J. Math. Anal. Appl. 385 (2012) 506–516

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A condition on delay for differential equations with discrete state-dependent delay

Alexander V. Rezounenko

Department of Mechanics and Mathematics, V.N. Karazin Kharkiv National University, 4, Svobody Sqr., Kharkiv, 61077, Ukraine

ARTICLE INFO

Article history:
Received 22 November 2010
Available online 1 July 2011
Submitted by J.J. Nieto

To Professor I.D. Chueshov on the occasion of his 60th birthday

Keywords:
Partial functional differential equation
State-dependent delay
Well-posedness
Global attractor

ABSTRACT

Parabolic differential equations with discrete state-dependent delay are studied. The approach, based on an additional condition on the delay function introduced in [A.V. Rezounenko, Differential equations with discrete state-dependent delay: uniqueness and well-posedness in the space of continuous functions, Nonlinear Anal. 70 (11) (2009) 3978–3986] is developed. We propose and study an analogue of the condition which is sufficient for the well-posedness of the corresponding initial value problem on the whole space of continuous functions C. The dynamical system is constructed in C and the existence of a compact global attractor is proved.

1. Introduction

Delay differential equations is one of the oldest branches of the theory of infinite-dimensional dynamical systems – theory which describes qualitative properties of systems, changing in time.

We refer to the classical monographs on the theory of ordinary delay equations (ODE) [2,7,11,12,22]. The theory of partial delay equations (PDE) is essentially less developed since such equations are infinite-dimensional in both time (as delay equations) and space (as PDEs) variables, which makes the analysis more difficult. We refer to some works which are close to the present research [3–6,28] and to the monograph [45].

A new class of equations with delays has recently attracted attention of many researchers. These equations have a delay term that may depend on the state of the system, i.e. the delay is `state-dependent` (SDD). Due to this type of delays such equations are inherently nonlinear and their study has begun in the case of ordinary differential equations [17,18,23–25,38,39] (for more details see also a recent survey [13], articles [8,19,26,27,40,41] and references therein).

Investigations of these equations essentially differ from the ones of equations with constant or time-dependent delays. The underlying main mathematical difficulty of the theory lies in the fact that delay terms with discrete state-dependent delays are not Lipschitz continuous on the space of continuous functions – the main space, on which the classical theory of equations with delays is developed (see [44] for an explicit example of the non-uniqueness and [13] for more details). It is a common point of view [13] that the corresponding *initial value problem* (IVP) is not generally well-posed in the sense of J. Hadamard [9,10] in the space of continuous functions (C). This leads to the search of (particular) classes of equations which may be well-posed in the space of continuous functions (C).

Results for partial differential equations with SDD have been obtained only recently in [29] (case of distributed delays, weak solutions), [16] (mild solutions, unbounded discrete delay), and [30] (weak solutions, bounded discrete and distributed delays), see also [32].

E-mail address: rezounenko@univer.kharkov.ua.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.06.070

The main goal of the present paper is to develop an alternative approach, based on an additional condition (see (H) below) introduced in [31]. We propose and study a *state-dependent* analogue of the condition which is sufficient for the well-posedness of the corresponding initial value problem in the space C. The presented approach includes the possibility that the state-dependent delay function does not satisfy the condition on a subset of the phase space C, but the IVP still be well-posed in the whole space C. This is our second goal which is to connect the approach developed for ODEs (a restriction to a subset of Lipschitz continuous functions) and the approach [31] of a different nature.

Having the well-posedness proved, we study the long-time asymptotic behavior of the corresponding dynamical system and prove the existence of a compact global attractor.

2. Formulation of the model with state-dependent discrete delay

Let us consider the following parabolic partial differential equation with delay

$$\frac{d}{dt}u(t) + Au(t) + du(t) = F(u_t), \tag{1}$$

where A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent, so $A : D(A) \to L^2(\Omega)$ generates an analytic semigroup, Ω is a smooth bounded domain in R^{n_0}, d is a non-negative constant. As usual for delay equations, we denote by u_t the function of $\theta \in [-r, 0]$ by the formula $u_t \equiv u_t(\theta) \equiv u(t+\theta)$. Here $r > 0$ is a fixed constant (the maximal value of the delay). We denote for short $C \equiv C([-r, 0]; L^2(\Omega))$. The norms in $L^2(\Omega)$ and C are denoted by $\|\cdot\|$ and $\|\cdot\|_C$ respectively.

The (nonlinear) delay term $F : C([-r, 0]; L^2(\Omega)) \to L^2(\Omega)$ has the form

$$F(\varphi) = B\big(\varphi\big(-\eta(\varphi)\big)\big), \tag{2}$$

where the (nonlinear) mapping $B : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz continuous

$$\big\|B\big(v^1\big) - B\big(v^2\big)\big\| \leqslant L_B \big\|v^1 - v^2\big\|, \quad \forall v^1, v^2 \in L^2(\Omega). \tag{3}$$

The function $\eta(\cdot) : C([-r, 0]; L^2(\Omega)) \to [0, r]$ represents the `state-dependent discrete delay`. It is important to notice that F is *nonlinear* unless η is constant (even in the case of linear B).

We consider Eq. (1) with the following initial condition

$$u|_{[-r,0]} = \varphi \in C \equiv C\big([-r, 0]; L^2(\Omega)\big). \tag{4}$$

Remark 1. The results presented in this paper could be easily extended to the case of a nonlinearity F of the form $F(\varphi) = \sum_k B^k(\varphi(-\eta^k(\varphi)))$ as well as to ODEs, for example, of the following form [33]

$$\dot{u}(t) + Au(t) + d \cdot u(t) = b\big(u\big(t - \eta(u_t)\big)\big), \quad u(\cdot) \in R^n,\ d \geqslant 0. \tag{5}$$

In the last case one simply needs to substitute $L^2(\Omega)$ by R^n and use $C \equiv C([-r, 0]; R^n)$ instead of $C([-r, 0]; L^2(\Omega))$. The function $b : R^n \to R^n$ is locally Lipschitz continuous and satisfies $\|b(w)\|_{R^n} \leqslant C_1\|w\|_{R^n} + C_b$ with $C_1, C_b \geqslant 0;$ A is a matrix.

Remark 2. As an example we could consider `non-local` delay term F (see (2)) with the following mapping

$$B(v)(x) \equiv \int_\Omega b\big(v(y)\big) f(x-y)\, dy, \quad x \in \Omega,$$

where $f : \Omega - \Omega \to R$ is a bounded and measurable function ($|f(z)| \leqslant M_f$, $\forall z \in \Omega - \Omega$) and $b : R \to R$ is a (locally) Lipschitz mapping, satisfying $|b(w)| \leqslant C_1|w| + C_b$ with $C_i \geqslant 0$. One can easily check that B satisfies (3) with $L_B \equiv L_b M_f |\Omega|$, where L_b is the Lipschitz constant of b, and $|\Omega| \equiv \int_\Omega 1\, dx$.

Another example is a `(local)` delay term F (see (2)) with $B(v)(x) \equiv b(v(x))$, $x \in \Omega$. An easy calculation shows that (3) is satisfied with $L_B \equiv L_b$.

The methods used in our work can be applied to other types of nonlinear and delay PDEs (as well as ODEs). We choose a particular form of nonlinear delay terms F for simplicity and to illustrate our approach on the diffusive Nicholson's blowflies equation (see the end of the article for more details).

3. The existence of mild solutions

In our study we use the standard

Definition 1. A function $u \in C([-r,T]; L^2(\Omega))$ is called a `mild solution` on $[-r,T]$ of the initial value problem (1), (4) if it satisfies (4) and

$$u(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)}\bigl\{F(u_s) - d\cdot u(s)\bigr\}\,ds, \quad t \in [0,T]. \tag{6}$$

We use the notation $C([a,b];X)$ for the space of the functions on $[a,b]$ which are continuous with respect to the strong topology on X (the norm topology).

