An Asymptotic Expansion for the Distribution of Hotelling’s $T^2$-Statistic under Nonnormality

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In this paper we obtain an asymptotic expansion for the distribution of Hotelling’s $T^2$-statistic under nonnormality when the sample size is large. In the derivation we find an explicit Edgeworth expansion of the multivariate $t$-statistic. Our method is to use the Edgeworth expansion and to expand the characteristic function of $T^2$.

1. INTRODUCTION

Let $\mathbf{x}$ be a $p \times 1$ random vector with mean vector $\mu$ and covariance matrix $\Sigma$. Let all $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be a sample of $n$ ($n > p$) independent observations of $\mathbf{x}$. Then, Hotelling $T^2$-statistic is defined by

$$T^2 = n(\bar{\mathbf{x}} - \mu)' S^{-1}(\bar{\mathbf{x}} - \mu).$$

(1.1)

where $\bar{\mathbf{x}} = n^{-1} \sum_{j=1}^{n} \mathbf{x}_j$ and $S = (n-1)^{-1} \sum_{j=1}^{n} (\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'$. The statistic is used for testing hypotheses about the mean vector $\mu$ and for obtaining confidence regions for the unknown $\mu$. Under the normality, it is well known (see, e.g., Anderson, 1984) that $[T^2/(n-1)][(n-1)/p]$ is distributed as an $F$-distribution with $p$ and $n-p$ degrees of freedom. The distribution of $T^2$ has been studied under some other underlying distributions including a mixture of two normal populations and an elliptical population by Srivastava and Awan (1982), Kaba and Gupta (1990), Iwashita (1994), etc.

It is easily seen that the limiting distributions of $T^2$ as $n \to \infty$ is a $\chi^2$-distribution with $p$ degrees of freedom. The purpose of this paper is to obtain an asymptotic expansion for the distribution of $T^2$ under nonnormality. The derivation is divided into two parts. The first part of the derivation is to find an explicit Edgeworth expansion of the multivariate $t$-statistic. The
second part is to expand the characteristic function of $T^2$, based on the Edgeworth expansion.

2. THE MAIN RESULT

Using the multivariate Student's $t$ statistic defined by $u = \sqrt{n} S^{-1/2} \bar{y}$, we can write Hotelling $T^2$-statistic as $T^2 = u' u$. Let $y = \Sigma^{-1/2}(x - \mu)$ and $y_j = \Sigma^{-1/2}(x_j - \mu)$. Then the statistic $u$ and hence $T^2$ are based on two standardized quantities defined by

$$ z = \sqrt{n} \bar{y}, \quad V = \sqrt{u' (W - I_p)}, $$

where $\bar{y} = n^{-1} \sum_{j=1}^n y_j$, $W = n^{-1} \sum_{j=1}^n W_j$, and $W_j = y_jy_j'$. In fact, $u$ is of the form

$$ u = \sqrt{n} (W - \bar{y} \bar{y}')^{-1/2} \bar{y} (1 - n^{-1})^{1/2} $$

$$ = \sqrt{n} \{ h(y, W) - h(0, I_p) \} (1 - n^{-1})^{1/2}; \tag{2.1} $$

where $h$ is a smooth function. From the last expression in (2.1) it can be seen (Bhattacharya and Ghosh, 1978) that $u$ has a valid Edgeworth expansion under appropriate conditions of $y = (y_1, ..., y_p)$. Following Bhattacharya and Ghosh (1978) we make the following assumptions which allow an expansion with a remainder $o(n^{-1/2})$.

**Assumption 1.** $E(|y|^{2k+2}) < \infty$.

**Assumption 2.** The distribution of $y = (y_1, ..., y_p)$ has an absolutely continuous component with a positive density on some nonempty open set $U$ such that 1, $y_1$, ..., $y_p$, $y_1^2$, ..., $y_p^2$ are linearly independent on $U$.

**Remark 1.** It is known (Hal, 1978) that for the case $p = 1$, the moment condition on $y$ may be weakened to $E(|y|^{k+2}) < \infty$. For a related discussion, see Bhattacharya and Ghosh (1988), Babu and Bai (1993).

