The growth in time of higher Sobolev norms of solutions to Schrödinger equations on compact Riemannian manifolds

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Abstract

In this paper, we shall estimate the growing speed for higher Sobolev norms of the solutions to Schrödinger equations on Riemannian manifolds \(d \geq 2\), under some bilinear Strichartz estimate assumptions.
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1. Introduction

Suppose \((M, g)\) is any compact Riemannian manifold of dimension \(d \geq 2\), without boundary. In this paper, we study the Cauchy problem for the cubic nonlinear Schrödinger equation posed on \(M\),

\[
\begin{aligned}
iu_t + \Delta u &= |u|^2 u, \\
u(0, x) &= u_0(x),
\end{aligned}
\]

\(\text{ }(1.1)\)
where the solution $u$ is a complex-valued function on $\mathbb{R} \times M$, and $\Delta$ denotes the Laplace–Beltrami operator associated to the metric $g$ on $M$. It is classical that smooth solutions of (1.1) satisfy the following two conservation laws,

$L^2$-mass:  
$$ \int_M |u(t, x)|^2 \, dx = \int_M |u_0(x)|^2 \, dx $$  
and

energy:  
$$ E(t) = \int_M |\nabla u(t, x)|^2 \, dx + \frac{1}{2} \int_M |u(t, x)|^4 \, dx = E_0. $$

In many cases, one can prove that for $u_0 \in H^s(M), s \geq 1$, there exists a unique global solution $u \in C(\mathbb{R}, H^s(M))$ to this problem (see Theorem 1.2). Using the conservation of mass and energy (1.2), (1.3), one can see that the $H^1$-norm of the solution is controlled by some constant. A natural question is to understand what happens to the $H^s$-norm, $s > 1$, when $|t| \to \infty$?

By the proof of the local existence, one can see that there is some $T = T(\|u_0\|_{H^1})$, for any $t \in [0, T]$,

$$ \|u(t)\|_{H^s} \leq C \|u_0\|_{H^s}. $$

Then, by the pacing argument, it is easy to get some exponential bound

$$ \|u(t)\|_{H^s} \leq C e^{C|t|}. $$

The first breakthrough improving this exponential rate was done by Bourgain in [3] where he proved that, in $\mathbb{T}^2$, (1.5) could be improved to a polynomial bound, i.e.

$$ \|u(t)\|_{H^s} \leq C|t|^{2(s-1)+}, \quad \text{for } |t| \to \infty, $$

which could also be extended to $\mathbb{R}^2$. Then in 1997, Staffilani [11] improved the bound to be $|t|^{(s-1)+}$ (in $\mathbb{R}^2$).

Here, we are interested in obtaining an abstract result and find some conditions to ensure these growth estimates.

First of all, let us give the main condition of this paper, which is some bilinear Strichartz estimates.

**Definition 1.1.** Let $0 \leq s_0 < 1$. We say that $S(t) = e^{it\Delta}$, the flow of the linear Schrödinger equation on $M$ stated above, satisfies property $(P_{s_0})$ if for all dyadic numbers $N$, $L$, and $u_0, v_0 \in L^2(M)$ localized on dyadic intervals of order $N, L$ respectively, i.e.

$$ 1_{N \leq \sqrt{-\Delta} < 2N}(u_0) = u_0 \quad \text{and} \quad 1_{L \leq \sqrt{-\Delta} < 2L}(v_0) = v_0, $$

the following estimate holds:

$$ \|S(t)u_0S(t)v_0\|_{L^2((0,1) \times M)} \leq C \left( \min(N, L) \right)^{s_0} \|u_0\|_{L^2(M)} \|v_0\|_{L^2(M)}. $$
In fact, such kind of bilinear Strichartz estimates was established and used by several authors in the context of the wave equations and of the Schrödinger equations. For example, in [8], Burq, Gérard, Tzvetkov showed that for Zoll surface of dimension 2, especially for $S^2$, $s_0 = \frac{1}{4} +$. Then in [9], they proved that $(P_{s_0})$ holds for $s_0 = \frac{1}{2} +$ for $S^2$ and $s_0 = \frac{3}{4} +$ for $S^2 \times S^1$. Also the results from Bourgain [5,6] and [7] proved that $s_0 = 0 +$ for $T^2$, $s_0 = \frac{1}{2} +$ for $T^3$, and $s_0 = \frac{2}{3} +$ for $\tilde{T}^3 = \mathbb{R}^3 / \prod_{j=1}^3 (\mathbb{Z}/a_j \mathbb{Z})$, where $a_j$ are pairwise irrational numbers. And Anton in [1] proved that $s_0 = \frac{3}{4} +$ for general manifolds with boundary and manifolds without boundary equipped with a Lipschitz metric $g$ for dimension $d = 2, 3$. Recently, this result is improved by Blair, Smith and Sogge in [2] to be $2 +$.

Under the condition $(P_{s_0})$, we have the following local well-posedness result (see the works by Bourgain [4], Ginibre [10], Burq, Gérard, Tzvetkov [8]).

**Theorem 1.2.** Suppose that there exist $C > 0$ and $0 \leq s_0 < 1$ such that condition $(P_{s_0})$ holds, then the Cauchy problem (1.1) is locally well-posed in $H^s(M)$ for any $s > s_0$. Moreover, if $s \geq 1$, then the solution is global.

We will prove this result briefly in Section 3 for the completeness of the paper.

The main result of this paper is:

**Theorem 1.3.** For any even integer $s > 1$, under the condition $(P_{s_0})$, if $u \in C(\mathbb{R}, H^s(M))$ is the solution to Cauchy problem (1.1) with the initial data $u_0 \in H^s(M)$, then the following estimate exists

$$
\|u(t)\|_{H^s(M)} \leq C |t|^{A(s-1)}, \quad \text{for } |t| \to \infty,
$$

(1.9)

here the constant $C$ is dependent on the $H^1$-norm control of the solution and

$$
\frac{1}{A} = \begin{cases}
(1 - \frac{d-2}{2(d-2s_0)})(1 - \frac{d-2}{d-s_0-1}) - , & d \geq 3, \\
(1 - s_0) - , & d = 2.
\end{cases}
$$

(1.10)

**Remark 1.4.** 1. From Theorem 1.2 and also from [8], one can see that the Cauchy problem (1.1) is local well-posed for $H^s$, $s > s_0$. Then the expression of $A$ shows that the lowest regularity of the local well-posedness threshold of the equation has some relationship with the growing speed of the Sobolev norm of high regularity. And if $s_0 \to 1$, $A$ would tend to infinity.

2. It is obvious that for a fixed $A$, if $s \to 1$, the norm would be controlled by some constant, and on the other hand, if $s \to \infty$, the speed would tend to infinity.

3. It is easy to see that on $\mathbb{R}^2$, $s_0 = 0 +$, $A = 1 +$, which meets the result of Staffilani [11].

4. The methods in this article would probably apply to any real $s > 1$, modulo some technical complications that we prefer to avoid for the sake of conciseness.

