S_\varepsilon(0,\tilde{t}_1)\cap S'_\varepsilon(0,\tilde{t}_1)} c(y)\mathrm{d}y - k_d\int_0^{\tilde{t}_1} c(\xi_u)\mathrm{d}u - k_d\int_0^{\tilde{t}_1} c(\xi'_u)\mathrm{d}u\Bigg|\Bigg] \\
&\le 4\|f\|^2\mathrm{E}_{0,x}\left[\left|\,k_d\int_0^{\tilde{t}_1} c(\xi_u)\mathrm{d}u - s_d(\varepsilon)\int_{S_\varepsilon(0,\tilde{t}_1)} c(y)\mathrm{d}y\right|\right] \\
&\quad + \|f\|^2\|c\| s_d(\varepsilon)\mathrm{E}_{0,x}\left[\lambda\left(S_\varepsilon(0,\tilde{t}_1)\cap S'_\varepsilon(0,\tilde{t}_1)\right)\right],
\end{aligned} \tag{24}$$

where in the second inequality we used the bound $|\mathrm{e}^{-x} - \mathrm{e}^{-y}| \le |x-y|$ for $x, y \ge 0$.

On the one hand, by [5] ($d = 2, 3$) and [13], p. 1009–1010 ($d \ge 4$), we have

$$\begin{aligned}
&|\log\varepsilon|\ \mathrm{E}_{0,x}\left[\lambda\left(S_\varepsilon(0,\tilde{t}_1)\cap S'_\varepsilon(0,\tilde{t}_1)\right)\right] = O(|\log\varepsilon|^{-1}) \quad \text{if } d = 2, \\
&\varepsilon^{-1}\mathrm{E}_{0,x}\left[\lambda\left(S_\varepsilon(0,\tilde{t}_1)\cap S'_\varepsilon(0,\tilde{t}_1)\right)\right] = O(\varepsilon) \quad \text{if } d = 3, \\
&\varepsilon^{2-d}\mathrm{E}_{0,x}\left[\lambda\left(S_\varepsilon(0,\tilde{t}_1)\cap S'_\varepsilon(0,\tilde{t}_1)\right)\right] = O(\varepsilon^2|\log\varepsilon|) \quad \text{if } d \ge 4.
\end{aligned} \tag{25}$$

On the other hand,

$$\begin{aligned}
&\mathrm{E}_{0,x}\left[\left|\,k_d\int_0^{\tilde{t}_1} c(\xi_u)\mathrm{d}u - s_d(\varepsilon)\int_{S_\varepsilon(0,\tilde{t}_1)} c(y)\mathrm{d}y\right|\right] \\
&\le \mathrm{E}_{0,x}\left[\left(k_d\int_0^{\tilde{t}_1} c(\xi_u)\mathrm{d}u - s_d(\varepsilon)\int_{S_\varepsilon(0,\tilde{t}_1)} c(y)\mathrm{d}y\right)^2\right]^{1/2}.
\end{aligned} \tag{26}$$

Proposition 1 ensures that the right-hand side of (26) is bounded by $K\|c\|\,|\log\varepsilon|^{-1}$ if $d = 2$ and by $K\|c\|\,\varepsilon|\log\varepsilon|$ if $d \ge 3$. Together with (24) and (25), this completes the proof of Lemma 2. □

4. Tightness of the sequence $X^{(n),B}$

Let $C^2_+(\mathbb{R}^d)$ denote the set of all nonnegative functions of class C^2 on $\mathbb{R}^d$. By Theorem II.4.1 in [16], the tightness of the sequence of the laws of the superprocesses $X^{(n),B}$ will follow if we can prove that the sequence of the laws of $\langle X^{(n),B}, \phi\rangle$ is tight, for every $\phi \in C^2_+(\mathbb{R}^d)$ with compact support. Note that condition (i) in Theorem II.4.1 of [16] holds thanks to the domination $X^{(n),B} \le X^0$. Recall that X^0 is the usual super-Brownian motion without obstacles.

Let us first introduce the **P**-negligible set $\Theta \subset \Omega$ outside which the desired tightness will hold.

Definition 1 (*Good environments*). Let Θ be the union over all choices of the integer $p \ge 1$ and of the rational numbers $q_1, \dots, q_p$ of the **P**-negligible sets on which the sequence $(X^{(n)}_{q_1}, \dots, X^{(n)}_{q_p})$ does not converge in distribution to $(X^*_{q_1}, \dots, X^*_{q_p})$ as $n \to \infty$. We call good environment any environment which does not belong to Θ.

To simplify notation, we again write $X^{(n)}$ for $X^{(n),B}$ (as in the last section, B is fixed) and prove tightness only on the time interval [0, 1]. Let us fix $\phi \in C^2_+(\mathbb{R}^d)$ with compact support. The tightness of the sequence $\langle X^{(n)}, \phi\rangle$ is a consequence of the following lemma.

