Generalized Weierstrass representation for surfaces in Heisenberg spaces

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We establish a spinorial representation for surfaces immersed with prescribed mean curvature in Heisenberg space. This permits to obtain minimal immersions starting with a harmonic Gauss map whose target is either the Poincaré disc or a hemisphere of the round sphere.

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1. Introduction

A widely known theorem by Ruh and Wilms asserts that constant mean curvature immersions in Euclidean space have harmonic Gauss map with target given by some Grassmannian space. This is the key point for the machinery of integrable systems to be applied for the construction and characterization of examples of minimal and constant mean curvature surfaces as may be found at references on the subject as [3,8] and [9].

A related result attributed to K. Kenmotsu [10] states that constant mean curvature surfaces immersed in Euclidean three space admit an integral representation in terms of its Gauss map which resembles the classical Weierstrass representation for minimal surfaces. This approach was revisited recently by I. Taimanov and D. Berdinsky in the context of another Riemannian ambient as three-dimensional Lie groups for instance. The method settled in [12,1] and [13] permits to obtain a spinorial representation of prescribed mean curvature in terms of Dirac-type equations.

Using different strategies, authors as B. Daniel, I. Fernández, P. Mira, M.L. Leite, J. Ripoll (see [2,4–6]) among others were able to define natural notions of Gauss map in the context of three-dimensional homogeneous spaces with four-dimensional isometry group. Quite recently, we obtained in [7] a similar result for the Riemannian three-dimensional spheres known as Berger spheres.

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1 L.J. Alías was partially supported by MEC project PCI2006-A7-0532, MICINN project MTM2009-10418, and Fundación Séneca project 04540/CERMI06, Spain. This research is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Regional Agency for Science and Technology (Regional Plan for Science and Technology 2007–2010).

2 J.H.S. de Lira was partially supported by CNPq, FUNCAP and PRONEX (Brazil) and MEC project PCI2006-A7-0532 (Spain).

3 J.A. Hinojosa was partially supported by PICDT-CAPES, Brazil.

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doi:10.1016/j.difgeo.2011.11.001
This motivates us to search for analogs of these results for spacelike surfaces immersed in three-dimensional Lorentzian homogeneous spaces. A natural candidate after the flat case, represented by Lorentz–Minkowski, is to consider the Lorentzian version of the nilpotent three-dimensional Lie group known as Heisenberg space. In this paper, we deal with both Riemannian and Lorentzian structures in the Heisenberg group in order to explore analogies and differences between the two settings.

Our main result gives a representation of a minimal immersion in terms of integrals involving a harmonic Gauss map whose target is a disc endowed either with the hyperbolic metric in the Riemannian case or with the spherical metric in the Lorentzian case. We point out that results about the harmonicity of the Gauss map and an integral representation of minimal surface in Riemannian Heisenberg group were previously obtained by B. Daniel [2] and by F. Mercuri, S. Montaldo and P. Piu [11].

This paper is organized as follows. In Section 2, we present some elementary geometric facts concerning Heisenberg space. In Section 3, we obtain a generalized Weierstrass–Kenmotsu representation for surfaces with prescribed mean curvature in Heisenberg space in terms of spinors satisfying a Dirac-type equation. In that section we combine the methods in [10] and [1]. Section 4 contains the results about the harmonicity of the Gauss map for minimal surfaces and the corresponding integral representation formulae. Finally, in Section 5 we present some interesting examples obtained by applying our integral representation formulae.

2. Heisenberg spaces

The three-dimensional Heisenberg spaces are the Lie groups indexed by a real positive parameter τ which may be represented by matrices of the form

\[
\mathbb{H}_3(\tau) = \begin{pmatrix} 1 & \sqrt{2} \tau x_1 & x_3 + \tau x_1 x_2 \\ 0 & 1 & \sqrt{2} \tau x_2 \\ 0 & 0 & 1 \end{pmatrix} : (x_1, x_2, x_3) \in \mathbb{R}^3.
\]

We denote by e the unit element in \( \mathbb{H}_3(\tau) \). In terms of exponential coordinates \((x_1, x_2, x_3)\), the group product is given by

\[
(x_1, x_2, x_3) \ast (x'_1, x'_2, x'_3) = (x_1 + x'_1, x_2 + x'_2, x_3 + x'_3 + \tau x_1 x'_2 - \tau x'_1 x_2).
\]

The Lie algebra \( T_e \mathbb{H}_3(\tau) \) is spanned by the vectors

\[
\partial_{x_1}|_e := \hat{e}_1, \quad \partial_{x_2}|_e := \hat{e}_2, \quad \partial_{x_3}|_e := \hat{e}_3.
\]

Denoting by \( E_1, E_2 \) and \( E_3 \) the left-invariant vector fields generated respectively by \( \hat{e}_1, \hat{e}_2 \) and \( \hat{e}_3 \), one has

\[
E_1 = \partial_{x_1} - \tau x_2 \partial_{x_3}, \quad E_2 = \partial_{x_2} + \tau x_1 \partial_{x_3}, \quad E_3 = \partial_{x_3}.
\]

We define a left-invariant metric in \( \mathbb{H}_3(\tau) \), which we denote by \( \langle \cdot, \cdot \rangle \), declaring that these vector fields are orthogonal and satisfy

\[
\langle E_1, E_1 \rangle = \langle E_2, E_2 \rangle = 1, \quad \langle E_3, E_3 \rangle = \pm 1 =: \epsilon.
\]

This metric is represented in coordinates by the line element

\[
d\mathbf{s}^2 = dx_1^2 + dx_2^2 + \epsilon (\tau x_2 dx_1 - \tau x_1 dx_2 + dx_3)^2.
\]

From now on, the Heisenberg group \( \mathbb{H}_3(\tau) \) endowed with this metric is denoted by \( \mathbb{H}_3(\epsilon, \tau) \). We denote by \( \nabla \) and \( R \) the Levi-Civita connection and the curvature tensor in \( \mathbb{H}_3(\epsilon, \tau) \), respectively.