The proof of this result relies on Bourgain’s strategy. Let us describe it briefly. First we shall prove some nonlinear estimates (see [8]), in simpler version here, which helps to prove Theorem 1.2 by fixed point argument. Next, we shall deal with the growth of $\|u(t)\|_{H^s(M)}$ for some even integer $s$ following the idea from [4].

The paper is organized as follows. In Section 2, we shall give some notations and define some functional spaces, in Section 3, some nonlinear estimates will be proved, and we shall give the proof of the local well-posedness results in the end. Finally, in Section 4 we prove Theorem 1.3.
2. Notations

Suppose \((M, g)\) is the Riemannian manifold mentioned at the beginning of the paper, and thus the spectrum of \(\Delta\) is discrete. Let \(e_k \in L^2(M), k \in \mathbb{N}\), be an orthonormal basis of eigenfunctions of \(-\Delta\) associated to eigenvalues \(\mu_k\). Denote by \(P_k\) the orthogonal projector on \(e_k\). The Sobolev space \(H^s(M)\) is therefore equipped with the norm (with \(\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}\))

\[
\|u\|_{H^s(M)}^2 = \sum_k \langle \mu_k \rangle^s \|P_ku\|^2_{L^2(M)}.
\]

(2.1)

Next, we shall give the definition of Bourgain spaces:

**Definition 2.1.** The space \(X^{s,b}(\mathbb{R} \times M)\) is the completion of \(C_0^\infty(\mathbb{R}_t; H^s(M))\) for the norm

\[
\|u\|_{X^{s,b}(\mathbb{R} \times M)}^2 = \sum_k \|\tau + \mu_k\|^b \|\widehat{P_ku}(\tau)\|^2_{L^2(\mathbb{R}_t; L^2(M))} = \|e^{-it\Delta}u(t, \cdot)\|^2_{H^b(\mathbb{R}_t; H^s)},
\]

(2.2)

where \(\widehat{P_ku}(\tau)\) denotes the Fourier transform of \(P_ku\) with respect to the time variable. And we denote it as \(X^{s,b}\) when there is no confusion.

Then, for \(1 \geq T > 0\), we denote by \(X^{s,b}_T(M)\) the space of restrictions of elements of \(X^{s,b}(\mathbb{R} \times M)\) endowed with the norm

\[
\|u\|_{X^{s,b}_T(M)} = \inf \{\|\tilde{u}\|_{X^{s,b}(\mathbb{R} \times M)}, \tilde{u}|_{(-T,T) \times M} = u\}.
\]

(2.3)

The following proposition (see Ginibre [10] and Burq, Gérard, and Tzvetkov [8]) gathers basic properties of this space.

**Proposition 2.2.**

1. \(u \in X^{s,b}(\mathbb{R} \times M) \iff e^{-it\Delta}u(t, \cdot) \in H^b(\mathbb{R}, H^s(M)).\)
2. For \(b > \frac{1}{2}\), \(X^{s,b}(\mathbb{R} \times M) \hookrightarrow C(\mathbb{R}, H^s(M)),\) and \(X^{s,b}_T(M) \hookrightarrow C((-T, T), H^s(M)).\)
3. \(X^{0,\frac{1}{2}}(\mathbb{R} \times M) \hookrightarrow L^4(\mathbb{R}, L^2(M)).\)
4. For \(s_1 \leq s_2\) and \(b_1 \leq b_2\), \(X^{s_2,b_2}(\mathbb{R} \times M) \hookrightarrow X^{s_1,b_1}(\mathbb{R} \times M).\)

Then, from Lemma 2.3 of [8], the condition \((P_{s_0})\) is equivalent to the following statement:

For any \(b > \frac{1}{2}\) and any \(f, g \in X^{s,b}(\mathbb{R} \times M)\) satisfying

\[
\mathbb{1}_{N \leq \sqrt{-\Delta} < 2N}(f) = f, \quad \mathbb{1}_{L \leq \sqrt{-\Delta} < 2L}(g) = g,
\]

one has

\[
\|fg\|_{L^2(\mathbb{R} \times M)} \leq C \left(\min(N, L)\right)^{s_0} \|f\|_{X^{0,b}(\mathbb{R} \times M)} \|g\|_{X^{0,b}(\mathbb{R} \times M)}.
\]

(2.4)
3. The crucial nonlinear estimates in $X^{s,b}$

Let us first recall a result from Burq, Gérard, and Tzvetkov (Proposition 2.5 of [8]) with a slightly simpler proof.

**Lemma 3.1.** If condition $(P_{s_0})$ holds, then for any $1 \geq s > s_0$, there exists some $\frac{1}{4} < b < \frac{1}{2}$, such that for any $f, g \in X^{0,b}(\mathbb{R} \times M)$, satisfying

$$\frac{1}{N} \leq \sqrt{-\Delta} < 2N (f) = f \quad \text{and} \quad \frac{1}{L} \leq \sqrt{-\Delta} < 2L (g) = g,$$

one has

$$\|fg\|_{L^2(\mathbb{R} \times M)} \leq C \left( \min(N, L) \right)^{\frac{s}{2}} \|f\|_{X^{s,b}(\mathbb{R} \times M)} \|g\|_{X^{s,b}(\mathbb{R} \times M)}. \quad (3.1)$$

**Proof.** Without loss of generality, we assume that $N \leq L$.

First, by the equivalent relationship (2.4), we have

$$\|fg\|_{L^2(\mathbb{R} \times M)} \leq C \left( \min(N, L) \right)^{s_0} \|f\|_{X^{0, \frac{1}{2} + s_0}(\mathbb{R} \times M)} \|g\|_{X^{0, \frac{1}{2} + s_0}(\mathbb{R} \times M)}$$

$$= CN^{s_0} \|f\|_{X^{0, \frac{1}{2} + s_0}} \|g\|_{X^{0, \frac{1}{2} + s_0}}; \quad (3.2)$$

hence, by bilinear interpolation between (3.2) and (3.3), we get the result of the lemma, where $b = \frac{1}{4} \left( 1 + \frac{d-2x}{d-2s_0} \right) + \frac{d-2x}{d-2s_0} \epsilon_0$, and $(d - 2s) \epsilon_0 < s - s_0$. \[\square\]

**Remark 3.2.** A special case for the above lemma is $s = 1$, in which $b = \frac{1}{4} \left( 1 + \frac{d-2x}{d-2s_0} \right) + \epsilon_1$, and $\epsilon_1 = \frac{d-2}{d-2s_0} \epsilon_0$.

Now, let us state the nonlinear estimates.

