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Kasparov Products and Dual Algebras

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Fundamental to the analytic K-homology theory of G. Kasparov [7, 8] is the construction of the external product in K-homology

$$K^i(A) \otimes K^j(B) \to K^{i+j}(A \otimes B).$$

This construction is modeled on the "sharp product" of elliptic operators over compact manifolds [2], and involves some deep functional-analytic considerations which at first sight may appear somewhat *ad hoc*.

A different approach to Kasparov's theory has recently been expounded by N. Higson [5], following the lead of W. Paschke [9]. He constructs a "dual algebra" $\mathfrak{D}(A)$ for any separable C^*-algebra A, in such a way that $K^i(A)$ is canonically identified with the ordinary K-theory of the dual algebra, $K_{1-i}(\mathfrak{D}(A))$. Higson's treatment covers the exactness and excision properties of K-homology, but stops short of the Kasparov product; it is natural to ask whether the product itself can be given a "dual" interpretation, in terms of the external product in ordinary K-theory. It is the purpose of this article to show that this can indeed be done.

A more leisurely exposition of K-theory and K-homology from this perspective will appear in [6].

1. K-THEORY OF GRADED C^*-ALGEBRAS

In this section we briefly review and reformulate a construction of K-theory for graded C^*-algebras, due to van Daele [12, 13]. We will consider $\mathbb{Z}/2$-graded *complex* C^*-algebras. The whole discussion also applies to the real case with only minor changes.

(1.1) Definition. A *supersymmetry* in a graded, unital C^*-algebra is an odd, self-adjoint, unitary element. We let $SS(A)$ denote the space of supersymmetries in A; it is equipped with the topology induced by the C^*-norm.

We will confine our attention to those unital C^*-algebras which contain supersymmetries. From the point of view of K-theory this is no loss, as there is a simple stabilization (see [12]) which converts any given C^*-algebra to a K-theoretically equivalent one containing supersymmetries. In fact, this stabilization amounts to the (graded) tensor product with the Clifford

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algebra $\mathbb{C}_{1,1}$ (which is just the algebra of 2×2 matrices equipped with the "diagonal and off-diagonal" grading).

Let the unital C^*-algebra A be given. Let $V(A)$ denote the disjoint union

$$V(A) = \bigsqcup_{k \in \mathbb{N}} \pi_0 SS(M_k(A)).$$

Then $V(A)$ is a semigroup, in fact a commutative semigroup, under the operation of direct sum. Notice that, by construction, $V(A)$ comes equipped with a natural semigroup-homomorphism $V(A) \to \mathbb{N}$.

(1.2) Definition. We let $G(A)$ be the Grothendieck group associated to the semigroup $V(A)$.

The semigroup homomorphism $V(A) \to \mathbb{N}$ gives rise to a group homomorphism d: $G(A) \to \mathbb{Z}$. It is easily seen that van Daele's definition of K-theory is equivalent to the following:

(1.3) Definition. For a unital C^*-algebra A (containing supersymmetries), we define $K_1(A) = \mathrm{Ker}(d\colon G(A) \to \mathbb{Z})$.

The associated short exact sequence of abelian groups

$$0 \to K_1(A) \to G(A) \to \mathbb{Z} \to 0$$

is of course split (since $\mathbb{Z}$ is free), but there is in general no canonical choice of splitting. Such a choice of splitting amounts to the choice of a "reference supersymmetry" in A, and allows us to view $K_1(A)$ as a direct summand of $G(A)$. This is the approach taken in [12], but it is then necessary to investigate the dependence of the definition on the choice of reference supersymmetry. For this reason we prefer the form of the general definition given above. It will however prove to be the case that the dual algebras in which we are interested are equipped with a canonical choice of reference supersymmetry.

