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# Products of Zero-One Processes and the Multilane Highway Crossing Problem*

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## 1. INTRODUCTION

For each $i$, $i = 1, 2, \cdots, n$, let $X^{(i)}(t)$ be a stochastic process which has only the values 0 or 1 on $-\infty < t < +\infty$. The process

$$Y(t) = \prod_{i=1}^{n} X^{(i)}(t) \tag{1}$$

is then also a process on 0 or 1. Furthermore if the processes $X^{(i)}(t)$ are statistically independent and stationary, $Y(t)$ is also stationary. We would like to know how, for certain simple types of processes, the stochastic properties of $Y(t)$ depend upon the properties of the $X^{(i)}(t)$.

Problems of this type can arise in a wide variety of practical situations. Suppose, for example, we have a machine. Let $X^{(i)}(t) = 0$ represent the event that the $i$th component of the machine works (or fails to work) at time $t$, and $X^{(i)}(t) = 1$ represent the event that it fails (works). Perhaps the $i$th component is a tube in an electronic circuit which burns out after some time and, after a random repair time, is replaced by a new one. The $X^{(i)}(t)$ might in this case be an alternating renewal process [1]. The event $Y(t) = 0$ is then the event that some components work (fail) at time $t$, and $Y(t) = 1$ the event that all components fail (work).

The motivation for the present investigation of this class of problems, however, arises from the study of certain highway traffic intersection (crossing or merging) problems. If a driver must yield the right of way at some intersection, he is looking for some suitable gap in the approaching traffic. For various practical reasons it is usually convenient to represent a traffic

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stream as a sequence of alternating blocks and gaps. A block is generated by a series of approaching cars spaced so close together (a platoon) that it is impossible for any driver to merge or cross within the block. The stochastic properties of the positions of cars within these platoons is likely to be quite complicated but also irrelevant. We ,therefore, postulate some probability distribution for the total length of the block, a quantity which one can measure experimentally without also measuring the individual positions within the block.

An approaching stream of traffic is then represented by a process $X(t)$ with $X(t) = 0$ corresponding to a block and $X(t) = 1$ a gap. If a driver wishes to cross a two-lane highway, he must find simultaneous gaps in two streams. He is, therefore, looking for time points where $Y(t) = X^{(1)}(t)\, X^{(2)}(t)$ is one.

Since the manner in which the block lengths in each lane depends upon the positions of cars within the blocks does not enter the problem explicitly, it is not necessary that they be defined in the same way for all lanes. Thus we might consider a gap to occur in lane $i$ only if two cars are spaced a distance at least $L_i$ apart, but we need not take all $L_i$ equal. Perhaps a driver must find simultaneous gaps in streams of quite different physical properties, a stream of cars and a stream of pedestrians.

We are not so much concerned here with what can be evalutated in principle, but rather in what can be done explicitly and easily, starting with the simplest problems and then perhaps later generalizing to somewhat more complicated ones.

A few special problems of the above type have been treated before. The simplest example is the following. Suppose we have two statistically independent streams of point cars, each defining a Poisson process, and we wish to find a spacing of at least $L$ simultaneously in both streams. One possible way to approach this problem is to represent each stream as an alternating sequence of blocks and gaps. The block and gaps are formed in each stream if we cover each car with a segment of length $L$ and let the gaps be the intervals not covered by any such segments. There will be a spacing of at least $L$ simultaneously in both streams if and only if there is simultaneously a gap in both block and gap streams.

Although this is an example of the type of problem described above, the simplest way to analyze it [2] is to observe that the superposition of two independent Poisson streams is also a Poisson stream. We will find a simultaneous spacing of length $L$ in both streams if and only if the combined stream has a gap of length $L$. The two stream problem is thus reduced to a single stream problem, the analysis of which is well known.

