The Characters of Quarternary Scaling Functions and Their Generators with Twelve-scale Dilation Factor

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Abstract
In this paper, the notion of matrix-valued multiresolution analysis is introduced. A procedure for designing biorthogonal matrix-valued quarternary wave wrap functions is presented and their characters are researched by virtue of time-frequency analysis method, matrix theory and operator theory. Three biorthogonality formulas concerning the wave wrap functions are obtained. Finally, new Riesz bases of four dimensional matrix-valued function space are derived by designing a series of subspaces of biorthogonal matrix-valued wave wrap functions.

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1. Introduction
In order to improve the locality of wavelets, Coifman and Meyer introduced the notion of univariate orthogonal wavelet packets. Wavelet packets, due to their nice characteristics, have been widely applied to signal processing [1], image compression [2], and fractal [3] and so on. The introduction for biorthogonal wavelet packets attributes to Cohen and Daubechies Yang and Cheng [4] constructed a-scale orthogonal multiwavelet packets which were more flexible in applications. Vector-valued wavelets are a class of special multiwavelets. Chen [4] introduced the notion for orthogonal vector-valued wavelets. For example, prefiltering is usually required for discrete multiwavelet transforms but not necessary for discrete vector-valued wavelet transforms. Examples of vector-valued signals are video images. Therefore, studying vector-valued wavelets is useful in multiwavelet theory and representations of signals. Shen [6] generalized the univariate orthogonal wavelet packets to the case of multivariate orthogonal wavelets such that they may be used in a wider field. Thus, it is significant and necessary to generalize the concept of univariate wavelet packets to the case of multivariate matrix-valued wavelets. The goal of this paper is to give the
definition and the construction of matrix-valued wavelet packets and construct several new Riesz bases of space $L^2(R^4, C^{rrr})$.

2. Matrix-valued multiresolution analysis

Let $R$ and $C$ be all real and all complex numbers, respectively. $Z$ and $N$ denote the set of integers and positive integers, respectively. Set $Z_+ = \{0\} \cup N$, and $r \in N$, as well as $r \geq 2$, $Z^4 = \{(n_1, n_2, n_3) : n_v \in Z, \ n_v = 1, 2, 3, 4\}$, $Z^4_+ = \{(n_1, n_2, n_3, n_4) : n_v \in Z_+, \ n_v = 1, 2, 3, 4\}$. Suppose that $B$ is an $4 \times 4$ matrix whose all entries are integers and all eigenvalues is larger than one in modulus. The absolute value of the determinant of matrix $B$ is denoted by $d$, i.e., $|\text{det}(B)| = d$. Order $B^T$ stands for the transpose of matrix, and $B^{T \ -1}$ is the inverse of the transpose of matrix $B$. For $\Lambda$, $\Lambda_1$, $\Lambda_2 \subset R^4$, Set $B\Lambda = \{Bx : x \in \Lambda\}$, $\Lambda_1 + \Lambda_2 = \{x_1 + x_2 : x_1 \in \Lambda_1, x_2 \in \Lambda_2\}$, $\Lambda_1 - \Lambda_2 = \{x_1 + x_2 : x_1 \in \Lambda_1, x_2 \in \Lambda_2\}$. It is known that there exist $d$ elements $\xi_1, \xi_2, \ldots, \xi_{d-1} \in Z^4_+$, by the finite group theory such that

$$Z^4 = \bigcup_{\xi \in \Lambda_0} (\xi + BZ^4),$$

$$(\xi + BZ^4) \cap (\xi_2 + BZ^4) = \emptyset,$$

where $\Lambda_0 = \{\xi_1, \xi_2, \ldots, \xi_{d-1}\}$ denotes the set of all different representative elements in the quotient group $Z^4/(BZ^4)$ and $\xi_1, \xi_2$ denote two arbitrary distinct elements in $\Lambda_0$. Set $\xi_0 = \{0\}$, where $\{0\}$ is the null of the set $Z^4_+$. Let $\Lambda = \Lambda_0 - \{0\}$ and $\Lambda, \Lambda_0$ be two index sets. The space $L^2(R^4, C^{rrr})$ is defined to be the set of all matrix-valued functions $\Upsilon(x)$, i.e.,

$$L^2(R^4, C^{rrr}) := \left\{ \begin{array}{ll}
\gamma_{11}(x) & \gamma_{12}(x) \ldots \gamma_{1r}(x) \\
\gamma_{21}(x) & \gamma_{22}(x) \ldots \gamma_{2r}(x) \\
\ldots & \ldots \ldots \ldots \\
\gamma_{r1}(x) & \gamma_{r2}(x) \ldots \gamma_{rr}(x) \\
\end{array} \right\} \in L^2(R^4), \ (l, j = 1, 2, \ldots, r).$$