Proposition 1. *(See [31].) Assume the mapping B is Lipschitz continuous (see (3)) and the delay function $\eta(\cdot): C([-r,0]; L^2(\Omega)) \to [0,r] \subset R_+$ is continuous.*

Then for any initial function $\varphi \in C$, the initial value problem (1), (4) has a global mild solution which satisfies $u \in C([-r,+\infty); L^2(\Omega))$.

The existence of a mild solution is a consequence of the continuity of $F: C \to L^2(\Omega)$ (see (1)) which gives the possibility to use the standard method based on the Schauder fixed point theorem (see e.g. [45, Theorem 2.1, p. 46]). The solution is also global (is defined for all $t \geqslant -r$) since (3) implies $\|F(\varphi)\| \leqslant L_B\|\varphi\|_C + \|B(0)\|$ and one can apply, for example, [45, Theorem 2.3, p. 49]. For the existence of solutions to delay PDEs and references see e.g. [34].

Remark 3. It is important to notice that even in the case of ordinary differential equations (even scalar) a mapping $F: C([-r_0,0];R) \to R$ of the form $\widetilde{F}(\varphi) = \tilde{f}(\varphi(-r(\varphi)))$ has a very unpleasant property. The authors in [20, p. 3] write "Notice that the functional $\widetilde{F}$ is defined on $C([-r_0,0];R)$, but it is clear that it is neither differentiable nor locally Lipschitz continuous, whatever the smoothness of $\tilde{f}$ and r." As a consequence, the Cauchy problem associated with equations with such a nonlinearity "... is `not` well-posed in the space of continuous functions, due to the non-uniqueness of solutions whatever the regularity of the functions $\tilde{f}$ and r" [20, p. 2]. See also a detailed discussion in [13].

Remark 4. For a study of solutions to equations with a state-dependent delay in the space $C([-r,0];E)$ with E not necessarily finite-dimensional Banach space, see e.g. [1].

In this work we concentrate on conditions for the IVP (1), (4) to be `well-posed`.

4. Main results: uniqueness, well-posedness and asymptotic behavior

As in the previous section, we assume that $\eta: C \to [0,r]$ is continuous and B is Lipschitz. Unlike to the existence of solutions, the uniqueness is an essentially more delicate question in the presence of discrete state-dependent delay (see a classical example of the non-uniqueness in [44]).

Let us remind an important additional assumption on the delay function η, as it was introduced in [31]:

- $\exists \eta_{ign} > 0$ such that η "ignores" values of $\varphi(\theta)$ for $\theta \in (-\eta_{ign}, 0]$ i.e.

$$\exists \eta_{ign} > 0\colon\ \forall \varphi^1, \varphi^2 \in C\colon\ \forall \theta \in [-r, -\eta_{ign}] \quad \Rightarrow \quad \varphi^1(\theta) = \varphi^2(\theta) \quad \Rightarrow \quad \eta\bigl(\varphi^1\bigr) = \eta\bigl(\varphi^2\bigr). \tag{H}$$

For examples of delay functions satisfying (H) and the proof of the *uniqueness* of mild solutions (given by Proposition 1) as well as the well-posedness of the IVP (1), (4), see [31].

Remark 5. It is important to notice that, discussing the condition (H) and its dependence on the value η_{ign}, we see that in the case $\eta_{ign} > r$, one has that the delay function η ignores `all` values of $\varphi(\theta)$, $\forall \theta \in [-r,0]$, so $\eta(\varphi) \equiv const$, $\forall \varphi \in C$ i.e. Eq. (1) becomes an equation with `constant (!)` delay. On the other hand, the analogue of assumption (H) with $\eta_{ign} = 0$, is trivial since $\varphi^1(\theta) = \varphi^2(\theta)$ for all $\theta \in [-r,0]$ means $\varphi^1 = \varphi^2$ in C, so $\eta(\varphi^1) = \eta(\varphi^2)$.

Remark 6. It is worth mentioning that the classical case of `constant` delay (see the previous remark) and the corresponding theory form the basis for the discussed approach, but is different to the approach of non-vanishing delays. In our case the delay η may vanish (we do **not** assume the existence of $r_0 > 0$ such that $\eta(\varphi) \geqslant r_0$, $\forall \varphi$).

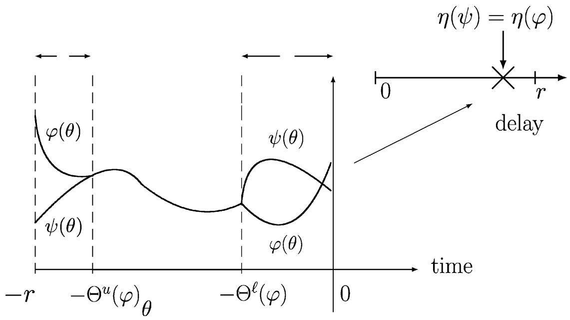

Fig. 1.

In the above condition (H) the semi-interval $(-\eta_{ign}, 0]$ is fixed (we remind that the value η_{ign} could be arbitrary small).

Our goal is to extend the approach based on the condition (H) to a wider class of state-dependent delay functions where the value η_{ign} is not a constant any more, but a function of the state. Moreover, as an easy additional extension, we also allow the upper bound of the delayed segment to be state-dependent. More precisely, we consider two functions $\Theta^u, \Theta^\ell : C \to [0, r]$, satisfying

$$\forall \varphi \in C \quad \Rightarrow \quad 0 \leqslant \Theta^\ell(\varphi) \leqslant \Theta^u(\varphi) \leqslant r.$$

Now we are ready to introduce [33] the following *state-dependent* condition for the state-dependent delay function $\eta : C \to [0, r]$ (cf. (H)):

- η "ignores" values of $\varphi(\theta)$ for $\theta \notin [-\Theta^u(\varphi), -\Theta^\ell(\varphi)]$ i.e.

$$\forall \psi \in C \text{ such that } \forall \theta \in \big[-\Theta^u(\varphi), -\Theta^\ell(\varphi)\big] \quad \Rightarrow \quad \psi(\theta) = \varphi(\theta) \quad \Rightarrow \quad \eta(\psi) = \eta(\varphi). \tag{$\widehat{\mathrm{H}}$}$$

The above condition means that state-dependent delay function η "ignores" all values of its argument φ outside of $[-\Theta^u(\varphi), -\Theta^\ell(\varphi)] \subset [-r, 0]$ and this delayed segment $[-\Theta^u(\varphi), -\Theta^\ell(\varphi)]$ is `state-dependent`. We could illustrate this property in Fig. 1.

Remark 7. The condition (H) is a particular case of $(\widehat{\mathrm{H}})$ with $\Theta^\ell(\varphi) \equiv \eta_{ign}$ and $\Theta^u(\varphi) \equiv r$, $\forall \varphi \in C$.

Example. It is easy to present many examples of (delay) functions η, which satisfy assumption $(\widehat{\mathrm{H}})$. The simplest one is

$$\eta(\varphi) = p_1\big(\varphi\big(-\chi\big(\varphi(-r)\big)\big)\big) \quad \text{with } p_1 : L^2(\Omega) \to [0, r] \tag{7}$$

and given $\chi : L^2(\Omega) \to [0, r]$. Here $\Theta^\ell(\varphi) \equiv \chi(\varphi(-r))$ and $\Theta^u(\varphi) = r$. It is easy to see that the above delay function η (7) ignores values of φ at points $\theta \in (-r, -\chi(\varphi(-r))) \cup (-\chi(\varphi(-r))), 0]$ and uses just two values of φ at points $\theta = -r$, $\theta = -\chi(\varphi(-r))$. In our notations, the delayed segment $[-\Theta^u(\varphi), -\Theta^\ell(\varphi)] = [-r, -\chi(\varphi(-r))]$ is state-dependent.