In a more general version of this problem [3, 4], the two traffic streams are again taken to be Poisson processes, but one is looking simultaneously for an empty interval of length at least $L_1$ in lane 1 and $L_2$ in lane 2 but $L_1 \neq L_2$. Again, one could approach this problem by representing each stream as alternating blocks and gaps. Then look for a simultaneous gap in both streams. This will turn out to be a very cumbersome way to threat the problem, however, and is not the way it was analyzed previously.

Despite the fact that in these examples we do not find it convenient to analyze the crossing problem by representing each traffic stream in terms of blocks and gaps, there are many other special cases in which such an approach does lead to fairly simple results and perhaps also a more realistic representation of the physical process in question.

## 2. Mean Lengths of Blocks and Gaps

Suppose the processes $X^{(i)}(t)$ are stationary and statistically independent. Let the successive lengths of the blocks and gaps in the $i$th stream be

$$-B_{j-1}^{(i)}, G_{j-1}^{(i)}, B_j^{(i)}, G_j^{(i)}, B_{j+1}^{(i)}, G_{j+1}^{(i)}, -$$

i.e., $X_i(t) = 1$ for a time $G_j^{(i)}$, then $X_i(t) = 0$ for a time $B_{j+1}^{(i)}$, etc. Let—$B_k$, $G_k$, $B_{k+1}$,—be the sequence of blocks and gap lengths in the process $Y(t)$.

For quite general classes of stationary processes (having certain ergodic properties but otherwise no restriction on their stochastic structure), one can say that over a sequence of $N$ blocks and gaps, the fraction of time spent in gaps is

$$\frac{\sum_{k=1}^{N} G_k^{(i)}}{\sum_{k=1}^{N} (G_k^{(i)} + B_k^{(i)})} \to \frac{E\{G^{(i)}\}}{E\{G^{(i)}\} + E\{B^{(i)})} \tag{2}$$

for $N \to \infty$. Where $E\{G^{(i)}\} \equiv E\{G_k^{(i)}\}$ is independent of $k$. This is also the probability that one will find a gap in lane $i$ at some arbitrarily selected time $t$.

The probability that one finds a gap at time $t$ in the $Y$-process should similarly be equal to

$$\frac{E\{G\}}{E\{G\} + E\{B\}}. \tag{2a}$$

But since the $n$ streams $X^{(i)}(t)$ are statistically independent of each other, the probability that there is a gap in the $Y$-process is the product of the probabilities that there are gaps in each of the $X^{(i)}$-processes. Thus

$$\frac{E\{G\}}{E\{B\} + E\{G\}} = \prod_{i=1}^{n} \frac{E\{G^{(i)}\}}{E\{G^{(i)}\} + E\{B^{(i)}\}}. \tag{3}$$

This formula gives only the *relative* mean lengths of the blocks and gaps for the $Y$-process in terms of those for the $X^{(i)}$-processes, but we can apply a similar argument to find $E\{G\}$ alone.

Suppose that in (2), instead of adding the total time spent in gaps, we added the values of

$$\max\{G_k^{(i)} - g, 0\}$$

for some constant $g \geqslant 0$, to obtain

$$\frac{\int_g^\infty (x-g)\, dF^{(i)}(x)}{E\{G^{(i)}\} + E\{B^{(i)}\}} = \frac{\int_g^\infty [1 - F^{(i)}(x)]\, dx}{E\{G^{(i)}\} + E\{B^{(i)}\}} \tag{4}$$

in which $F^{(i)}(g)$ is the marginal distribution function for the gaps in lane $i$. If we had extended the length of each block in the $X^{(i)}$-process by an amount $g$, to obtain a new block and gap process, (4) represents the mean time spent in the gaps of the new $i$th process. But if we superimpose these processes to obtain a new $Y$-process, a gap in the new $Y$-process occurs if and only if gaps of length at least $g$ occur simultaneously in all of the original $X^{(i)}$ processes, and consequently a gap of length at least $g$ occurs also in the original $Y$-process. We thus conclude by the same arguments as for (3) that

$$\frac{\int_g^\infty [1 - F(x)]\, dx}{E\{G\} + E\{B\}} = \prod_{i=1}^n \frac{\int_g^\infty [1 - F^{(i)}(x)]\, dx}{E\{G^{(i)}\} + E\{B^{(i)}\}},$$

where $F(x)$ is the marginal distribution function for the gaps $G$ of the $Y$-process.