Lemma 4. *If $\omega \notin \Theta$, then for every $\varepsilon > 0$ there exist $k = k(\varepsilon) \ge 1$ and $n_0 = n_0(\omega, \varepsilon, k)$ such that for all $n \ge n_0$,*

$$\mathbb{P}_\mu\left[\bigcup_{i=0}^{k-1}\left\{\sup_{\frac{i}{k}\le t\le\frac{i+1}{k}}\left|\langle X^{(n)}_t, \phi\rangle - \langle X^{(n)}_{\frac{i}{k}}, \phi\rangle\right| > \varepsilon\right\}\right] < \varepsilon.$$

Lemma 4 easily implies that the sequence $\langle X^{(n)}, \phi\rangle$ is tight. Indeed, let us fix a good environment and $\eta > 0$. By Lemma 4, there exist $k(\eta)$ and $n_0(\omega, \eta, k)$ such that for all $n \ge n_0$,

$$\mathbb{P}_\mu\left[\bigcup_{i=0}^{k-1}\left\{\sup_{\frac{i}{k}\le t\le\frac{i+1}{k}}\left|\langle X^{(n)}_t, \phi\rangle - \langle X^{(n)}_{\frac{i}{k}}, \phi\rangle\right| > \frac{\eta}{3}\right\}\right] < \eta. \tag{27}$$

If $n \ge n_0$ is fixed, on the complement of the event considered in (27), we have for every $s, t \in [0, 1]$

$$|t-s| \le \frac{1}{k} \quad \Rightarrow \quad \left|\langle X^{(n)}_t, \phi\rangle - \langle X^{(n)}_s, \phi\rangle\right| \le \eta$$

and therefore $w(\langle X^{(n)}, \phi\rangle, \frac{1}{k}, 1) \le \eta$, using the notation of Ethier and Kurtz [17] for the modulus of continuity of the process $\langle X^{(n)}, \phi\rangle$. Thus, for all $n \ge n_0$,

$$\begin{aligned}&\mathbb{P}_\mu\left[w\left(\langle X^{(n)}, \phi\rangle, \frac{1}{k}, 1\right) \le \eta\right]\\ &\quad\ge \mathbb{P}_\mu\left[\left(\bigcup_{i=0}^{k-1}\left\{\sup_{\frac{i}{k}\le t\le\frac{i+1}{k}}\left|\langle X^{(n)}_t, \phi\rangle - \langle X^{(n)}_{\frac{i}{k}}, \phi\rangle\right| > \frac{\eta}{3}\right\}\right)^c\right] \ge 1-\eta.\end{aligned}$$

In addition, ϕ is bounded so that the first condition of Theorem 3.7.2 in [17] is trivially fulfilled, hence Corollary 3.7.4 of [17] implies that for any good environment, the sequence of the laws of $\langle X^{(n)}, \phi\rangle$ under $\mathbb{P}_\mu$ is tight.

Let us now turn to the proof of Lemma 4.

Proof of Lemma 4. We fix a good environment. Let $\varepsilon > 0$. The process $(\langle X_t^*, \phi\rangle)_{t\geq 0}$ is continuous, therefore there exists $k_0(\varepsilon)$ such that for all $k \geq k_0$,

$$\mathbb{P}_\mu\left[\sup_{0\leq i\leq k-1}\left|\langle X^*_{\frac{i+1}{k}}, \phi\rangle - \langle X^*_{\frac{i}{k}}, \phi\rangle\right| \geq \frac{\varepsilon}{2}\right] < \frac{\varepsilon}{3}. \tag{28}$$

There exists $K = K(\varepsilon) \geq 1$ such that

$$\mathbb{P}_\mu\left[\sup_{0\leq t\leq 1}\langle X_t^0, 1\rangle \geq K\right] < \frac{\varepsilon}{3}. \tag{29}$$

By a trivial domination argument, the bound (29) remains valid if we replace X^0 by $X^{(n)}$ (in fact for any environment). In the following, we fix the constant $K \geq 1$ such that (29) holds.

We now have the following result:

Lemma 5. *There exists a constant $C = C(\phi, K)$ such that for every integer $k \geq 1$ and every measure $\gamma \in \mathcal{M}_f(\mathbb{R}^d)$ satisfying $\langle \gamma, 1\rangle \leq K$,*

$$\mathbb{P}_\gamma\left[\sup_{0\leq s\leq \frac{1}{k}}\left|\langle X_s^0, \phi\rangle - \langle X_0^0, \phi\rangle\right| > \frac{\varepsilon}{2}\right] \leq \frac{C}{k^2}.$$