**Proposition 3.3.** Under the condition of $(P_{s_0})$, let $s > s_0$. There exist $(b, b') \in \mathbb{R}^2$ satisfying

$$0 < b' < \frac{1}{2} < b, \quad b + b' < 1, \quad (3.4)$$

and $C > 0$ such that for every triple $\{u_j\}$ $(j = 1, 2, 3)$ in $X^{s,b}(\mathbb{R} \times M)$,

$$\|u_1 \tilde{u}_2 u_3\|_{X^{s-b'}(\mathbb{R} \times M)} \leq C \|u_1\|_{X^{s,b}(\mathbb{R} \times M)} \|u_2\|_{X^{s,b}(\mathbb{R} \times M)} \|u_3\|_{X^{s,b}(\mathbb{R} \times M)}. \quad (3.5)$$
**Proof.** Assume by density that \( u_j \in C_0^\infty(\mathbb{R} \times M) \). By duality argument, we just need to prove

\[
\left| \int_{\mathbb{R} \times M} \tilde{u}_0 u_1 \tilde{u}_2 u_3 \, dx \, dt \right| \leq C \| u_0 \|_{X^{s,b'}(\mathbb{R} \times M)} \prod_{j=1}^3 \| u_j \|_{X^{s,b}(\mathbb{R} \times M)},
\]

for some \( b, b' \) to be decided later, and \( u_0 \in X^{-s,b'}(\mathbb{R} \times M) \).

Then, let \( N_0, N_1, N_2, N_3 \) be dyadic integers, i.e. \( N_j = 2^{n_j}, n_j \in \mathbb{N}, j = 0, 1, 2, 3, \) and \( N := (N_0, N_1, N_2, N_3) \).

Denote \( J \) the left-hand side of (3.6), then

\[
J \lesssim \sum_N I(N),
\]

where

\[
I(N) = \left| \int_{\mathbb{R} \times M} \tilde{u}_0^{N_0} u_1^{N_1} \tilde{u}_2^{N_2} u_3^{N_3} \, dx \, dt \right|,
\]

with

\[
u_j^{N_j} = \frac{1}{2\pi} \sum_{N_j \leq (\mu_j)^{\frac{1}{2}} < 2N_j} \int_{\mathbb{R}} \hat{P}_k u_j(\tau) e^{i\tau \tau} \, d\tau, \quad \text{for } j = 0, 1, 2, 3.
\]

First, by Lemma 2.8 of [8], there exists \( \tilde{C} > 0 \) such that if \( N_0 \geq \tilde{C}(N_1 + N_2 + N_3) \), then for every \( p > 0 \), and \( b' > 0, b > \frac{1}{2} \), there exists

\[
I(N) \lesssim N_0^{-p} \| u_0 \|_{X^{-s,b'}(\mathbb{R} \times M)} \prod_{j=1}^3 \| u_j \|_{X^{s,b}(\mathbb{R} \times M)}.
\]

Hence, divide \( J \) into two parts, \( J_1 \) and \( J_2 \), where the summation is restricted to \( N_0 \geq \tilde{C}(N_1 + N_2 + N_3) \) in \( J_1 \) and other possibilities are in \( J_2 \). Thus for \( J_1 \), as \( N_1, N_2, N_3 \) could be controlled by \( N_0 \), we can choose \( p \) large enough such that the above sequence is geometric summable, hence we just need to deal with \( J_2 \).

By symmetry argument and because the presence of complex conjugates will play no role here, we can assume \( N_1 \geq N_2 \geq N_3 \), so \( N_0 \leq 3\tilde{C}N_1 \).

For \( s > s_0 \), there exists \( s_1 \leq s \), such that \( s_0 < s_1 < s \) (for example when \( s \leq 1 \) we can choose \( s_1 = s - \epsilon' \), where \( \epsilon' < s - s_0 \)). Therefore, by Hölder and Lemma 3.1, there exists \( b' = \frac{1}{4}(1 + \frac{d-2s_1}{d-2s_0}) + \epsilon_1 \), such that

\[
I(N) \lesssim \| u_0^{N_0} u_2^{N_2} \|_{L^2(\mathbb{R} \times M)} \| u_1^{N_1} u_3^{N_3} \|_{L^2(\mathbb{R} \times M)}
\]
\begin{align}
\lesssim & \left(\min(N_0, N_2)\right)^{s_1} \left(\min(N_1, N_3)\right)^{s_1} \prod_{j=0}^{3} \|u_j^N\|_{X^{0,b'}} \\
\lesssim & \left(N_2 N_3\right)^{s_1-s} \left(\frac{N_0}{N_1}\right)^{s} \|u_0^{N_0}\|_{X^{-r,b'}} \prod_{j=1}^{3} \|u_j^{N_j}\|_{X^{r,b'}}.
\end{align}

(3.9)

Here we choose \( b = 1 - \epsilon_1 - b' = \frac{1}{4} \left(3 - \frac{d-2}{d-2b_0}\right) - 2\epsilon_1. \)

Therefore,

\begin{equation}
\sum_{N_0 \leq \tilde{C}(N_1+N_2+N_3)} I(N) \lesssim \sum_{N_0 \leq 3\tilde{C}N_1} \left(\frac{N_0}{N_1}\right)^{s} \alpha(N_1) \beta(N_0) \prod_{j=2}^{3} \|u_j\|_{X^{r,b}},
\end{equation}

where \( \alpha(N_1) = \|u_1^{N_1}\|_{X^{r,b}} \) and \( \beta(N_0) = \|u_0^{N_0}\|_{X^{-r,b'}}. \) As \( N_0 \leq 3\tilde{C}N_1, \) there is some \( l_0 > 0 \) depending only on \( \tilde{C}, \) such that \( N_1 = 2^l N_0, \) for \( l \geq -l_0. \) Hence by Cauchy–Schwarz inequality in \( N_0, \)

\begin{align}
\sum_{N_0 \leq 3\tilde{C}N_1} \left(\frac{N_0}{N_1}\right)^{s} \alpha(N_1) \beta(N_0) &= \sum_{l \geq -l_0} \sum_{N_0 \leq 3\tilde{C}N_1} \left(\frac{N_0}{N_1}\right)^{s} \alpha(2^l N_0) \beta(N_0) \\
&\lesssim \sum_{l \geq -l_0} 2^{-sl} \left\{ \sum_{N_0 \leq 3\tilde{C}N_1} \alpha(2^l N_0)^2 \right\}^\frac{1}{2} \left\{ \sum_{N_0 \leq 3\tilde{C}N_1} \beta(N_0)^2 \right\}^\frac{1}{2} \\
&\lesssim \|u_0\|_{X^{-r,b'}} \|u_1\|_{X^{r,b}},
\end{align}

(3.10)

which finishes the proof for the proposition. \( \square \)

**Remark 3.4.** For \( s \geq 1, \) and the special case \( u_1 = u_2 = u_3 = u, \) we can have another form of estimate, which follows the same proof by choosing \( s_1 = 1 - \epsilon_1. \) Then (3.9) is changed to be

\begin{align}
I(N) \lesssim \|u_0^{N_0} u_2^{N_2}\|_{L^2(\mathbb{R} \times M)} \|u_1^{N_1} u_3^{N_3}\|_{L^2(\mathbb{R} \times M)} \\
\lesssim \left(\min(N_0, N_2)\right)^{s_1} \left(\min(N_1, N_3)\right)^{s_1} \prod_{j=0}^{3} \|u_j^N\|_{X^{0,b'}} \\
\lesssim \left(N_2 N_3\right)^{-\epsilon_1} \left(\frac{N_0}{N_1}\right)^{s} \|u_0^{N_0}\|_{X^{-r,b'}} \|u_1^{N_1}\|_{X^{r,b}} \prod_{j=2}^{3} \|u_j^{N_j}\|_{X^{1,b'}}.
\end{align}