In the same way, one has

$$\eta(\varphi) = \sum_{k=1}^{N} p_k\big(\varphi\big(-\chi^k\big(\varphi(-r)\big)\big)\big) \quad \text{with } p_k, \chi^k : L^2(\Omega) \to [0, r].$$

In this case $[-\Theta^u(\varphi), -\Theta^\ell(\varphi)] = [-r, -\min_k\{\chi^k(\varphi(-r))\}]$. A slightly more general example is

$$\eta(\varphi) = \sum_{k=1}^{N} p_k\big(\varphi\big(-\chi^k\big(\varphi\big(-r^k\big)\big)\big)\big) \quad \text{with } p_k, \chi^k : L^2(\Omega) \to [0, r],\ \min r^k \in (0, r].$$

Here $\Theta^u(\varphi) = \max\{r^1, \dots, r^N, \chi^1(\varphi(-r^1)), \dots, \chi^N(\varphi(-r^N))\}$ and

$$\Theta^\ell(\varphi) = \min\big\{r^1, \dots, r^N, \chi^1\big(\varphi\big(-r^1\big)\big), \dots, \chi^N\big(\varphi\big(-r^N\big)\big)\big\}.$$

Examples of integral delay terms are as follows

$$\eta(\varphi) = \int_{-\chi^2(\varphi(-r^2))}^{-\chi^1(\varphi(-r^1))} p_1\big(\varphi(\theta)\big) g(\theta)\, d\theta, \quad \text{and} \quad \eta(\varphi) = p_1\left(\int_{-\chi^2(\varphi(-r^2))}^{-\chi^1(\varphi(-r^1))} \varphi(\theta) g(\theta)\, d\theta\right).$$

Similar to the previous example, $\Theta^u(\varphi)=\max\{r^1,r^2,\chi^1(\varphi(-r^1)),\chi^2(\varphi(-r^2))\}$ and

$$\Theta^\ell(\varphi)=\min\bigl\{r^1,r^2,\chi^1\bigl(\varphi\bigl(-r^1\bigr)\bigr),\chi^2\bigl(\varphi\bigl(-r^2\bigr)\bigr)\bigr\}.$$

Remark 8. It is interesting to notice that an assumption similar to the existence of upper function $\Theta^u(\cdot)$ is used in [42] for ODEs with SDD (locally bounded delay). On the other hand, an assumption similar to (H) is used in [14,41] for neutral ODEs (see (A4)(ii) in [14]), but together with another assumption on SDD to be bounded from below by a constant $r_0>0$ (cf. Remark 6).

Following [31, Theorem 1] we have the first result.

Theorem 1. *Let both functions $\Theta^u,\Theta^\ell : C\to[0,r]$ be continuous and $\Theta^\ell(\varphi)>0$ for all $\varphi\in C$. Assume the delay function $\eta: C\to[0,r]$ is continuous and satisfies assumption $(\widehat{\mathrm{H}})$; the mapping B is Lipschitz continuous (see (3)).*

Then for any initial function $\varphi\in C$, the initial value problem (1), (4) has a `unique` *mild solution $u:[-r,\infty)\to L^2(\Omega)$ (given by Proposition 1).*

If we define the `evolution operator` *$S_t: C\to C$ by the formula $S_t\varphi\equiv u_t$, where u is the unique mild solution of (1), (4) with initial function φ, then the pair (S_t,C) constitutes a dynamical system i.e. the following properties are satisfied:*

(1) *$S_0=Id$ (identity operator in C);*
(2) *$\forall t,\tau\geqslant 0\Rightarrow S_tS_\tau=S_{t+\tau}$;*
(3) *$t\mapsto S_t$ is a strongly continuous mapping;*
(4) *for any $t\geqslant 0$ the evolution operator S_t is continuous in C.*

The proof follows the line of [31, Theorem 1] taking into account that condition $\Theta^\ell(\varphi)>0$, $\forall\varphi\in C$ implies that for any fixed $\varphi\in C$, due to the continuity of $\Theta^\ell: C\to[0,r]$, there exists a neighborhood $U(\varphi)\subset C$ such that for all $\psi\in U(\varphi)$ one has $\Theta^\ell(\psi)\geqslant\frac{1}{2}\Theta^\ell(\varphi)>0$. That means that in $U(\varphi)\subset C$ we have the (state-independent) condition (H) with $\eta_{ign}=\frac{1}{2}\Theta^\ell(\varphi)>0$ and all the arguments presented in [31, Theorem 1] could be directly applied to this case. To obtain the global result we use proposition 1 which provides the global existence of solutions for any $\xi\in C$ and assume the existence of $\tau>0$ such that for the initial function $\varphi=S_\tau\xi$ the uniqueness of solutions fails. This leads to a contradiction since $\Theta^\ell(\varphi)>0$ and the local result proved above gives the uniqueness. The same arguments give the continuity of S_t for any $t>0$. We remark that as in [31, p. 3981], properties 1, 2 are consequences of the uniqueness of mild solutions. Property 3 is given by Proposition 1 since the solution is a continuous function. □

Remark 9. We do `not` assume that the functions Θ^u,Θ^ℓ (which are used in $(\widehat{\mathrm{H}})$ to present the delayed segment $[-\Theta^u(\varphi),-\Theta^\ell(\varphi)]$) are the functions presenting the smallest possible delayed segment. More precisely, it is possible that there exist two other functions $\widetilde{\Theta}^u,\widetilde{\Theta}^\ell$ such that for all $\varphi\in C$ one has $0\leqslant\Theta^\ell(\varphi)\leqslant\widetilde{\Theta}^\ell(\varphi)\leqslant\widetilde{\Theta}^u(\varphi)\leqslant\Theta^u(\varphi)\leqslant r$ and the same delay η satisfies $(\widehat{\mathrm{H}})$ with $\widetilde{\Theta}^u,\widetilde{\Theta}^\ell$ as well.

Our next step in studying the *state-dependent* condition $(\widehat{\mathrm{H}})$ is an attempt to avoid the condition $\Theta^\ell(\varphi)>0$, $\forall\varphi\in C$. We are going to consider the general case $\Theta^\ell(\varphi)\geqslant 0$, $\forall\varphi\in C$ with a `non-empty` set $Z\equiv\{\varphi\in C\colon\ \Theta^\ell(\varphi)=0\}\neq\emptyset$.

Theorem 2. *Assume the mapping B is Lipschitz continuous (see (3)).*

Moreover, let the following conditions be satisfied:

(1) *both functions $\Theta^u,\Theta^\ell: C\to[0,r]$ are continuous;*
(2) *there exists $L>0$ such that*

$$Z\equiv\bigl\{\varphi\in C\colon\ \Theta^\ell(\varphi)=0\bigr\}\subset C\mathcal{L}_L\equiv\left\{\varphi\in C\colon\ \sup_{t\neq s}\frac{\|\varphi(t)-\varphi(s)\|}{|t-s|}\leqslant L\right\};$$

(3) *delay function $\eta: C\to[0,r]$ is continuous and satisfies assumption $(\widehat{\mathrm{H}})$;*
(4) *for all $\varphi\in Z$ one has $\eta(\varphi)>0$;*
(5) *there exist $\omega>0$ and $L_\eta>0$ such that for all $\varphi,\psi\in U_\omega(Z)\equiv\{\chi\in C\colon\ \exists\nu\in Z\colon\ \|\chi-\nu\|_C\leqslant\omega\}$ one has $|\eta(\varphi)-\eta(\psi)|\leqslant L_\eta\cdot\|\varphi-\psi\|_C$.*

Then for any initial function $\varphi\in C$, the initial value problem (1), (4) has a `unique` *mild solution $u(t)$, $t\geqslant 0$ (given by Proposition 1). Moreover, the pair (S_t,C) constitutes a dynamical system (see Theorem 1).*

Proof of Theorem 2. We begin with an observation that any solution u at any time moment $t\geqslant 0$ satisfies either $u_t\in Z$ or $u_t\in C\setminus Z$. Moreover any solution is global (by Proposition 1) and for different time moments $t^1,t^2\geqslant 0$ one may have $u_{t^1}\in Z$ and $u_{t^2}\in C\setminus Z$. Hence we split our proof into two cases: $\varphi\in Z$ and $\varphi\in C\setminus Z$.