Consider the homomorphism $G(A) \to G(A \oplus A)$ induced by the diagonal map $a \mapsto (a, a)$. This homomorphism is split injective, and it is not hard to see that its cokernel is canonically isomorphic to $K_1(A)$; one has a short exact sequence

$$0 \to G(A) \to G(A \oplus A) \to K_1(A) \to 0 \qquad (*)$$

where the quotient map sends a supersymmetry (F, F') over $A \oplus A$ to the formal difference $[F] - [F']$ in $K_1(A)$.

In [12], after the definition of K_1 has been extended in the usual way to non-unital algebras, the higher K-theory groups $K_n(A)$, $n \geqslant 2$, are defined by suspension: $K_n(A) = K_1(C_0(\mathbb{R}^{n-1} \otimes A)$. In terms of the groups G, we may give an equivalent definition as follows.

(1.4) Proposition. *Let A be a unital C^*-algebra. Put $G_n(A) = G(C(S^{n-1}) \otimes A)$. Then there is a split short exact sequence*

$$0 \to G(A) \to G_n(A) \to K_n(A) \to 0$$

where the map $G(A) \to G_n(A)$ is induced by the inclusion of A as constant A-valued functions on the sphere. Equivalently, there is a split short exact sequence

$$0 \to K_n(A) \to G_n(A) \to G(A) \to 0$$

where the map $G_n(A) \to G(A)$ comes from evaluation at the north pole.

Notice that the exact sequence $(*)$ is the case $n = 1$ of this result. Finally, we may give one last reformulation. Let D^n denote the n-disc, and note the standard inclusion $S^{n-1} \to D^n$. It is clear that G is a homotopy functor, and so, since D^n is contractible, $G(C(D^n) \otimes A) \cong G(A)$. Thus we obtain, by rearranging the exact sequence above,

(1.5) Proposition. *The group $K_n(A)$ is the cokernel of the natural split injection*

$$G(C(D^n) \otimes A) \to G(C(S^{n-1}) \otimes A)$$

induced by the restriction of functions from D^n to S^{n-1}.

In our discussion of products on higher K-homology groups we will also need to make use of the following fact.

(1.6) Proposition. *The group $K_{n+m+1}(A)$ is (canonically isomorphic to) the kernel of the natural map*

$$G(C(S^m \times S^n) \otimes A) \to G_{m+1}(A) \oplus G_{n+1}(A)$$

induced by the inclusions (at the north pole) of the two factors of the product, $S^m \hookrightarrow S^m \times S^n$ and $S^n \hookrightarrow S^m \times S^n$.

Proof. We need a form of excision for the G-groups (which may, for instance, be deduced from the corresponding result for the K-groups, or proved directly): if Y is a retract of X, then the kernel of the split surjection $G(C(X) \otimes A) \to G(C(Y) \otimes A)$ is isomorphic to the kernel of the split surjection $G(C(X/Y) \otimes A) \to G(A)$. Now S^m is a retract of $S^m \times S^n$, and S^n is a retract of $(S^m \times S^n)/S^m$; moreover, $((S^m \times S^n)/S^m)/S^n = S^m \wedge S^n = S^{m+n}$. The result follows from the second part of 1.4. ▮

This argument comes from [1, Corollary 2.4.8].

2. THE JOIN CONSTRUCTION AND THE EXTERNAL PRODUCT

Let X and Y be polyhedra, embedded in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. We regard both as embedded in $\mathbb{R}^{m+n+1}$, by embedding $\mathbb{R}^m$ and $\mathbb{R}^n$ as skew affine subspaces of $\mathbb{R}^{m+n+1}$. The (external) *join* $X * Y$ of X and Y is the polyhedron in $\mathbb{R}^{m+n+1}$ made up of the union of all line segments from points of X to points of Y (see [11], Chapter 2). As a topological space, $X * Y$ is the quotient of $X \times Y \times [0, 1] \sqcup X \sqcup Y$ by the equivalence relation which identifies every point $(x, y, 0)$ in the product with the corresponding $x \in X$, and every point $(x, y, 1)$ in the product with the corresponding $y \in Y$. We note from [11] the standard calculations

$$D^p * D^q = D^{p+q+1}, \qquad D^p * S^q = D^{p+q+1}, \qquad S^p * S^q = S^{p+q+1}$$

where D^p denotes the p-disc and S^p denotes the p-sphere.