The Lie bracket is determined by

\[
[E_1, E_2] = 2\tau E_3, \quad [E_1, E_3] = [E_2, E_3] = 0.
\]

Using this, one proves

**Lemma 1.** The left-invariant frame \([E_1, E_2, E_3]\) satisfies

\[
\nabla_{E_1} E_1 = 0, \quad \nabla_{E_2} E_1 = -\tau E_3, \quad \nabla_{E_3} E_1 = -\epsilon \tau E_2,
\]

\[
\nabla_{E_1} E_2 = \tau E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_3} E_2 = \epsilon \tau E_1,
\]

\[
\nabla_{E_1} E_3 = -\epsilon \tau E_2, \quad \nabla_{E_2} E_3 = \epsilon \tau E_1, \quad \nabla_{E_3} E_3 = 0.
\]

The symbols \( \Gamma^k_{ij} \) in \( \mathbb{H}_3(\epsilon, \tau) \) defined by

\[
\nabla_{E_i} E_j = \Gamma^k_{ij} E_k
\]
Lemma 2. The curvature tensor in \( \mathbb{H}_3(\epsilon, \tau) \) is given by
\[
R(E_1, E_2)E_3 = R(E_2, E_3)E_1 = R(E_3, E_1)E_2 = 0,
\]
\[
R(E_1, E_2)E_2 = -3\epsilon \tau^2 E_1, \quad R(E_2, E_3)E_3 = \tau^2 E_2,
\]
\[
R(E_3, E_1)E_1 = \epsilon \tau^2 E_3.
\]  

We also notice that

\[ \nabla Xz = X z, \]

Thus, one has
\[
\langle Xz, Xz \rangle = \langle Xz, Xz \rangle = 0, \quad \langle Xz, Xz \rangle = \frac{1}{2} e^{i\omega}. \]

The Gauss–Weingarten equations for \( X \) are
\[
\nabla_{Xz} Xz = 2\omega Xz + \frac{\epsilon}{2} q N, \]
\[
\nabla Xz Xz - \nabla_{Xz} Xz = 0, \quad \nabla Xz Xz + \nabla_{Xz} Xz = H e^{i\omega} N,
\]
where the mean curvature \( H \) and the Hopf function \( q \) are the coefficients of the complexified second fundamental form of \( \Sigma \) with respect to the unit normal vector field
\[
N = 2i e^{-2\omega} Xz \times Xz.
\]

Here, the cross product is given by
\[
E_1 \times E_2 = \epsilon E_3, \quad E_2 \times E_3 = E_1, \quad E_3 \times E_1 = E_2.
\]

Using the left-invariant frame, we write
\[
Xz = \sum_{a=1}^{3} Z^a E_a X, \quad Xz = \sum_{a=1}^{3} \bar{Z}^a E_a X,
\]
for some complex-valued functions \( Z^a(z, \bar{z}) \). Thus, (8) and (12) become respectively
\[
(Z^1)^2 + (Z^2)^2 + \epsilon (Z^3)^2 = 0, \]
\[
|Z^1|^2 + |Z^2|^2 + \epsilon |Z^3|^2 = \frac{1}{2} e^{2\omega},
\]
and
\[
N = 2i e^{-2\omega} \left( (Z^2 Z^3 - Z^2 \bar{Z}^3) E_1 + (Z^1 \bar{Z}^3 - Z^1 Z^3) E_2 + \epsilon (Z^1 Z^2 - Z^1 \bar{Z}^2) E_3 \right).
\]

Using Lemma 1, one proves that the Gauss–Weingarten equations (9), (10) and (11) are respectively equivalent to
\[
\partial z Z^1 + 2\epsilon \tau Z^2 Z^3 = 2\omega Z^1 + i\epsilon q e^{-2\omega} (\bar{Z}^2 Z^3 - Z^2 \bar{Z}^3),
\]
\[
\partial z Z^2 - 2\epsilon \tau Z^1 Z^3 = 2\omega Z^2 + i\epsilon q e^{-2\omega} (Z^1 \bar{Z}^3 - \bar{Z}^1 Z^3),
\]
\[
\partial z Z^3 = 2\omega Z^3 + i\epsilon q e^{-2\omega} (\bar{Z}^1 Z^2 - Z^1 \bar{Z}^2)
\]
and
Proposition 1. Let \( X : \Sigma \to \mathbb{H}_3(\epsilon, \tau) \) be an isometric immersion. Then, the field of spinors \( \psi = (\psi_1, \psi_2) \) defined by Eqs. (14) and (27) satisfy the Dirac equation \( \mathbf{D}\psi = 0 \), where

\[
\mathbf{D} = \begin{pmatrix}
0 & \partial_z \\
-\partial_z & 0
\end{pmatrix} + \begin{pmatrix}
U & 0 \\
0 & V
\end{pmatrix}
\]

with potentials

\[
-\partial_z Z^1 + \partial_z \bar{Z}^1 = 0, \quad (21)
\]

\[
-\partial_z Z^2 + \partial_z \bar{Z}^2 = 0, \quad (22)
\]

\[
-\partial_z Z^3 + \partial_z \bar{Z}^3 + 2\epsilon (Z^1 \bar{Z}^2 - \bar{Z}^1 Z^2) = 0, \quad (23)
\]

and

\[
\partial_z Z^1 + \partial_z \bar{Z}^1 + 2\epsilon \tau (Z^2 \bar{Z}^2 + \bar{Z}^2 Z^2) = 2iH(\bar{Z}^2 Z^3 - Z^2 \bar{Z}^3), \quad (24)
\]

\[
\partial_z Z^2 + \partial_z \bar{Z}^2 - 2\epsilon \tau (Z^3 \bar{Z}^2 + \bar{Z}^2 Z^3) = 2iH(Z^1 \bar{Z}^3 - \bar{Z}^1 Z^3), \quad (25)
\]

\[
\partial_z Z^3 + \partial_z \bar{Z}^3 = 2iH(\bar{Z}^1 Z^2 - Z^1 \bar{Z}^2). \quad (26)
\]

We then define \( \psi_1 \) and \( \psi_2 \) by

\[
Z^1 = \frac{1}{2}(\bar{\psi}^2 - \epsilon \psi_1^2), \quad Z^2 = \frac{i}{2}(\bar{\psi}^2 + \epsilon \psi_1^2), \quad Z^3 = \psi_1 \bar{\psi}_2, \quad (27)
\]

that is,

\[
Z^1 + iZ^2 = -\epsilon \psi_1^2, \quad \bar{Z}^1 + i\bar{Z}^2 = \psi_2^2. \quad (28)
\]

