Generalized Polynomial Identities, III

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Herstein [3] has proved that, for any simple ring $R$ with involution ($\ast$) which is not a ring of quaternions, every element which commutes with all symmetric elements is central. This theorem has an interpretation in the theory of generalized identities of rings with involution: If $R$ is simple and $[r, X + X^\ast]$ is a generalized identity of $(R, \ast)$, then either $R$ is a PI-ring (ring with polynomial identity) of degree $\leq 2$, or $r \in \text{Cent}(R)$. Such an interpretation leads one to believe, in view of [6-8], that (i) Herstein’s theorem has a proof in the theory of generalized identities with involution, and (ii) Herstein’s theorem can be generalized in this theory. The object of this paper is to unify some results of [6-8] in order to develop a method of handling problems of this type, and to prove a very general form of Herstein’s theorem as an application. Immediately we run into a minor problem because Herstein’s theorem is for rings without 1, whereas [6-8] treat rings with 1. There are systematic ways to extend such results on rings with 1 to rings without 1 (cf. [6]); we shall be content instead to sketch a method of developing a parallel theory of generalized identities in rings with involution without 1 and shall quote results of [6-8] which were proved without using the existence of 1.

Throughout this paper, $R$ is a ring with center $C$. An involution is an automorphism of degree 1 or 2; let $(R, \ast)$ denote the ring $R$ with involution ($\ast$). An involution ($\ast$) of $R$ induces an automorphism on $C$; we say that ($\ast$) is of the first (resp. second) kind if this automorphism has degree 1 (resp. degree 2) on $C$. An ideal of $(R, \ast)$ is an ideal of $R$ which is invariant under ($\ast$). $(R, \ast)$ is prime if $AB \neq 0$ for all nonzero ideals $A, B$ of $(R, \ast)$; $(R, \ast)$ is semiprime if $A^2 \neq 0$ for all nonzero ideals $A$ of $(R, \ast)$. $(R, \ast)$ is semiprime iff $R$ is semiprime, as is easy to show. Let $R_1$ be the ring with 1 formally adjoined to $R$ (i.e., the additive group $\mathbb{Z} \oplus R$, endowed with multiplication $(n_1, r_1)(n_2, r_2) = (n_1n_2, n_1r_2 + n_2r_1 + r_1r_2)$). $R$ is identified with $0 \oplus R$, an ideal of $R_1$, inducing an identification of $C_1$ with $\text{Cent}(R_1)$. An involution ($\ast$) on $R$ induces an involution on $R_1$, given by $(n, r)^* = (n, r^*)$; clearly $R$ is an ideal of $(R_1, \ast)$.

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Form \((R_1\{X\}, \ast)\) as in [6, Sect. 4], calling its elements \emph{generalized polynomials}. Any generalized polynomial \(f\) can be written (not necessarily uniquely) as the sum of "monomials" of the form \(h = r_1 Y_1 r_2 Y_2 \cdots r_t Y_t r_{t+1}, \) where \(Y \in \{X_1, X_1^*, X_2, X_2^*, \ldots\}.\) Call such \(Y\) the "indeterminates" of \(h\), call \(Y_1 Y_2 \cdots Y_t\) the \textit{fingerprint} of \(h\), and call each \(r_i\) a \textit{coefficient} of \(h\). Write \(f = \sum_{u=1}^{\infty} h_u\), suitable monomials \(h_u\). We also write \(f(X_1, X_1^*, \ldots, X_m, X_m^*)\) to denote that \(\bigcup_u \{\text{indeterminates of } h_u\} \subseteq \{X_1, X_1^*, \ldots, X_m, X_m^*\}\). The \textit{degree} of \(h_u\) in the \(i\)th indeterminate is the sum of the degrees of \(X_i\) and \(X_i^*\) in \(h_u\); \(\text{deg}(h_u) = \sum_i\) (degree of \(h_u\) in the \(i\)th indeterminate). A \textit{generalized monomial} of \(f\) is the sum of those \(h_u\) with the same fingerprint. (Note that we get the same (nonzero) generalized monomials, regardless of the choice of the \(h_u\).) Say \(f\) is \textit{weakly homogeneous} if the fingerprints of all generalized monomials of \(f\) have the same degree; \(f\) is \textit{multilinear} if the fingerprint of each generalized monomial has degree 1 in each indeterminate "occurring" in \(f\). For example, \(r_1 X_1 r_2 X_2^* + X_2 X_1^* r_3\) is multilinear, whereas \(X_1 X_1^*\) is not multilinear. Define \((R, \ast)(f) = \{f(r_1, r_1^*, \ldots, r_m, r_m^*) \mid r_i \in R\}\) and \(R(f) = \{f(r_1, r_2, \ldots, r_{2m-1}, r_{2m}) \mid r_i \in R\}\). Clearly, \((R, \ast)(f) \subseteq R(f)\). We say that \(f\) is \textit{special} (on \((R, \ast)\)) if \((R, \ast)(f) = R(f)\). Also, \(f\) is a GI (generalized identity) of \((R, \ast)\) if \((R, \ast)(f) = 0\); \(f\) is \((R, \ast)-\text{proper}\) if some generalized monomial of \(f\) is not a GI of \((R, \ast)\). Clearly, if \(f(X_1, X_1^*, \ldots, X_m, X_m^*)\) is a special GI of \((R, \ast)\), then \(f(X_1, \ldots, X_{2m})\) is a GI of \(R\).