Examples of matrix-valued signals are video images where $\gamma_{lj}(x)$ denotes the pixel on the $l$th row and the $j$th column at the point $x$. $\|\Upsilon\|$ stands for the norm of the matrix-valued function $\Upsilon(x) \in L^2(R^4, C^{rrr})$,

$$\|\Upsilon\| := \left( \sum_{l,j=1}^r \int_{R^4} \left| \gamma_{l,j}(x) \right|^2 \ dx \right)^{1/2},$$

(1)

For any $h \in L^2(R^4, C^{rrr})$, its integration and its Fourier transform are defined, respectively, as follows,

$$\int_{R^4} \Upsilon(x) \ dx = (\int_{R^4} \gamma_{l,v}(x) \ dx)_{l,v=1}^r$$

(2)
\[ \hat{\Upsilon}(\xi) := \int_{\mathbb{R}^4} \Upsilon(x) \cdot \exp\{-ix \cdot \xi\} dx, \] (3)

where \( x \cdot \xi \) stands for the inner product of real \( r \)-dimensional vectors \( x \) and \( \xi \). For any matrix-valued functions \( h, \lambda \in L^2(\mathbb{R}^r, C_r^{r \times r}) \), their symbol inner product is defined to be

\[ \langle \lambda, h \rangle := \int_{\mathbb{R}^4} \lambda(x) h(x) \lambda(x)^* \, dx, \] (4)

where * means the transpose and the complex conjugate.

**Definition 1.** We say that a pair of matrix-valued functions \( \Upsilon(x), \tilde{\Upsilon}(x) \in L^2(\mathbb{R}^4, C_r^{r \times r}) \) are biorthogonal, if their translates satisfy

\[ \langle \Upsilon(\cdot), \tilde{\Upsilon}(\cdot - k) \rangle = \delta_{0,k} I_r, \quad k \in \mathbb{Z}^4, \] (5)

where \( I_r \) denotes the \( r \times r \) identity matrix and \( \delta_{0,k} \) is Kronecker symbol.

**Definition 2.** A sequence of matrix-valued functions \( \{\Upsilon_k(x)\}_{k \in \mathbb{Z}^4} \subset U \subset L^2(\mathbb{R}^4, C_r^{r \times r}) \) is called a Riesz basis in \( U \) if it satisfy (i) for any \( \Psi(x) \in U \), there exists a unique \( r \times r \) matrix sequence \( \{P_k\}_{k \in \mathbb{Z}^4} \in l^2(\mathbb{Z}^4)^{r \times r} \) such that

\[ \Psi(x) = \sum_{k \in \mathbb{Z}^4} P_k \Upsilon_k(x), \] (6)

where \( l^2(\mathbb{Z}^4)^{r \times r} \) is equals to \( \{B : Z^4 \to C_r^{r \times r}, \|B\|_2 = (\sum_{l,j=1}^{r} \sum_{k \in \mathbb{Z}^4} |b_{l,j}(k)|^2)^{\frac{1}{2}} < +\infty\} \), (ii) and there exist constants \( 0 < C_1 \leq C_2 < +\infty \) such that, for any matrix sequence \( \{P_k\}_{k \in \mathbb{Z}^4} \), the following equality holds, i.e.,

\[ C_1 \|\{P_k\}\|_* \leq \left\| \sum_{k \in \mathbb{Z}^4} P_k \hat{h}_k(x) \right\| \leq C_2 \|\{P_k\}\|_*, \] (7)

where \( \|\{P_k\}\|_* \) denotes the norm of the sequence of constant matrices \( \{P_k\}_{k \in \mathbb{Z}^4} \). For example, it is known that

\[ \|\{P_k\}\|_* = (\sum_{k \in \mathbb{Z}^4} \|P_k\|_F^2)^{\frac{1}{2}} \] where \( \|P_k\|_F \); for every \( k \in \mathbb{Z}^4 \), denotes the Frobenius norm of a matrix \( P_k \).