The proof of Lemma 5 is deferred to the end of the section. Let us define

$$A_n = \left\{\sup_{0\leq t\leq 1}\langle X_t^{(n)}, 1\rangle \geq K\right\}.$$

Then,

$$\begin{aligned}
&\mathbb{P}_\mu\left[\bigcup_{i=0}^{k-1}\left\{\sup_{\frac{i}{k}\leq t\leq\frac{i+1}{k}}\left|\langle X_t^{(n)}, \phi\rangle - \langle X^{(n)}_{\frac{i}{k}}, \phi\rangle\right| > \varepsilon\right\}\right]\\
&\leq \mathbb{P}_\mu[A_n] + \mathbb{P}_\mu\left[\sup_{0\leq i\leq k-1}\left|\langle X^{(n)}_{\frac{i+1}{k}}, \phi\rangle - \langle X^{(n)}_{\frac{i}{k}}, \phi\rangle\right| > \frac{\varepsilon}{2}\right]\\
&\quad + \mathbb{P}_\mu\left[A_n^c \cap \left\{\sup_{0\leq i\leq k-1}\left\{\sup_{\frac{i}{k}\leq t\leq\frac{i+1}{k}}\left(\langle X_t^{(n)}, \phi\rangle - \langle X^{(n)}_{\frac{i}{k}}, \phi\rangle\right)\right\} > \varepsilon\right\}\right]\\
&\quad + \mathbb{P}_\mu\left[A_n^c \cap \left\{\sup_{0\leq i\leq k-1}\left\{\sup_{\frac{i}{k}\leq t\leq\frac{i+1}{k}}\left(\langle X^{(n)}_{\frac{i}{k}}, \phi\rangle - \langle X_t^{(n)}, \phi\rangle\right)\right\} > \varepsilon\right\}\right.\\
&\qquad\qquad \left.\cap\left\{\sup_{0\leq i\leq k-1}\left|\langle X^{(n)}_{\frac{i+1}{k}}, \phi\rangle - \langle X^{(n)}_{\frac{i}{k}}, \phi\rangle\right| \leq \frac{\varepsilon}{2}\right\}\right]\\
&= a_n + b_n + c_n + d_n.
\end{aligned}$$

From (29), we have

$$a_n < \frac{\varepsilon}{3}.$$

Moreover, from the definition of a good environment, if $k \geq k_0$,

$$\limsup_{n\to\infty} b_n \leq \mathbb{P}_\mu\left[\sup_{0\leq i\leq k-1}\left|\langle X^*_{\frac{i+1}{k}},\phi\rangle - \langle X^*_{\frac{i}{k}},\phi\rangle\right| \geq \frac{\varepsilon}{2}\right] < \frac{\varepsilon}{3},$$

by (28). Thus if $k \geq k_0$, there exists $n_0(\omega,\varepsilon,k)$ such that for all $n \geq n_0$, $b_n(k) \leq \frac{\varepsilon}{3}$. Then,

$$\begin{aligned} c_n &\leq \sum_{i=0}^{k-1}\mathbb{P}_\mu\left[\langle X^{(n)}_{\frac{i}{k}},1\rangle \leq K;\ \sup_{\frac{i}{k}\leq t\leq\frac{i+1}{k}}\left(\langle X^{(n)}_t,\phi\rangle - \langle X^{(n)}_{\frac{i}{k}},\phi\rangle\right) > \varepsilon\right] \\ &= \sum_{i=0}^{k-1}\mathbb{E}_\mu\left[\mathbb{I}_{\left\{\langle X^{(n)}_{\frac{i}{k}},1\rangle\leq K\right\}}\ \mathbb{P}_{X^{(n)}_{\frac{i}{k}}}\left[\sup_{0\leq t\leq\frac{1}{k}}\left(\langle X^{(n)}_t,\phi\rangle - \langle X^{(n)}_0,\phi\rangle\right) > \varepsilon\right]\right]. \end{aligned}$$

The last equality is obtained by applying the Markov property to $X^{(n)}$ at time $\frac{i}{k}$. By a domination argument, we have for all $\gamma \in \mathcal{M}_f(\mathbb{R}^d)$ such that $\langle\gamma,1\rangle \leq K$,

$$\begin{aligned} \mathbb{P}_\gamma\left[\sup_{0\leq t\leq\frac{1}{k}}\left(\langle X^{(n)}_t,\phi\rangle - \langle X^{(n)}_0,\phi\rangle\right) > \varepsilon\right] &\leq \mathbb{P}_\gamma\left[\sup_{0\leq t\leq\frac{1}{k}}\left(\langle X^0_t,\phi\rangle - \langle X^0_0,\phi\rangle\right) > \varepsilon\right] \\ &\leq \frac{C}{k^2} \end{aligned}$$

by Lemma 5. It follows that

$$c_n \leq k\cdot\frac{C}{k^2} = \frac{C}{k} < \frac{\varepsilon}{6}$$

if $k \geq k_1(\varepsilon)$. Finally,

$$\begin{aligned} d_n \leq \sum_{i=0}^{k-1}\mathbb{P}_\mu\Bigg[&\sup_{\frac{i}{k}\leq t\leq\frac{i+1}{k}}\langle X^{(n)}_t,1\rangle \leq K;\ \sup_{\frac{i}{k}\leq t\leq\frac{i+1}{k}}\left(\langle X^{(n)}_{\frac{i}{k}},\phi\rangle - \langle X^{(n)}_t,\phi\rangle\right) > \varepsilon; \\ &\sup_{0\leq i\leq k-1}\left|\langle X^{(n)}_{\frac{i+1}{k}},\phi\rangle - \langle X^{(n)}_{\frac{i}{k}},\phi\rangle\right| \leq \frac{\varepsilon}{2}\Bigg]. \end{aligned}$$

We fix $i \in \{0,\ldots,k-1\}$ and consider the stopping time

$$T_i := \inf\left\{t \geq \frac{i}{k} : \langle X^{(n)}_t,\phi\rangle \leq \langle X^{(n)}_{\frac{i}{k}},\phi\rangle - \varepsilon\right\}.$$