The resulting expressions for the metric and for the unit normal \( N \) are respectively

\[
e^{2\omega} = (|\psi_2|^2 + \epsilon |\psi_1|^2)^2
\]

and

\[
N = e^{-\omega}((\psi_1 \psi_2 + \bar{\psi}_1 \bar{\psi}_2)E_1 + i(\bar{\psi}_1 \bar{\psi}_2 - \psi_1 \psi_2)E_2 + (|\psi_1|^2 - \epsilon |\psi_1|^2)E_3). \quad (29)
\]

When \( \epsilon = -1 \), we may suppose without loss of generality that \(|\psi_2| > |\psi_1|\), what yields

\[
e^{\omega} = |\psi_2|^2 + \epsilon |\psi_1|^2. \quad (30)
\]

We now want to determine functions \( U \) and \( V \) so that

\[
\partial_z \psi_1 = V \psi_2, \quad \partial_z \psi_2 = -U \psi_1. \quad (31)
\]

Adding (21) and (22) multiplied by \( i \), we obtain

\[
\partial_z (\epsilon \psi_1^2) + \partial_z (\psi_2^2) = 0
\]

what after eliminating the common factor \( 2\psi_1 \psi_2 \) becomes

\[
eV - U = 0. \quad (32)
\]

Summing up (24) to (25) multiplied by \( i \), one has

\[
\partial_z (-\psi_1^2) + \partial_z (\psi_2^2) = -2i\epsilon \tau (-\epsilon \psi_1^2 \bar{\psi}_1 \psi_2 + \psi_1 \bar{\psi}_2 \psi_2^2) = 2H(\psi_1 \bar{\psi}_2 \psi_2^2 + \epsilon \psi_1^2 \bar{\psi}_1 \psi_2).
\]

what is equivalent up to the common factor \( -2\psi_1 \psi_2 \) to

\[
eV + U = -H(|\psi_2|^2 + \epsilon |\psi_1|^2) - i\epsilon \tau (|\psi_2|^2 - \epsilon |\psi_1|^2). \quad (33)
\]

Solving (32) and (33) for \( U \) and \( V \), one finds

\[
U = -\frac{1}{2}(H(|\psi_2|^2 + \epsilon |\psi_1|^2) + i\epsilon \tau (|\psi_2|^2 - \epsilon |\psi_1|^2)), \quad (34)
\]

\[
V = -\frac{\epsilon}{2}(H(|\psi_2|^2 + \epsilon |\psi_1|^2) + i\epsilon \tau (|\psi_2|^2 - \epsilon |\psi_1|^2)). \quad (35)
\]

These calculations are summarized in

\[
\text{Proposition 1.}
\]
\[ U = -\frac{1}{2} H \big( |\psi_2|^2 + \epsilon |\psi_1|^2 \big) + i \epsilon \tau \big( |\psi_2|^2 - \epsilon |\psi_1|^2 \big) \] (37)

and

\[ V = -\frac{\epsilon}{2} H \big( |\psi_2|^2 + \epsilon |\psi_1|^2 \big) + i \epsilon \tau \big( |\psi_2|^2 - \epsilon |\psi_1|^2 \big) \]. (38)

We may describe the immersion \( X \) in terms of exponential coordinates as

\[ X(z) = (x_1(z), x_2(z), x_3(z)). \]

Hence, writing

\[ X_2 = \phi_1 \partial_{x_1} + \phi_2 \partial_{x_2} + \phi_3 \partial_{x_3}, \quad \text{where } \phi_k = \frac{\partial x_k}{\partial z}, \quad k = 1, 2, 3 \] (39)

and comparing (39) and (14) in view of (2) and (27), one gets

\[ \phi_1 = \frac{1}{2} (\bar{\psi}_2 - \epsilon \psi_1^2), \quad \phi_2 = \frac{i}{2} (\bar{\psi}_2 + \epsilon \psi_1^2), \quad \phi_3 = \frac{\tau}{2} (i(\bar{\psi}_2 + \epsilon \psi_1^2) x_1 - (\bar{\psi}_2 - \epsilon \psi_1^2) x_2) + \psi_1 \bar{\psi}_2. \] (40)

Therefore, we conclude that

\[ x_1(z) = x_1(z_0) + 2 \text{Re} \int_{z_0}^{z} \frac{1}{2} (\bar{\psi}_2 - \epsilon \psi_1^2) dz. \] (41)

\[ x_2(z) = x_2(z_0) + 2 \text{Re} \int_{z_0}^{z} \frac{i}{2} (\bar{\psi}_2 + \epsilon \psi_1^2) dz. \] (42)

\[ x_3(z) = x_3(z_0) + 2 \text{Re} \int_{z_0}^{z} \left( \frac{\tau}{2} (i(\bar{\psi}_2 + \epsilon \psi_1^2) x_1 - (\bar{\psi}_2 - \epsilon \psi_1^2) x_2) + \psi_1 \bar{\psi}_2 \right) dz. \] (43)

We now prove a converse of this fact. For that, we use the following standard fact from Complex Analysis.

**Lemma 3.** Let \( \phi \) be a \( C^2 \) function in a simply connected Riemann surface \( \Sigma \) so that

\[ \text{Im} \partial_{\overline{z}} \phi = 0. \]

Then, the form \( \phi \, dz \) has no real periods and then

\[ 2 \partial_{\overline{z}} \int_{z_0}^{z} \text{Re}(\phi \, dz) = \phi. \]

The following auxiliar result is easily verified.