Let us consider $\varphi \in C$ which is an initial condition (see (4)). We start with the simple case $\varphi \notin Z$. By definition of Z, we have $\Theta^{\ell}(\varphi) > 0$. If there exists such $\tau > 0$ that $S_\tau \varphi \in Z$, we might turn this case to the proof of the case $\varphi \in Z$ by taking $S_\tau \varphi$ as φ. Otherwise it can be proved by the same argument as in the proof of Theorem 1 (the state-independent condition (H) is satisfied locally).

The rest of the proof is devoted to the case $\varphi \in Z$ and is organized as follows. For all $t \geqslant 0$ we denote by $u^k(t)$ *any* solution of (1), (4) with the initial function φ^k and by $u(t)$ *any* solution of (1), (4) with the initial function φ. We choose two arbitrary solutions and look for an estimate for $\|u^k(t) - u(t)\|$ in terms of $\|\varphi^k - \varphi\|_C$. The final estimate (see (18) below) will provide the uniqueness and continuous dependence.

We recall some estimates similar to estimates (6)–(13) in [31]. We use the variation of constants formula for parabolic equations (with $\widetilde{A} \equiv A + d \cdot E$, where E is the identity operator from $L^2(\Omega)$ to $L^2(\Omega)$)

$$u(t) = e^{-\widetilde{A}t} u(0) + \int_0^t e^{-\widetilde{A}(t-\tau)} B\bigl(u\bigl(\tau - \eta(u_\tau)\bigr)\bigr)\, d\tau, \tag{8}$$

$$u^k(t) = e^{-\widetilde{A}t} u^k(0) + \int_0^t e^{-\widetilde{A}(t-\tau)} B\bigl(u^k\bigl(\tau - \eta\bigl(u^k_\tau\bigr)\bigr)\bigr)\, d\tau. \tag{9}$$

Since the operator A is positive and $d \geqslant 0$ we use (see [6,15]) $\|e^{-\widetilde{A}t}\| \leqslant 1$ and $\|e^{-\widetilde{A}(t-\tau)}\| \leqslant 1$ to get

$$\begin{aligned}\bigl\|u^k(t) - u(t)\bigr\| &\leqslant \bigl\|u^k(0) - u(0)\bigr\| + \int_0^t \bigl\|B\bigl(u^k\bigl(\tau - \eta\bigl(u^k_\tau\bigr)\bigr)\bigr) - B\bigl(u\bigl(\tau - \eta(u_\tau)\bigr)\bigr)\bigr\|\, d\tau \\ &\leqslant \bigl\|\varphi^k(0) - \varphi(0)\bigr\| + J_1^k(t) + J_2^k(t),\end{aligned} \tag{10}$$

where we denote (for $s \geqslant 0$)

$$J_1^k(s) \equiv \int_0^s \bigl\|B\bigl(u^k\bigl(\tau - \eta\bigl(u^k_\tau\bigr)\bigr)\bigr) - B\bigl(u\bigl(\tau - \eta\bigl(u^k_\tau\bigr)\bigr)\bigr)\bigr\|\, d\tau, \tag{11}$$

$$J_2^k(s) \equiv \int_0^s \bigl\|B\bigl(u\bigl(\tau - \eta\bigl(u^k_\tau\bigr)\bigr)\bigr) - B\bigl(u\bigl(\tau - \eta(u_\tau)\bigr)\bigr)\bigr\|\, d\tau. \tag{12}$$

Using the Lipschitz property (3) of B, one easily gets

$$J_1^k(t) \leqslant L_B \int_0^t \bigl\|u^k\bigl(\tau - \eta\bigl(u^k_\tau\bigr)\bigr) - u\bigl(\tau - \eta\bigl(u^k_\tau\bigr)\bigr)\bigr\|\, d\tau \leqslant L_B t \max_{s \in [-r,t]} \bigl\|u^k(s) - u(s)\bigr\|. \tag{13}$$

Estimates (10), (13) and property $J_2^k(s) \leqslant J_2^k(t)$ for $s \leqslant t \leqslant t_0$ give

$$\max_{t \in [0,t_0]} \bigl\|u^k(t) - u(t)\bigr\| \leqslant \bigl\|\varphi^k(0) - \varphi(0)\bigr\| + L_B t_0 \max_{s \in [-r,t_0]} \bigl\|u^k(s) - u(s)\bigr\| + J_2^k(t_0).$$

Hence

$$\max_{s \in [-r,t_0]} \bigl\|u^k(s) - u(s)\bigr\| \leqslant \bigl\|\varphi^k - \varphi\bigr\|_C + L_B t_0 \max_{s \in [-r,t_0]} \bigl\|u^k(s) - u(s)\bigr\| + J_2^k(t_0). \tag{14}$$

Now we study properties of J_2^k which essentially `differ` from the ones in [31] since (H) is not satisfied. The Lipschitz property of B implies

$$J_2^k(t_0) \leqslant L_B \int_0^{t_0} \bigl\|u\bigl(\tau - \eta\bigl(u^k_\tau\bigr)\bigr) - u\bigl(\tau - \eta(u_\tau)\bigr)\bigr\|\, d\tau. \tag{15}$$

Since $\varphi \in Z$, property (4) gives $\eta(\varphi) > 0$. Due to the continuity of η (see (3)), there exists $\alpha > 0$ such that for all $\psi \in U_\alpha(\varphi) \equiv \{\psi \in C\colon\ \|\varphi - \psi\|_C \leqslant \alpha\}$ one has

$$\eta(\psi) \geqslant \frac{3}{4}\eta(\varphi) > 0. \tag{16}$$

We choose $\alpha < \omega$ (see property (5). By definition, a solution is continuous (with respect to the norm topology on $L^2(\Omega)$), hence for any two solutions $u(t)$ and $u^k(t)$ there exist two time moments $t_\varphi, t_{\varphi^k} > 0$ such that for all $t \in (0, t_\varphi]$ one has $u_t \in U_\alpha(\varphi)$ and for all $t \in (0, t_{\varphi^k}]$ one has $u_t^k \in U_\alpha(\varphi)$.

Remark 10. More precisely, we assume that there exists $N_\alpha \in N$ such that for all $k \geqslant N_\alpha$ one has $\varphi^k \in U_{\alpha/2}(\varphi)$ and hence there exists time moment $t_{\varphi^k} \in (0, t_0]$ such that for all $t \in (0, t_{\varphi^k}]$ one has $u_t^k \in U_\alpha(\varphi)$. This assumption (on N_α) is not restrictive since for the uniqueness of solutions we have $\varphi^k = \varphi$ while for the continuity with respect to initial data (see below) we have $\varphi^k \to \varphi$ in C.

