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Equilibrium Points of Rational n-Person Games

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Von Neumann in [9] introduced two-person games with rational payoff functions and extended the minimax theorem. Since then, other very efficient and elegant approaches to prove it in the extended case have been considered (for example, Bellman [1]). Loomis in [4] and Shapley in [8] also considered such games. In this paper, we prove the existence of an equilibrium point in an n-person game with rational payoff functions.

Consider the following auxiliary result:

LEMMA. *Let Σ be a nonempty, compact, and convex set in a Euclidean space and let ϕ be a real continuous function defined on $\Sigma \times \Sigma$. If for each $\tau \in \Sigma$, $\phi(\cdot, \tau)$ is quasi-concave, then there exists a $\bar{\sigma} \in \Sigma$ such that*

$$\phi(\bar{\sigma}, \bar{\sigma}) = \max_{\sigma \in \Sigma} \phi(\sigma, \bar{\sigma}).$$

Proof. For a given $\tau \in \Sigma$, consider the set

$$\theta(\tau) = \{\sigma \in \Sigma : \phi(\sigma, \tau) = \max_{\rho \in \Sigma} \phi(\rho, \tau)\}$$

which is nonempty since ϕ is continuous. Moreover, by the quasi-concavity of $\phi(\cdot, \zeta)$, $\theta(\zeta)$ is convex. This determines a multivalued function $\theta: \Sigma \to \Sigma$. We will show that θ is upper-semicontinuous. Indeed, consider any two convergent sequences $\zeta(n) \to \zeta$ and $\sigma(n) \to \sigma$ of points of Σ, such that $\sigma(n) \in \theta(\zeta(n))$ for each n. Therefore, we have the following

$$\phi(\sigma(n), \zeta(n)) \to \phi(\sigma, \zeta)$$

and

$$\max_{\rho \in \Sigma} \phi(\rho, \zeta(n)) \to \max_{\rho \in \Sigma} \phi(\rho, \zeta)$$

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since both functions are continuous. On the other hand, the condition $\sigma(n) \in \theta(\zeta(n))$ for each n, implies that both previous limits coincide. Hence, $\sigma \in \theta(\zeta)$, which in other words, says that the function θ is upper-semi-continuous.

Using Kakutani's fixed point theorem, we obtain the existence of a point $\bar{\sigma} \in \theta(\bar{\sigma})$. Such a point satisfies the condition of the lemma. Q.E.D.

Now consider an n-person game $\Gamma = \{\Sigma_i, F_i; i \in N\}$ where the strategy sets Σ_i are compact and convex in Euclidean spaces. The payoff functions $F_i = M_i/N_i$ are continuous rational functions where M_i and N_i are concave and convex in $\sigma_i \in \Sigma_i$ for each $\tau_{N-\{i\}} \in \Sigma_{N-\{i\}} = \mathsf{X}_{j\in N-\{i\}} \Sigma_j$, respectively. Besides, N_i is positive.

THEOREM. *Γ has an equilibrium point.*

Proof. For each player $i \in N$, consider the function

$$G_i(\sigma_i, \tau) = N_i(\tau)\, M_i(\sigma_i, \tau_{N-\{i\}}) - M_i(\tau)\, N_i(\sigma_i, \tau_{N-\{i\}})$$

with $\tau \in \Sigma = \mathsf{X}_{j\in N} \Sigma_j$. This function is concave in $\sigma_i \in \Sigma_i$ for each $\tau \in \Sigma$ since it is the sum of two concave functions.

$$\phi(\sigma, \tau) = \sum_{i\in N} G_i(\sigma_i, \tau)$$

is concave in $\sigma \in \Sigma$, too. By the previous lemma there is a $\bar{\sigma} \in \Sigma$, such that

$$\phi(\bar{\sigma}, \bar{\sigma}) = \max_{\sigma\in\Sigma} \phi(\sigma, \bar{\sigma})$$

or equivalently

$$G_i(\bar{\sigma}_i, \bar{\sigma}) \geqslant G_i(\sigma_i, \bar{\sigma})$$

for each $i \in N$ and $\sigma_i \in \Sigma_i$. Substituting the M's and N's, we have

$$0 \geqslant N_i(\bar{\sigma})\, M_i(\sigma_i, \bar{\sigma}_{N-\{i\}}) - M_i(\bar{\sigma})\, N_i(\sigma_i, \bar{\sigma}_{N-\{i\}}).$$

Therefore, $\bar{\sigma}$ is an equilibrium point of Γ. Q.E.D.