We want to mimic this construction in C^*-algebra theory. Let A and B be C^*-algebras. For definiteness, let us say that we will take the tensor product $A \otimes B$ in the spatial (minimal) C^*-norm; if A and B are graded we of course take the graded tensor product, but we think it unnecessary to introduce a special notation.

(2.1) DEFINITION. The join of the unital C^*-algebras A and B is the C^*-algebra $A * B$ defined as follows: $A * B$ consists of those continuous functions f: $[0, 1] \to A \otimes B$, such that $f(0) \in A \otimes 1$ and $f(1) \in 1 \otimes B$. The algebra $A * B$ is equipped with the norm and involution it inherits as a closed subalgebra of $C[0, 1] \otimes A \otimes B$.

It should be apparent that, for algebras of continuous functions on polyhedra X and Y as above, $C(X) * C(Y) = C(X * Y)$. The notation $*$ is also often used for free products of C^*-algebras, but we will not follow this usage here.

We will obtain products, both in K-theory and K-homology, from the following fundamental construction. Pick functions s, c: $[0, 1] \to [0, 1]$ such that $s(0) = 0$, $c(1) = 0$, and $s^2 + c^2 = 1$; as suggested by the notation, a natural choice is $s(x) = \sin \pi x/2$, $c(x) = \cos \pi x/2$, but others are possible.

(2.2) DEFINITION. Let A and B be unital (graded) C^*-algebras. Then there is defined an *external product* homomorphism

$$\times : G(A) \otimes G(B) \to G(A * B)$$

by

$$[F] \otimes [G] \mapsto c \otimes F \otimes 1 + s \otimes 1 \otimes G.$$

The homomorphism so defined is independent of the choice of functions s and c.

The verification that we do obtain a well-defined homomorphism in this way is straightforward.

Let us use this construction to define the K-theory external product. Notice that if A, A', B and B' are C^*-algebras, then there is an inclusion

$$(A' \otimes A) * (B' \otimes B) \hookrightarrow (A' * B') \otimes A \otimes B.$$

In particular, for polyhedra X and Y, we have an inclusion

$$(C(X) \otimes A) * (C(Y) \otimes B) \hookrightarrow C(X * Y) \otimes A \otimes B.$$

If $X = S^{m-1}$, $Y = S^{n-1}$, then $X * Y = s^{m+n-1}$. Using the inclusion above together with the external product of 2.2 we get a product

$$G_m(A) \otimes G_n(B) \to G_{m+n}(A \otimes B).$$

Since the join of a sphere and a disc is a disc, we find that if $x \in G_m(A)$ extends to an element of $G(C(D^m) \otimes A)$, then $x \times y$, for any $y \in G_n(B)$, extends to an element of $G(C(D^{m+n}) \otimes A \otimes B)$. Thus, by 1.5, the product on the G-groups passes to a product on K-theory,

$$K_m(A) \otimes K_n(B) \to K_{m+n}(A \otimes B).$$

Let us verify that this description of the external product agrees with that given by Atiyah in [1]. Let $A = C(X)$, $B = C(Y)$. Recall that the group

$$K_m(A) = K^{-m}(X) = K^0(X \times \mathbb{R}^m)$$

is generated by complexes of vector bundles over $X \times \mathbb{R}^m$, which are exact outside a compact set; by "rolling up," any such complex is equivalent to a complex of length two. Such a complex may in turn be thought of as a "cycle" consisting of a $\mathbb{Z}/2$-graded vector bundle over $X \times \mathbb{R}^m$, equipped with an odd self-adjoint endomorphism D such that $D^2 > 0$ outside a compact set. In terms of this definition of K-theory, it is easy to describe Atiyah's external product: given (V_X, D_X) over $X \times \mathbb{R}^m$ and (V_Y, D_Y) over $Y \times \mathbb{R}^n$, their external product is given by the (external, graded) tensor product bundle $V = V_X \otimes V_Y$ over $X \times Y \times \mathbb{R}^{m+n}$, and the endomorphism $D = D_X \otimes 1 + 1 \otimes D_Y$, whose square is positive outside a compact set.