\textbf{Remark 1.} Every multilinear generalized monomial is special.

\textbf{Theorem A} [6, Theorem 7]. \textit{If }\((R, \ast)\textit{ is prime and }\ast\textit{ is of the second kind, then every multilinear GI of }\((R, \ast)\textit{ is special.}\)

Following Baxter and Martindale [2], say a (left) \(R\)-module \(M\) is \((R, \ast)\)-\textit{faithful} if \(rM \neq 0\) or \(r^*M \neq 0\) for each nonzero \(r\) in \(R\). \((R, \ast)\) is \textit{primitive} if some irreducible \(R\)-module \(M\) is \((R, \ast)\)-faithful; in this case, call \(M\) a \textit{faithful, irreducible }\((R, \ast)\)-module, and note that \(P = \{r \in R \mid rM = 0\}\) is a primitive ideal of \(R\) such that \(P \cap P^* = 0\).

\textbf{Proposition B} [6, Proposition 6]. \textit{If }\((R, \ast)\textit{ is primitive, then either }R\textit{ is primitive or every GI of }\((R, \ast)\textit{ is special.}\)

\textbf{Theorem C} [8, Theorem 5]. \textit{Any prime ring with involution }\((R, \ast)\textit{ can be embedded in a primitive ring with involution satisfying each multilinear GI of }\((R, \ast)\textit{.}\)

\textbf{Remark 2.} If \(D\) is a division ring and \(R\) is a dense subring of \(\text{End}_DM_D\), then \(D = \text{End}_R M \subseteq \text{End}_D M\) and \(R \subseteq \text{End}_D M\); if \(F\) is a maximal subfield of \(D\), then the subring \(R'\) of \(\text{End}_D M\) generated by \(R\) and \(F\) has the following well-known properties (cf. [1]): (i) \(F = \text{Cent}(R')\); (ii) \(M\) is a faithful irreducible
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(iii) $R'$ satisfies every multilinear GI of $R$.

Theorem C can be refined a bit to produce

**Theorem 1.** Suppose $(R, *)$ is prime. Then either (i) every multilinear GI of $(R, *)$ is special or (ii) $(R, *)$ can be embedded in a ring with involution $(R'', *)$ satisfying every multilinear GI of $(R, *)$, such that $R''$ is a dense subring (with 1) of the ring of linear transformations of a vector space over some field $F = \text{Cent}(R'')$, and $(*)$ is of the first kind on $R''$.

**Proof.** We use Martindale's central closure, developed in [5]. Embed $(R, *)$ in a primitive ring with involution $(A, *)$ satisfying each multilinear GI of $(R, *)$. By Proposition B, we are done unless $A$ is primitive. The central closure $A'$ of $A$ is primitive and has an involution, which we also call $(*)$. By Theorem A, we are done unless $(*)$ is of the first kind on $A'$.

There exists a division ring $D$ and a $D$-vector space $M$, such that $A'$ is a dense subring of $\text{End}_D M$. Let $F$ be a maximal subfield of $D$, and, as in Remark 2, let $R''$ be the subring of $\text{End}_F M$ generated by $A'$ and $F$. If $C' = \text{Cent}(A')$, then $R'' \approx A' \otimes_{C'} F$. (This is a property of the central closure.) Hence $(*)$ induces an involution on $R''$, given by $(\sum a_i \alpha_i)^* = \sum a_i^* \alpha_i$, $a_i \in A'$, $\alpha_i \in F$. If $(*)$ is of the second kind, then we are done by Theorem A; hence, we may assume $(*)$ is of the first kind.

Q.E.D.

Theorem C and Theorem 1 lead us to study primitive rings with involution. For added accuracy, we need a notion of *height* of a generalized polynomial, as in [7]. Although we could use the definition of [7], we give another definition which leads to the same results, as one can easily verify: If $v$ is the smallest number of monomials whose sum is $f$, and if $\deg f = d$, then $ht(f) = dv$.