(3.11)

Then we can get an estimate as follows:

\begin{equation}
\|u^2\|_{X^{s,-b'}(\mathbb{R} \times M)} \leq C \|u\|_{X^{s,b}(\mathbb{R} \times M)} \|u\|^2_{X^{1,b}(\mathbb{R} \times M)}.
\end{equation}

(3.12)

On the other hand, because \( b' \) and \( b \) only depend on \( s_1, \) so in this case we can choose \( \tilde{b}' = \frac{1}{4} (1 + \frac{d-2}{d-2b_0}) + (\frac{1}{2} \frac{1}{d-2b_0}) + 1) \epsilon_1 \) and \( \tilde{b} = \frac{1}{4} (3 - \frac{d-2}{d-2b_0}) - \epsilon_2, \) where \( \epsilon_2 = (2 + \frac{1}{2} \frac{1}{d-2b_0}) \epsilon_1. \) This estimate will be used later.
Remark 3.5. It is obvious that $b$ corresponding to $s$, $s < 1$, is smaller than $\tilde{b}$.

Then, by the above estimates, we can prove Theorem 1.2.

Proof of Theorem 1.2. The proof for local well-posedness could be referred to [8], but to be complete, we shall state it briefly.

Denote $S(t) = e^{it\Delta}$ the free evolution. By Duhamel formula, the function $u \in C^\infty(\mathbb{R} \times M)$ solves (1.1) if and only if it also solves

$$u(t) = S(t)u_0 - i \int_0^t S(t-s)\{|u(\tau)|^2 u(\tau)\} d\tau.$$  \hfill (3.13)

Then by Proposition 2.11 of [8], it has: take some $\psi \in C_c^\infty(\mathbb{R})$, which equals to 1 on $[-1, 1]$, then

$$\|\psi(t)S(t)u_0\|_{X^{1,\tilde{b}}(\mathbb{R} \times M)} \lesssim \|u_0\|_{H^{s}(M)}$$  \hfill (3.14)

and

$$\left\|\psi(t/T) \int_0^t S(t-s)F(\tau) d\tau\right\|_{X^{s,\tilde{b}}(\mathbb{R} \times M)} \lesssim T^{1-b-b'} \|F\|_{X^{s,-b'}(\mathbb{R} \times M)},$$  \hfill (3.15)

provided $0 < b' < \frac{1}{2} < b$, $b + b' < 1$, $0 < T \leq 1$.

Therefore, we can see that:

First,

$$\|\psi(t)S(t)u_0\|_{X^{1,\tilde{b}}(M)} \lesssim \|u_0\|_{H^{s}(M)},$$  \hfill (3.16)

and secondly,

$$\left\|\int_0^t S(t-\tau)|u(\tau)|^2 u(\tau) d\tau\right\|_{X^{1,\tilde{b}}(M)} \lesssim T^{1-b-b'} \|u\|^3_{X^{1,\tilde{b}}}.$$  \hfill (3.17)

Hence, by the fixed point theorem, since $b(s) + b'(s) < 1$, there exists some $T = T(\|u_0\|_{H^s}) \sim \|u_0\|_{H^s(M)}^{-\frac{1}{1-b-b'}}$, such that the solution exists on $[0, T]$, and

$$\|u\|_{X^{s,b(s)}(M)} \lesssim \|u_0\|_{H^{s}(M)}.$$  \hfill (3.18)

The uniqueness and continuous dependence on initial value could be proved in the same way.

Then, consider the cases $s \geq 1$. First, for $s = 1$, by the above statement, we can see that there is some $T = T(\|u_0\|_{H^1})$, such that the solution exists on $[0, T]$, and

$$\|u\|_{X^{1,b(s)}(M)} \lesssim \|u_0\|_{H^1(M)}.$$  \hfill (3.19)
Because of the conservation laws (1.2) and (1.3), and $X^{s,b}_T(M) \hookrightarrow C((-T,T),H^s(M))$, we can see that the norm $\|u\|_{X^{s,b}_T(M)}$ could be controlled uniformly by $C(\|u_0\|_{H^1(M)})$, hence there is some lower bound for $T$, and also the solution is global.

Next, for $s > 1$, by the estimate in Remark 3.4, we can see that there is some $T = T(\|u_0\|_{H^1(M)}) \sim \|u_0\|_{H^1(M)}^{-\frac{s-1}{2-s'}}$, such that the solution exists on $[0,T]$, and

$$\|u\|_{X^{s,b}_T(M)} \lesssim \|u_0\|_{H^s(M)}. \quad (3.20)$$

As $T$ depends only on the $H^1$-norm of the data, and because we have already showed that it could be controlled uniformly, by elementary iterating process, the solution is global. \qed

4. The proof for Theorem 1.3

The idea of the proof for this theorem is inspired by Bourgain’s work in [4]. He dealt with the case of $T^2$, with $s_0 = 0+$.

Without loss of generality, just consider the case $t \to +\infty$. Choose a sequence $\{t_j\}$, such that $0 < c < I_j = t_{j+1} - t_j \leq T$, hence $t_j \to +\infty$ as $j \to +\infty$. For example we can just choose $t_j$ to be $jT$.

We omit the proof for the following lemma, which could be shown by induction easily.

**Lemma 4.1.** If there is some $\gamma = \gamma(s_0,d)$, such that there exists

$$\|u(t_{j+1})\|_{H^{s'}(M)} \leq \|u(t_j)\|_{H^{s'}(M)} + C\|u(t_j)\|_{H^s(M)}^{1-\frac{s'}{s}}, \quad (4.1)$$

then we have

$$\|u(t)\|_{H^{s'}(M)} \leq C|t|^{\frac{s-1}{s'}}, \quad \text{for } t \text{ large enough.} \quad (4.2)$$

Hence, by this lemma, let $A = \frac{1}{\gamma}$, then the result of Theorem 1.3 follows. Therefore, we just need to find out some proper $\gamma$ satisfying (4.1).

Calculate the $\gamma$ in Lemma 4.1. By the conditions of the theorem, $s$ is an even integer, so there is $s'$ such that $s = 2s'$.