Then, the ith term of the previous sum is bounded by

$$\begin{aligned} &\mathbb{P}_\mu\left[T_i \leq \frac{i+1}{k},\ \langle X^{(n)}_{T_i},1\rangle \leq K,\ \sup_{T_i\leq t\leq T_i+\frac{1}{k}}\left(\langle X^{(n)}_t,\phi\rangle - \langle X^{(n)}_{T_i},\phi\rangle\right) \geq \frac{\varepsilon}{2}\right] \\ &\quad = \mathbb{E}_\mu\left[\mathbb{I}_{\left\{T_i\leq\frac{i+1}{k},\langle X^{(n)}_{T_i},1\rangle\leq K\right\}}\ \mathbb{P}_{X^{(n)}_{T_i}}\left[\sup_{0\leq t\leq\frac{1}{k}}\left(\langle X^{(n)}_t,\phi\rangle - \langle X^{(n)}_0,\phi\rangle\right) \geq \frac{\varepsilon}{2}\right]\right] \end{aligned}$$

by the strong Markov property at time T_i. Using Lemma 5 once again, we see that this quantity is bounded by C/k^2, hence for $k \geq k_1(\varepsilon)$,

$$d_n \leq \frac{C}{k} \leq \frac{\varepsilon}{6}.$$

Combining the preceding estimates, we obtain that for $k = k_0(\varepsilon) \vee k_1(\varepsilon)$, and every $n \geq n_0(\omega, \varepsilon, k)$,

$$\mathbb{P}_\mu \left[\bigcup_{i=0}^{k-1} \left\{ \sup_{\frac{i}{k} \leq t \leq \frac{i+1}{k}} \left| \langle X_t^{(n)}, \phi \rangle - \langle X_{\frac{i}{k}}^{(n)}, \phi \rangle \right| > \varepsilon \right\} \right] < \varepsilon.$$

This completes the proof of Lemma 4. $\square$

Proof of Lemma 5. Let $\gamma \in \mathcal{M}_f(\mathbb{R}^d)$ be such that $|\gamma| := \langle \gamma, 1 \rangle \leq K$. Recall that the process $(\langle X_t^0, 1 \rangle)_{t \geq 0}$ is a martingale. From the maximal inequality applied to the nonnegative submartingale $(\langle X_t^0, 1 \rangle - |\gamma|)^4$,

$$\mathbb{P}_\gamma \left[\sup_{0 \leq t \leq \frac{1}{k}} \langle X_t^0, 1 \rangle > 2K \right] \leq \frac{1}{(2K - |\gamma|)^4} \mathbb{E}_\gamma \left[\left(\langle X_{\frac{1}{k}}^0, 1 \rangle - |\gamma| \right)^4 \right].$$

We claim that

$$\mathbb{E}_\gamma \left[\left(\langle X_{\frac{1}{k}}^0, 1 \rangle - |\gamma| \right)^4 \right] = \frac{24}{k^3} |\gamma| + \frac{12}{k^2} |\gamma|^2. \tag{30}$$

To prove this claim, recall that $Y_t = \langle X_t^0, 1 \rangle$ is a Feller diffusion, whose semigroup Laplace transform is given by

$$\mathbb{E}\left[\exp -\lambda Y_t |\, Y_0 = y \right] = \exp\left(-\frac{\lambda y}{1 + \lambda t} \right),$$

for $\lambda \geq 0$. Thus,

$$\begin{aligned} \mathbb{E}\left[\exp -\lambda (Y_t - y) |\, Y_0 = y \right] &= \exp\left(\frac{\lambda^2 t y}{1 + \lambda t} \right) \\ &= 1 + \lambda^2 t y - \lambda^3 t^2 y + \lambda^4 t^3 y + \frac{\lambda^4 t^2 y^2}{2} + o(\lambda^4), \end{aligned}$$

as $\lambda \to 0$. From this expansion of the Laplace transform, we derive that

$$\mathbb{E}\left[(Y_t - y)^4 \,|Y_0 = y \right] = 24 t^3 y + 12 t^2 y^2,$$

which proves our claim (30). It follows that

$$\mathbb{P}_\gamma \left[\sup_{0 \leq t \leq \frac{1}{k}} \langle X_t^0, 1 \rangle > 2K \right] \leq \frac{12 |\gamma| (|\gamma| + 2)}{(2K - |\gamma|)^4 \, k^2}.$$

Let us denote by $A_{K,k}$ the event $\left\{ \sup_{0 \leq t \leq \frac{1}{k}} \langle X_t^0, 1 \rangle > 2K \right\}$ and by $B_{K,k}$ the event $\left\{ \sup_{0 \leq t \leq \frac{1}{k}} \left| \langle X_t^0, \phi \rangle - \langle X_0^0, \phi \rangle \right| > \frac{\varepsilon}{2} \right\}$. Then,

$$\mathbb{P}_\gamma [B_{K,k}] \leq \mathbb{P}_\gamma [A_{K,k}] + \mathbb{P}_\gamma [A_{K,k}^c \cap B_{K,k}] \leq \frac{c_0}{k^2} + \mathbb{P}_\gamma [A_{K,k}^c \cap B_{K,k}], \tag{31}$$

where c_0 is a constant depending on K.