**Definition 3.** A matrix-valued multiresolution analysis of \( L^2(\mathbb{R}^4, C_r^{r \times r}) \) is a nested sequence of closed subspaces \( \{V_j\}_{j \in \mathbb{Z}} \) such that (i) \( \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \); (ii) \( \bigcap_{j \in \mathbb{Z}} V_j = \{O\} \) and \( \bigcup_{j \in \mathbb{Z}} V_j \) is dense in \( L^2(\mathbb{R}^4, C_r^{r \times r}) \), where \( O \) denotes an \( r \times r \) zero matrix; (iii) \( \lambda(x) \in V_j \iff \lambda(Bx) \in V_j \), \( \forall j \in \mathbb{Z} \); (v) there exists \( \Gamma(x) \in V_0 \), called a matrix-valued scaling function, such that \( \{\Gamma_k(x) := \Gamma(x - n), n \in \mathbb{Z}^4\} \) form a Riesz basis for subspace \( V_0 \).

Since \( \Gamma(x) \in V_0 \subset V_1 \), by definition 3 and (6), there exists a finitely supported sequence of
constant $r \times r$ matrix $\{P_k\}_{k \in Z^4} \in l^2(Z^4)^{r \times r}$ such that

$$\Gamma(x) = \sum_{k \in Z^4} P_k \Gamma(Bx - k). \quad (8)$$

Equation (8) is said to be a refinement equation. Let

$$\mathcal{P}(\xi) = \frac{1}{d} \sum_{k \in Z^4} P_k \cdot \exp\{-ik \cdot \xi\}, \quad \xi \in R^4. \quad (9)$$

where $\mathcal{P}(\xi)$, which is a $2\pi Z^4$ periodic function, is called a symbol of $G(x)$. Thus, equation (8) becomes

$$\hat{\Gamma}(\xi) = \mathcal{P}(B^{-T}\xi) \hat{\Gamma}(B^{-T}\xi), \quad \xi \in R^4. \quad (10)$$

Let $W_j, j \in Z$ be the direct complementary subspace of $V_j$ in $V_{j+1}$. Assume that there exist $d-1$ matrix-valued functions $L^2(R^4, C^{r \times r})$, $\rho \in \Lambda$, such that their translates and dilations form a Riesz basis of $W_j$, i.e.,

$$W_j = \text{clos}_{\xi(\cdot, \cdot, c, \cdot)}(\text{Span}\{F_\rho(B^{j'-k}) : k \in Z^4, \mu \in \Lambda\}), \quad (11)$$

where $j \in Z$. Since $F_\rho(x) \in W_0 \subset V_1, \rho \in \Lambda$, there exist $d-1$ finite supported sequences of constant $r \times r$ matrix $\{D^{(\mu)}_{\rho}\}_{k \in Z^4}$ such that

$$F_\rho(x) = \sum_{\nu \in Z^4} D^{(\mu)}_{\nu} \Gamma(Bx - \nu), \quad \mu \in \Lambda. \quad (12)$$

Taking the Fourier transform for both sides of (12) gives

$$\hat{F}_\rho(B\xi) = \mathcal{D}^{(\mu)}(\xi) \hat{\Gamma}(\xi), \quad \xi \in R^4, \quad \mu \in \Lambda. \quad (13)$$

where

$$\mathcal{P}^{(\mu)}(\xi) = \frac{1}{d} \sum_{\nu \in Z^4} P^{(\mu)}_{\nu} \cdot \exp\{-i\nu \cdot \xi\}, \quad \mu \in \Lambda. \quad (14)$$

If $F(x), \tilde{F}(x) \in L^2(R^4, C^{r \times r})$ are a pair of biorthogonal matrix-valued scaling functions, then it follows by Definition 1 that

$$\left\langle F(\cdot), \tilde{F}(\cdot - k) \right\rangle = \delta_{0,k} \mathcal{I}_r, \quad k \in Z^4. \quad (15)$$