Because of the conservation of $L^2$-norm (1.2), we just need to calculate $\|(\sqrt{-\Delta})^s u\|_{L^2(M)}$, 

$$\left\| \Delta^{s'}u(t_{j+1}) \right\|^2_{L^2(M)} - \left\| \Delta^{s'}u(t_j) \right\|^2_{L^2(M)} = \int_{t_j}^{t_{j+1}} \partial_t \left( \left\| \Delta^{s'}u(t) \right\|^2_{L^2} \right) dt$$

$$= 2 \text{Re} \int_{t_j}^{t_{j+1}} \int_M \partial_t \Delta^{s'}\bar{u}(t) \Delta^{s'}u(t) \, dx \, dt$$
\[
2 \text{Re } i \int_{t_j}^{t_{j+1}} \int_{M} \Delta^t \tilde{u}(t) \left( \Delta \Delta^t u(t) - \Delta^t \left( |u(t)|^2 u(t) \right) \right) \, dx \, dt
\]

= \[2 \text{Im } \int_{t_j}^{t_{j+1}} \int_{M} \Delta^t \tilde{u}(t) \Delta^t \left( |u(t)|^2 u(t) \right) \, dx \, dt.
\]

There are three kinds of terms:

(i) \(I = \text{Im } \int_{t_j}^{t_{j+1}} \int_{M} \left| \Delta^t \tilde{u}(t) \right|^2 |u(t)|^2 \, dx \, dt\), which equals to 0.

(ii) \(II = \text{Im } \int_{t_j}^{t_{j+1}} \int_{M} \left( \Delta^t \tilde{u}(t) \right)^2 |u(t)|^2 \, dx \, dt\), which we will mainly deal with.

(iii) \(III = \text{Im } \int_{t_j}^{t_{j+1}} \int_{M} \left( \Delta^t \tilde{u}(t) \right) \partial^\alpha_1 u_1(t) \partial^\alpha_2 u_2(t) \partial^\alpha_3 u_3(t) \, dx \, dt\), here the partial differential is with respect to the space variable \(x\), one of the \(u_i\) is \(\tilde{u}\), and the other two are \(u\), and \(|\alpha_i| < s\), \(\sum_{i=1}^{3} |\alpha_i| = s\), and at most one \(|\alpha_i| = 0\).

4.1. Deal with (iii)

As from the calculation above, we can see that calculation for \(\tilde{u}\) and \(u\) makes no difference, we could suppose \(|\alpha_1| \geq |\alpha_2| \geq |\alpha_3|\).

We should consider two cases:

(a) \(|\alpha_3| > 0\).

(b) \(|\alpha_3| = 0\) and \(|\alpha_1| \geq |\alpha_2| > 0\).

Hence, \(III \lesssim III_1 + III_2\), where \(III_1\) is restricted to the case (a), and \(III_2\) is restricted to case (b).

(a) \(III_1 \lesssim \left\| \Delta^t u \right\|_{X^{-s_0 + \epsilon_3, b_1}} \left\| \partial^{\alpha_1} u_1 \partial^{\alpha_2} u_2 \partial^{\alpha_3} u_3 \right\|_{X^{s_0 + \epsilon_3, -b_1'}} \)

\[
\lesssim \left\| \Delta^t u \right\|_{X^{-s_0 + \epsilon_3, b_1}} \prod_{j=1}^{3} \left\| \partial^{\alpha_j} u \right\|_{X^{s_0 + \epsilon_3, b_1}}
\]

\[
\lesssim \left\| u \right\|_{X^{s_0 + \epsilon_3, -b_1'}} \prod_{j=1}^{3} \left\| u \right\|_{X^{s_0 + \epsilon_3, b_1}}
\]

\[
\lesssim \left\| u \right\|_{X^{s_0 + \epsilon_3, -b_1'}} \prod_{j=1}^{3} \left\| u \right\|_{X^{s_0 + \epsilon_3, b_1}} \tag{4.4}
\]

by Proposition 3.3. Here \(\epsilon_3 > 0\) is small enough so that \(s_0 + \epsilon_3 < 1\). \(b_1'\) and \(b_1\) are the parameters given by Proposition 3.3 with respect to \(s_0 + \epsilon_3\), and \(\tilde{b}\) is the one in Remark 3.4. Remark 3.5 shows that \(b_1 < \tilde{b}\).

Since

\[
\left\| u(t) \right\|_{X^{s_0 + \epsilon_3, b_1'}} \lesssim \left\| u(t) \right\|_{X^{s_0 + \epsilon_3, b_1'}}^{1 - \frac{s_0 + \epsilon_3}{s_1 - 1}} \left\| u(t) \right\|_{X^{s_0 + \epsilon_3, b_1}}^{\frac{s_0 + \epsilon_3}{s_1 - 1}} \lesssim \left\| u(t_1) \right\|_{H^s}^{1 - \frac{s_0 + \epsilon_3}{s_1 - 1}} \left\| u(t_1) \right\|_{H^s}^{\frac{s_0 + \epsilon_3}{s_1 - 1}} \lesssim \left\| u(t_1) \right\|_{H^s}^{- \frac{s_0 + \epsilon_3}{s_1 - 1}},
\]

we could suppose

\[
\text{we could suppose (iii)

and because $s_0 + \epsilon_3 < 1$, $|\alpha_i| + s_0 + \epsilon_3 < s$,

$$
\|u(t)\|_{X^{\beta,\delta}_{T}} \lesssim \|u(t)\|_{X^{\alpha+\epsilon_3,\delta}_{T}}^{1-\frac{|\alpha_j|+s_0+\epsilon_3-1}{s-1}} \|u(t)\|_{X^{\alpha,\delta}_{T}}^{\frac{|\alpha_j|+s_0+\epsilon_3-1}{s-1}} \lesssim \|u(t)\|_{H^s}^{\frac{|\alpha_j|+s_0+\epsilon_3-1}{s-1}}.
$$

Inserting these two estimates into (4.4) gives

$$
\|u(t)\|_{H^s} \lesssim \|u(t)\|_{H^s} \frac{\sum_{j=1}^{3} |\alpha_j|+s_0+\epsilon_3-3}{s-1} = \|u(t)\|_{H^s}^{2-\frac{2-1-s_0-\epsilon_3}{s-1}}.
$$

(b) As the same estimate as (a), it gives

$$
\|u(t)\|_{H^s} \lesssim \|u(t)\|_{H^s} \frac{\sum_{j=1}^{2} |\alpha_j|+s_0+2\epsilon_3-2}{s-1} = \|u(t)\|_{H^s}^{2-\frac{1-s_0-\epsilon_3}{s-1}}.
$$

Therefore,

$$
\|u(t)\|_{H^s} \lesssim \|u(t)\|_{H^s}^{2-\frac{1-s_0-\epsilon_3}{s-1}}.
$$

4.2. Study of (ii)

$$
\|u(t)\|_{H^s} \lesssim \|u(t)\|_{H^s}^{2-\frac{1-s_0-\epsilon_3}{s-1}}.
$$

Lemma 4.2. There exists $c$ satisfying (4.25), such that

$$
\|u(t)\|_{H^s} \lesssim \|u(t)\|_{H^s}^{2-\frac{1-s_0-\epsilon_3}{s-1}}.
$$

Proof. We first prove the result in the whole space $X^{s,b}$, then for the $X^{s,b}_{T}$, the result follows by restriction as before.
Denote $\tilde{u}_1 = \Delta^s \tilde{u}$, $u_2 = u$ and $u_3 = u$. By duality, we just need to prove for every $u_0 \in X^{0, \tilde{b}}$, there exists

$$\left| \int_{\mathbb{R} \times M} \tilde{u}_0 \tilde{u}_1 u_2 u_3 \, dx \, dt \right| \lesssim \|u_0\|_{X^{0, \tilde{b}}} \|u_1\|_{X^{-c, \tilde{b}}} \|u_2\|_{X^{1, \tilde{b}}} \|u_3\|_{X^{1, \tilde{b}}}.$$  \hspace{1cm} (4.11)