The connection with our description of $K_m(A) = K_1(C(S^{m-1}) \otimes A \otimes \mathbb{C}_{1,1})$ is now simple, and is as follows. An element of $K_m(A)$ is given by a function $F_X\colon X \times S^{m-1} \to SS(M_k(\mathbb{C}_{1,1}))$ for some k. To this function we associate the (trivial) graded vector bundle $V_X = \mathbb{C}^k \oplus \mathbb{C}^k$ on $X \times \mathbb{R}^m$, together with the endomorphism $D_X(x, r, \theta) = rF(x, \theta)$, where we have used polar coordinates (r, θ), $r \in \mathbb{R}^+$, $\theta \in S^{m-1}$, on $\mathbb{R}^m$. Suppose that F_X and F_Y are given, construct the associated cycles (V_X, D_X) and (V_Y, D_Y), and form their external product (V, D) as described above. We may coordinatise $z \in S^{m+n-1} \subset \mathbb{R}^{m+n}$ by (x, y, θ), where $x \in S^{m-1}$, $y \in S^{n-1}$, $\theta \in [0, \pi/2]$ and $z = x \cos\theta + y \sin\theta$; this choice of coordinates expresses the identity $S^{m+n-1} = S^{m-1} * S^{n-1}$. Clearly then, for $z \in S^{m+n-1}$,

$$D(z) = \cos\theta (D_X \otimes 1) + \sin\theta (1 \otimes D_Y)$$

and thus (V, D) is the cycle corresponding to our definition of the external product $F = F_1 \times F_2$. This shows that our definition agrees with Atiyah's.

3. DUALITY AND *K*-HOMOLOGY THEORY

Now we recall the rudiments of the "duality" approach to K-homology, as described in [4, 5]. It makes for a small simplification to discuss K-homology only for *ungraded* C^*-algebras, but the discussion can in fact be carried through in general (cf. [6]).

Let A be an ungraded separable C^*-algebra (unital or not) and let $\sigma\colon A \to \mathfrak{B}(\mathscr{H})$ be a representation of A on a Hilbert space. We make the direct sum $H = \mathscr{H} \oplus \mathscr{H}$ into a graded Hilbert space in the natural way, and allow A to act by the direct sum $\rho = \sigma \oplus \sigma$.

We define $\mathfrak{D}_\rho(A)$ to be the graded C^*-subalgebra of $\mathfrak{B}(H)$ consisting of those operators T such that $T\rho(a) - \rho(a)\,T$ is compact for all $a \in A$. We define $\mathfrak{K}\mathfrak{D}_\rho(A)$ to be the ideal in $\mathfrak{D}_\rho(A)$ consisting of those T for which $T\rho(a)$ and $\rho(a)\,T$ are both compact. And we define $\mathfrak{Q}\mathfrak{D}_\rho(A)$ to be the quotient:

$$\mathfrak{Q}\mathfrak{D}(A) = \mathfrak{D}(A)/\mathfrak{K}\mathfrak{D}(A).$$

Except for the replacement of $\mathscr{H}$ by $H = \mathscr{H} \oplus \mathscr{H}$, which has the effect of tensoring with $\mathbb{C}_{1,1}$ and therefore does not change the K-theory, these are exactly the dual algebras introduced in [5], where $\mathfrak{D}_\rho(A)$ was denoted $D_\rho(A)$, and $\mathfrak{K}\mathfrak{D}_\rho(A)$ was denoted $D_\rho(A, A)$.