This along with (3) gives

$$\frac{\int_g^\infty [1 - F(x)]\, dx}{E\{G\}} = \prod_{i=1}^n \frac{\int_g^\infty [1 - F^{(i)}(x)]\, dx}{E\{G^{(i)}\}}. \tag{5}$$

We might also have derived (5) by a slightly different argument. If we choose an arbitrary time point, the left side of (5) represents the distribution function of the gap one lands in, provided one lands in a gap. The right-hand side is the product of corresponding quantities for the individual lanes. The equation does not contain any properties of the block lengths. If the block lengths are zero, the $X^{(i)}$ processes are point proecsses, and the $Y$-process is the superposition of the $X^{(i)}$-processes. Formulas similar to (5) have previously been derived for the superposition of independent renewal processes [1, 5, 6].

If we differentiate (5) with respect to $g$ and set $g = 0$, we obtain

$$\frac{1}{E\{G\}} = \sum_{k=1}^{n} \frac{1}{E\{G^{(k)}\}}. \tag{6}$$

Equation (6) gives the mean gap $E\{G\}$ in terms of the mean gaps of the $X^{(i)}$-process. This along with (5) determines the marginal distribution function $F(x)$ of the gaps; and (6) combined with (3) gives the mean block length, namely

$$E\{B\} = \frac{\prod_{i=1}^{n} [1 + E\{B^{(i)}\}/E\{G^{(i)}\}] - 1}{\sum_{i=1}^{n} 1/E\{G^{(i)}\}}. \tag{7}$$

For the special case of identical lanes

$$E\{B^{(i)}\} = E\{B^{(1)}\} \qquad \text{and} \qquad E\{G^{(i)}\} = E\{G^{(1)}\},$$

this simplifies to

$$E\{B\} = E\{B^{(1)}\} \frac{[1 + E\{B^{(1)}\}/E\{G^{(1)}\}]^n - 1}{nE\{B^{(1)}\}/E\{G^{(1)}\}},$$

$$E\{G\} = \frac{E\{G^{(i)}\}}{n}. \tag{8}$$

The mean block length $E\{B\}$ is a monotone inceasing function of $n$ but the mean gap length $E\{G\}$ is a monotone decreasing function of $n$. For large $n$ the block length increases nearly exponentially with $n$ but the gap lengths decrease only as $n^{-1}$.

In the special case of only two identical streams, the average time from the start of one block to the start of the next block is

$$E\{B\} + E\{G\} = \tfrac{1}{2} [E\{B^{(1)}\} + E\{G^{(1)}\}] [1 + E\{B^{(1)}\}/E\{G^{(1)}\}]. \tag{9}$$

This is greater than or less than the corresponding quantity $E\{B^{(1)}\} + E\{G^{(1)}\}$ for a single stream accordingly as the blocks $B^{(1)}$ are on the average larger or smaller than the gaps $G^{(1)}$. If the gaps are much larger than the blocks, the dominant effect of superposition is that a short block in one lane is contained within a long gap of the other so that the combined process has two gaps plus a block where the single stream had only one gap. The value of $B + G$ is therefore, on the average, shorter than $B^{(1)} + G^{(1)}$. On the other hand, if the blocks are much larger than the gaps, the dominant effect is that a gap in one lane may be completely covered by a block in the other lane. The block in the combined stream may therefore extend over several block-gap sequences of the single streams.