Denote

$$u^{N_j}_{j} = \frac{1}{2\pi} \sum_{\substack{N_j \leq (\mu_k)^{1/2} < 2N_j}} \int \mathcal{P}_k u_j e^{i\tau x} \, dx \, dt,$$

and

$$u^{N_j L_j}_{j} = \frac{1}{2\pi} \sum_{\substack{N_j \leq (\mu_k)^{1/2} < 2N_j, \: L_j \leq (\tau + \mu_k)^{1/2} < 2L_j}} \int \mathcal{P}_k u_j e^{i\tau x} \, dx \, dt,$$

where $N_j$ and $L_j$ are dyadic integers, and denote

$$N := (N_0, N_1, N_2, N_3) \quad \text{and} \quad L := (L_0, L_1, L_2, L_3).$$

Let $J = \left| \int_{\mathbb{R} \times M} \tilde{u}_0 \tilde{u}_1 u_2 u_3 \, dx \, dt \right|$, then it gives

$$J \lesssim \sum_L \sum_N I(L, N), \hspace{1cm} (4.12)$$

where

$$I(L, N) = \left| \int_{\mathbb{R} \times M} \tilde{u}_0^{N_0 L_0} \tilde{u}_1^{N_1 L_1} u_2^{N_2 L_2} u_3^{N_3 L_3} \, dx \, dt \right|. \hspace{1cm} (4.13)$$

Also

$$J \lesssim \sum_N I(N), \hspace{1cm} (4.14)$$

with

$$I(N) = \left| \int_{\mathbb{R} \times M} \tilde{u}_0^{N_0} \tilde{u}_1^{N_1} u_2^{N_2} u_3^{N_3} \, dx \, dt \right|. \hspace{1cm} (4.15)$$

Then by (3.8), we can restrict our argument to $N_0 \leq C'(N_1 + N_2 + N_3)$, and the contribution of $N_0 > C'(N_1 + N_2 + N_3)$ could be controlled by $\|u_0\|_{X^{0, \tilde{b}}} \|u_1\|_{X^{-p', \tilde{b}}} \|u_2\|_{X^{1, \tilde{b}}} \|u_3\|_{X^{1, \tilde{b}}}$, with arbitrary $p' > 0$. 


Case a. \( N_2 \geq N_1^\delta \), or \( N_3 \geq N_1^\delta \), where \( 0 < \delta < 1 \) is a small parameter that will be fixed later.

So \( N_0 \leq 3C' \max\{N_1, N_2, N_3\} \).

Without loss of generality, take \( N_3 \geq N_1^\delta \) for example, and the other case could be treated in the same way.

\[
I(N) = \left| \int_{\mathbb{R} \times M} \tilde{u}_0^{-N_0} \tilde{u}_1^{-N_1} u_2^{-N_2} u_3^{-N_3} \, dx \, dt \right| \lesssim \left\| \tilde{u}_0^{-N_0} u_2^{-N_2} \right\|_{L^2(\mathbb{R} \times M)} \left\| u_1^{-N_1} u_3^{-N_3} \right\|_{L^2(\mathbb{R} \times M)}
\]

\[
\lesssim (\min(N_0, N_2))^{s_0} (\min(N_1, N_3))^{s_0} \prod_{j=0}^{3} \left\| u_j^{N_j} \right\|_{X^{0, \delta}}^3 \prod_{j=2}^{3} \left\| u_j^{N_j} \right\|_{X^{1, \delta}}^3
\]

\[
\lesssim (N_2 N_3)^{s_0} (N_2 N_3)^{-1} \prod_{j=0}^{1} \left\| u_j^{N_j} \right\|_{X^{0, \delta}}^3 \prod_{j=2}^{3} \left\| u_j^{N_j} \right\|_{X^{1, \delta}}^3
\]

\[
\lesssim (N_0 N_1 N_2 N_3)^{-\epsilon_4} (N_0 N_1)^{\epsilon_4} (N_2 N_3)^{s_0-1+\epsilon_4} \prod_{j=0}^{1} \left\| u_j^{N_j} \right\|_{X^{0, \delta}}^3 \prod_{j=2}^{3} \left\| u_j^{N_j} \right\|_{X^{1, \delta}}^3
\]

\[
\lesssim (N_0 N_1 N_2 N_3)^{-\epsilon_4} \max\{N_1, N_2, N_3\}^{\epsilon_4} (N_2 N_3)^{s_0-1+\epsilon_4} \prod_{j=0}^{1} \left\| u_j^{N_j} \right\|_{X^{0, \delta}}^3 \prod_{j=2}^{3} \left\| u_j^{N_j} \right\|_{X^{1, \delta}}^3
\]

\[
\lesssim (N_0 N_1 N_2 N_3)^{-\epsilon_4} N_1^\delta (s_0-1) + \epsilon_5 N_2^\delta s_0-1 + \epsilon_5 \prod_{j=0}^{1} \left\| u_j^{N_j} \right\|_{X^{0, \delta}}^3 \prod_{j=2}^{3} \left\| u_j^{N_j} \right\|_{X^{1, \delta}}^3
\]

\[
\lesssim (N_0 N_1 N_2 N_3)^{-\epsilon_4} \left\| u_0 \right\|_{X^{0, \delta}} \left\| u_1 \right\|_{X^{-\delta(1-s_0)+\epsilon_5, \delta}} \prod_{j=2}^{3} \left\| u_j^{N_j} \right\|_{X^{1, \delta}}, \quad (4.16)
\]

with \( \epsilon_5 = 4\epsilon_4 \) and \( \epsilon_4 \) is taken to make sure that \(-\delta(1 - s_0) + \epsilon_5 < 0\), i.e. \( 0 < \epsilon_4 < \frac{\delta(1-s_0)}{4} \).

Case b. \( N_2, N_3 \leq N_1^\delta \).

Hence \( N_0 \leq 3C' N_1 \).

By Parseval’s formula,

\[
I(L, N) = \left| \int_{\mathbb{R} \times M} \tilde{u}_0^{-N_0} \tilde{u}_1^{-N_1} u_2^{-N_2} u_3^{-N_3} \, dx \, dt \right| = \left| \int_{[t_0 + t_1 + t_2 + t_3 = 0] \times M} \tilde{u}_0^{-N_0} (\tau_0) \tilde{u}_1^{-N_1} (\tau_1) u_2^{-N_2} (\tau_2) u_3^{-N_3} (\tau_3) \, dx \, d\tau \right|. \quad (4.17)
\]
Subcase b1. \(|\tau_2|, |\tau_3| \ll N_1^2\).