**Theorem 2.** There is an increasing function $\varphi: \mathbb{Z}^+ \cup \{0\} \to \mathbb{Z}^+ \cup \{0\}$ with the following property: Suppose $(R, *)$ is primitive with faithful, irreducible module $M$, and let $D = \text{End}_{R, P} M$, where $P = \{r \in R \mid rM = 0\}$, and suppose $f(X_1, X_1^*, \ldots, X_m, X_m^*)$ is a multilinear GI of $(R, *)$. For every generalized monomial $f_u$ of $f$ and for all $r_1, \ldots, r_m$ in $R$, $\dim(f_u(r_1, r_1^*, \ldots, r_m, r_m^*))M \leq \varphi(ht(f))$.

**Proof.** If every multilinear GI of $(R, *)$ is special, then we are done by [7, Theorem 1]. Otherwise, by Theorem 1, we may assume that $P = 0$ and $D$ is a field; then we are done by [7, Lemma 3].

Q.E.D.

Note for any ring $R'$ satisfying each multilinear GI of $R$, that $C \subseteq \text{Cent}(R')$.

**Proof:** For any $c$ in $C$, $[c, X_i]$ is a GI of $R$, hence of $R'$, so $c \in \text{Cent}(R')$. Now, in view of the above results, it is highly desirable to have a method to pass from an arbitrary GI to a multilinear GI. In fact, the standard multilinearization procedure (cf. [6, Sect. 4; 6, Proposition 1]) yields

**Remark 3.** Suppose $f$ is a generalized polynomial, and $h$ is a generalized
monomial of \( f \) such that \( \deg h = \deg f \geq 1 \). There is a multilinear generalized polynomial \( f' \) satisfying: (i) \( \deg f' = \deg f \); (ii) all coefficients of \( f' \) are coefficients of \( f \); (iii) \( (R, \ast)(f') \subseteq (R, \ast)(f) \); (iv) there is a generalized monomial \( h' \) of \( f' \) which can be “specialized” to \( h \), for which in particular \( (R, \ast)(h) \subseteq (R, \ast)(h') \); (v) \( \text{ht}(f') \leq (\text{ht} f)! \).

A generalized polynomial \( f \) is called a classical polynomial if all of its coefficients are in \( C \). In this case, we can write \( f = \sum c_i h_i \), \( c_i \in C \), each \( h_i \) is a monomial which is equal to its fingerprint (i.e., \( h_i \) are classical monomials, with coefficients \( 1 \)). Any classical GI of \( (R, \ast) \) is merely called an identity of \( (R, \ast) \). A famous classical polynomial is \( S_k(x_1, \ldots, x_k) = \sum (sgn) x_{11} \cdots x_{kk} \), summed over all permutations \( \sigma \) of \( (1, \ldots, k) \).

Clearly any \( (R, \ast) \)-proper classical polynomial must have a coefficient \( c \) with \( Rc \neq 0 \). But \( Rc \) and \( \{ r \in R \mid cr = 0 \} \) are ideals of \( (R, \ast) \) whose product is 0. If \( (R, \ast) \) is prime, then \( cr \neq 0 \) for all \( r \neq 0 \) in \( R \).

A semiprime ring \( R \) is a PI-ring if \( S_{2m} \) is an identity of \( R \); the smallest such \( m \) is the degree of \( R \). This definition coincides with the usual definition of degree, if \( R \) is central simple, by the Anitsur–Levitzki theorem. We are ready to begin to generalize Herstein’s theorem. Let \( [a, b] \) denote \( ab - ba \), and let \( \lceil x \rceil \) denote “the greatest integer in \( x \).”

**THEOREM 3.** Let \( (R, \ast) \) be a prime ring with involution, and suppose \( f(x_1, x_1^*, \ldots, x_m^*, x_m^*) \) is an \( R \)-proper classical polynomial of degree \( d > 0 \), such that \( [r, f(r_1, r_1^*, \ldots, r_m^*, r_m^*)] = 0 \) for all \( r_1, \ldots, r_m \) in \( R \). Then either (i) \( r \in C \), or (ii) \( R \) is a PI-ring of degree \( \leq \max(2, d) \) (and if \( [r, f] \) is a special GI of \( (R, \ast) \) then \( R \) has degree \( \leq \lceil d/2 \rceil \)).