We say that $F_\rho(x), \tilde{F}_\rho(x) \in L^2(R^4, C^{r \times r}), \rho \in \Lambda$ are a pair of biorthogonal matrix-valued wavelets associated with a pair of biorthogonal matrix-valued scaling functions $\Gamma(x)$ and $\hat{\Gamma}(x)$, if the family $\{\Gamma(\cdot - k), \rho \in Z^4, \rho \in \Lambda\}$ is a $k \in Z^4, \rho \in \Lambda$ Riesz basis of subspace $W_0$, and

$$\left\langle \Gamma(\cdot), \tilde{F}_\rho(\cdot - u) \right\rangle = 0, \quad \rho \in \Lambda, \quad u \in Z^4, \quad (16)$$

$$\left\langle \tilde{\Gamma}(\cdot), F_\rho(\cdot - k) \right\rangle = 0, \quad \rho \in \Lambda, \quad k \in Z^4, \quad (17)$$

$$\left\langle F_\rho(\cdot), \tilde{F}_\rho(\cdot - k) \right\rangle = \delta_{\rho,0} \delta_{0,k} \mathcal{I}_r, \rho, \mu \in \Lambda, k \in Z^4. \quad (18)$$

Set
\[
W_j^{(\rho)} = \text{clos}_{L^2(R^d, C_m)} \left\{ W_\rho (B^j \cdot m) : m \in Z^d \right\},
\]
where \( \rho \in \Lambda, j \in \mathbb{Z} \). By the definition of \( W_j \) and (11), (15)-(18), we obtain
\[
L^2(R^d, C_m) = \bigcup_{j \in \mathbb{Z}} W_j = \bigcup_{j \in \mathbb{Z}} (\bigcup_{\rho \in \Lambda} W_j^{(\rho)}),
\]
where \( \bigcup \) denotes the direct sum. Similar to (8) and (12), there exist \( m \) finite supported sequences of \( r \times r \) constant matrices \( \{ \tilde{P}_k \}_{k \in \mathbb{Z}^d} \) and \( \{ \tilde{D}_k^{(\rho)} \}_{k \in \mathbb{Z}^d}, \rho \in \Lambda \) such that \( \tilde{\Gamma}(x) \) and \( \tilde{F}(x) \) satisfy the following refinement equation:
\[
\tilde{\Gamma}(x) = \sum_{k \in \mathbb{Z}^d} \tilde{P}_k \tilde{\Gamma}(Bx - k),
\]
\[
\tilde{F}_\rho(x) = \sum_{k \in \mathbb{Z}^d} \tilde{D}_k^{(\rho)} \tilde{\Gamma}(Bx - k), \quad \rho \in \Lambda.
\]

3. Properties of vector-valued wavelet packets

Let
\[
\Psi_0(x) = \Gamma(x), \quad \Psi_\rho(x) = F_\rho(x), \quad \tilde{\Psi}_0(x) = \tilde{\Gamma}(x),
\]
\[
\tilde{\Psi}_\rho(x) = F_\rho(x), \quad M_k^{(0)} = P_k, \quad M_k^{(\rho)} = D_k^{(\rho)},
\]
\[
\tilde{M}_k^{(0)} = \tilde{P}_k, \quad \tilde{M}_k^{(\rho)} = \tilde{D}_k^{(\rho)}, \quad \rho \in \Lambda, k \in \mathbb{Z}^d, B = 12 I_r.
\]

Then, equation (8) and (12) can be jointly written as follows:
\[
\Psi_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} M_k^{(\rho)} \Psi_0(Bx - \nu), \quad \rho \in \Lambda_0.
\]

For any \( \alpha \in \mathbb{Z}^d \) and the given matrix-valued biorthogonal scaling functions \( \Psi_0(x) \) and \( \tilde{\Psi}_0(x) \), iteratively define, respectively,
\[
\Psi_{\alpha}(x) = \Psi_{12\beta + \rho}(x) = \sum_{\nu \in \mathbb{Z}^d} M_k^{(\rho)} \Psi_\beta(12x - u),
\]
\[
\tilde{\Psi}_{\alpha}(x) = \tilde{\Psi}_{12\beta + \rho}(x) = \sum_{\nu \in \mathbb{Z}^d} \tilde{M}_k^{(\rho)} \tilde{\Psi}_\beta(12x - \nu),
\]
where \( \beta \in \mathbb{Z}^d \) is the unique element such that \( \alpha = 12 \beta + \rho, \quad \rho \in \Lambda_0 \) holds.