**Lemma 4.** Let \( \psi_1, \psi_2 \) be complex-valued functions satisfying Dirac equation (36) with potentials (37) and (38). Then,

\[ \frac{1}{2} \partial_{\overline{z}} (\bar{\psi}_2 - \epsilon \psi_1^2) = H \big( |\psi_2|^2 + \epsilon |\psi_1|^2 \big) \text{Re}(\psi_1 \psi_2) - \epsilon \tau \big( |\psi_2|^2 - \epsilon |\psi_1|^2 \big) \text{Im}(\psi_1 \psi_2), \] (44)

\[ \frac{i}{2} \partial_{\overline{z}} (\bar{\psi}_2 + \epsilon \psi_1^2) = H \big( |\psi_2|^2 + \epsilon |\psi_1|^2 \big) \text{Im}(\psi_1 \psi_2) + \epsilon \tau \big( |\psi_2|^2 - \epsilon |\psi_1|^2 \big) \text{Re}(\psi_1 \psi_2), \] (45)

\[ \partial_{\overline{z}}(\psi_1 \bar{\psi}_2) = -\frac{1}{2}(\epsilon H + i \tau) \big( |\psi_2|^4 - |\psi_1|^4 \big). \] (46)

We are then able to prove the following integral representation theorem.

**Theorem 1.** Let \( H \) be a given function in a simply connected Riemann surface \( \Sigma \). Let \( \psi_1 \) and \( \psi_2 \) be complex-valued functions in \( \Sigma \), satisfying the condition \( |\psi_2|^2 + \epsilon |\psi_1|^2 > 0 \) and the Dirac equations
\[ \partial_2 \psi_1 = \frac{-i}{2} (H(|\psi_2|^2 + \varepsilon |\psi_1|^2) + i\varepsilon \tau (|\psi_2|^2 - \varepsilon |\psi_1|^2)) \psi_2. \]  
(47)

\[ \partial_2 \psi_2 = \frac{1}{2} (H(|\psi_2|^2 + \varepsilon |\psi_1|^2) + i\varepsilon \tau (|\psi_2|^2 - \varepsilon |\psi_1|^2)) \psi_1. \]  
(48)

Then, the map
\[ X = (x_1, x_2, x_3) : \Sigma \to \mathbb{H}_3(\varepsilon, \tau), \]  
(49)

with components
\[ x_1(z) = 2 \text{Re} \int_{z_0}^z \frac{1}{2} (\bar{\psi}_2^2 - \varepsilon \psi_1^2) \, dz, \]  
(50)

\[ x_2(z) = 2 \text{Re} \int_{z_0}^z \frac{i}{2} (\bar{\psi}_2^2 + \varepsilon \psi_1^2) \, dz, \]  
(51)

\[ x_3(z) = 2 \text{Re} \int_{z_0}^z \left( \tau \left( \frac{i}{2} (\bar{\psi}_2^2 + \varepsilon \psi_1^2) x_1 - \frac{1}{2} (\bar{\psi}_2^2 - \varepsilon \psi_1^2) x_2 \right) + \psi_1 \bar{\psi}_2 \right) \, dz. \]  
(52)

is a conformal immersion with mean curvature \(H\). Moreover, the induced metric in \(\Sigma\) is given by
\[ (|\psi_2|^2 + \varepsilon |\psi_1|^2)^2 |dz|^2. \]  
(53)

**Proof.** Eqs. (44) and (45) imply that
\[ \text{Im} \partial_2 (\bar{\psi}_2^2 - \varepsilon \psi_1^2) = 0 \quad \text{and} \quad \text{Im} \partial_2 (i\bar{\psi}_2^2 + i\varepsilon \psi_1^2) = 0. \]  
(54)

Moreover, one has
\[ \frac{i}{2} (\bar{\psi}_2^2 + \varepsilon \psi_1^2) \partial_2 x_1 - \frac{1}{2} (\bar{\psi}_2^2 - \varepsilon \psi_1^2) \partial_2 x_2 = \frac{i}{4} (\bar{\psi}_2^2 + \varepsilon \psi_1^2) (\psi_2^2 - \varepsilon \bar{\psi}_1^2) + \frac{i}{4} (\bar{\psi}_2^2 - \varepsilon \psi_1^2) (\psi_2^2 + \varepsilon \bar{\psi}_1^2) \]
\[ = \frac{i}{2} (|\psi_2|^4 - |\psi_1|^4). \]

Thus, we conclude that
\[ \text{Im} \partial_2 \left( \tau \left( \frac{i}{2} (\bar{\psi}_2^2 + \varepsilon \psi_1^2) x_1 - \frac{1}{2} (\bar{\psi}_2^2 - \varepsilon \psi_1^2) x_2 \right) + \psi_1 \bar{\psi}_2 \right) = 0. \]

Hence, by Lemma 3, the functions \(x_1\), \(x_2\) and \(x_3\) are well-defined and satisfy
\[ \frac{\partial x_1}{\partial z} = \frac{1}{2} (\bar{\psi}_2^2 - \varepsilon \psi_1^2), \]  
\[ \frac{\partial x_2}{\partial z} = \frac{i}{2} (\bar{\psi}_2^2 + \varepsilon \psi_1^2), \]  
\[ \frac{\partial x_3}{\partial z} = \tau \left( \frac{i}{2} (\bar{\psi}_2^2 + \varepsilon \psi_1^2) x_1 - \frac{1}{2} (\bar{\psi}_2^2 - \varepsilon \psi_1^2) x_2 \right) + \psi_1 \bar{\psi}_2. \]

Therefore, it follows that
\[ X_2 = \partial_2 x_1 \partial_1 + \partial_2 x_2 \partial_2 + \partial_2 x_3 \partial_3 = \frac{1}{2} (\bar{\psi}_2^2 - \varepsilon \psi_1^2) E_1 + \frac{i}{2} (\bar{\psi}_2^2 + \varepsilon \psi_1^2) E_2 + \psi_1 \bar{\psi}_2 E_3 \]
and then
\[ \langle X_2, X_2 \rangle = 0 \quad \text{and} \quad \langle X_2, X_3 \rangle = \frac{1}{2} (|\psi_2|^2 + \varepsilon |\psi_1|^2)^2, \]