At this point, we remark the fact that the previous theorem cannot immediately be extended to the case when M_i is quasi-concave and N_i is convex. The reason is that the sum of a quasi-concave and a concave functions is not necessarily quasi-concave. Indeed, consider, as an example, the functions defined in [0, 2]:

$$
\begin{aligned}
f(x) &= \frac{1}{\epsilon}x + \left(1 - \frac{1}{\epsilon}\right) && \text{if} \quad x \in [1-\epsilon, 1], \\
&= -\frac{1}{\epsilon}x + \left(1 + \frac{1}{\epsilon}\right) && \text{if} \quad x \in [1, 1+\epsilon], \\
&= 0 && \text{if} \quad x \in [0, 1-\epsilon] \cup [1+\epsilon, 2],
\end{aligned}
$$

$$
g(x) = x, \quad x \in [0, 2]
$$

for some $0 < \epsilon < 1$. It is easy to see that $f + g$ is not quasi-concave.

As a direct application of the Theorem, we have the following result:

COROLLARY 1. *The extension* $\Gamma^* = \{\tilde{\Sigma}_i, H_i; i \in N\}$ *of any finite game* $\Gamma = \{\Sigma_i, A_i/B_i, i \in N\}$ *with*

$$
H_i(x_1, ..., x_n) = \frac{E_i(x_1, ..., x_n)}{F_i(x_1, ..., x_n)},
$$

$x_i \in \tilde{\Sigma}_i$ *and where* E_i *and* F_i *are the expectations of* A_i *and* B_i, *respectively, has an equilibrium point when* $B_i > 0$.

If B_i is identically one, then an equilibrium point of Γ^* becomes an equilibrium point for the mixed extension $\tilde{\Gamma} = \{\tilde{\Sigma}_i, E_i; i \in N\}$. On the other hand, in the case of zero-sum two-person games we have von Neumann's theorem.

As a further application of the main results, we have:

COROLLARY 2. *For any game* $\Gamma = \{\Sigma_i, F_i; i \in N\}$ *with compact and convex strategy sets* Σ_i *in enclidean spaces and where the payoff functions* $F_i = M_i/N_i$ *are rational functions with* M_i *and* $N_i > 0$ *concave and convex in* $\sigma_i \in \Sigma_i$, *respectively, there exists a point* $\bar{\sigma} \in \Sigma$ *such that for each* $i \in N$ *and each* $\sigma_i \in \Sigma_i$

$$
\frac{\min_{\rho_{e(i)}} M_i(\bar{\sigma}_i, \rho_{e(i)}, \bar{\sigma}_{f(i)})}{\max_{\rho_{e(i)}} N_i(\bar{\sigma}_i, \rho_{e(i)}, \bar{\sigma}_{f(i)})} \geqslant \frac{\min_{\rho_{e(i)}} M_i(\sigma_i, \rho_{e(i)}, \bar{\sigma}_{f(i)})}{\max_{\rho \in e(i)} N_i(\sigma_i, \rho_{e(i)}, \bar{\sigma}_{f(i)})}
$$

with

$$
e(i) = N - (f(i) \cup \{i\}), \qquad f(i) \subset N - \{i\},
$$

and

$$
\rho_{e(i)} \in \Sigma_{e(i)} = \mathop{\times}_{j \in e(i)} \Sigma_j .
$$

Proof. The minimum of concave functions is concave and the maximum of convex functions is convex. Therefore, the theorem applied to the new game with maximum and minimum in the payoff functions, gives the desired result immediately. Q.E.D.

We note that by using the Fan–Glicksberg generalization of Kakutani's fixed-point theorem, given in [2, 3], the technique allows us to have the existence of an equilibrium point for rational games defined on suitable linear topological spaces, and therefore, for the mixed extension of continuous games can be obtained in a similar manner.

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