**Proof.** Clearly \( [r, f(x_1, x_1^*, \ldots, x_m^*, x_m^*)] \) is a GI of \( (R, \ast) \); by Remark 3, we may assume \( f \) is multilinear, so that \( d = m \). Thus, by Theorem C, we may assume that \( (R, \ast) \) is primitive. Write \( f = \sum c_i h_i \), \( c_i \in C \), classical monomials \( h_i \). Let \( M \) be a faithful, irreducible \( (R, \ast) \)-module, and \( P = \{ r \in R \mid rM = 0 \} \); let \( D = \text{End}_{R/P} M \).

**Case I.** \( [r, f] \) is a special GI of \( (R, \ast) \). Then \( [r, f(x_1, \ldots, x_{2m})] \) is a GI of \( R \), and thus of the anti-isomorphic rings \( R/P \) and \( R/P^\ast \). Let \( t = \lceil d/2 \rceil \). Viewing \( R/P \) as a dense subring of \( \text{End}_{M_D} \), we obtain \( F \) and \( R' \) as in Remark 2. \( R' \) has natural \( R_1 \)-bimodule actions, given by \( (n, r)x = nx + (r + P)x \) and \( x(n, r) = nx + x(r + P) \), for \( n \) in \( Z \), \( r \) in \( R \), \( x \) in \( R' \); we use these actions to evaluate \( f \) on \( R' \). Let \( \{ y_k \} \) be an \( F \)-basis of \( M \), and define \( e_{ij} \) by \( e_{ij}(y_j) = y_i \) and \( e_{ij}(y_k) = 0 \) if \( k \neq j \).

Suppose \( M \) has \( F \)-dimension \( > t \). Then \( f(e_{11}, e_{12}, e_{22}, \ldots) = c_{11}e_{11}, c_1 \) being the coefficient of \( x_1 \cdots x_m \) in \( f \). Hence \( c_1[r, e_{11}] = [r, c_{11}] = 0 \). Likwise, \( c_u[r, e_{ij}] = 0 \) for each coefficient \( c_u \); by symmetry, \( c_u[r, e_{ij}] = 0 \) for every \( e_{ij} \).

By the density theorem, \( (c_{ij}^* + P) \in \text{Cent}(R/P) \), for each \( c_{ij} \).

Now \( R/P \) has degree \( n \) iff \( R/P^\ast \) has degree \( n \), in which case \( R \) has degree \( n \).
Therefore, we have proved that either $R$ has degree $\leq [d/2]$, or $c_u r + P \in \text{Cent}(R/P)$. Similarly, we are done unless $c_u r + P^* \in \text{Cent}(R/P^*)$. Thus, either $R$ has degree $\leq [d/2]$ or $c_u r \in C$, for all coefficients $c_u$, implying (since $f$ is $(R, *)$-proper) $r \in C$.

**Case II.** The GI $[r, f]$ of $(R, *)$ is not special. By Theorem 1, we may assume that $R$ is primitive and $D = C$, a field. First assume that $\dim M$ is finite. Then $R$ is central simple, and the nature of $(*)$ is well known. In fact, we may assume that either $(*)$ is the transpose or the canonical symplectic involution with respect to a suitable set of matric units $e_{ij}$. If $\dim M > m$, then for some $\mu$ in $C$ (depending only on $f$), $e_{ij} + \mu e_{ij} \in (R, *)(f)$ for all $i \neq j$. (For example, if $(*)$ is the transpose, send each $X_i$ to $e_{1i} ;$ if $(*)$ is canonical symplectic, send $X_1$ to $e_{11}$, $X_2$ to $e_{12}$, $X_3$ to $e_{22}$, etc.) If $\dim M > 2$, we conclude that $r \in C$. Hence, if $M$ is finite-dimensional, then either $r \in C$ or $R$ has PI-degree $\leq \max(2, m)$.

Therefore, we may assume that $M$ is infinite-dimensional. Suppose soc $R = 0$. Then, by Theorem 2, $[r, c_u h_u]$ is an identity of $(R, *)$ for all $u$; by Remark 1, $[r, c_u X_1 \cdots X_m]$ is a GI of $R$, for each $u$. Embedding $R$ in $R'$ as in Remark 2, and sending $X_2, \ldots, X_m$ each to 1, we have $0 = [r, c_u X_1] = c_u [r, r_1]$ for each $r_1$ in $R'$. Since $f$ is $(R, *)$-proper, $[r, r_1] = 0$ for all $r_1$ in $R$, so $r \in \text{Cent}(R)$, and we are done.