In addition, $M_t := \langle X_t^0, \phi\rangle - \langle X_0^0, \phi\rangle - \int_0^t \mathrm{d}r \langle X_r^0, \frac{1}{2}\Delta\phi\rangle$ is a continuous martingale with quadratic variation $2\int_0^t \mathrm{d}r\langle X_r^0, \phi^2\rangle$. By the Dubins–Schwarz theorem (see Theorem V.1.7 in [18]), there exists a standard one-dimensional Brownian motion W such that $M_t = W_{\langle M\rangle_t}$ for all $t \geq 0$ a.s. On the event $A_{K,k}^c$, we have

$$\left|\int_0^t \mathrm{d}r\left\langle X_r^0, \frac{1}{2}\Delta\phi\right\rangle\right| \leq t\|\Delta\phi\|K \leq \frac{c_1}{k} \quad \text{if } t \in [0, k^{-1}]$$

and

$$\langle M\rangle_t \leq \frac{4\|\phi\|^2 K}{k} = \frac{c_2}{k} \quad \text{if } t \in [0, k^{-1}],$$

where c_1 and c_2 are constants depending on ϕ and on K. Choose k_0 such that $c_1 k^{-1} < \frac{1}{4}\varepsilon$ for every $k \geq k_0$. Then for all $k \geq k_0$,

$$\begin{aligned}
\mathbb{P}_\gamma[A_{K,k}^c \cap B_{K,k}] &= \mathbb{P}_\gamma\left[A_{K,k}^c \cap \left\{\sup_{0\leq t\leq 1/k}\left|M_t + \int_0^t \mathrm{d}r\left\langle X_r^0, \frac{1}{2}\Delta\phi\right\rangle\right| > \frac{\varepsilon}{2}\right\}\right] \\
&\leq \mathbb{P}_\gamma\left[A_{K,k}^c \cap \left\{\sup_{0\leq t\leq 1/k}|M_t| > \frac{\varepsilon}{4}\right\}\right] \\
&\leq \mathrm{P}\left[\sup_{0\leq t\leq (c_2/k)}|W_t| > \frac{\varepsilon}{4}\right] \\
&\leq \frac{c_3}{k^2},
\end{aligned}$$

where c_3 is a constant depending on ϕ, K and ε. Together with (31), this completes the proof of Lemma 5. □

5. Proofs of Theorem 1 and Corollary 1

The proof of Theorem 1 in the case when D is bounded is easily obtained from the results of the previous sections. Let us take $B = D$ and let $\mathcal{E}$ denote the union of the **P**-negligible set on which there exist rational numbers $t_1, \ldots, t_p$ such that $(X_{t_1}^{(n)}, \ldots, X_{t_p}^{(n)})$ does not converge to $(X_{t_1}^*, \ldots, X_{t_p}^*)$ and of the **P**-negligible set on which the sequence $X^{(n),B}$ is not tight. The set $\mathcal{E}$ is also **P**-negligible and on $\mathcal{E}^c$, Theorem 3.7.8 of [17] allows us to conclude that $X^{(n)} \xrightarrow{(d)} X^*$ when $n \to \infty$.

We can now use the previous result to complete the proof of Theorem 1 when D is a domain of $\mathbb{R}^d$ which is not necessarily bounded.

Proof of Theorem 1 (*For a general domain D*). Let μ be a finite measure on D and suppose first that the support of μ is bounded. Under $\mathbb{P}_\mu$, the superprocesses $X^{(n)}$ are stochastically dominated by the superprocess X^0, whose range

$$\mathcal{R}(X^0) = \overline{\bigcup_{t\geq 0}\operatorname{supp} X_t^0}$$

is almost surely compact since its initial value has compact support. Consequently, for every $\varepsilon > 0$, there exists a bounded open subset B of D containing the support of μ such that, for every environment and every $n \geq 1$,

$$\mathbb{P}_\mu[\mathcal{R}(X^{(n),D}) \subset B] \geq 1 - \varepsilon$$

and

$$\mathbb{P}_\mu[\mathcal{R}(X^{*,D}) \subset B] \geq 1 - \varepsilon.$$

From these inequalities, we can deduce that

$$d(\mathbb{P}_\mu^{(n),D}, \mathbb{P}_\mu^{(n),B}) \leq 2\varepsilon, \qquad n \geq 1; \tag{32}$$

$$d(\mathbb{P}_\mu^{*,D}, \mathbb{P}_\mu^{*,B}) \leq 2\varepsilon, \tag{33}$$

where d is the Prohorov metric on $\mathcal{M}_1(D_{\mathcal{M}_f(D)}[0,\infty))$. By the results of the last two sections, with $\mathbf{P}$-probability 1 there exists an integer $n_0(\omega)$ such that for all $n \geq n_0$,

$$d(\mathbb{P}_\mu^{(n),B}, \mathbb{P}_\mu^{*,B}) \leq \varepsilon.$$

Together with (32) and (33), this yields

$$d(\mathbb{P}_\mu^{(n),D}, \mathbb{P}_\mu^{*,D}) \leq 5\varepsilon \quad \text{for all } n \geq n_0(\omega),$$

hence we can conclude that $\mathbb{P}_\mu^{(n),D}$ converges towards $\mathbb{P}_\mu^{*,D}$ on a set of $\mathbf{P}$-probability 1.