Remark 11. It is important to notice that we take `any` solution from the set of solutions of IVP (1), (4) with the initial function φ (and denote it by $u(t)$) and take `any` solution from the set of solutions of IVP (1), (4) with the initial function φ^k (and denote it by $u^k(t)$) i.e. the values t_φ, t_{φ^k} may depend on the choice of `two` solutions.

Hence (16) implies that for all $\tau \in [0, t_1]$, with $t_1 \leqslant \min\{t_\varphi; t_{\varphi^k}; \frac{3}{4}\eta(\varphi)\}$ one gets $\tau - \eta(u_\tau) \leqslant 0, \tau - \eta(u_\tau^k) \leqslant 0$ and $u(\tau - \eta(u_\tau)) = \varphi(\tau - \eta(u_\tau))$, $u(\tau - \eta(u_\tau^k)) = \varphi(\tau - \eta(u_\tau^k))$. Hence, see (15) and properties (2), (5),

$$J_2^k(t_1) \leqslant L_B \int_0^{t_1} \left\| \varphi\left(\tau - \eta\left(u_\tau^k\right)\right) - \varphi\left(\tau - \eta(u_\tau)\right) \right\| d\tau \leqslant L_B L \int_0^{t_1} \left| \eta\left(u_\tau^k\right) - \eta(u_\tau) \right| d\tau$$
$$\leqslant L_B L L_\eta t_1 \max_{s \in [-r, t_1]} \left\| u^k(s) - u(s) \right\|.$$

Finally, we get (see the last estimate and (14))

$$\left(1 - L_B t_1 [1 + L L_\eta]\right) \max_{s \in [-r, t_1]} \left\| u^k(s) - u(s) \right\| \leqslant \left\| \varphi^k - \varphi \right\|_C.$$

Choosing small enough $t_1 > 0$ (to have $1 - L_B t_1 [1 + L L_\eta] > 0$) i.e.

$$t_1 \equiv \min\left\{ t_\varphi; t_{\varphi^k}; \frac{3}{4}\eta(\varphi); q\left(L_B[1 + L L_\eta]\right)^{-1} \right\} \quad \text{for any fixed } q \in (0, 1), \tag{17}$$

we get

$$\max_{s \in [-r, t_1]} \left\| u^k(s) - u(s) \right\| \leqslant \left(1 - L_B t_1 [1 + L L_\eta]\right)^{-1} \left\| \varphi^k - \varphi \right\|_C. \tag{18}$$

It is easy to see that (18) particularly implies the *uniqueness* of mild solutions to I.V.P. (1), (4) in case when $\varphi^k = \varphi$.

It gives us the possibility to define the `evolution operator` $S_t : C \to C$ by the formula $S_t \varphi \equiv u_t$, where $u(t)$ is the unique mild solution of (1), (4) with initial function φ.

Our next goal is to prove that pair (S_t, C) constitutes a `dynamical system` (see the properties (1)–(4) as they are formulated in Theorem 1). As in [31, p. 3981], properties 1, 2 are consequences of the uniqueness of mild solutions. Property 3 is given by Proposition 1 since the solution is a continuous function.

Let us prove property 4. We consider any sequence $\{\varphi^k\}_{k=1}^\infty \subset C$, which converges (in space C) to φ. Denote by $u^k(t)$ the (unique!) mild solution of (1), (4) with the initial function φ^k and by $u(t)$ the (unique!) mild solution of (1), (4) with the initial function φ.

One could think that (18) already provides the continuity with respect to initial data, but there is an important technical property used in developing (18), i.e. the choice of t_1 (see (17) and Remark 11). In contrast to the previous study, now we have infinite set of functions $\{\varphi^k\}_{k=1}^\infty \subset C$, so it may happen that $t_1 = t_1^k \to 0$ when $k \to \infty$.

We recall (see the text after (16)) that `two` time moments $t_\varphi, t_{\varphi^k} > 0$ have been chosen such that for all $t \in (0, t_\varphi]$ one has $u_t \in U_\alpha(\varphi)$ and for all $t \in (0, t_{\varphi^k}]$ one has $u_t^k \in U_\alpha(\varphi)$. Now our goal is to show that an infinite number of moments t_φ, $\{t_{\varphi^k}\}_{k=1}^\infty$ could be chosen in such a way that

$$t_2 \equiv \inf\{t_\varphi, t_{\varphi^1}, t_{\varphi^2}, \ldots, t_{\varphi^k}, \ldots\} > 0 \quad \text{and} \quad u_t, u_t^k \in U_\alpha(\varphi) \quad \text{for all } t \in (0, t_2].$$

To get this, we use the standard proof of the existence of a mild solution by a fixed point argument (see e.g. [45, p. 46, Theorem 2.1]). More precisely, let U be an open subset of C and $\widetilde{F} : [0, b] \times U \to L^2(\Omega)$ be continuous. Here we use notations of [45, p. 46] chosen as follows. Constants $\delta > 0$ and $N > 0$ are such that $\|\widetilde{F}(\psi)\| \leqslant N$ for all $\psi \in \overline{B_\delta(\varphi)} \equiv \{\psi \in C: \|\psi - \varphi\|_C \leqslant \delta\}$. Since the operator $\widetilde{A}$ is positive we use $\|e^{-\widetilde{A}t}\| \leqslant M = 1$ for all $t \geqslant 0$. The time moment $t' < r$ is chosen so that if $0 \leqslant t \leqslant t'$ then $\|\varphi(t + \theta) - \varphi(\theta)\| < \delta/3$ for all $\theta \in [-r, 0]$ and $\|e^{-\widetilde{A}t}\varphi(0) - \varphi(0)\| < \delta/3$. Set $t_3 \equiv \min\{t'; b; \delta/(3N); \delta\}$ and $Y_1 \equiv \{y \in C([-r, t_3]; L^2(\Omega)): \ y(0) = \varphi(0))\}$. For $\varphi \in C$ and any $y \in Y_1$ we consider the extension function $\hat{y}$ as follows

$$\hat{y}(s) \equiv \left[\begin{array}{ll} \varphi(s) & \text{for } s \in [-r, 0]; \\ y(t) & \text{for } s \in (0, t_3]. \end{array} \right.$$

Let $Y_2 \equiv \{y \in Y_1\colon\ \hat{y}_t \in \overline{B_\delta(\varphi)}$ for $t \in [0, t_3]\}$. Consider a mapping G on Y_2 as follows

$$G(y)(t) \equiv e^{-\widetilde{A}t}\varphi(0) + \int_0^t e^{-\widetilde{A}(t-\tau)}\widetilde{F}(\hat{y}_\tau)\,d\tau.$$

One can check (see [45, pp. 46, 47, Theorem 2.1]), that G maps Y_2 into Y_2. The solution is given by a fixed point $y = G(y)$. For our goal it is sufficient to choose $\delta \leqslant \alpha$ and $t_2 \leqslant t_3$ to get $u_t, u_t^k \in U_\alpha(\varphi)$ for all $t \in (0, t_2]$. Here we use φ^k instead of φ when necessary. The crucial point here is the possibility to choose t' (and hence t_3 and t_2) independent of $k \in N$. The choice of $t' < r$ so that if $0 \leqslant t \leqslant t'$ then $\|\varphi(t+\theta) - \varphi(\theta)\| < \delta/3$ and $\|\varphi^k(t+\theta) - \varphi^k(\theta)\| < \delta/3$ for `all` $k \in N$ and all $\theta \in [-r, 0]$ is possible due to the convergence of φ^k (to φ in C). Since the points of a convergent sequence form a precompact set in C they are equicontinuous. Now estimate (18) can be applied to our case and this completes the proof of property (4) and Theorem 2. □

Discussing assumptions of Theorem 2, let us present a constructive *example* of the function Θ^ℓ which satisfies assumption (2). Consider any compact and convex set $K_C \subset C\mathcal{L}_L \subset C$. For example, for any compact and convex set $K \in L^2(\Omega)$, the set $\{\varphi \in C\mathcal{L}_L\colon\ \forall\theta \in [-r, 0], \varphi(\theta) \in K\}$ is compact (by Arzela–Ascoli theorem) and convex. First, constructing Θ^ℓ, we set $\Theta^\ell(\varphi) = 0$ for all $\varphi \in K_C$. Second, we take any $p \in (0, r]$ and set $\Theta^\ell(\varphi) = p$ for all $\varphi \in C$ such that $\mathrm{dist}_C(\varphi, K_C) \geqslant 1$. Third, for any $\varphi \in C$ such that $\mathrm{dist}_C(\varphi, K_C) \in (0, 1)$ we set $\Theta^\ell(\varphi) = p \cdot \mathrm{dist}_C(\varphi, K_C) \in (0, p)$. By construction, Θ^ℓ satisfies (2).