On the other hand, suppose soc $R \neq 0$. By the structure theorem on involutions of primitive rings with socle (cf. [4, p. 82, Theorem 1]), we may have chosen $M$ to be self-dual relative to some nondegenerate scalar product $g : M \times M \to F$, such that $(*)$ can be identified with the adjoint relative to $g$. Since $F$ is a field, we will choose the identity map to be the anti-automorphism of $F$ used in [4, p. 83, line 1]. By [4, p. 83, Theorem 2], $g$ is Hermitian or skew Hermitian.

We claim that, for any vector $y$ in $M$, $ry$ and $y$ are $F$-dependent. Indeed, suppose that $ry$ and $y$ are $F$-independent. Write

$$f'(X_1, X_1^*, \ldots, X_{2m}, X_{2m}^*) = f(X_1^* X_2, (X_1^* X_2)^*, \ldots, X_{2m-1}^* X_{2m}, (X_{2m-1}^* X_{2m})^*) = f(X_1^* X_2, X_2^* X_1, \ldots, X_{2m-1}^* X_{2m}, X_{2m}^* X_{2m-1}).$$

Since $(R, *)(f') \subseteq (R, *)(f)$, $[r, f']$ is a GI of $(R, *)$. Moreover, $f'$ has the same coefficients as $f$, so we may assume that $c X_{2m}^* X_{2m-1} \cdots X_2^* X_1$ is not a GI of $(R, *)$, where $c$ is the appropriate coefficient. Thus, $cy' \neq 0$ for all $y'$ in $M$. Now let $y_0 = ry$ and $V_0 = Fy$, and make the following inductive definitions (for $i \geq 1$): $V_i = V_{i-1} + F y_{i-1}$. Pick $x_{2i-1}$ such that $x_{2i-1} V_{i-1} = 0$ and $x_{2i-1} y_{i-1} = y_{i-1}$. Then choose $y'_{i-1}$ such that $g(y'_{i-1}, y_{i-1}) = 1$, and choose $y_{i} \notin V_i$, such that $g(V_i, y_i) = 0$. Define $x_{2i}$ by $x_{2i} : z \mapsto g(z, y_i) y'_{i-1}$, for $z$ in $M$. Then $x_{2i} V_i = 0$. Also, $x_{2i}^* : z \mapsto y_{i} g(y'_{i-1}, z) = g(y_{i-1}, z) y_i$, so
$x_m^* y_{l-1} = y_l$. But then 

$$[r, f'(x_1, x_1^*, ..., x_{2m}, x_{2m}^*)] = cx_2^* x_{2m-1} \cdots x_2^* x_1 y_0 = cy_m \neq 0,$$

contrary to $[r, f']$ being a GI of $(R, \ast)$.

Hence, given $y$ in $M$, one can find $\mu$ in $F$ with $ry = \mu y$. Moreover, $\mu$ is independent of $y$. (Indeed, if $r_1 = \mu_1 y_1$ and $r_2 = \mu_2 y_2$, and if $\mu_1 \neq \mu_2$, then $y_1$ and $y_2$ are $F$-independent. But then $\mu_1 y_1 + \mu_2 y_2 = r(y_1 + y_2) = \mu_0 (y_1 + y_2)$ for some $\mu_0$ in $F$, and thus $\mu_1 = \mu_2 = \mu_0$.) Therefore $r \in C$.

Q.E.D.

**Corollary.** If $(R, \ast)$ is prime and if $r$ commutes with all symmetric (resp. antisymmetric) elements, then either $r \in C$ or $R$ is a PI-ring of degree $\leq 2$.

**Proof.** Set $f = x_1 + x_1^*$ or $f = x_1 - x_1^*$ in Theorem 3.

Theorem 3 can itself be generalized very naturally. To do this, we need an involution analog of a generalization of [6, Theorem 4].

**Theorem 4.** Suppose $(R, \ast)$ is prime, and $f(X_1, X_1^*, ..., X_m, X_m^*)$ is weakly homogeneous, with $\deg f \geq 1$ and $(R, \ast)(f) \subseteq C$. Then either $f$ is a GI of $(R, \ast)$ or $R$ is a PI-ring of degree $\leq \varphi((ht[f, X_{m+1}]))htf$, $\varphi$ as in Theorem 2.

**Proof.** We are done unless there is a generalized monomial $f_n (of)$ which is not a GI of $(R, \ast)$. But $[X_{m+1}, f]$ is a GI of $(R, \ast)$, and, by Remark 3, there is a multilinear GI $[X_{m+1}, f]$ of $(R, \ast)$, with generalized monomial $f' X_{m+1}$ such that $(R, \ast)(f') \subseteq (R, \ast)(f_n)$. Let $t = (ht[X_{m+1}, f])/ \geq ht[X_{m+1}, f']$.