what implies that \(X\) is a conformal immersion, since \(|\psi_2|^2 + \varepsilon |\psi_1|^2 > 0\). Moreover, a unit normal field along \(X\) may be given by
\[ N = \frac{1}{|\psi_2|^2 + \varepsilon |\psi_1|^2} \left( 2 \text{Re}(\psi_1 \psi_2) E_1 + 2 \text{Im}(\psi_1 \psi_2) E_2 + (|\psi_1|^2 - \varepsilon |\psi_2|^2) E_3 \right). \]
It remains to prove that $X$ has mean curvature $H$. In order to do that, one computes
\[
\nabla_{X_1} X_2 = \frac{1}{2} (\bar{\psi}_2 - \epsilon \psi_1^2) z E_1 + \frac{i}{2} (\bar{\psi}_2 + \epsilon \psi_1^2) z E_2 + (\psi_1 \bar{\psi}_2) z E_3 + \frac{1}{2} (\bar{\psi}_2 - \epsilon \psi_1^2) \nabla_{X_1} E_1
\]
\[+ \frac{i}{2} (\bar{\psi}_2 + \epsilon \psi_1^2) \nabla_{X_1} E_2 + \psi_1 \bar{\psi}_2 \nabla_{X_2} E_3.
\]
However, bearing in mind (5), it results that
\[
\nabla_{X_1} E_1 = \frac{i \tau}{2} (\psi_2^2 + \epsilon \psi_1^2) E_3 - \epsilon \tau \psi_1 \bar{\psi}_2 E_2,
\]
\[
\nabla_{X_1} E_2 = \frac{\tau}{2} (\psi_2^2 - \epsilon \psi_1^2) E_3 + \epsilon \tau \psi_1 \bar{\psi}_2 E_1,
\]
\[
\nabla_{X_1} E_3 = -\epsilon \tau (\psi_2^2 - \epsilon \psi_1^2) E_2 - \frac{i \tau}{2} (\psi_2^2 + \epsilon \psi_1^2) E_1
\]
and then
\[
\frac{1}{2} (\bar{\psi}_2 - \epsilon \psi_1^2) \nabla_{X_1} E_1 + \frac{i}{2} (\psi_2^2 + \epsilon \psi_1^2) \nabla_{X_1} E_2 + \psi_1 \bar{\psi}_2 \nabla_{X_2} E_3
\]
\[= \frac{i \tau}{2} (|\psi_2|^2 - \epsilon |\psi_1|^2) (\psi_1 \bar{\psi}_2 - \psi_1 \bar{\psi}_2) E_1 - \epsilon \tau (|\psi_2|^2 - \epsilon |\psi_1|^2) (\psi_1 \bar{\psi}_2 + \psi_1 \bar{\psi}_2) E_2 + \frac{i \tau}{2} (|\psi_2|^4 - |\psi_1|^4) E_3
\]
\[= \epsilon \tau (|\psi_2|^2 - \epsilon |\psi_1|^2) (\mathcal{L}(\psi_1 \psi_2) E_1 - \Re(\psi_1 \psi_2) E_2 + \epsilon (|\psi_2|^2 - \epsilon |\psi_1|^2) E_3).
\]
Gathering these expressions and those in Lemma 4, one gets
\[
\nabla_{X_1} X_2 = \frac{1}{2} H (|\psi_2|^2 + \epsilon |\psi_1|^2) (2 \Re(\psi_1 \psi_2) E_1 + 2 \Im(\psi_1 \psi_2) E_2 + (|\psi_1|^2 - \epsilon |\psi_2|^2) E_3)
\]
\[= \frac{1}{2} H (|\psi_2|^2 + \epsilon |\psi_1|^2)^2 N = \frac{1}{2} H e^{2\omega} N,
\]
what concludes the proof of the theorem. \qed

We point out that given $\psi_1, \psi_2$ solving (36) with potentials (37) and (38), the fields $e^{\frac{i}{2}} \psi_1$ and $e^{\frac{i}{2}} \psi_2$, where $\vartheta \in \mathbb{R}$, also satisfy the same equation. Thus, this gives an associated family of surfaces with same prescribed mean curvature $H$.

**Corollary 1.** Let $\psi = (\psi_1, \psi_2)$ be a field of spinors solving (47) and (48). Then, for any $\vartheta \in \mathbb{R}$, the spinor fields
\[
\psi^\vartheta = (e^{\frac{i}{2}} \psi_1, e^{\frac{i}{2}} \psi_2)
\]
are also solutions of the same system. Then, given a conformal immersion $X: \Sigma \rightarrow \mathbb{H}_3(\epsilon, \tau)$ with prescribed mean curvature $H$ and represented by $\psi$, there exists an associated family of immersions $X_0$ represented by $\psi^\vartheta$ such that $X_0 = X$. All surfaces in such a family are isometric and have same prescribed mean curvature $H$.

Finally, we state the following result, whose Riemannian version in $\mathbb{H}_3(1, 1)$ is already known (see [1] and [2]).

**Proposition 2.** The quadratic differential
\[
Q \, dz^2 = \left( \frac{q}{2} + 2 \tau^2 \left( \frac{z^2}{H + i \tau} \right)^2 \right) dz^2
\]
is holomorphic in a Riemann surface $\Sigma$ immersed with constant mean curvature $H$ in $\mathbb{H}_3(\epsilon, \tau)$.

**Proof.** Writing Codazzi equation in terms of the introduced notation we give
\[
\frac{q}{2} = \langle R(X_2, X_2)X_2, N \rangle + \epsilon e^{2\omega} H_2.
\]
Some computation involving the expression of the curvature tensor in Lemma 2 yields the following expression
\[
\langle R(X_2, X_2)X_2, N \rangle = 2 \epsilon \tau^2 (|\psi_2|^4 - |\psi_1|^4) \psi_1 \bar{\psi}_2.
\]
On the other hand, one has, using (34) and (35),
\[
\partial_2 Z^3 = \partial_2 (\psi_1 \bar{\psi}_2) = \psi_1 \partial_2 \bar{\psi}_2 + \bar{\psi}_2 \partial_2 \psi_1 = -\bar{U} |\psi_1|^2 + V |\psi_2|^2 = -\frac{\epsilon}{2} (|\psi_2|^4 - |\psi_1|^4)(H + i \tau).
\]
Therefore,
\[ \partial_2(\bar{Z}^3)^2 = -\epsilon (|\psi_2|^4 - |\psi_1|^4)(H + i\epsilon \tau)\psi_1\psi_2. \]
Thus, we obtain
\[ \langle R(X_2, X_2)X_2, N \rangle = -\frac{2\tau^2}{H + i\epsilon \tau} \partial_2(\bar{Z}^3)^2, \]
what may be written as
\[ \left( \frac{q}{2} + 2\tau^2 \frac{(\bar{Z}^3)^2}{H + i\epsilon \tau} \right) = \frac{1}{2}e^{2\omega}H_z + 2\tau^2 \left( \frac{1}{H + i\epsilon \tau} \right) (\bar{Z}^3)^2. \]
(57) This finishes the proof. \( \square \)