We shall denote the $j$th cumulants of $y$ by $\kappa^{(i_1, ..., i_j)}$. From our definition of $y$, note that $\kappa^{(i)} = 0$, $\kappa^{(i, j)} = \delta_{ij}$, where $\delta_{ij}$ denotes the Kronecker's delta. Our result depends on the third and fourth order cumulants through

$$ \kappa_3^{(1)} = \left\{ \sum_{i, j, k} (\kappa^{(i, j, k)})^2 \right\}^{1/2}, $$

$$ \kappa_3^{(2)} = \left\{ \sum_j \left( \sum_k (\kappa^{(j, k, k)})^2 \right) \right\}^{1/2} = \left\{ \sum_{i, j, k} \kappa^{(i, i, j, j)} \kappa^{(j, j, k, k)} \right\}^{1/2}, \tag{2.2} $$

$$ \kappa_4^{(1)} = \sum_{i, j} \kappa^{(i, i, j, j)}. $$
Theorem. Under Assumption 1 with \( k = 2 \) and Assumption 2 the distribution function of \( T^2 \) can be expanded as

\[
P(T^2 \leq x) = G_p(x) + \frac{1}{n^2} \sum_{j=0}^{3} \beta_j G_{p+j}(x) + o(n^{-1})
\]

\[
= G_p(x) - \frac{2x}{np} g_p(x) \left\{ \beta_1 + \beta_2 + \beta_3 
+ \left( \frac{\beta_2 + \beta_3}{p+2} \right) \frac{x^2}{(p+2)(p+4)} \right\} + o(n^{-1}),
\]

uniformly for all positive real numbers \( x \), where \( G_p(x) \) and \( g_p(x) \) are the distribution function and the density function of a \( \chi^2 \)-variate with \( p \) degrees of freedom, respectively. The coefficients \( \beta_j \)'s are given by

\[
\begin{align*}
\beta_0 &= -\frac{1}{4} p^2 + \frac{1}{4} (\kappa_3^{(1)})^2 - \frac{1}{2} \kappa_4^{(1)}, \\
\beta_1 &= -\frac{1}{2} p - \frac{1}{2} (\kappa_3^{(1)})^2 + \frac{1}{2} \kappa_4^{(1)}, \\
\beta_2 &= \frac{1}{2} p (p+2) - \frac{1}{4} (\kappa_3^{(2)})^2 - \frac{1}{2} \kappa_4^{(1)}, \\
\beta_3 &= \frac{3}{2} (\kappa_3^{(1)})^2 + \frac{1}{2} (\kappa_3^{(2)})^2.
\end{align*}
\]

Remark 2. Kano (1995) has obtained the same asymptotic expansion as in the theorem, based on a different method. The original manuscripts by Kano and myself were independently completed almost at the same time.

Remark 3. The result depends on the cumulants up to the fourth order of \( y \). So, it is conjectured that for any \( p \), Assumption 1 with \( k = 2 \) may be weakened to \( E|y|^4 < \infty \). More precisely, as the referee pointed out, under Assumption 2 the joint distribution of \((Z_1, \ldots, Z_{2p})\) has an absolute continuous component, where \( Z_1 = y_{11} + y_{21}, \ldots, Z_p = y_{p1} + y_{p2}, Z_{p+1} = y_{12}^2 + y_{22}, \ldots, Z_{2p} = y_{2p1} + y_{2p2} \). Then, a condition similar to (2.9) of Babu and Bai (1993) (on the conditional characteristic function of \((Z_1, \ldots, Z_p)\) given the rest variables) holds. Thus, if the result of Babu and Bai (1993) can be extended to the multivariate case, applying it to our problem, it can be conjectured that the asymptotic expansion for the distribution of Hotelling’s \( T^2 \)-statistic can be established to any order under the minimal moment condition.

If all the cumulants of the third order are zero, then \( \kappa_3^{(1)} = \kappa_3^{(2)} = 0 \) and hence

\[
\begin{align*}
\beta_0 &= -\frac{1}{4} p^2 - \frac{1}{2} \kappa_4^{(1)}, \\
\beta_1 &= -\frac{1}{2} p + \frac{1}{2} \kappa_4^{(1)}, \\
\beta_2 &= \frac{1}{2} p (p+2) - \frac{1}{4} \kappa_4^{(1)}, \\
\beta_3 &= 0.
\end{align*}
\]
More specially, when $x$ has an elliptical distribution $E_{p}(\mu, A)$ with the characteristic function $\exp(\psi(\mu))$, then we have

$$
\beta_0 = -\frac{1}{2} p \{ p + (p + 2) \kappa \}, \quad \beta_1 = -\frac{1}{2} p \{ 1 - (p + 2) \kappa \}, \\
\beta_2 = \frac{1}{4} p (p + 2)(1 - \kappa), \quad \beta_3 = 0,
$$

where $\kappa$ is the kurtosis parameter defined by $\kappa = \psi''(0)/\langle \psi'(0) \rangle^2 - 1$. This result was obtained by Iwashita (1994).