**Definition 4.** Two sets of matrix-valued function \( \{ \Psi_{12\beta + \rho}(x), \beta \in \mathbb{Z}^d_+, \rho \in \Lambda_0 \} \) and \( \{ \tilde{\Psi}_{12\beta + \rho}(x), \beta \in \mathbb{Z}^d_+, \rho \in \Lambda_0 \} \) are said to be wavelet wrapss with respect to a pair of biorthogonal matrix-valued scaling functions \( \Gamma(x) \) and \( \tilde{\Gamma}(x) \), respectively, where \( \Psi_{12\beta + \rho}(x) \) and \( \tilde{\Psi}_{12\beta + \rho}(x) \) are given by (24) and (25), respectively.

Applying the Fourier transform for (24) and (25) yields,
\[
\hat{\Psi}_{12\beta + \rho}(12\xi) = M^{(\rho)}(\xi) \hat{\Psi}_\beta(\xi), \quad \rho \in \Lambda_0,
\]
\[ \hat{\Psi}_{12\beta+\rho}(12\xi) = \mathcal{M}^{(\rho)}(\xi) \hat{\Psi}_\rho(\xi), \quad \rho \in \Lambda_0, \]  

(27)

Where

\[ \mathcal{M}^{(\rho)}(\xi) = \frac{1}{12^\rho} \sum_{\nu \in \mathbb{Z}^4} M^{(\rho)}(\xi) \cdot \exp\{-i k \cdot \xi\}, \quad \mu \in \Lambda_0, \xi \in \mathbb{R}^4, \]

**Theorem 1.** Suppose \{\Psi_\alpha(x), \alpha \in \mathbb{Z}_+^4\} and \{\tilde{\Psi}_\alpha(x), \alpha \in \mathbb{Z}_+^4\} are wavelet packets with respect to a pair of biorthogonal multiple vector-valued scaling functions \(\Psi_0(x)\) and \(\tilde{\Psi}_0(x)\). Then, for \(\alpha \in \mathbb{Z}_+^4\), we have

\[ \langle \Psi_\alpha(\cdot), \tilde{\Psi}_\alpha(-k) \rangle = \delta_{0,k} I_r, \quad k \in \mathbb{Z}^4. \]

(29)

**Proof.** The result (29) follows from (15) as \(\alpha = 0\). Assume that (29) holds when \(|\alpha| = \sum_{i=1}^4 \alpha_i < \eta\), where \(\eta\) is a positive integer, and \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}_+^4\). For the case of \(\alpha \in \mathbb{Z}_+^4\), \(|\alpha| = \eta\), we will prove (29) holds. Order \(\alpha = 12\beta + \rho\) where \(\beta \in \mathbb{Z}_+^4, \rho \in \Lambda_0\). Since all eigenvalues of \(A\) is large than one in modulus, then \(|\beta| < |\alpha|\). By induction assumption, we have:

\[ \langle \Psi_\alpha(\cdot), \tilde{\Psi}_\alpha(-k) \rangle = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \hat{\Psi}_{12\beta+\rho}(\xi) \hat{\tilde{\Psi}}_{12\beta+\rho}(\xi)^* \cdot \exp\{i k \cdot \xi\} d\xi \]

\[ = \frac{1}{(2\pi)^4} \int_{(0,2\pi)^4} I_r e^{ik \cdot \xi} d\xi = \delta_{0,k} I_r. \]

Therefore, the result is established.

**Theorem 2.** Suppose that \{\Psi_\alpha(x), \alpha \in \mathbb{Z}_+^4\} and \{\tilde{\Psi}_\alpha(x), \alpha \in \mathbb{Z}_+^4\} are wavelet packets with respect to a pair of Biorthogonal multiple vector-valued functions \(\Psi_0(x)\) and \(\tilde{\Psi}_0(x)\), respectively. Then, for \(\beta \in \mathbb{Z}_+^4, n \in \mathbb{Z}^4\), we have

\[ \langle \Psi_{12\beta+\mu}(\cdot), \tilde{\Psi}_{12\beta+\rho}(-n) \rangle = \delta_{0,n} \delta_{\mu,\rho} I_r, \quad \mu, \rho \in \Lambda_0. \]

**Proof.**

\[ \langle \Psi_{12\beta+\mu}(\cdot), \tilde{\Psi}_{12\beta+\rho}(-n) \rangle = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \hat{\Psi}_{12\beta+\mu}(\xi) \hat{\tilde{\Psi}}_{12\beta+\rho}(\xi)^* \cdot \exp\{i n \cdot \xi\} d\xi \]

\[ = \frac{1}{(2\pi)^4} \int_{(0,2\pi)^4} \delta_{\mu,\rho} I_r e^{in \cdot \xi} d\xi = \delta_{0,n} \delta_{\mu,\rho} I_r. \]

This completes the proof of the theorem.