From the above formulas, we can evaluate from (2a) the probability that one will find, at an arbitrary time $t$, a gap in the $Y$-process or a block. If we find a block, however, these formulas do not tell us how long we must wait before we see a gap. This depends upon second moments of $B$, the distribution of which we have not found because this does depend upon the more detailed structural properties of the processes $X^{(i)}$.

It is possible to make some classifications of the structure of the $Y$-process for certain special types of the $X^{(i)}$-processes. If the $X^{(i)}$-processes are alternating renewal processes; i.e., the $B_k^{(i)}$ and $G_k^{(i)}$ are independent random variables, then the $Y$-process is a form of semi-Markov process. If at the start of any gap $G_k$ of the $Y$-process, one specifies the age of the gaps in all lanes, then one can uniquely give the distribution of the remaining life of all these gaps, and of all subsequent events.

In the special case of this in which the gaps all have an exponential distribution, the future is independent of the age of the existing gaps. The $Y$-process is then also an alternating renewal process with exponentially distributed gaps (but perhaps a rather complicating distribution of block lengths). The $Y$-process, however, would be completely determined by the distribution of the blocks, the distribution of the gaps being uniquely defined already through (6) and its exponential form.

## 3. Exponentially Distributed Blocks and Gaps

We consider now the special case in which

$$\begin{aligned} P\{G^{(i)} > t\} &= \exp(-\alpha_i t) \\ P\{B^{(i)} > t\} &= \exp(-\beta_i t) \end{aligned} \tag{10}$$

for some constants $\alpha_i$ and $\beta_i$.

This is a particularly simple case because of the special property of exponentially distributed random variables that the future life of the random variable is independent of how long it has already lived. Not only is this mathematically one of the simplest cases (perhaps even simpler than to find simultaneous gaps in the Poisson streams), but for some applications to highway crossing, the exponential distributions are probably as realistic as any simple type of distribution. There is some experimental evidence [7, 8] to suggest that spacings between cars not in platoons are approximately exponentially distributed, and that the number of cars in a platoon has approximately a geometric distribution (which implies that the duration of a platoon should be nearly exponentially distributed). Almost any heuristic argument or simple model of queueing in traffic also leads to distributions that could be approximated reasonably well with exponential distributions.

For this special model, the gaps of the $Y$-process will be exponentially distributed with a distributive function

$$P\{G > t\} = \prod_{1}^{n} P\{G^{(i)} > t\} = \exp\left(-t\sum_{i=1}^{n} \alpha_i\right).$$

To find the distribution of the block lengths $B$ of the process $Y$, we observe that the vector process $\{X^{(1)}(t), X^{(2)}(t), \cdots X^{(n)}(t)\}$ is a semi-Markov process on a space of $2^n$ points. If we know at any time $t$ whether or not each stream is in a block or gap state, we will know the probability for the life-time of this state and also which state it will go to next. The block length $B$ is the recurrence time for the state $\{1, 1, \cdots, 1\}$ in the process $\{X^{(1)}(t), \cdots, X^{(n)}(t)\}$.

We shall consider in detail only the case $n = 2$, although the methods used here are quite standard and can be generalized to arbitrary $n$.

We number the states $\{1, 1\}$, $\{1, 0\}$, $\{0, 1\}$, $\{0, 0\}$ as states 0, 1, 2, and 3 respectively and let

$$p_k(t) = P\{\text{state is } k \text{ at time } t \text{ and has not visited state 0 during } (0, t)\}.$$

Following the usual method [9], we obtain for $p_k(t)$ the differential equation

$$\frac{dp(t)}{dt} = p(t)\, R \tag{11}$$

with $p(t)$ denoting the vector

$$p(t) = (p_1(t), p_2(t), p_3(t)) \tag{12}$$

and $R$ the matrix

$$R = \begin{pmatrix} -(\alpha_1 + \beta_2) & 0 & \alpha_1 \\ 0 & -(\alpha_2 + \beta_1) & \alpha_2 \\ \beta_1 & \beta_2 & -(\beta_1 + \beta_2) \end{pmatrix}. \tag{13}$$

This describes the situation that if the system is in state 1, say, at time $t$ (having not yet visited state 0), it will leave this state at a rate $-(\alpha_1 + \beta_2)$, enter state 0 (an absorbing state) at a rate $\beta_2$ because the block in lane 2 ends, or enter state 3 at a rate $\alpha_1$ because the gap in lane 1 ends.