In a special case $p = 1$ the coefficients $\beta_i$'s are given by

$$
\beta_0 = \frac{1}{2} \kappa_2^2 - \frac{1}{4} \kappa_4 - \frac{1}{2}, \quad \beta_1 = -\frac{1}{2} \kappa_2^2 + \frac{1}{2} \kappa_4, \\
\beta_2 = -\frac{1}{2} \kappa_2^2 - \frac{1}{2} \kappa_4 + \frac{1}{2}, \quad \beta_3 = \frac{1}{8} \kappa_2^2,
$$

where $\kappa_1 = E(y^3)$ and $\kappa_4 = E(y^4) - 3$.

3. PROOF OF THE MAIN RESULT

First we consider an explicit Edgeworth expansion of $u$ with a remainder term $o(n^{-1})$ under Assumption 1 with $k = 2$ and Assumption 2. Let us denote the $j$th cumulants of $u$ by $\kappa^{(i)}(u)$, Then it is known (see, e.g., Bhattacharya and Ghosh, 1978; Hall, 1992) that the cumulants of $u$ take the form

$$
\kappa^{(i)}(u) = \frac{1}{\sqrt{n}} q^{(i)} + O(n^{-3/2}),
$$

$$
\kappa^{(i, j)}(u) = \delta_{ij} + \frac{2}{\sqrt{n}} q^{(i, j)} + O(n^{-2}),
$$

$$
\kappa^{(i, j, k)}(u) = \frac{6}{\sqrt{n}} q^{(i, j, k)} + O(n^{-3/2}),
$$

$$
\kappa^{(i, j, k, l)}(u) = \frac{24}{n} q^{(i, j, k, l)} + O(n^{-2}),
$$

and in general, for $j \geq 2$

$$
\kappa^{(i, \ldots, j)}(u) = O(n^{-j+2/2}).
$$

Furthermore, we have

$$
\sup_{B \subseteq \mathbb{R}} \left| \mathbb{P}(u \in B) - \int_{B} \psi_{4, a}(x) \, dx \right| = o(n^{-1}), \quad (3.1)
$$
where the supremum is over all Borel sets $B$, and

$$
\psi_{\lambda,\beta}(x) = \phi_{\sigma}(x) \left\{ 1 + \frac{1}{\sqrt{n}} \left( \sum_{i} q^{(i)} H^{(i)}(x) + \sum_{j, k} q^{(j, k)} H^{(j, k)}(x) \right) \right. \\
+ \frac{1}{n} \left( \sum_{i, j} q^{(i, j)} + \frac{1}{2} q^{(i)} \delta_{i} \right) H^{(i)}(x) \\
+ \sum_{i, j, k} (q^{(i, j, k)} + q^{(i)} q^{(j, k)}) H^{(i, j, k)}(x) \\
+ \left. \frac{1}{2} \sum_{i, j, k, l, e, f} q^{(i, j, k, l, e, f)} H^{(i, j, k, l, e, f)}(x) \right\}. 
$$

(3.2)

Here for $x = (x_1, ..., x_p)$,

$$
\phi_{\sigma}(x) = \prod_{i=1}^{p} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2}, \\
H^{(i, ..., j)}(x) \phi_{\sigma}(x) = (-1)^j (\partial/\partial x_a) \cdots (\partial/\partial x_j) \phi_{\sigma}(x).
$$

In general, it is a hard work to express $q^{(i)}$, $q^{(i, j)}$, etc. in a simple form. After much simplification, we can obtain the expression:

$$
q^{(i)} = -\frac{1}{2} \sum_{u, a} \kappa^{(i, a, a)}, \\
q^{(i, j)} = \frac{1}{2} \sum_{u, b} \kappa^{(i, j, a, b)} + \frac{1}{2} \sum_{u, a} \kappa^{(i, a, b)} \kappa^{(j, a, b)} + \frac{1}{2} (p + 1) \delta_{i}, \\
q^{(i, j, k)} = -\frac{1}{2} \kappa^{(i, j, k)}, \\
q^{(i, j, k, l)} = -\frac{1}{4} \kappa^{(i, j, k, l)} + \frac{1}{2} \sum_{u} \kappa^{(i, a, b)} \kappa^{(j, a, b)} + \frac{1}{4} \delta_{i} \delta_{j}.
$$