As for asymptotic behavior, we study of the long-time behavior of the dynamical system (S_t, C), constructed in Theorems 1 and 2. Similar to [31, Theorem 2] we have the following result.

Theorem 3. *Assume all the assumptions of Theorem* 1 *or* 2 *are satisfied and additionally mapping B* (*see* (2)) *is bounded. Then the dynamical system (S_t, C) has a compact global attractor $\mathcal{A}$ which is a compact set in all spaces $C_\delta \equiv C([-r, 0]; D(A^\delta))$, $\forall\delta \in [0, \frac{1}{2})$.*

Lemma. *Let all the assumptions of Theorem* 2 *be satisfied and the mapping B is bounded. Then there exists a constant $\widetilde{L} > 0$ such that the global attractor $\mathcal{A}$* (*see Theorem* 3) *is a subset of $C\mathcal{L}_{\widetilde{L}}$* (*cf. condition* (2) *in Theorem* 2).

Remark 12. Lemma gives a possibility to consider system (1), (4) with a state-dependent delay function η which does not ignore values of its argument φ for all points $\varphi \in \mathcal{A}$.

Proof of Lemma. Consider any solution $u_t \in \mathcal{A}$. Let us denote $f(t) \equiv F(u_t)$ and prove that f is Hölder continuous.

We recall that for a dissipative dynamical system [37], (S_t, C) we have a set $B_1 \subset C$ (a ball $B_1 \equiv \{\psi\colon\ \|\psi\|_C \leqslant R_0\}$) such that for all $\varphi \in C$ there exists $t_\varphi \geqslant 0$ such that $S_t\varphi \in B_1$ for all $t \geqslant t_\varphi$ (for more details, see e.g. [6,37]). We will need the following property, proved in [31, estimate (29) with $\delta = 0$]

$$\big\|u(t_1) - u(t_2)\big\| \leqslant L_0|t_1 - t_2|^{1/2} \tag{19}$$

for any solution, belonging to B_1 (in particular, for any solution belonging to the attractor $\mathcal{A} \subset B_1$).

In (19) the constant L_0 is independent of solution u.

One can check that

$$\big\|f(t_1) - f(t_2)\big\| \leqslant L_B \cdot \big\|u\big(t_1 - \eta(u_{t_1})\big) - u\big(t_2 - \eta(u_{t_2})\big)\big\|. \tag{20}$$

Using (19), the Lipschitz property of η (see (5) in Theorem 2), we get from (20) that

$$\begin{aligned}
\big\|f(t_1) - f(t_2)\big\| &\leqslant L_B L_0 \cdot \big|t_1 - \eta(u_{t_1}) - \big(t_2 - \eta(u_{t_2})\big)\big|^{1/2} \\
&\leqslant L_B L_0 \cdot \big(|t_1 - t_2| + \big|\eta(u_{t_1}) - \eta(u_{t_2})\big|\big)^{1/2} \\
&\leqslant L_B L_0 \cdot \big(|t_1 - t_2| + L_\eta\|u_{t_1} - u_{t_2}\|\big)^{1/2} \\
&\leqslant L_B L_0 \cdot \big(|t_1 - t_2| + L_\eta L_0|t_1 - t_2|^{1/2}\big)^{1/2} \\
&\leqslant L_B L_0 \cdot \big(|t_1 - t_2|^{1/2} + (L_\eta L_0)^{1/2}|t_1 - t_2|^{1/4}\big).
\end{aligned}$$

Finally, for $|t_1 - t_2| < 1$ one has

$$\big\|f(t_1) - f(t_2)\big\| \leqslant L_B L_0 \cdot \big\{1 + (L_\eta L_0)^{1/2}\big\}|t_1 - t_2|^{1/4}. \tag{21}$$

Let us consider any $\psi \in \mathcal{A}$. It is well known that the attractor consists of whole trajectories i.e. $u_s \in \mathcal{A}$, $\forall s \in R$. We take any $t_0 > r > 0$ and get $\varphi \in \mathcal{A}$ such that $S_{t_0}\varphi = \psi$. Consider the variation of constants formula for parabolic equations (with $\widetilde{A} \equiv A + d \cdot E$, see (8))

$$u(t) = e^{-\widetilde{A}t}\varphi(0) + \int_0^t e^{-\widetilde{A}(t-\tau)}F(u_\tau)\,d\tau. \tag{22}$$

To estimate the first term in the above formula (22) we first prove that

$$\left\| e^{-\widetilde{A}t_1}v - e^{-\widetilde{A}t_2}v \right\| \leqslant \frac{1}{et_1}|t_1 - t_2| \cdot \|v\|, \quad 0 < t_1 < t_2. \tag{23}$$

Since A and $\widetilde{A}$ are densely-defined self-adjoint positive linear operators in $L^2(\Omega)$ with compact resolvent we consider the orthonormal basis (of $L^2(\Omega)$) $\{e_k\}_{k=1}^{\infty}$ which consists of eigenvectors of the operator $\widetilde{A}$, so

$$\widetilde{A}e_k = \lambda_k e_k, \quad 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \to +\infty.$$

Consider for any $v \in L^2(\Omega)$ and any $t_2 > t_1 > 0$,

$$\left\| e^{-\widetilde{A}t_1}v - e^{-\widetilde{A}t_2}v \right\|^2 = \sum_{k=1}^{\infty} \left(e^{-\lambda_k t_1} - e^{-\lambda_k t_2}\right)^2 v_k^2, \tag{24}$$

where $v_k \equiv \langle v, e_k \rangle_{L^2(\Omega)}$. It is easy to see that for any $\mu > 0$ one has

$$\left| e^{-\mu t_1} - e^{-\mu t_2} \right| \leqslant |t_2 - t_1| \cdot \max_{\tau \in [t_1, t_2]} \mu e^{-\mu\tau} = |t_2 - t_1| \cdot \mu e^{-\mu t_1}$$

and $\sup_{k \in N} |e^{-\lambda_k t_1} - e^{-\lambda_k t_2}| \leqslant |t_2 - t_1| \cdot \sup_{\mu > 0} \mu e^{-\mu t_1} = \frac{1}{et_1}|t_1 - t_2|$. We substitute this and $\|v\|^2 = \sum_{k=1}^{\infty} v_k^2$ into (24) to get (23).

The estimate (23) gives the uniform (w.r.t. $\varphi \in \mathcal{A}$) Lipschitz property of the first term $e^{-\widetilde{A}t}\varphi(0)$ in (22) for $t > t_0 > 0$. As discussed (see the text before (19)), we have $\|\varphi(0)\| \leqslant R_0$ for all $\varphi \in \mathcal{A}$ due to the dissipativeness of the dynamical system $(S_t; C)$ (for more details, see [31, estimate (23)]).