If time 0 represents a time at which the process just left state 0, the probability that it will enter state $k$ is proportional to the rates of transition from state 0 to $k$. The probability vector $p(0)$, immediately after the process leaves state 0, is, therefore

$$p(0) = \left(\frac{\alpha_2}{\alpha_1 + \alpha_2}, \frac{\alpha_1}{\alpha_1 + \alpha_2}, 0\right). \tag{14}$$

With this as the initial conditions for (11), $p_k(t)$ represents the probability that the process is in state $k$ at time $t$ and that the block of the $Y$-process which starts at time 0 has not yet ended. The block length $B$, therefore has a distribution function

$$F_B(t) = P\{B \leqslant t\} = 1 - p_1(t) - p_2(t) - p_3(t). \tag{15}$$

If we let $e$ denote the column vector

$$e = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{16}$$

we can also write (15) as

$$F_B(t) = 1 - p(t)\, e. \tag{17}$$

Rather than evaluate these quantities directly, we consider the moment generating functions. Let $\bar{p}(s)$ be the Laplace transform of the vector $p(t)$

$$\bar{p}_k(s) = \int_0^\infty e^{-st}\, p_k(t)\, dt \tag{18}$$

and $M_B(s)$ the moment generating function of $B$. Then

$$\begin{aligned} M_B(s) &= \int_0^\infty e^{-st}\, dF_B(t) = -\int_0^\infty e^{-st}\, d[p(t)\, e] \\ &= 1 - s\bar{p}(s)\, e, \end{aligned} \tag{19}$$

and the moments of $B$ are

$$E\{B^n\} = (-1)^n \frac{d^n}{ds^n} M_B(s)\Big|_{s=0} = (-1)^{n-1}\, n\, \frac{d^{n-1}}{ds^{n-1}} [\bar{p}(s)\, e]\Big|_{s=0}. \tag{20}$$

To evaluate $\bar{p}(s)$, we take the Laplace transform of (11)

$$\bar{p}(s)\, R = \int_0^\infty e^{-st} \left(\frac{dp}{dt}\right) dt = s\bar{p}(s) - p(0)$$

or

$$\bar{p}(s) = p(0)\,[sI - R]^{-1} \tag{21}$$

in which $I$ is the $3 \times 3$ identity matrix. Since the inverse of a $3 \times 3$ matrix can be evaluated explicitly (though perhaps not in a neat form), Eqs. (21), (13) and (14) determine $M_B(s)$ in (19).

The evaluation of $E\{B\}$ from the formulas gives

$$E\{B\} = \frac{(1 + \alpha_1/\beta_1)\,(1 + \alpha_2/\beta_2) - 1}{\alpha_1 + \alpha_2},$$

which agrees with the more general formula (7). The evaluation of the second moments, already rather tedious, gives

$$E\{B\} = \frac{1}{\beta_1{}^2\beta_2{}^2(\alpha_1 + \alpha_2)}\left\{\alpha_2\beta_1(\alpha_1 + \beta_1) + \alpha_1\beta_2(\alpha_2 + \beta_2) + \frac{\alpha_1\alpha_2(\alpha_1 + \beta_1)\,(\alpha_2 + \beta_2)}{\alpha_1 + \alpha_2 + \beta_1 + \beta_2}\right\}.$$