**Case I.** $[X_{m+1}, f']$ is not a special GI of $(R, \ast)$. Then, by Theorem 1, we may assume that $(R, \ast) \subseteq (R', \ast)$, with $R'$ a dense subring of the ring of endomorphisms of some vector space $M$ over a field $F$. For any $x_1, ..., x_m$ in $R$, with $x_{m+1} = 1$, dim$(f'_n(x_1, x_1^*, ..., x_m, x_m^*)M) = \deg(f'_n(x_1, x_1^*, ..., x_m, x_m^*)x_{m+1}M) \leq \varphi(t)$, by Theorem 2. Applying this argument for each $f_n$, we see that dim$(f(x_1, x_1^*, ..., x_m, x_m^*)M) \leq \varphi(t) ht(f)$ for each $x_1, ..., x_m$ in $R$. Thus, if $(R, \ast)(f) \neq 0, M$ has $F$-dimension $\leq \varphi(t) h(f)$, so $R$ is a PI-ring of degree $\leq \varphi(t) h(f)$.

**Case II.** $[X_{m+1}, f']$ is a special GI of $(R, \ast)$. By Theorem C, we may assume that $(R, \ast)$ is primitive. Let $M$ be a faithful, irreducible $(R, \ast)$-module, and let $P = \{ r \in R | rM = 0 \}$. Assume $f(X_1, ..., X_{2m})$ is not a GI of $R$ (for otherwise we are done). Since $R$ is the subdirect product of $R/P$ and $R/P^*$, we may assume that $f(X_1, ..., X_{2m})$ is not a GI of $R/P$. Moreover, $R/P$ and $R/P^*$ are anti-isomorphic, so it suffices to prove that $R/P$ is a PI-ring of degree $\leq \varphi(t) h(f)$. In other words, replacing $f$ by $f(X_1, ..., X_{2m})$, we may assume that $P = 0$. Let $R'$, $M$, and $F$ be as in Remark 2. As in the proof of Case I, dim $M \leq \varphi(t) h(f)$, so $R$ is PI of degree $\leq \varphi(t) h(f)$. Q.E.D.

**Remark 4.** If a generalized polynomial $f$ is homogeneous in the first indeterminate and $f \neq (R, \ast)(f) \subseteq C$, then there are $x_2, ..., x_m$ in $R$, such that $f''(X_1, X_1^*) = f(X_1, X_1^*, x_2^*, ..., x_m, x_m^*)$ has the property $0 \neq (R, \ast)(f'') \subseteq$
Moreover, \( f'' \) is homogeneous and \( \text{ht}(f'') \leq \text{ht}(f) \); by Theorem 4, \( R \) is a PI-ring of degree \( \leq \varphi(t) \text{ht}(f) \). Thus, in Theorem 4 (and in its consequences) we can replace the condition "\( f \) is weakly homogeneous" by "\( f \) is homogeneous in some indeterminate." Note that this trick can be used to decrease dramatically the bound of the PI-degree of \( R \), given in Theorem 4.

**Theorem 5.** If \( (R, \ast) \) is semiprime and \( f(X_1, X_1^*, ..., X_m, X_m^*) \) is a weakly homogeneous generalized polynomial of degree \( \geq 1 \) with \( (R, \ast)(f) \subseteq C \), then \( (R, \ast) \) is a subdirect product of a semiprime PI-ring with involution, of degree \( \leq \varphi((\text{ht}[X_{m+1}, f])!) \text{ht}(f) \), and a semiprime ring with involution, of which \( f \) is a GI.

**Proof.** Take \( (R, \ast) \) as a subdirect product of prime rings with involution, and apply Theorem 4.

**Theorem 6.** Suppose \( (R, \ast) \) is prime, \( f_1(X_1, X_1^*, ..., X_m, X_m^*) \) is a weakly homogeneous generalized polynomial of degree \( d_1 \), and \( f_2(X_1, X_1^*, ..., X_u, X_u^*) \) is an \( R \)-proper classical polynomial of degree \( d_2 \). If \( [f_1(a_i, a_i^*, ..., a_m, a_m^*), f_2(b_1, b_1^*, ..., b_u, b_u^*)] = 0 \), all \( a_i, b_i \) in \( R \), then one of the three following possibilities must hold: (i) \( d_1 = 0 \), and \( f_1 \) is a constant in \( C \); (ii) \( R \) is a PI-ring of degree \( \leq \max(\varphi((\text{ht}[f_1, X_{m+1}]!) \text{ht}(f_1), d_2, 2) \); (iii) \( f_1 \) is a GI of \( R \).