Finally, if the support of μ is unbounded, we can replace μ by the measure $\tilde{\mu}$ defined as the restriction of μ to a large ball centered at the origin. Using once again the domination of $X^{(n),D}$ (for all $n \geq 1$) and of $X^{*,D}$ by X^0, the law of $X^{(n),D}$ under $\mathbb{P}_\mu$ can be approximated uniformly in n by the law of $X^{(n),D}$ under $\mathbb{P}_{\tilde{\mu}}$, and similarly for $X^{*,D}$. The desired result then follows from the bounded support case. We leave the details to the reader. □

We end this section with the proof of Corollary 1.

Proof of Corollary 1. Let us argue by contradiction and suppose that there exist $\delta > 0$ and a sequence $\{\varepsilon_k, k \in \mathbb{N}\}$ decreasing to zero such that for all $k \geq 1$,

$$\mathbf{P}\left[d\left(\mathbb{P}_\mu^{\varepsilon_k,D}, \mathbb{P}_\mu^{*,D}\right) > \delta\right] \geq \delta. \tag{34}$$

By extracting a subsequence, we can always choose ε_k such that

$$\sum_{k=1}^{\infty} |\log \varepsilon_k|^{-1} < \infty \quad \text{if } d = 2,$$

or

$$\sum_{k=1}^{\infty} \varepsilon_k \, |\log \varepsilon_k| < \infty \quad \text{if } d \geq 3.$$

But the latter condition is the only requirement for the sequence of superprocesses $X^{\varepsilon_k,D}$ to converge in distribution to $X^{*,D}$ with $\mathbf{P}$-probability 1, yielding a contradiction with (34). □

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Appendix. Proof of (11)

The bound (11) is a consequence of the following lemma.

Lemma 6. *There exists a function* $\varphi : \mathbb{R}^2 \to [0, \infty]$ *such that* $\int_{\mathbb{R}^2} \mathrm{d}y \varphi(y) < \infty$ *and for every* $y \in \mathbb{R}^2$ *and* $\varepsilon \in (0, \frac{1}{2})$,

$$\left| \mathrm{P}_0 \left[y \in S_\varepsilon(0,1) \right] - \frac{\pi}{|\log \varepsilon|} \int_0^1 \mathrm{d}s p_s(y) \right| \leq \frac{\varphi(y)}{|\log \varepsilon|^2},$$

where $p_s(y) = (2\pi s)^{-1} \exp\left\{-|y|^2/(2s)\right\}$ *is the Brownian transition density.*

Remark 3. The convergence of $|\log \varepsilon| \mathrm{P}_0[y \in S_\varepsilon(0,1)]$ towards $\pi \int_0^1 \mathrm{d}s p_s(y)$ as ε tends to 0 was first obtained by Spitzer [19]. See also [5,20] for related results.

Before proving Lemma 6, let us use it to derive (11). If c is a bounded nonnegative measurable function on $\mathbb{R}^2$ such that $\|c\| \leq 1$, then for every $\varepsilon \in (0, \frac{1}{2}]$ and $z \in \mathbb{R}^2$,

$$\begin{aligned} &\left| \mathrm{E}_z \left[|\log \varepsilon| \int_{S_\varepsilon(0,1)} c(y) \mathrm{d}y - \pi \int_0^1 c(\xi_s) \mathrm{d}s \right] \right| \\ &\quad = \left| \int \mathrm{d}y c(z+y) \left(|\log \varepsilon| \mathrm{P}_0 \left[y \in S_\varepsilon(0,1) \right] - \pi \int_0^1 \mathrm{d}s p_s(y) \right) \right| \\ &\quad \leq |\log \varepsilon|^{-1} \int \mathrm{d}y \varphi(y), \end{aligned}$$

which is the desired result.

Let us hence establish Lemma 6. The following proof is due to J.-F. Le Gall (personal communication).

Proof of Lemma 6. If $|y| \leq 10\varepsilon$, simple estimates show that the bound of Lemma 6 holds with $\varphi(y) = C\left((\log |y|)^2 + 1\right)$ for a suitable constant C. So we assume that $|y| > 10\varepsilon$. We put

$$\tau_\varepsilon(y) = \inf\{t \geq 0 : |\xi_t - y| \leq \varepsilon\}$$

in such a way that $\{y \in S_\varepsilon(0,1)\} = \{\tau_\varepsilon(y) \leq 1\}$. Let a_ε be an arbitrary point of the circle of radius ε centered at the origin, and

$$f(\varepsilon) = \mathrm{E}_{a_\varepsilon} \left[\int_0^1 \mathrm{d}s \mathbb{I}_{\{|\xi_s| \leq \varepsilon\}} \right].$$

A straightforward calculation gives

$$f(\varepsilon) = \varepsilon^2 |\log \varepsilon| + O(\varepsilon^2) \tag{35}$$

as $\varepsilon \to 0$.