**Theorem 3** [5]. For any \(\alpha, \beta \in \mathbb{Z}_+^4\), we have
\[ \langle \Psi_\alpha (\cdot), \tilde{\Psi}_\beta (\cdot - n) \rangle = \delta_{0,n} \delta_{\alpha,\beta} I_r. \]  

(30)

**Proof.** When \( \alpha = \beta \), (30) follows by Theorem 1. as \( \alpha \neq \beta \), it follows from Lemma 4 that (30) holds, too. Assuming that \( \alpha \) is not equal to \( \beta \), as well as at least one of \( \{ \alpha, \beta \} \) doesn’t belong to \( \Gamma_0 \), we rewrite \( \alpha, \sigma \) as \( \alpha = 12 \alpha_1 + \rho_1, \beta = 12 \beta_1 + \mu_1 \), where \( \rho_1, \mu_1 \in \Lambda_0 \). **Case 1.** If \( \alpha_1 = \beta_1 \), then \( \rho_1 \neq \mu_1 \). (30) follows by virtue of (25), (26):

\[
(2\pi)^4 \left\langle \Psi_\alpha (\cdot), \tilde{\Psi}_\beta (\cdot - k) \right\rangle
= \int_{\mathbb{R}^d} \tilde{\Psi}_\alpha (\rho_1 + \gamma (\cdot)) \tilde{\Psi}_\beta (\rho_1 + \gamma (\cdot))^* \exp \{ ik \cdot \xi \} d\xi
= \int_{[0,2\pi]^d} \delta_{\rho_1,\gamma} I_r \cdot \exp \{ ik \cdot \xi \} d\xi = 0.
\]

**Case 2** If \( \alpha_1 \neq \beta_1 \), order \( \alpha_1 = 12 \alpha_2 + \rho_2, \beta_1 = 12 \beta_2 + \mu_2 \), where \( \alpha_2, \beta_2 \in \mathbb{Z}_+^4 \), and \( \rho_2, \mu_2 \in \Lambda_0 \). Provided that \( \alpha_2 = \beta_2 \), then \( \rho_2 \neq \mu_2 \). Similar to Case 1, (30) can be established. After taking finite steps (denoted by \( \kappa \)), we obtain \( \alpha_\kappa \in \Lambda_0 \), and \( \rho_\kappa, \mu_\kappa \in \Lambda_0 \). Thus, after taking finite steps (denoted by \( \kappa \)), we obtain \( \alpha_\kappa = \lambda_\kappa \), then \( \rho_\kappa \neq \mu_\kappa \). (30) follows. If \( \alpha_\kappa \neq \beta_\kappa \), then we obtain that

\[
\langle \Psi_\alpha (\cdot), \tilde{\Psi}_\beta (\cdot - k) \rangle = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^d} \tilde{\Psi}_\alpha (\xi) \tilde{\Psi}_\beta (\xi)^* \cdot e^{ik \cdot \xi} d\xi
= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^d} \mathcal{M}^{(\rho_1)} (\xi/12) \mathcal{M}^{(\rho_2)} (\xi/144) \tilde{\Psi}_\alpha (\xi/144)
\cdot \tilde{\mathcal{M}}^{(\mu_1)} (\xi/12) \tilde{\mathcal{M}}^{(\mu_2)} (\xi/144) \cdot e^{ik \cdot \xi} d\xi = \cdots
= \frac{1}{(2\pi)^4} \int_{[0,2\pi]^d} \left\{ \prod_{i=1}^{\kappa} Q^{(\mu_1)} (\xi/12) \right\} \cdot \left\{ \prod_{i=1}^{\kappa} Q^{(\mu_2)} (\xi/144) \right\} \cdot \exp \{ -ik \cdot \xi \} d\xi = 0.
\]

Therefore, for any \( \alpha, \sigma \in \mathbb{Z}_+^4 \), (28) is established.

4. Conclusion

The concept of biorthogonal multiple vector-valued wavelet packets of space \( L^2 (\mathbb{R}^d, C^{rr^r}) \) is introduced. Their properties are studied by virtue of time-frequency analysis method, matrix theory and operator theory, and three biorthogonality formulas concerning these wavelet packets are given.

References