**Proof.** We may assume \( f_2 \) is multilinear. Suppose \( R \) is not a PI-ring of degree \( \leq \max(d_2, 2) \). For any nonzero \( r \) in \( (R, \ast)(f_1) \), we have \( [r, f_2] \) is a GI of \( (R, \ast) \). Hence, by Theorem 3, \( (R, \ast)(f_1) \subseteq C \). Therefore, by Theorem 4, either \( f_1 \) is a constant in \( C \) or \( f_1 \) is a GI of \( (R, \ast) \) or \( R \) is a PI-ring of degree \( \leq \varphi((\text{ht}[f_1, X_{m+1}]!) \text{ht}(f_1) \). Q.E.D.

**Theorem 7.** Suppose \( (R, \ast) \) is semiprime, with \( f_1, f_2 \) as in Theorem 5. Then \( (R, \ast) \) is a subdirect product of a semiprime PI-ring with involution \( (R_0, \ast) \) and a semiprime ring with involution \( (R_1, \ast) \), such that either \( f_1 \) is a constant whose image in \( R_1 \) lies in \( \text{Cent}(R_1) \), or \( f_1 \) is a GI of \( (R_1, \ast) \). (We can also bound the degree of \( R_0 \) by a function of \( \deg f_2 \) and \( \text{ht} f_1 \).)

**Proof.** Write \( (R, \ast) \) as a subdirect product of prime images, and apply Theorem 6.

Although the results of this paper are given only for rings with involution, they imply results for arbitrary rings, as is seen by the following standard trick:

Let \( R \) be any ring and introduce the exchange involution \( (\circ) \) on \( R \oplus R^\circ \), given by \( (r_1, r_2^\circ) = (r_2, r_1^\circ) \), where \( R^\circ \) is the opposite ring of \( R \). If \( R \) is prime (resp. semiprime) then \( (R \oplus R^\circ, \circ) \) is prime (resp. semiprime), and every GI of \( (R \oplus R^\circ, \circ) \) is special (as is easy to see). Hence we have

**Theorem 3'.** Let \( R \) be a prime ring, and suppose \( f(X_1, ..., X_m) \) is an \( R \)-proper
polynomial of degree $d$, such that $[r, f(r_1, \ldots, r_m)] = 0$ for all $r_1, \ldots, r_m$ in $R$. Then either $r \in C$ or $R$ is a PI-ring of degree $\leq [d/2]$.

**Theorem 6'.** Suppose $R$ is prime, $f_1$ is a weakly homogeneous generalized polynomial of degree $d_1$, and $f_2$ is an $R$-proper classical polynomial such that $[f_1, f_2]$ is a GI of $R$. Then either (i) $d_1 = 0$ and $f_1$ is a constant in $C$; (ii) $R$ is a PI-ring (of degree bounded by a function of $ht f_1$ and $deg f_2$); or (iii) $f_1$ is a GI of $R$.

**Theorem 7'.** Suppose $R$ is semiprime, with $f_1, f_2$ given as in Theorem 6'. Then $R$ is a subdirect product of a semiprime PI-ring $R_0$ (of degree bounded by a function of $ht f_1$ and $deg f_3$) and a semiprime ring $R_1$, such that either $f$ is a constant whose image in $R_1$ lies in $\text{Cent}(R_1)$, or $f$ is a GI of $R_1$.

Finally, we note that the situation of Theorem 6 can be generalized still further. Namely, suppose $(R, *)$ is prime and $f_1$ and $f_2$ are both weakly homogeneous generalized polynomials of $(R, *)$ such that $[f_1, f_2]$ is a GI of $(R, *)$. Under what condition can we conclude that $(R, *)$ satisfies a proper GI, or better yet, a PI? The only positive results I have involve technical assumptions about linear independence of various coefficients.

**Appendix**

The referee has given an interesting alternative method to obtain the results of this paper, based on the ideas of Martindale (Prime rings with involution and generalized polynomial identities, *J. Algebra* 22 (1972), 502–516), which we shall refer to as [M]. We shall state the referee's result (Theorem M) and see how it relates to Theorem 2 and similar notions.

Suppose $M$ is a vector space over a field $F$, and $T = \text{End}_F M$. Let $P$ be a dense $F$-subalgebra of $T$, and suppose $P$ has an involution ($*$). Call a subset $R$ of $P$ weakly $*$-dense if the following property is satisfied:

For any $F$-independent elements $x_1, \ldots, x_k$ of $T$, either $\sum Fx_i \cap \text{soc } T \neq 0$ or, given $F$-independent elements $y_1, \ldots, y_m$ of $M$ and a finite-dimensional subspace $U_0$ of $M$, we can find an element $a$ in $R$, such that $x_i a = x_i a^* = 0$, $2 \leq i \leq k$, $x_i a^* = 0$, and $x_i a y_1, \ldots, x_i a y_m$ are $F$-independent modulo $U_0$.