4. Gauss map

From now on, we denote by \( S^2(1) \) the Euclidean unit sphere in \( T_e\mathbb{H}^3(1, \tau) \). In the Lorentzian case, we denote
\[ \mathbb{H}^2(-1) = \{(a, b, c) \in T_e\mathbb{H}^3(-1, \tau); a^2 + b^2 - c^2 = -1, c > 0 \}. \]
When \( \epsilon = 1 \), we consider the stereographic projection with respect to the North pole \( \hat{e}_3 \) with images in the extended plane \( \hat{C} \). In Lorentzian case, as usual, the stereographic projection is taken with respect to \(-\hat{e}_3\) onto the Euclidean unit disc \( \mathbb{D} \). In what follows, we denote both \( S^2(1) \) and \( \mathbb{H}^2(-1) \) by \( \mathbb{Q}^2 \). In both cases, the stereographic projection is given by
\[ \pi_\epsilon(a, b, c) = \frac{a}{1 - \epsilon c} + i \frac{b}{1 - \epsilon c} \]
and its inverse by
\[ \pi_\epsilon^{-1}(w) = \frac{1}{1 + \epsilon |w|^2} (2\text{Re}(w), 2\text{Im}(w), |w|^2 - \epsilon). \]
(59)

Now, one defines the functions
\[ f = \bar{\psi}_2, \quad g = \frac{\psi_1}{\psi_2} \]
in such a way that
\[ Z^1 = \frac{1}{2}(1 - \epsilon g^2)f, \quad Z^2 = \frac{i}{2}(1 + \epsilon g^2)f, \quad Z^3 = gf. \]
Thus,

Lemma 5. Given an isometric immersion \( X: \Sigma \to \mathbb{H}_3(\epsilon, \tau) \), one has
\[ e^{2\omega} = |f|^2 (1 + \epsilon |g|^2)^2, \]
\[ N = \frac{1}{1 + \epsilon |g|^2} (2\text{Re}(g)E_1 + 2\text{Im}(g)E_2 + (|g|^2 - \epsilon)E_3). \]
(62)
The function \( g \) has an expected geometric interpretation.

Lemma 6. Given an isometric immersion \( X: \Sigma \to \mathbb{H}_3(\epsilon, \tau) \) with Gauss map \( \eta: \Sigma \to \mathbb{Q}^2 \), one has \( g = \pi_\epsilon(\eta) \).

Dirac equation \( (36) \) is rewritten in terms of \( f \) and \( g \) as

Lemma 7. The functions \( f \) and \( g \) satisfy
\[ f_z = |f|^2 \bar{g}(H(1 + \epsilon |g|^2) - i\epsilon(1 - \epsilon |g|^2)), \]
\[ g_z = -\frac{f}{2}(\epsilon H(1 + \epsilon |g|^2)^2 + i\epsilon(1 - \epsilon |g|^2)^2). \]
(63)
Moreover, if \( X: \Sigma \to \mathbb{H}_3(\epsilon, \tau) \) is a minimal immersion, then
\[ g_{z\zeta} - i\epsilon \tau \bar{f}(1 - \epsilon |g|^2)\bar{g}g_z = 0. \]
Proof. The first two equations are mere restatements of (47) and (48) in terms of \( f \) and \( g \). Concerning the third one, we compute
\[
g_{zz} = -\frac{i}{2} \bar{\tau} \bar{f} (1 - \epsilon |g|^2)^2 + i \epsilon \bar{\tau} \bar{f} (1 - \epsilon |g|^2) (\bar{g} \bar{g}_z + \bar{g}_z g) \\
= \frac{\epsilon}{2} \bar{\tau}^2 |f|^2 g (1 - \epsilon |g|^2)^3 - \frac{\epsilon}{2} \bar{\tau}^2 |f|^2 g (1 - \epsilon |g|^2)^3 + i \epsilon \bar{\tau} (1 - \epsilon |g|^2) \bar{g}_z \\
= i \epsilon \bar{\tau} (1 - \epsilon |g|^2) \bar{g}_z.
\]
This finishes the proof of the lemma. \( \square \)

Remark 1. The counterpart of Corollary 1 in terms of \( x, \bar{x} \) and as in Lemma 7 may be stated as follows: for any \( \vartheta \in \mathbb{R} \), the functions
\[
f^{\vartheta} = e^{-i \vartheta} f \quad \text{and} \quad g^{\vartheta} = e^{i \vartheta} g
\]
are solutions of the system (63)-(64). Thus, these pairs of functions describe an associated family of immersions with same prescribed mean curvature and isometric induced metrics.

Eq. (65) has the following interpretation.

Corollary 2. Let \( X : \Sigma \rightarrow \mathbb{H}_3(\epsilon, \tau) \) be a minimal isometric immersion. If \( \epsilon = 1 \), suppose that the Gauss map \( \eta \) is not horizontal and has images contained in the southern hemisphere. Then, \( g = \pi_\vartheta(\eta) \) is a harmonic map with target being the disc \( \mathbb{D} \) endowed with the metric
\[
d_\mathbb{D}^2 = \frac{|dw|^2}{(1 - \epsilon |w|^2)^2}.
\]
Thus, \( g \) satisfies the equation
\[
g_{zz} + \frac{2 \epsilon \bar{g}}{1 - \epsilon |g|^2} g_z g_z = 0.
\]
We remark that \( (\mathbb{D}, d_\mathbb{D}^2) \) is the Poincaré disc model for the hyperbolic plane \( \mathbb{H}^2 \), whereas \( (\mathbb{D}, d_{\mathbb{D}^2 - 1}) \) is isometric to an open hemisphere of the unit sphere \( \mathbb{S}^2 \).