For the case of identical lanes $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, these simplify to

$$E\{B\} = \frac{(2 + \alpha/\beta)}{2\beta}, \qquad \mathrm{Var}\,\{B\} = \frac{4 + 6(\alpha/\beta) + (\alpha/\beta)^2}{4\beta^2}. \tag{22}$$

For an exponentially distributed random variable, the variance and the square of the mean are equal. Here we find that

$$\frac{\mathrm{Var}\,B}{E^2\{B\}} = 1 + \frac{2\alpha\beta}{(2\beta + \alpha)^2} \geqslant 1, \tag{23}$$

which means that, compared with the exponential distribution, the distribution of $B$ has more probability at short values of $B$ and large values. For $\alpha/\beta \ll 1$, the individual streams are mostly gaps; the blocks of the combined stream will usually coincide with the blocks of one stream or the other. The latter, however, are exponentially distributed, so the ratio (23) goes to 1 for $\alpha/\beta \to 0$. If $\beta/\alpha \to 0$, both streams are mostly blocks. The occurrence of a gap in the combined stream is a rare event as shown in (22) by the fact that

$$E\{B\} \sim \frac{\alpha}{2\beta^2} \to \infty.$$

For rather general types of processes, the time to the first occurrence of a rare event is usually almost exponential. This is indicated here by the fact that for $\beta/\alpha \to 0$, (23) again goes to 1.

The probability $p(t)$ can be found either from inversion of the generating function (19), which is a rational function of $s$, or from Eq. (11) directly.

If we let $D$ denote the diagonal matrix

$$D = \begin{pmatrix} (\alpha_1/\beta_1)^{1/2} & 0 & 0 \\ 0 & (\alpha_2/\beta_2)^{1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$D^{-1}RD = \begin{pmatrix} -(\alpha_1+\beta_2) & 0 & (\alpha_1\beta_1)^{1/2} \\ 0 & -(\alpha_2+\beta_1) & (\alpha_2\beta_2)^{1/2} \\ (\alpha_1\beta_1)^{1/2} & (\alpha_2\beta_2)^{1/2} & -(\beta_1+\beta_2) \end{pmatrix} \tag{24}$$

is symmetric. The eigenvalues of this matrix (and of $R$) are therefore real. The eigenvectors are also real and orthogonal. From the secular equation one can also show that the eigenvalues are negative.

Let $-s_j, j = 1, 2, 3$ be the eigenvalues of $D^{-1}RD$ and $y_j$ the corresponding normalized eigenvectors, i.e.,

$$y_j D^{-1}RD = -s_j y_j, \qquad y_j \cdot y_k = \delta_{jk}$$

with $y_j \cdot y_k$ denoting the usual inner product. The solution of (11) can then be written as

$$p(t) = \sum_{j=1}^{3} [p(0)\,D \cdot y_j]\, e^{-s_j t} y_j D^{-1} \tag{25}$$

and from (17)

$$F_B(t) = 1 - \sum_{j=1}^{3} [p(0)\,D \cdot y_j]\, e^{-s_j t} [y_j D^{-1} \cdot e]. \tag{26}$$

The distribution $1 - F_B(t)$ is therefore a sum of at most three negative exponentials.

The evaluation of the eigenvalues and eigenvectors of a $3 \times 3$ matrix is a straightforward but not elegant exercise which we shall not pursue here. The problem, however, does simplify considerably in a few special cases.

## 4. Special Cases

(a) *Identical lanes.* For two identical lanes, $R$ is invariant to the interchange of states 1 and 2. One eigenvector is obviously

$$y_1 = 2^{-1/2}(1, -1, 0) \qquad \text{for} \qquad s_1 = \alpha + \beta.$$

The problem can then be reduced to a two-dimensional one. The results are

$$y_{2,3} = C_\pm\left(-1, -1, \frac{-\alpha+\beta\pm\gamma}{2(\alpha\beta)^{1/2}}\right), \qquad s_{2,3} = \frac{\alpha+3\beta\pm\gamma}{2}$$

with

$$\gamma^2 = \alpha^2 + 6\alpha\beta + \beta^2$$

$$C_\pm{}^2 = \frac{2\alpha\beta}{\pm\gamma(-\alpha+\beta\pm\gamma)},$$

and

$$F_B(t) = 1 - \textstyle\sum_\pm \frac{1}{2}\,[1 \mp (\alpha+\beta)/\gamma]\exp[-(\alpha+3\beta\pm\gamma)\,t/2].$$