Lower bound. An application of the strong Markov property at time $\tau_\varepsilon(y)$ shows for every $u \in (0,1]$, that

$$\mathrm{E}_0\left[\int_0^u \mathrm{d}s \mathbb{I}_{\{|\xi_s - y| \le \varepsilon\}}\right] \le \mathrm{P}_0\left[\tau_\varepsilon(y) \le u\right] f(\varepsilon).$$

On the other hand,

$$\mathrm{E}_0\left[\int_0^u \mathrm{d}s \mathbb{I}_{\{|\xi_s - y| \le \varepsilon\}}\right] = \int_0^u \mathrm{d}s \int_{|z-y|\le\varepsilon} \mathrm{d}z\ p_s(z),$$

and thus

$$\begin{aligned}
&\left|\mathrm{E}_0\left[\int_0^u \mathrm{d}s \mathbb{I}_{\{|\xi_s - y| \le \varepsilon\}}\right] - \pi\varepsilon^2 \int_0^u \mathrm{d}s p_s(y)\right| \\
&\quad \le \int_0^u \frac{\mathrm{d}s}{2\pi s} \int_{|z-y|\le\varepsilon} \mathrm{d}z \left|\exp\left\{-\frac{|z|^2}{2s}\right\} - \exp\left\{-\frac{|y|^2}{2s}\right\}\right| \\
&\quad \le \int_0^u \frac{\mathrm{d}s}{2\pi s} \int_{|z-y|\le\varepsilon} \mathrm{d}z \exp\left\{-\frac{|y|^2}{4s}\right\} \left|\frac{|z|^2 - |y|^2}{2s}\right| \\
&\quad \le \frac{\varepsilon^3}{2}|y| \int_0^u \frac{\mathrm{d}s}{s^2} \exp\left\{-\frac{|y|^2}{4s}\right\} \\
&\quad \le \varepsilon^3\ \Psi_1(y),
\end{aligned} \tag{36}$$

where the function

$$\Psi_1(y) = |y| \int_0^1 \frac{\mathrm{d}s}{s^2} \exp\left\{-\frac{|y|^2}{4s}\right\}$$

is easily seen to be integrable over $\mathbb{R}^2$.

By combining the preceding estimates, we arrive at

$$\mathrm{P}_0\left[\tau_\varepsilon(y) \le u\right] \ge \frac{\pi\varepsilon^2}{f(\varepsilon)} \int_0^u \mathrm{d}s p_s(y) - \frac{\varepsilon^3}{f(\varepsilon)} \Psi_1(y) \tag{37}$$

and using (35) it readily follows that

$$\mathrm{P}_0\left[y \in S_\varepsilon(0,1)\right] - \frac{\pi}{|\log\varepsilon|} \int_0^1 \mathrm{d}s p_s(y) \ge \frac{\varphi_1(y)}{|\log\varepsilon|^2}$$

with a nonnegative function φ_1 such that $\int \mathrm{d}y \varphi_1(y) < \infty$.

Upper bound. This part is a little more delicate. We rely on the same idea of applying the strong Markov property at time $\tau_\varepsilon(y)$, but we need to be more careful in our estimates. For every $v > 0$, we have

$$\begin{aligned}
\mathrm{E}_0\left[\int_0^{1+v} \mathrm{d}s \mathbb{I}_{\{|\xi_s - y| \le \varepsilon\}}\right] &= \mathrm{E}_0\left[\mathbb{I}_{\{\tau_\varepsilon(y) \le 1+v\}} \mathrm{E}_{\xi_{\tau_\varepsilon(y)}}\left[\int_0^s \mathrm{d}r\ \mathbb{I}_{\{|\xi_r - y| \le \varepsilon\}}\right]_{s=1+v-\tau_\varepsilon(y)}\right] \\
&= \mathrm{E}_0\left[\mathbb{I}_{\{\tau_\varepsilon(y) \le 1+v\}} \int_0^{1+v-\tau_\varepsilon(y)} \mathrm{d}r \mathrm{P}_{a_\varepsilon}\left[|\xi_r| \le \varepsilon\right]\right],
\end{aligned}$$

where a_ε is as previously a fixed point of the circle of radius ε centered at the origin. We can rewrite the previous expression as

$$\mathrm{E}_0\left[\int_0^{1+v} \mathrm{d}r\, \mathbb{I}_{\{\tau_\varepsilon(y)\le 1+v-r\}} \mathrm{P}_{a_\varepsilon}\left[|\xi_r|\le\varepsilon\right]\right] = \int_0^{1+v} \mathrm{d}r\, \mathrm{P}_0\left[\tau_\varepsilon(y)\le 1+v-r\right]\mathrm{P}_{a_\varepsilon}\left[|\xi_r|\le\varepsilon\right].$$

We apply this calculation with $v = v_\varepsilon = |\log\varepsilon|^{-1}$. For $r \in [0, v_\varepsilon]$, $\mathrm{P}_0\left[\tau_\varepsilon(y)\le 1+v_\varepsilon-r\right]$ is bounded from below by $\mathrm{P}_0\left[\tau_\varepsilon(y)\le 1\right]$, and thus

$$\mathrm{P}_0\left[\tau_\varepsilon(y)\le 1\right]\int_0^{v_\varepsilon} \mathrm{d}r \mathrm{P}_{a_\varepsilon}\left[|\xi_r|\le\varepsilon\right] \le \mathrm{E}_0\left[\int_0^{1+v_\varepsilon} \mathrm{d}s \mathbb{I}_{\{|\xi_s-y|\le\varepsilon\}}\right] - \int_{v_\varepsilon}^{1+v_\varepsilon} \mathrm{d}r \mathrm{P}_{a_\varepsilon}\left[|\xi_r|\le\varepsilon\right]\mathrm{P}_0\left[\tau_\varepsilon(y)\le 1+v_\varepsilon-r\right].$$