Proof. By hypothesis, when \( \epsilon = 1 \), one has \( |g|^2 < 1 \). Therefore, the image of \( g \) lies within \( \mathbb{D} \). On the other hand, when \( \epsilon = -1 \), we have \( |g| < 1 \) merely by definition. Hence, in view of (65), it follows that
\[
g_{zz} = i \epsilon \bar{\tau} \bar{f} (1 - \epsilon |g|^2) \bar{g}_z g = i \epsilon \bar{\tau} \bar{f} (1 - \epsilon |g|^2) \bar{g} 1 \epsilon \bar{f} (1 - \epsilon |g|^2) \bar{g}_z g = -2 \epsilon \bar{g} 1 \epsilon |g|^2 \bar{g}_z g.
\]
This equation, in turn, corresponds to the harmonicity of the map \( g : \Sigma \rightarrow (\mathbb{D}, d_\mathbb{D}^2) \). \( \square \)

For further reference, we restate Theorem 1 in terms of the data \( f \) and \( g \), obtaining

Theorem 2. Let \( f \) and \( g \) be functions defined in a simply connected Riemann surface \( \Sigma \), so that \( f \) never vanishes, satisfying (63) and (64), that is,
\[
f_z = |f|^2 \bar{g} (H(1 + \epsilon |g|^2) - i \epsilon \tau (1 - \epsilon |g|^2)), \quad g_z = -\frac{f}{2} (\epsilon H(1 + \epsilon |g|^2) + i \tau (1 - \epsilon |g|^2))^2.
\]
Then, the map
\[
X = (x_1, x_2, x_3) : \Sigma \rightarrow \mathbb{H}_3(\epsilon, \tau),
\]
defined by
\[
x_1(z) = 2 \text{Re} \int_{z_0}^z \frac{1}{2} (1 - \epsilon g^2) f \, dz, \quad (70)
\]
\[
x_2(z) = 2 \text{Re} \int_{z_0}^z \frac{i}{2} (1 + \epsilon g^2) f \, dz.
\]
Theorem 3. Since (77) and (78) correspond to Eqs. (69) for what implies that never vanishes since.

Proof. We notice that the function is a minimal conformal immersion whose Gauss map is \( g \).

Then, the map is a conformal immersion with mean curvature \( H \) and Gauss map \( g \).

We now present a result that makes evident that the harmonicity of a map \( g : \Sigma \to (\mathbb{D}, ds^2_\mathbb{D}) \) is the full compatibility condition for the first order system in either Theorem 1 or Theorem 2.

Theorem 3. Let \( g : \Sigma \to (\mathbb{D}, ds^2_\mathbb{D}) \) be a harmonic never holomorphic map, where

\[
ds^2_\Sigma = \frac{|dw|^2}{(1 - \epsilon|w|^2)^2}.
\]

Then, the map

\[ X = (x_1, x_2, x_3) : \Sigma \to \mathbb{H}_3(\epsilon, \tau), \]

defined by

\[
x_1(z) = -2 \text{Re} \int_{z_0}^z \frac{i}{\tau(1 - \epsilon|g|^2)^2} \bar{g}_z dz,
\]

\[
x_2(z) = 2 \text{Re} \int_{z_0}^z \frac{1}{\tau(1 - \epsilon|g|^2)^2} \bar{g}_z dz,
\]

\[
x_3(z) = -2 \text{Re} \int_{z_0}^z \left( \frac{\tau}{2} (i(1 + \epsilon g^2)x_1 - (1 - \epsilon g^2)x_2) + g \right) \frac{2i}{\tau(1 - \epsilon|g|^2)^2} \bar{g}_z dz
\]

is a minimal conformal immersion whose Gauss map is \( g \).

Proof. We notice that the function \( f \) defined by

\[
f := -\frac{2i}{\tau(1 - \epsilon|g|^2)^2} \bar{g}_z
\]

never vanishes since \( g \) is never holomorphic. Moreover, the derivative \( g_z \) satisfies (64) for \( H = 0 \), that is,

\[
g_z = -\frac{i}{2} \overline{\tau f}(1 - \epsilon|g|^2)^2.
\]

On the other hand, the harmonicity of \( g \) allows us to compute

\[
f_z = -\frac{2i}{\tau(1 - \epsilon|g|^2)^2} \bar{g}_z (1 - \epsilon|g|^2)^2 + 2\epsilon \bar{g}_z (1 - \epsilon|g|^2)(g \bar{g}_z + \bar{g} g_z)
\]

\[
= -\frac{2i}{\tau(1 - \epsilon|g|^2)^2} (2\epsilon \bar{g}_z g_z (1 - \epsilon|g|^2) + 2\epsilon \bar{g}_z (1 - \epsilon|g|^2)(g \bar{g}_z + \bar{g} g_z))
\]

\[
= -\frac{4i}{\tau(1 - \epsilon|g|^2)^2} \bar{g} g_z g_z = -\frac{4i}{\tau(1 - \epsilon|g|^2)^2} \bar{g} \left( -\frac{i\tau}{2} \overline{\tau f}(1 - \epsilon|g|^2)^2 \right) \left( \frac{i}{2} f(1 - \epsilon|g|^2)^2 \right)
\]

\[-i\epsilon \tau |f|^2 \bar{g}(1 - \epsilon|g|^2)^2.
\]

what implies that

\[
f_z = -i\epsilon \tau |f|^2 \bar{g}(1 - \epsilon|g|^2)^2.
\]

Since (77) and (78) correspond to Eqs. (69) for \( H = 0 \), the result follows from Theorem 2. The integral formulae (73)–(75) stem from (70)–(72) after replacing (76) in these latter formulae.
5. Examples

5.1. Geodesics

Let \( \alpha : I \subset \mathbb{R} \rightarrow (\mathbb{D}, ds^2) \) be a geodesic with unit speed in \((\mathbb{D}, ds^2)\). Up to isometry, we may suppose that the trace of \( \alpha \) is a Euclidean diameter of \( \mathbb{D} \) and, in terms of Cartesian coordinates in \( \mathbb{D} \), it can be written as

\[ \alpha(u) = (\tan_\epsilon(u), 0) \]

where

\[ \tan_\epsilon(u) = \begin{cases} \tanh(u) & \text{if } \epsilon = 1, \\ \tan(u) & \text{if } \epsilon = -1. \end{cases} \]