Next we note that $T^2$ has a valid expansion. This is an immediate consequence from (3.1). In fact, taking $B = B x = \{ u; u \leq x \}$ in (3.1), we have

$$
P(T^2 \leq x) = \int_{B} \psi_{\lambda,\beta}(u) \, du + o(n^{-1}), 
$$

(3.4)

uniformly for all positive real numbers $x$. A final result will be obtained by computing the integral in (3.4). However, it is difficult to compute it.
directly. As an indirect method we consider to expand the characteristic function of $T^2$,

$$C_T(t) = E\{\exp(itT^2)\}$$

$$= \int \exp(it'x'x) f_x(x) dx,$$

and to use the uniqueness theorem of Fourier transform. Considering the transformation from $x$ to $u = \varphi^{-1}x$, we have

$$C_T(t) = \varphi^r E_u \left[ 1 + \frac{1}{\sqrt{n}} \left\{ \sum_i q^{(i)} H^{(i)}(\varphi u) + \sum_{i,j,k} q^{(i,j,k)} H^{(i,j,k)}(\varphi u) \right\} \right.$$  

$$+ \frac{1}{\sqrt{n}} \left\{ \sum_{i,j} \left( q^{(i,j)} + \frac{1}{2} q^{(j)} q^{(j)} \right) H^{(i,j)}(\varphi u) \right. \right.$$  

$$+ \sum_{i,j,k,l} \left( q^{(i,j,k,l)} + q^{(j)} q^{(j)} q^{(j)} \right) H^{(i,j,k,l)}(\varphi u) \right. \right.$$  

$$\left. \left. + \frac{1}{2} \sum_{i,j,k,l,e,f} q^{(i,j,k,l,e,f)} H^{(i,j,k,l,e,f)}(\varphi u) \right\} + o(n^{-1}), \quad (3.5) \right.$$  

where $\varphi = (1 - 2it)^{-1/2}$. The validity of this reduction is based on the fact that (3.1) holds and for a polynomial $h(x)$, $I(x) = \int_{-\infty}^{\infty} \exp(-x^2) h(x) dx$ is analytic on the half-plane such that the real part of $\alpha$ is positive. The expectation in (3.5) is taken under the normal random vector $u$ whose elements are independently distributed as $N(0,1)$.

It is easily seen that

$$E_u[H^{(i)}(\varphi u)] = 0,$$

$$E_u[H^{(i,j,k)}(\varphi u)] = 0,$$

$$E_u[H^{(i,j)}(\varphi u)] = (\varphi^2 - 1) \delta_{ij},$$

$$E_u[H^{(i,j,k,l)}(\varphi u)] = (\varphi^2 - 1)^2 \sum_{[3]} \delta_{ij} \delta_{kl},$$

$$E_u[H^{(i,j,k,l,e,f)}(\varphi u)] = (\varphi^2 - 1)^3 \sum_{[15]} \delta_{ij} \delta_{kl} \delta_{ef},$$

$$E_u[H^{(i,j,k,l,e,f)}(\varphi u)] = (\varphi^2 - 1)^3 \sum_{[15]} \delta_{ij} \delta_{kl} \delta_{ef}.$$
where \( \sum_{[n]} \) denotes a sum of \( n \) similar terms, determined by suitable permutations of the indices. Substituting these results to (3.5), we have

\[
C_T(t) = \varphi^2 \left[ 1 + \frac{1}{n} \left( \alpha_1(\varphi^2 - 1) + \alpha_2(\varphi^2 - 1)^2 + \alpha_3(\varphi^2 - 1)^3 \right) \right] + o(n^{-1}), \tag{3.6}
\]

where

\[
\begin{align*}
\alpha_1 &= \sum_i q_{i}^{(k,i)} + \frac{1}{2} \sum_j (q_{j}^{(i)})^2, \\
\alpha_2 &= 3 \sum_{i,j} q_{i,j}^{(k,i,j)} + 3 \sum_{i,j} q_{i,j}^{(i,j,i)}, \\
\alpha_3 &= \frac{9}{2} \sum_{i,j,k} q_{i,j,k}^{(k,i,j,k)} + 3 \sum_{i,j,k} (q_{i,j,k}^{(i,j,k)})^2.
\end{align*}
\]  

Using (2.3) and (3.3) we can simplify \( \alpha_j \)'s. The final result can be obtained by formally inverting the characteristic function (3.6). This completes the proof of the theorem.

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REFERENCES