**Proposition.** $P$ is weakly $*$-dense.

**Proof.** Same as [M, Theorem 4.6], except that in the last paragraph we take $a = xtr^*$.

Let $A_0(f)$ denote the $F$-subspace of $P$ generated by the coefficients of a generalized polynomial $f$. Let Statement A be the assertion: "There is a
generalized polynomial \( f' \) with coefficients in \( A_0(f) \), equal to \( f \) in \( (P\{X\}, \ast) \), such that each monomial of \( f' \) has at least one coefficient in \( \text{soc} T \)."

**Theorem M.** Every generalized multilinear identity of \((P, \ast)\) satisfies statement \( A \).

*Proof.* We follow the same procedure as in [M, Theorem 3.5]. Given a generalized multilinear identity \( f \) of \((P, \ast)\), write \( f \) in \((P\{X\}, \ast)\) as \( f_1 + f_2 \) with \( A_0(f_i) \subseteq A_0(f) \), \( i = 1, 2 \), such that every monomial of \( f_1 \) has a coefficient in \( \text{soc} T \) under which stipulation \( A_0(f_i) \) has the smallest possible dimension. Take a base \( G \) of \( A_0(f) \), consisting only of elements of \( f_2 \). The proof now ends analogously to [M, Theorem 3.5]. (Note that an intricate result of Amitsur, stated in [M, Theorem 3.3], is needed.) Q.E.D.

A slightly stronger result can be obtained by slightly modifying [7, Lemma 3]. Form Statement \( A' \) by adding to the end of Statement \( A \) the phrase, "with rank bounded by a function of \( ht(f) \)."

**Theorem M'.** Every generalized multilinear identity of \((P, \ast)\) satisfies statement \( A' \).

*Proof.* Use the notation and proof of [7, Lemma 3]. We actually obtain the apparently stronger statement, that if \( f \) is \((V, (u_t))\)-valued for suitable \( u_t \), then Statement \( A' \) holds for every generalized monomial \( f_n \) of \( f \). Setting up induction on \( ht(f) \), we may assume that \( f = f' \). But then, again by induction, \( f_1 \) satisfies Statement \( A' \), implying \( f_1 X_2 \ast X_{1\omega_{11}} \) satisfies Statement \( A' \). By symmetry, \( f_n \) (and thus \( f_2 \)) satisfy Statement \( A' \). Q.E.D.

(Note: A standard ultraproduct argument shows that Theorem M implies Theorem M'.) Define Statement \( A'' \) as: "For every generalized multilinear \( f_n \) of \( f \), and for every \( x_1, \ldots, x_n \) in \( P \), the rank of \( f_n(x_1, x_1\ast, \ldots, x_n, x_n\ast) \) is bounded by a function of \( ht(f) \)."

Clearly Statement \( A' \) implies Statement \( A'' \), because the generalized monomials of \( f \) and \( f' \) are the same. Thus, Theorem M' implies Theorem 2, and we have two additional methods to obtain the results of this paper. (In fact, Theorem M' gives faster proofs of the other theorems than Theorem 2.)

We claim, conversely, that Statement \( A'' \) implies Statement \( A' \). Indeed, we may assume that \( f \) is a generalized monomial, and that, for all \( x_1, \ldots, x_m \) in \( P \), rank \( f(x_1, x_1\ast, \ldots, x_m, x_m\ast) \leq \varphi(ht(f)) \). Let \( k = 2\varphi(ht(f)) + 1 \). Then \( S_k(f(X_1, X_1\ast, \ldots, X_m, X_m\ast)X_{m+1}, \ldots, f(X_1, X_1\ast, \ldots, X_m, X_m\ast)X_{m+k}) \) is a generalized identity of \((P, \ast)\), so Statement \( A' \) follows from Theorem M'.

Incidentally, it is impossible to strengthen Theorem M to the sentence, "If \( f \) is a generalized identity of \((P/\text{soc} P, \ast)\), then Statement \( A \) holds; just let \( P \) be the \( F \)-subalgebra of \( \text{End}_F M \) generated by the socle, with \( f = \)
$X_1X_2 - X_2X_1$. One can get theorems of this type by imposing some sort of "absoluteness" condition on $f$, such as $f$ being a generalized identity of $(P'/\text{soc } P', *)$ for every ultrapower $P'$ of $P$.

References