To prove that the second term in (22) is Lipschitz for $t > t_0$ we need

Definition 2. (See [15, Definition 1.3.1].) We call a linear operator A in a Banach space X a `sectorial operator` if it is a closed densely defined operator such that, for some ψ in $(0, \pi/2)$ and some $M \geqslant 1$ and real a, the sector

$$S_{a,\psi} = \left\{\lambda:\ \psi \leqslant \left|arg(\lambda - a)\right| \leqslant \pi,\ \lambda \neq a\right\}$$

is in the resolvent set of A and

$$\left\|(\lambda - A)^{-1}\right\| \leqslant M/|\lambda - a| \quad \text{for all } \lambda \in S_{a,\psi}.$$

to use the following:

Proposition 2. *(See [15, Lemma 3.2.1].) Let $\widetilde{A}$ be a sectorial operator in Banach space X. Assume function $f : (0, T) \to X$ is locally Hölder continuous and $\int_0^{\rho} \|f(s)\|_X\, ds < \infty$ for some $\rho > 0$. Denote by $\Phi(t) \equiv \int_0^t e^{-\widetilde{A}(t-s)} f(s)\, ds$ for $t \in [0, T)$. Then function $\Phi(\cdot)$ is continuous on $[0, T)$, continuously differentiable on $(0, T)$, $\Phi(t) \in D(\widetilde{A})$ for $0 < t < T$ and $d\Phi(t)/dt + \widetilde{A}\Phi(t) = f(t)$ for $0 < t < T$ and $\Phi(t) \to 0$ in X as $t \to 0+$.*

Remark 13. Our operator $\widetilde{A}$ is sectorial since any self-adjoint, densely defined operator bounded from below in a Hilbert space is sectorial (see e.g. [15, Example 2, p. 26]).

We apply the above Proposition 2 to $f(t) \equiv F(u_t)$ and use (21). The property $\int_0^{\rho} \|f(s)\|_X\, ds < \infty$ for some $\rho > 0$ follows from the dissipativeness $\|u(t)\| \leqslant R_0$, the continuity of $F : C \to L^2(\Omega)$ and of the continuity of the mild solution u. One uses the continuous differentiability of Φ on $[t_0 - r, t_0] \subset (0, T)$ which implies that $\max_{t \in [t_0 - r, t_0]} \|\Phi'(t)\| \equiv M_{\Phi;1} < \infty$. In our case Φ represents the second term in (22) which is proved to be Lipschitz continuous with Lipschitz constant $M_{\Phi;1}$ independent of u.

We notice that the independence $M_{\Phi;1}$ of solutions belonging to the attractor follows from the detailed proof of Proposition 2 [15, Lemma 3.2.1] using the boundedness of $\mathcal{A} \subset C$.

On the other hand one can also use the following:

Proposition 3. *(See [15, Lemma 3.5.1].) Let $\widetilde{A}$ be a sectorial operator in the Banach space X and $f : (0, T) \to X$ satisfy*

$$\left\| f(t) - f(s) \right\| \leqslant K(s)(t - s)^{\gamma} \quad \textit{for } 0 < s < t < T < \infty,$$

where $K : (0, T) \to \mathbb{R}$ is continuous with $\int_0^T K(s)\, ds < \infty$.

Then for every $\beta \in [0,\gamma)$ *the function*

$$\Phi : (0,T) \ni t \mapsto \int_0^t e^{-\widetilde{A}(t-s)} f(s)\,ds \in X^\beta \equiv D\big(\widetilde{A}^\beta\big)$$

is continuously differentiable with

$$\left\| \frac{d\Phi(t)}{dt} \right\|_\beta \leqslant M t^{-\beta} \| f(t)\| + M \int_0^t (t-s)^{\gamma-\beta-1} K(s)\,ds \tag{25}$$

for $0 < t < T$. *Here* M *is a constant independent of* $\gamma, \beta, f(\cdot)$.

For the definition and properties of spaces X^β we refer to [15]. In our case we use Proposition 3 with $\gamma = \frac{1}{4}$, $K(s) \equiv L_B L_0 \cdot \{1+(L_\eta L_0)^{1/2}\}$ (see (21)) and $\beta = 0$ (it implies $X^\beta = X^0 = L^2(\Omega)$). The boundedness of the global attractor implies that $\|f(t)\| \leqslant C$ with C independent of solutions. Hence (25) gives the uniform boundedness of $\|\frac{d\Phi(t)}{dt}\| \leqslant \widehat{C}$ for $t \in [t_0 - r, t_0] \subset R_+$ and guarantees the independence $M_{\Phi;1}$ from solutions. The proof of Lemma is complete. □

Remark 14. One can also easily extend the method developed here to the case of non-autonomous nonlinear delay terms, for example, using another nonlinear function $\hat{b} : R \times R \to R$ (see Remark 2) instead of b to have $(\widehat{F}(t,u_t))(x) = \hat{b}(t, u(t-\eta(u_t), x))$ or $(\widehat{F}(t,u_t))(x) = \int_\Omega \hat{b}(t, u(t-\eta(u_t), y)) f(x-y)\,dy$ in Eq. (1).

As an application we can consider the diffusive Nicholson's blowflies equation (see e.g. [36]) with the state-dependent delay. More precisely, we consider Eq. (1) where $-A$ is the Laplace operator with the Dirichlet boundary conditions, $\Omega \subset R^{n_0}$ is a bounded domain with a smooth boundary, the function f (see Remark 2) can be, for example, $f(s) = \frac{1}{\sqrt{4\pi\alpha}} e^{-s^2/4\alpha}$, as in [35] (for the non-local in space nonlinearity) or Dirac delta-function to get the local in space nonlinearity, the nonlinear function b is given by $b(w) = p \cdot we^{-w}$. Function b is bounded, so for any continuous delay function η, satisfying $(\widehat{\mathrm{H}})$, the conditions of Theorems 1, 2 are valid. As a result, we conclude that the initial value problem (1), (4) is well-posed in C and the dynamical system (S_t, C) has a global attractor (Theorem 3).

As another application we propose the diffusive model of Hematopoiesis with the state-dependent delay:

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - du(t,x) + \frac{\beta u(t-\eta(u_t), x)}{1+u^m(t-\eta(u_t), x)}, \quad (t,x) \in \Omega,\ \beta \in \mathbb{R},\ m \geqslant 1$$

with the Neumann boundary conditions $\frac{\partial u(t,x)}{\partial n} = 0$, $x \in \partial\Omega$. For the case of constant delay $\eta(u_t) \equiv \tau > 0$ see [43]. The model of Hematopoiesis when u does not depend on the spatial variable $x \in \Omega$ was first proposed by Mackey and Glass [21]. One can easily check that the nonlinearity B (see (2)), given by $B(v) = \beta v(1+v^m)^{-1}$, is bounded and globally Lipschitz provided $m = 2k$, $k \in \mathbb{N}$.

In *conclusion*, we notice that in the current article the emphasis is on the delay term and its properties, not on the partial differential operator. We believe that the type of state-dependent delay studied here is important for partial differential equations modeling real world phenomena. We hope that the approach developed here to study the state-dependent delay will be successfully used for other delay PDEs in spite of the necessity to use essentially different methods to work with other partial differential operators/ boundary value problems.

Acknowledgments

The author is grateful to I.D. Chueshov and H.-O. Walther for useful discussions of an early version of the manuscript and to the reviewers for valuable comments and suggestions which have led to an improvement of the presentation.