(The coefficient of $\exp(-s_1 t)$ vanishes.) Since $|(\alpha+\beta)/\gamma| < 1$, both exponentials are multiplied by positive coefficients. $F_B(t)$ is a distribution with "decreasing failure rate," For large $t$, $1 - F_B(t)$ is nearly proportional to the exponential with the slower decay, namely the one with decay rate $(\alpha + 3\beta - \gamma)/2$.

(b) $\alpha_1 \to 0$. Suppose one lane is mostly gaps (lane 1 say), as would be the case if one lane of traffic was very light. One would then expect the presence of a lane 1 to have little effect. For $\alpha_1 = 0$ we see from (13) that

$$s_1 = \beta_2, \qquad s_2 = \beta_1, \qquad s_3 = \alpha_2 + \beta_2 + \beta_2,$$

but $p(0) = (1, 0, 0)$, which is a left eigenvector of $R$ for eigenvalue $-\beta_2$. The only nonvanishing term of (26) is the one for $j = 1$. Thus

$$F_B(t) \to 1 - \exp(-\beta_2 t) \qquad \text{for} \qquad \alpha_1 \to 0,$$

which is also the distribution function for the blocks in lane 2 alone.

From here one could go on, using perturbation methods, to determine the effect of a small but nonzero $\alpha_1$.

(c) $\beta_1 \to 0$. Suppose now that the traffic in lane one is very heavy and the blocked periods very long ($\beta_1$ small). If we set $\beta_1 = 0$, the roots become

$$s_1 = 0, \qquad s_2 = \alpha_1 + \beta_2, \qquad s_3 = \alpha_2 + \beta_2.$$

For $s_1 = 0$, however, $\exp(-s_1 t) = 1$ in (26) for every finite $t$, and $F_B(t)$ becomes an improper distribution. It is desirable, therefore, to compute, through a perturbation expansion, the lowest order nonvanishing contribution to $s_1$ for $\beta_1 \to 0$. This estimate requires a "second order" perturbation calculation of the eigenvalues of (24) because $\beta_1$ appears linearly in the diagonal elements, but the off-diagonal elements are proportional to $\beta^{1/2}$.

The result is

$$s_1 = \frac{\beta_2(\alpha_1 + \alpha_2 + \beta_2)}{(\alpha_1 + \beta_2)(\alpha_2 + \beta_2)} \beta_1 + o(\beta_1). \tag{27}$$

If in (26) we set $\beta_1 = 0$ everywhere except in the factor $\exp(-s_1 t)$, we obtain

$$\begin{aligned} F_B(t) = 1 &- \frac{\alpha_1}{(\alpha_1 + \alpha_2)}\left[1 + \frac{\alpha_2}{\alpha_1 + \beta_2}\right] \exp(-s_1 t) \\ &- \frac{\alpha_2}{(\alpha_1 + \alpha_2)}\left[1 - \frac{\alpha_1}{\alpha_1 + \beta_2}\right] \exp[-(\alpha_1 + \beta_2)t] \end{aligned} \tag{28}$$

in which one of the exponentials decays very rapidly compared with the other. $F_B(t)$ starts at $t = 0$ with the value $F_B(0) = 0$, but as $t$ increases the last term of (28) decays before the other terms have changed very much. For $1/(\alpha_1 + \beta_2) \ll t \ll 1/s_1$,

$$F_B(t) \sim 1 - \frac{\alpha_1}{(\alpha_1 + \alpha_2)}\left[1 + \frac{\alpha_2}{\alpha_1 + \beta_2}\right].$$

As $t$ increases further, the other exponential then gradually decays.