Let F be a supersymmetry in $\mathfrak{Q}\mathfrak{D}_\rho(A)$. We may lift it to an operator (which we will also denote by F) in $\mathfrak{D}_\rho(A)$, having the property that

$(F-F^*)\,\rho(a)$ and $(F^2-1)\,\rho(a)$ are compact for all $a \in A$. It follows that the triple (H, F, ρ) is a *Fredholm module* over A, that is a "cycle" for Kasparov's [7] K-homology group $K^0(A)$. By this construction we obtain a homomorphism

$$\alpha_\rho \colon K_1(\mathfrak{Q}\mathfrak{D}_\rho(A)) \to K^0(A).$$

The following fact explains our interest in the construction.

(3.1) PROPOSITION. *If the representation ρ is sufficiently large, then α_ρ is an isomorphism; and the quotient map $K_1(\mathfrak{D}_\rho(A)) \to K_1(\mathfrak{Q}\mathfrak{D}_\rho(A))$ is also an isomorphism.*

Proof. See [5, Theorem 1.5]. We also refer to [5] for a precise explanation of what is meant by "sufficiently large" in the statement of the proposition. ∎

Because of this proposition, we will sometimes omit explicit mention of the sufficiently large representation ρ from our notation for the dual algebras. It is worth noting that, for the dual algebra $\mathfrak{Q}\mathfrak{D}(A)$ associated to a sufficiently large representation, there is a canonical splitting of the short exact sequence

$$0 \to K_1(\mathfrak{Q}\mathfrak{D}(A)) \to G(\mathfrak{Q}\mathfrak{D}(A)) \to \mathbb{Z} \to 0$$

which defines the group K_1. Recall that a Fredholm module (H, F, ρ) is called *degenerate* if $F^2 = 1$, $F = F^*$, and $F\rho(a) = \rho(a)\,F$ for all $a \in A$. Analogously, we will call a supersymmetry $F \in \mathfrak{D}_\rho(A)$ *degenerate* if it commutes "on the nose" with $\rho(A)$. An example of such a degenerate supersymmetry is the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Now a simple Eilenberg swindle argument gives the following

(3.2) LEMMA. *All degenerate supersymmetries in $\mathfrak{D}_\rho(A)$, for ρ sufficiently large, give rise to the same element of $G(\mathfrak{D}_\rho(A))$.*

We can therefore split the exact sequence, by sending the generator of $\mathbb{Z}$ to the canonical degenerate class in $G(\mathfrak{Q}\mathfrak{D}(A))$. This allows us to regard $K_1(\mathfrak{Q}\mathfrak{D}(A))$ as a summand in $G(\mathfrak{Q}\mathfrak{D}(A))$: thus every supersymmetry in $\mathfrak{Q}\mathfrak{D}(A)$ gives rise to a K-theory class, with degenerates giving rise to the zero class; this is the beginning of the construction of an inverse map from K-homology to the K-theory of the dual algebra.

Finally we recall the statement of the Kasparov technical theorem [7, 8, 3]:

(3.3) THEOREM. *Let H be a separable Hilbert space and let E_1 and E_2 be separable C^*-subalgebras of $\mathfrak{B}(H)$ which are "essentially orthogonal," that is, $E_1 \cdot E_2 \subseteq \mathfrak{K}$. Let Δ be a separable linear subspace of $\mathfrak{B}(H)$ which derives*

E_1, in the sense that $[\Delta, E_1] \subseteq E_1$. Then there is a positive operator $X \in \mathfrak{B}(H)$ with $\|X\| \leqslant 1$ which essentially separates E_1 and E_2 and essentially commutes with Δ, that is:

(i) $(1-X) \cdot E_1 \subseteq \mathfrak{K}$;

(ii) $X \cdot E_2 \subseteq \mathfrak{K}$;

(iii) $[X, \Delta] \subseteq \mathfrak{K}$.

The result also holds in the graded case, where we interpret $[.,.]$ as graded commutator; in this case the operator X may be taken to be even.