References

[1] O. Arino, E. Sanchez, A saddle point theorem for functional state-dependent delay differential equations, Discrete Contin. Dyn. Syst. 12 (4) (2005) 687–722.
[2] N.V. Azbelev, V.P. Maksimov, L.F. Rakhmatullina, Introduction to the Theory of Functional Differential Equations, Nauka, Moscow, 1991.
[3] L. Boutet de Monvel, I.D. Chueshov, A.V. Rezounenko, Inertial manifolds for retarded semilinear parabolic equations, Nonlinear Anal. 34 (1998) 907–925.
[4] I.D. Chueshov, On a certain system of equations with delay, occurring in aeroelasticity, J. Sov. Math. 58 (1992) 385–390.
[5] I.D. Chueshov, A.V. Rezounenko, Global attractors for a class of retarded quasilinear partial differential equations, C. R. Acad. Sci. Paris Ser. I 321 (1995) 607–612; detailed version: Math. Phys. Anal. Geom. 2 (3) (1995) 363–383.
[6] I.D. Chueshov, Introduction to the Theory of Infinite-Dimensional Dissipative Systems, Acta, Kharkov, 1999 (in Russian); English transl.: Acta, Kharkov, 2002, see http://www.emis.de/monographs/Chueshov.
[7] O. Diekmann, S. van Gils, S. Verduyn Lunel, H.-O. Walther, Delay Equations: Functional, Complex, and Nonlinear Analysis, Springer-Verlag, New York, 1995.

[8] Yong-Hong Fan, Lin-Lin Wang, Multiplicity of periodic solutions for a delayed ratio-dependent predator–prey model with Holling type III functional response and harvesting terms, J. Math. Anal. Appl. 365 (2010) 525–540.
[9] J. Hadamard, Sur les problèmes aux derivees partielles et leur signification physique, Bull. Univ. Princeton (1902) 13.
[10] J. Hadamard, Le problème de Cauchy et les èquations aux derivees partielles linéaires hyperboliques, Hermann, Paris, 1932.
[11] J.K. Hale, Theory of Functional Differential Equations, Springer-Verlag, Berlin, Heidelberg, New York, 1977.
[12] J.K. Hale, S.M. Verduyn Lunel, Theory of Functional Differential Equations, Springer-Verlag, New York, 1993.
[13] F. Hartung, T. Krisztin, H.-O. Walther, J. Wu, Functional differential equations with state-dependent delays: Theory and applications, in: A. Canada, P. Drabek, A. Fonda (Eds.), Handbook Of Differential Equations, Ordinary Differential Equations, vol. 3, Elsevier Science B.V., North-Holland, 2006, pp. 435–545.
[14] F. Hartung, On differentiability of solutions with respect to parameters in neutral differential equations with state-dependent delays, preprint, 2010.
[15] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer-Verlag, Berlin, Heldelberg, New York, 1981.
[16] E. Hernandez, A. Prokopczyk, L. Ladeira, A note on partial functional differential equations with state-dependent delay, Nonlinear Anal. Real World Appl. 7 (4) (2006) 510–519.
[17] T. Krisztin, O. Arino, The 2-dimensional attractor of a differential equation with state-dependent delay, J. Dynam. Differential Equations 13 (2001) 453–522.
[18] T. Krisztin, A local unstable manifold for differential equations with state-dependent delay, Discrete Contin. Dyn. Syst. 9 (2003) 933–1028.
[19] W.-S. Li, Y.-K. Chang, J.J. Nieto, Solvability of impulsive neutral evolution differential inclusions with state-dependent delay, Math. Comput. Modelling 49 (2009) 1920–1927.
[20] M. Louihi, M.L. Hbid, O. Arino, Semigroup properties and the Crandall Liggett approximation for a class of differential equations with state-dependent delays, J. Differential Equations 181 (2002) 1–30.
[21] M.C. Mackey, L. Glass, Oscillation and chaos in physiological control system, Science 197 (1977) 287–289.
[22] A.D. Myshkis, Linear Differential Equations with Retarded Argument, 2nd edition, Nauka, Moscow, 1972.
[23] J. Mallet-Paret, R.D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags I, Arch. Ration. Mech. Anal. 120 (1992) 99–146.
[24] J. Mallet-Paret, R.D. Nussbaum, P. Paraskevopoulos, Periodic solutions for functional-differential equations with multiple state-dependent time lags, Topol. Methods Nonlinear Anal. 3 (1) (1994) 101–162.
[25] J. Mallet-Paret, R.D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags II, J. Reine Angew. Math. 477 (1996) 129–197.
[26] J. Mallet-Paret, R.D. Nussbaum, Superstability and rigorous asymptotics in singularly perturbed state-dependent delay-differential equations, J. Differential Equations 250 (2011) 4037–4084.
[27] J. Mallet-Paret, R.D. Nussbaum, Stability of periodic solutions of state-dependent delay-differential equations, J. Differential Equations 250 (2011) 4085–4103.
[28] A.V. Rezounenko, Approximate inertial manifolds for retarded semilinear parabolic equations, J. Math. Anal. Appl. 282 (2) (2003) 614–628.
[29] A.V. Rezounenko, J. Wu, A non-local PDE model for population dynamics with state-selective delay: local theory and global attractors, J. Comput. Appl. Math. 190 (1–2) (2006) 99–113.
[30] A.V. Rezounenko, Partial differential equations with discrete and distributed state-dependent delays, J. Math. Anal. Appl. 326 (2) (2007) 1031–1045; see also detailed preprint, http://arxiv.org/pdf/math.DS/0503470, March 22, 2005.
[31] A.V. Rezounenko, Differential equations with discrete state-dependent delay: uniqueness and well-posedness in the space of continuous functions, Nonlinear Anal. 70 (11) (2009) 3978–3986.
[32] A.V. Rezounenko, Non-linear partial differential equations with discrete state-dependent delays in a metric space, Nonlinear Anal. 73 (6) (2010) 1707–1714; see detailed preprint, arXiv:0904.2308v1, April 15, 2009.
[33] A.V. Rezounenko, Elementary Properties of Ordinary Differential Equations with State-Dependent Delays, Lecture Notes, Kharkiv University Press, Kharkiv, 2010.
[34] W.M. Ruess, Existence of solutions to partial differential equations with delay, in: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in: Lect. Notes Pure Appl. Math., vol. 178, 1996, pp. 259–288.
[35] J.W.-H. So, J. Wu, X. Zou, A reaction diffusion model for a single species with age structure. I. Travelling wavefronts on unbounded domains, Proc. R. Soc. Lond. Ser. A 457 (2001) 1841–1853.
[36] J.W.-H. So, Y. Yang, Dirichlet problem for the diffusive Nicholson's blowflies equation, J. Differential Equations 150 (2) (1998) 317–348.
[37] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, Berlin, Heidelberg, New York, 1988.
[38] H.-O. Walther, Stable periodic motion of a system with state dependent delay, Differential Integral Equations 15 (2002) 923–944.
[39] H.-O. Walther, The solution manifold and C^1-smoothness for differential equations with state-dependent delay, J. Differential Equations 195 (1) (2003) 46–65.
[40] H.-O. Walther, On a model for soft landing with state-dependent delay, J. Dynam. Differential Equations 19 (3) (2007) 593–622.
[41] H.-O. Walther, Linearized stability for semiflows generated by a class of neutral equations, with applications to state-dependent delays, J. Dynam. Differential Equations 22 (3) (September 2010) 439–462.
[42] H.-O. Walther, Differential equations with locally bounded delay, preprint.
[43] X. Wang, Z. Li, Dynamics for a type of general reaction–diffusion model, Nonlinear Anal. 67 (2007) 2699–2711.
[44] E. Winston, Uniqueness of the zero solution for differential equations with state-dependence, J. Differential Equations 7 (1970) 395–405.
[45] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, New York, 1996.