The two exponential terms of (28) also have a simple interpretation. According to (14), a block starts at $t = 0$ either with a block in lane 2 or in lane 1 (states 1 or 2 respectively). The last term of (28) contains the factor $\alpha_2/(\alpha_1 + \alpha_2)$, the probability that the block starts because lane 2 has a block. The probability that the system will still be in this state at time $t$ is $\exp[-(\alpha_1 + \beta_2)t]$, the probability that block in lane 2 still survives and the gap in lane 1 still survives. This is also the exponential factor of the last term of (28). When the system leaves this state, it can either go to state 3 because a block starts also in lane 1, or to state 0 (the end of block $B$) if the block in lane 2 ends. If the former happens, the blocked state is likely to live a long time (of order $1/\beta_1$). The probability that the latter occurs rather than the former is, however, the remaining factor in the last term of (28), $\beta_2/(\alpha_1 + \beta_2)$. Thus the last term of (28) represents the probability that the state will go from 0 to 1 and back to 0 but will do so in a time larger than $t$.

The coefficient of the other exponential term can now be interpreted as the probability that the block either started from a block in lane 1, or a block in lane 1 is formed before the state can return to state 0. If the latter occurs, it happens, on the average, within a time which is negligible compared with the life time of the block in lane 1 and can be considered to happen essentially at $t = 0$ also.

If after the block in lane 1 ends at time $B_1^{(1)}$ say, the subsequent gap in lane 1 is covered by a block in lane 2, the block $B$ continues through the next block of length $B_2^{(1)}$ in lane 1, etc. If $\beta_1 \ll \alpha_1$, $\beta_2$, and $\alpha_2$, lane 2 will (almost always) go through many block-gap sequences during the time $B_1^{(1)}$. The state of lane 2 at time $B_1^{(1)}$ and its future behavior will have nearly reached a statistical equilibrium and be statistically independent of the state of lane 2 at time 0 or the value of $B_1^{(1)}$. The probability $p$ that the gap in lane 1 will be covered by a block in lane 2 is approximately

$$p = \frac{\alpha_2}{(\alpha_2 + \beta_2)} \cdot \frac{\alpha_1}{(\alpha_1 + \beta_2)},$$

the probability that lane 2 is blocked at time $B_1^{(1)}$, $\alpha_2/(\alpha_2 + \beta_2)$, times the probability that the block in lane 2 will outlive the gap in lane 1, $\alpha_1/(\alpha_1 + \beta_2)$. When this happens, the time consumed in the transition is negligible compared with the time $B_2^{(1)}$ of the next block in lane 1. The length $B$ is the time to the first occurrence of a gap in lane 1 not covered by a block in lane 2, which is approximately

$$B = \sum_{j=1}^{M} B_j^{(1)},$$

in which $M$ is a random variable with a geometric distribution

$$P\{M = m\} = (1 - p)\, p^{m-1}.$$

Since the $B_j^{(1)}$ are exponentially distributed, it follows also that $B$ will be exponentially distributed (conditioned that it include at least one $B_j^{(1)}$). The mean of $B$ is

$$E\{B\} = \frac{E\{M\}}{\beta_1} = \frac{1}{\beta_1(1 - p)} = \frac{(\alpha_2 + \beta_2)(\alpha_1 + \beta_1)}{\beta_1\beta_2[\alpha_1 + \alpha_2 + \beta_2]},$$

which we recognize from (27) as $1/s_1$, the rate of decay of the remaining exponential in (28).

There are several other special cases in which the roots $s_j$ can be approximated by relatively simple formulas, but we shall not pursue this further. Suffice it to say that results given here are manageable, simple enough to be understandable, and, for certain applications, realistic enough to be useful.

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