From the bound (36), we have

$$\mathrm{E}_0\left[\int_0^{1+v_\varepsilon} \mathrm{d}s \mathbb{I}_{\{|\xi_s-y|\le\varepsilon\}}\right] \le \pi\varepsilon^2\int_0^1 \mathrm{d}s p_s(y) + \varepsilon^3\Psi_1(y) + v_\varepsilon\varepsilon^2 \mathrm{e}^{-|y|^2/10}.$$

On the other hand, by (37),

$$\begin{aligned}
&\int_{v_\varepsilon}^{1+v_\varepsilon} \mathrm{d}r \mathrm{P}_{a_\varepsilon}\left[|\xi_r|\le\varepsilon\right]\mathrm{P}_0\left[\tau_\varepsilon(y)\le 1+v_\varepsilon-r\right] \\
&\quad\ge \int_{v_\varepsilon}^{1+v_\varepsilon} \mathrm{d}r \mathrm{P}_{a_\varepsilon}\left[|\xi_r|\le\varepsilon\right]\frac{\varepsilon^2}{f(\varepsilon)}\left(\pi\int_0^{1+v_\varepsilon-r} \mathrm{d}s p_s(y) - \varepsilon\Psi_1(y)\right) \\
&\quad= \frac{\pi\varepsilon^2}{f(\varepsilon)}\left(\int_{v_\varepsilon}^{1+v_\varepsilon} \mathrm{d}r \mathrm{P}_{a_\varepsilon}\left[|\xi_r|\le\varepsilon\right]\right)\left(\int_0^1 \mathrm{d}s p_s(y) - \frac{\varepsilon}{\pi}\Psi_1(y)\right) \\
&\qquad - \frac{\pi\varepsilon^2}{f(\varepsilon)}\left(\int_{v_\varepsilon}^{1+v_\varepsilon} \mathrm{d}r \mathrm{P}_{a_\varepsilon}\left[|\xi_r|\le\varepsilon\right]\int_{1+v_\varepsilon-r}^1 \mathrm{d}s p_s(y)\right).
\end{aligned}$$

Straightforward estimates give

$$\int_{v_\varepsilon}^{1+v_\varepsilon} \mathrm{d}r \mathrm{P}_{a_\varepsilon}\left[|\xi_r|\le\varepsilon\right] = \varepsilon^2\left(\frac{1}{2}\log|\log\varepsilon| + O(1)\right)$$

and

$$\int_{v_\varepsilon}^{1+v_\varepsilon} \mathrm{d}r \mathrm{P}_{a_\varepsilon}\left[|\xi_r|\le\varepsilon\right]\int_{1+v_\varepsilon-r}^1 \mathrm{d}s p_s(y) \le \varepsilon^2\Psi_2(y),$$

where

$$\Psi_2(y) = \int_0^1 \mathrm{d}s \log\left(\frac{1}{1-s}\right) p_s(y)$$

is integrable over $\mathbb{R}^2$. Summarizing, we have

$$\mathrm{P}_0\left[\tau_\varepsilon(y)\le 1\right]\int_0^{v_\varepsilon} \mathrm{d}r \mathrm{P}_{a_\varepsilon}\left[|\xi_r|\le\varepsilon\right]$$

$$\leq \left(\pi\varepsilon^2 \int_0^1 \mathrm{d}s p_s(y)\right) \times \left(1 - \frac{\left(\frac{1}{2}\log|\log\varepsilon| - K\right)\varepsilon^2}{f(\varepsilon)}\right)$$
$$+ \left(\varepsilon^3 + O\left(\varepsilon^3 \frac{\log|\log\varepsilon|}{|\log\varepsilon|}\right)\right)\Psi_1(y) + v_\varepsilon \varepsilon^2 \mathrm{e}^{-|y|^2/10} + \frac{\pi\varepsilon^4}{f(\varepsilon)}\Psi_2(y).$$

Finally, it is easy to verify that

$$g(\varepsilon) \equiv \int_0^{v_\varepsilon} \mathrm{d}r \mathrm{P}_{a_\varepsilon}\left[|\xi_r| \leq \varepsilon\right] \geq \varepsilon^2 \left(|\log\varepsilon| - \frac{1}{2}\log|\log\varepsilon| - K'\right),$$

and so by dividing the two sides of the previous inequality by $g(\varepsilon)$, we obtain

$$\mathrm{P}_0\left[\tau_\varepsilon(y) \leq 1\right] \leq \frac{\pi}{|\log\varepsilon|}\int_0^1 \mathrm{d}s p_s(y) + \frac{\varphi_2(y)}{|\log\varepsilon|^2},$$

with a function φ_2 such that $\int \varphi_2(y)\mathrm{d}y < \infty$. This completes the proof of Lemma 6. □

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