Since

\[ | - \tau_0 + \mu_{k_0} | + | - \tau_1 + \mu_{k_1} | \geq | - \tau_0 - \tau_1 + \mu_{k_0} + \mu_{k_1} | \geq | \mu_{k_0} + \mu_{k_1} | - | \tau_0 + \tau_1 | \geq | \mu_{k_1} - | \tau_2 + \tau_3 | \geq \frac{1}{2} N_1^2, \]  

we have \(L_0 L_1 \gtrsim N_1^2\).

\[
I(L, N) \lesssim \left\| u_0^{N_0 L_0} \right\|_{L_2^4 L_2^\infty} \left\| u_1^{N_1 L_1} \right\|_{L_2^4 L_2^\infty} \left\| u_2^{N_2 L_2} \right\|_{L_2^4 L_2^\infty} \left\| u_3^{N_3 L_3} \right\|_{L_2^4 L_2^\infty} 
\lesssim (N_2 N_3)^{\frac{d}{2}} \prod_{j=0}^{3} \left\| u_j^{N_j L_j} \right\|_{L_2^4 L_2^\infty} 
\lesssim (N_2 N_3)^{\frac{d}{2}} \prod_{j=0}^{3} \left\| u_j^{N_j L_j} \right\|_{X_0^{\frac{1}{2}}} 
\lesssim (N_2 N_3)^{\frac{d}{2}} \frac{1}{(L_0 L_1 L_2 L_3)^{\frac{d}{2}-\frac{1}{2}}} \left\| u_0^{N_0 L_0} \right\|_{X_0^{\frac{1}{2}}} \left\| u_1^{N_1 L_1} \right\|_{X_0^{\frac{1}{2}}} \left\| u_2^{N_2 L_2} \right\|_{X_0^{\frac{1}{2}}} 
\lesssim (N_2 N_3)^{\frac{d}{2}-1} \frac{1}{(L_0 L_1 L_2 L_3)^{\frac{d}{2}-\frac{1}{2}}} \left\| u_0^{N_0 L_0} \right\|_{X_0^{\frac{1}{2}}} \left\| u_1^{N_1 L_1} \right\|_{X_0^{\frac{1}{2}}} \left\| u_2^{N_2 L_2} \right\|_{X_0^{\frac{1}{2}}} \left\| u_3^{N_3 L_3} \right\|_{X_0^{\frac{1}{2}}} \left\| u_4^{N_4 L_4} \right\|_{X_0^{\frac{1}{2}}}. \]  

Then

\[
\sum_L I(L, N) \lesssim \sum_{L_0, L_1} (N_2 N_3)^{\frac{d}{2}-1} \frac{1}{(L_0 L_1)^{\frac{d}{2}-\frac{1}{2}}} \left\| u_0^{N_0 L_0} \right\|_{X_0^{\frac{1}{2}}} \left\| u_1^{N_1 L_1} \right\|_{X_0^{\frac{1}{2}}} \left\| u_2^{N_2 L_2} \right\|_{X_0^{\frac{1}{2}}} \left\| u_3^{N_3 L_3} \right\|_{X_0^{\frac{1}{2}}} \left\| u_4^{N_4 L_4} \right\|_{X_0^{\frac{1}{2}}} 
\lesssim \sum_{L_0, L_1} (N_2 N_3)^{\frac{d}{2}-1} \frac{1}{(L_0 L_1)^{\frac{d}{2}-\frac{1}{2}}} \frac{1}{(L_0 L_1)^{\varepsilon_0}} \left\| u_0^{N_0 L_0} \right\|_{X_0^{\frac{1}{2}}} \left\| u_1^{N_1 L_1} \right\|_{X_0^{\frac{1}{2}}} \left\| u_2^{N_2 L_2} \right\|_{X_0^{\frac{1}{2}}} \left\| u_3^{N_3 L_3} \right\|_{X_0^{\frac{1}{2}}} \left\| u_4^{N_4 L_4} \right\|_{X_0^{\frac{1}{2}}} 
\lesssim (N_2 N_3)^{\frac{d}{2}-1} N_1^{-2(\frac{d}{2}-\frac{1}{2})-\varepsilon_0} \left\| u_0^{N_0} \right\|_{X_0^{\frac{1}{2}}} \left\| u_1^{N_1} \right\|_{X_0^{\frac{1}{2}}} \left\| u_2^{N_2} \right\|_{X_0^{\frac{1}{2}}} \left\| u_3^{N_3} \right\|_{X_0^{\frac{1}{2}}} \left\| u_4^{N_4} \right\|_{X_0^{\frac{1}{2}}} 
\lesssim (N_0 N_1 N_2 N_3)^{-\varepsilon_0} (N_2 N_3)^{\frac{d}{2}-1+\varepsilon_0} N_1^{-2(\frac{d}{2}-\frac{1}{2})-\varepsilon_0} \times (N_0)^{\varepsilon_0} \left\| u_0^{N_0} \right\|_{X_0^{\frac{1}{2}}} \left\| u_1^{N_1} \right\|_{X_0^{\frac{1}{2}}} \left\| u_2^{N_2} \right\|_{X_0^{\frac{1}{2}}} \left\| u_3^{N_3} \right\|_{X_0^{\frac{1}{2}}} \left\| u_4^{N_4} \right\|_{X_0^{\frac{1}{2}}} 
\lesssim (N_0 N_1 N_2 N_3)^{-\varepsilon_0} (N_1)^{2(\frac{d}{2}-1+\varepsilon_0)} N_1^{-2(\frac{d}{2}-\frac{1}{2})-\varepsilon_0} N_1^{\varepsilon_0}. 
\]
Then

\[ \sum_{L_3} I(L, N) \lesssim \sum_{L_3} (N_2 N_3)^{d - 1} \frac{1}{L_3^{\frac{d - 2}{4}}} \| u_0 \|_{X^{0, \delta}} \| u_1 \|_{X^{0, \delta}} \| u_2 \|_{X^{1, \delta}} \| u_3 \|_{X^{1, \delta}} \]

Therefore,

\[ I(L, N) \lesssim (N_2 N_3)^{d - 1} \frac{1}{(L_0 L_1 L_2 L_3)^{\frac{d - 2}{4}}} \| u_0 \|_{X^{0, \delta}} \| u_1 \|_{X^{0, \delta}} \| u_2 \|_{X^{1, \delta}} \| u_3 \|_{X^{1, \delta}} \]

(4.21)

As \( \tilde{b} = \frac{d}{4}(3 - \frac{d - 2}{d - 2\tilde{\epsilon}_0}) - \epsilon_2 \), where \( \epsilon_2 \) depends on \( \epsilon_1 \) which depends on \( \epsilon_0 \). Choose \( \delta \), such that

\( (\frac{d}{2} - 1)\delta < \tilde{b} - \frac{d}{4} \), and \( \epsilon_0, \epsilon_6 \) small enough, so that

\( -2(\tilde{b} - \frac{d}{4}) + 2\delta (\frac{d}{2} - 1) + (4 + 2\delta)\epsilon_6 < 0 \).