Hence, the map \( g : I \times \mathbb{R} \rightarrow (\mathbb{D}, ds^2) \) defined by

\[ g(u, v) = \alpha(u), \quad u \in I, \quad (79) \]

is a harmonic map. Then, Theorem 3 implies that this map defines a conformal minimal immersion \( X : I \times \mathbb{R} \rightarrow \mathbb{H}_3(\epsilon, \tau) \) whose coordinates are explicitly given by

\[ x_1(u, v) = \frac{1}{\tau} v, \]
\[ x_2(u, v) = \frac{1}{2\tau} \sin_\epsilon(2u), \]

with

\[ \sin_\epsilon(u) = \begin{cases} \sin(u) & \text{if } \epsilon = 1, \\ \sinh(u) & \text{if } \epsilon = -1, \end{cases} \]

and

\[ x_3(u, v) = \frac{2}{\tau} \sin_\epsilon(2u)v = 4\tau x_1(u, v)x_2(u, v). \]

More simply, this minimal surface corresponds to the graph of \( x_3 = 4\tau x_1x_2 \).

5.2. Minimal helicoidal surfaces

Let \((\varrho, \theta, h)\), \( \varrho > 0 \), be cylindrical coordinates in \( \mathbb{H}_3(\epsilon, \tau) \) defined by

\[ x_1 = \varrho \cos \theta, \quad x_2 = \varrho \sin \theta, \quad x_3 = h. \quad (80) \]

Given a constant \( a \), we define a variable \( \zeta \) by

\[ \zeta = h - a\theta \quad (81) \]

and then we consider in the half-plane \( \mathbb{R}^2_+ = \{(\varrho, \zeta) : \varrho > 0\} \) the graph \( \varrho \mapsto (\varrho, \zeta(\varrho)) \) of a solution of the ODE

\[ \frac{d\zeta}{d\varrho} = \frac{k}{\varrho} \frac{\sqrt{\varrho^2 + \epsilon(\tau \varrho^2 - a)^2}}{\sqrt{\varrho^2 - \epsilon k^2}}, \quad (82) \]

where \( k \) is a given constant.

If \( k = 0 \), then we obtain the surface described by

\[ h = a\theta + b, \]

for some constant \( b \). This surface corresponds to a minimal helicoid in \( \mathbb{H}_3(\epsilon, \tau) \) given in coordinates by

\[ x_1(\varrho, \theta) = \varrho \cos \theta, \]
\[ x_2(\varrho, \theta) = \varrho \sin \theta, \]
\[ x_3(\varrho, \theta) = a\theta + b. \]

From now on, we consider the case when \( k \neq 0 \), that is, when \( \frac{d\zeta}{d\varrho} \neq 0 \). For sake of brevity, we denote...
In this case, let \( l \) be a function of \( \varrho \) satisfying the ODE
\[
\frac{dl}{d\varrho} = -\frac{\epsilon (a - \tau \varrho^2)}{\varrho^2 + \epsilon (a - \tau \varrho^2)^2} \Psi(\varrho).
\] (83)
We then consider a change of variables \( F : U \subset \mathbb{R}^2 \to \mathbb{R}^2 \) of the form
\[
F(u, v) = (\varrho(u), \theta(u, v))
\] (84)
where
\[
\theta(u, v) = l(\varrho(u)) + v
\]
and the variable \( u \) is given by
\[
\left( \frac{du}{d\varrho} \right)^2 = \frac{\varrho^2}{(\varrho^2 - \epsilon k^2)(\varrho^2 + \epsilon (a - \tau \varrho^2)^2)}. \tag{85}
\]
We then define a map \( g : U \to \mathbb{D} \) given by
\[
g(u, v) = -k \frac{\varrho \Psi(\varrho) + i(a - \tau \varrho^2)}{\varrho^2 \Psi(\varrho) - k \varrho} e^{i \theta(u, v)}. \tag{86}
\]
Eqs. (83) and (85) guarantee that the coordinates \((u, v)\) are conformal in \( U \). It turns out that the map \( g : U \to (\mathbb{D}, ds_\mathbb{D}^2) \) defined in (86) is a harmonic map in the sense of (68). Thus, \( g \) represents a conformal minimal immersion whose local expression is given by (73)–(75).

In the particular case when \( a = 0 \), we have a minimal surface of revolution analog of the catenoid. Notice that both helicoid and catenoid belong to families of helicoidal minimal surfaces.

5.3. CMC surfaces in \( \mathbb{R}^3 \) and minimal surfaces in \( \mathbb{H}_3(-1, \tau) \)

Let \( \Sigma \) be a simply connected Riemann surface and let \( Y : \Sigma \to \mathbb{R}^3 \) be a conformal immersion with constant non-zero mean curvature. Suppose that the spherical image of \( \Sigma \) is contained in an open hemisphere of \( S^2 \), the southern hemisphere, say. Then, it follows from the classical result by Ruh and Wilms that if \( n : \Sigma \to S^2 \) is the Gauss map of \( Y \), then the map \( g : \Sigma \to (\mathbb{D}, ds_\mathbb{D}^2) \) is harmonic, where \( g = \pi_1 \circ n \). Thus, it follows from Theorem 3 that \( g \) represents a minimal immersion \( X : \Sigma \to \mathbb{H}_3(-1, \tau) \).

We conclude that there exists a correspondence between open portions of CMC surfaces in Euclidean space whose spherical image is contained in an open hemisphere and minimal surfaces in \( \mathbb{H}_3(-1, \tau) \).

In particular, given a simply connected open domain \( \Sigma \subset \mathbb{R}^2 \) and a real function \( h : \Sigma \to \mathbb{R} \) whose graph has constant non-zero mean curvature in \( \mathbb{R}^3 \), then the map
\[
g(z) = 2 \Lambda h_z, \quad z \in \Sigma,
\] (87)
represents a minimal surface in \( \mathbb{H}_3(-1, \tau) \), where
\[
\Lambda = \frac{\sqrt{1 + |dh|^2}}{1 - \sqrt{1 + |dh|^2}}. \tag{88}
\]

References