4. THE KASPAROV PRODUCT IS THE DUAL OF THE EXTERNAL PRODUCT

In this section we investigate the join of two dual algebras. Let A_1 and A_2 be (ungraded) separable C^*-algebras, and let ρ_1 and ρ_2 be representations of A_1 and A_2 on graded Hilbert spaces H_1 and H_2, as in the previous section. Let $\rho = \rho_1 \otimes \rho_2$ be the tensor product representation of $A = A_1 \otimes A_2$ on $H = H_1 \otimes H_2$ (graded tensor product!). Let D_j $(j = 1, 2)$ be separable C^*-subalgebras of $\mathfrak{D}_{\rho_j}(A_j)$. We will construct a C^*-homomorphism

$$\varphi: D_1 * D_2 \to \mathfrak{Q}\mathfrak{D}_\rho(A). \qquad (\dagger)$$

There is a choice involved in the construction of φ, but we will see that up to homotopy the result is independent of the choice.

The construction (which is modeled on Kasparov's, see [7]) begins by an application of the technical theorem. Define $E_1 = \mathfrak{K}(H_1) \otimes \rho_2(A_2) + \mathfrak{K}(H)$ and $E_2 = \rho_1(A_1) \otimes \mathfrak{K}(H_2) + \mathfrak{K}(H)$, and let Δ be the closed linear span of $(\rho_1(A_1) + D_1) \otimes 1$ and $1 \otimes (\rho_2(A_2) + D_2)$. Then E_1, E_2 and Δ satisfy the hypotheses of the graded technical theorem; let X be an operator of the kind produced by the theorem. (Note that the space of suitable operators X is affine, and hence connected.)

Recall that $D_1 * D_2$ is a subalgebra of $C([0, 1]) \otimes D_1 \otimes D_2$. Since the spectrum of X is contained in $[0, 1]$, we may define an operator $f(X) \in \mathfrak{B}(H)$ for each $f \in C[0, 1]$. Moreover, by construction X (and therefore each $f(X)$) commutes modulo compacts with $D_1 \otimes D_2$. It follows that the formula

$$\varphi: f \otimes d_1 \otimes d_2 \mapsto [f(X) \otimes d_1 \otimes d_2]$$

gives a well-defined C^*-homomorphism from $D_1 * D_2$ to the Calkin algebra $\mathfrak{Q}(H)$. We need to show that the image of this homomorphism is contained in $\mathfrak{Q}\mathfrak{D}_\rho(A)$.

For this purpose we consider separately the compactness of $[\varphi(f \otimes d_1 \otimes d_2), \rho(a)]$ in three cases:

(i) $f(x) = x$, and $d_1 = 1$;

(ii) $f(x) = 1 - x$, and $d_2 = 1$;

(iii) $f(0) = f(1) = 0$.

The join algebra is spanned by generators of these three sorts, so it is enough to consider them separately.

In the first case, we need to show that $X(l \otimes d_2)$ commutes compactly with $\rho(a)$; it is enough to consider $a = a_1 \otimes a_2$. Then (using $\sim$ to denote equality modulo compact operators)

$$[X(1 \otimes d_2), \rho(a)] \sim X(\rho_1(a_1) \otimes [d_2, \rho_2(a_2)]) \sim 0$$

using the second and third properties of X provided by the technical theorem, and the fact that d_2 commutes compactly with $\rho_2(a_2)$. This proves the first case, and the second is entirely analogous.

As for the last case, note that any function f with $f(0) = f(1) = 0$ can be uniformly approximated by functions of the form $x \mapsto x(1-x)\, g(x)$, with g continuous. The operator $g(X)$ commutes (modulo compacts) with the image of ρ. Consequently, it is sufficient to prove the third case for $f(x) = x(1-x)$. But now

$$X(1-X)\, d_1 \otimes d_2 \sim (X1 \otimes d_2) \cdot ((1-X)\, d_1 \otimes 1)$$

is equal (modulo compacts) to the product of two elements of $\mathfrak{D}_\rho(A)$, so itself must belong to $\mathfrak{D}_\rho(A)$.