Subcase b2. \( |\tau_2| \gtrsim N_2^2 \), or \( |\tau_3| \gtrsim N_1^2 \).

As \( N_i \leq N_1^\delta \), for \( i = 2, 3 \),

\[ |\tau_i + \mu_{k_i}| \gtrsim |\tau_i| - \mu_{k_i} \gtrsim N_1^2. \]

Hence \( L_2 \gtrsim N_1^2 \) or \( L_3 \gtrsim N_1^2 \).

Here we consider the case \( L_3 \gtrsim N_1^2 \), and the other one could be dealt with similarly.

Then

\[ \sum_{L} I(L, N) \lesssim \sum_{L_3} (N_2 N_3)^{d - 1} \frac{1}{N_1^{2(\tilde{b} - \frac{1}{4} - \epsilon_6)}} \| u_0 \|_{X^{0, \delta}} \| u_1 \|_{X^{0, \delta}} \| u_2 \|_{X^{1, \delta}} \| u_3 \|_{X^{1, \delta}} \]

(4.20)
\[ \lesssim (N_2 N_3)^{\frac{d}{2}} \frac{1}{N_1^{2(\beta - \frac{1}{4} - \epsilon_6)}} \| u_0^{N_0} \|_{X^{0,0}} \| u_1^{N_1} \|_{X^{0,0}} \prod_{j=2}^{3} \| u_j^{N_j} \|_{X^{1,0}} \]

\[ \lesssim (N_0 N_1 N_2 N_3)^{-\epsilon_6} (N_2 N_3)^{\frac{d}{2}-1+\epsilon_6} N_0^{\epsilon_6} \frac{1}{N_1^{2(\beta - \frac{1}{4} - \epsilon_6)-\epsilon_6}} \times \| u_0^{N_0} \|_{X^{0,0}} \| u_1^{N_1} \|_{X^{0,0}} \prod_{j=2}^{3} \| u_j^{N_j} \|_{X^{1,0}} \]

\[ \lesssim (N_0 N_1 N_2 N_3)^{-\epsilon_6} (N_1)^{2\delta\left(\frac{d}{2}-1+\epsilon_6\right)} \frac{1}{N_1^{2(\beta - \frac{1}{4} - \epsilon_6)-\epsilon_6}} \times N_1^{\epsilon_6} \| u_0^{N_0} \|_{X^{0,0}} \| u_1^{N_1} \|_{X^{0,0}} \prod_{j=2}^{3} \| u_j^{N_j} \|_{X^{1,0}} \]

\[ \lesssim (N_0 N_1 N_2 N_3)^{-\epsilon_6} (N_1)^{-2(\beta - \frac{1}{4})+2\delta\left(\frac{d}{2}-1\right)+2(\delta+4)\epsilon_6} \times \| u_0^{N_0} \|_{X^{0,0}} \| u_1^{N_1} \|_{X^{0,0}} \prod_{j=2}^{3} \| u_j^{N_j} \|_{X^{1,0}} \]

\[ \lesssim (N_0 N_1 N_2 N_3)^{-\epsilon_6} \| u_0^{N_0} \|_{X^{0,0}} \| u_1^{N_1} \|_{X^{0,0}} \prod_{j=2}^{3} \| u_j^{N_j} \|_{X^{1,0}}. \]

(4.22)

Conclusively,

\[-c = \max \left\{ -\delta(1-s_0) + \epsilon_5, -2\left(\beta - \frac{1}{4}\right) + 2\delta\left(\frac{d}{2} - 1\right) + (4 + 2\delta)\epsilon_6 \right\}. \]

Because both \( \epsilon_5 \) and \( \epsilon_6 \) are chosen to be small enough, there is some \( \epsilon_7 \) such that

\[-c = \max \left\{ -\delta(1-s_0) + \epsilon_7, -2\left(\beta - \frac{1}{4}\right) + 2\delta\left(\frac{d}{2} - 1\right) + \epsilon_7 \right\}. \] (4.23)

To equivalent this two terms, i.e. \(-\delta(1-s_0) = -2(\beta - \frac{1}{4}) + 2\delta\left(\frac{d}{2} - 1\right), \)

\[ \delta = \frac{2(\beta - \frac{1}{4})}{d-s_0-1}, \] (4.24)

which makes sense for \( s_0 < 1 \), and \( d \geq 2 \), and also satisfies \( \left(\frac{d}{2} - 1\right)\delta < \beta - \frac{1}{4} \).

As \( \beta = \frac{1}{4}(3 - \frac{d-2}{d-2s_0}) - \epsilon_2 \), put it into the expression of \( \delta \).

However, since we require that \( \delta < 1 \), for \( d \geq 3 \), take \( \delta = \frac{2(\beta - \frac{1}{4})}{d-s_0-1} \), and when \( d = 2 \), we can see that \(-\delta(1-s_0) > -2(\beta - \frac{1}{4}), \) for any \( \delta < 1 \), with \( \epsilon_2 \) small enough.
Therefore,
\[ c = \begin{cases} 
(1 - \frac{\frac{d-2}{2} - \frac{d-2}{2d-2s_0}}{d-1-s_0})-, & d \geq 3, \\
(1 - s_0)-, & d = 2. 
\end{cases} \] \hspace{1cm} (4.25)

Comparing Lemma 4.2 with the result of (iii), there is
\[ \tilde{c} = \begin{cases} 
(1 - \frac{\frac{d-2}{2} - \frac{d-2}{2d-2s_0}}{d-1-s_0})-, & d \geq 3, \\
(1 - s_0)-, & d = 2, 
\end{cases} \] \hspace{1cm} (4.26)
such that
\[ \|u(t_{j+1})\|_{H^s}^2 \leq \|u(t_j)\|_{H^s}^2 + C \|u(t_j)\|_{H^s}^{2 - \frac{1}{1-\gamma}}, \] \hspace{1cm} (4.27)
i.e.
\[ \|u(t_{j+1})\|_{H^s} \leq \|u(t_j)\|_{H^s} \sqrt{1 + C \|u(t_j)\|_{H^s}^{1-\frac{1}{1-\gamma}}}. \] \hspace{1cm} (4.28)
Because for \( j \) large enough, \( \|u(t_j)\|_{H^s}^{-1} \) would be very small, hence by the inequality \( \sqrt{1 + \epsilon} \leq 1 + C \epsilon \) for \( \epsilon \) small enough,
\[ \|u(t_{j+1})\|_{H^s} \leq \|u(t_j)\|_{H^s} + C \|u(t_j)\|_{H^s}^{1 - \frac{1}{1-\gamma}}, \] \hspace{1cm} (4.29)
thus \( \gamma = \tilde{c} \), and
\[ \frac{1}{A} = \gamma = \begin{cases} 
(1 - \frac{\frac{d-2}{2} - \frac{d-2}{2d-2s_0}}{d-1-s_0})-, & d \geq 3, \\
(1 - s_0)-, & d = 2. 
\end{cases} \] \hspace{1cm} (4.30)

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**References**