We have now completed the construction of the homomorphism φ advertised in equation (†). We note that the only element of indeterminacy in the construction lay in the choice of the operator X. Since any two choices of X are linearly homotopic, we see that φ is uniquely determined up to homotopy, as was previously claimed.

Since φ is determined up to homotopy, we obtain from it a homomorphism on the level of the G-groups

$$G(D_1 * D_2) \to G(\mathfrak{Q}\mathfrak{D}_\rho(A))$$

which is independent of all choices. Composing this with the external product of 2.2 we obtain a homomorphism

$$G(D_1) \otimes G(D_2) \to G(\mathfrak{Q}\mathfrak{D}_\rho(A)).$$

Finally, it is apparent that the G-group of any C^*-algebra is the direct limit of the G-groups of its separable subalgebras. We may therefore pass to the direct limit to obtain a product homomorphism

$$\times : G(\mathfrak{D}_{\rho_1}(A_1)) \otimes G(\mathfrak{D}_{\rho_2}(A_2)) \to G(\mathfrak{Q}\mathfrak{D}(A)). \tag{‡}$$

Now (3.1), for sufficiently large representations, the K-homology group $K^0(A_i)$ is isomorphic to $K_1(\mathfrak{D}(A_i))$, that is the kernel of the "dimension" map d: $G(\mathfrak{D}(A_i)) \to \mathbb{Z}$. It is evident that $d(x \times y) = d(x)\, d(y)$, so the product (‡) on the G-groups passes to a product on K-homology. The following is the main observation of this paper.

(4.1) THEOREM. *The product on K-homology, obtained from the external product on K-theory of the dual algebras by the procedure outlined above, coincides with the Kasparov product.*

The proof is immediate. Indeed, following through the construction above, we see that it gives as the product of two supersymmetries $F_j \in \mathfrak{D}_{\rho_j}(A_j)$, the supersymmetry

$$F = (1 - X^2)^{1/2}\, F_1 \otimes 1 + X 1 \otimes F_2$$

in $\mathfrak{Q}\mathfrak{D}_\rho(A)$. But this is exactly Kasparov's expression for the "sharp product" of two Fredholm modules.

Remark. One can also give a similar treatment of the "slant product" between K-homology and K-theory, but we will not do this here.

Finally let us treat briefly the case of products on higher K-homology:

$$K^{-n_1}(A_1) \otimes K^{-n_2}(A_2) \to K^{-n_1-n_2}(A_1 \otimes A_2). \tag{$**$}$$

Let D_j be separable subalgebras of $\mathfrak{Q}\mathfrak{D}_{\rho_j}(A_j)$ as above. Then $K^{-n_j}(A_j)$ is the direct limit (taken over separable subalgebras D_j) of the groups

$$H_j = \operatorname{Ker}(G_{n_j+1}(D_j) \to G(D_j)).$$

There is an inclusion of algebras

$$(C(S^{n_1}) \otimes D_1) * (C(S^{n_2}) \otimes D_2) \to C(S^{n_1} \times S^{n_2}) \otimes (D_1 * D_2).$$

Using 1.6 this gives us a product

$$H_1 \otimes H_2 \to K_{n_1+n_2+1}(D_1 * D_2) \to K_{n_1+n_2+1}(\mathfrak{Q}\mathfrak{D}_\rho(A)) = K^{-n_1-n_2}(A).$$

Taking the direct limit over separable D_j, as before, we obtain the product ($**$) that was required.

Remark. It is interesting to contrast the use of the join construction in the higher external product in K-theory (Section 2) and in K-homology (above). In K-theory the join parameter $t \in [0, 1]$ becomes an extra "spatial" variable (a suspension coordinate). In K-homology the join parameter is represented "spectrally" by the extra operator X. Compare [10] for a related discussion.

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