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Asymptotic stability for anisotropic Kirchhoff systems ☆

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Abstract

We study the question of asymptotic stability, as time tends to infinity, of solutions of dissipative anisotropic Kirchhoff systems, involving the $p(x)$-Laplacian operator, governed by time-dependent nonlinear damping forces and strongly nonlinear power-like variable potential energies. This problem had been considered earlier for potential energies which arise from restoring forces, whereas here we allow also the effect of amplifying forces. Global asymptotic stability can then no longer be expected, and should be replaced by local stability. The results are further extended to the more delicate problem involving higher order damping terms.

Keywords: Dissipative anisotropic $p(x)$-Kirchhoff systems; Time-dependent nonlinear damping forces; Strongly nonlinear potential energies; Local and global asymptotic stability

1. Introduction

In this paper we investigate the asymptotic behavior of solutions of the dissipative anisotropic $p(x)$-Kirchhoff systems of the form

$$\begin{cases} u_{tt} - M\big(\mathscr{I}u(t)\big)\Delta_{p(x)}u + Q(t,x,u,u_t) + f(x,u) = 0 & \text{in } \mathbb{R}_0^+ \times \Omega, \\ u(t,x) = 0 & \text{on } \mathbb{R}_0^+ \times \partial\Omega, \end{cases} \tag{1.1}$$

where $u = (u_1, \dots, u_N) = u(t,x)$ is the vectorial displacement, $N \geqslant 1$, $\mathbb{R}_0^+ = [0,\infty)$, Ω is a bounded domain of $\mathbb{R}^n$, M is given by

$$M(\tau) = a + b\gamma\tau^{\gamma-1}, \quad \tau \geqslant 0, \tag{1.2}$$

with $a, b \geqslant 0$, $a+b > 0$ and $\gamma > 1$, and $\mathscr{I}u(t) = \int_\Omega \{|Du(t,x)|^{p(x)}/p(x)\}\,dx$ is the natural associated $p(x)$-Dirichlet energy integral. In the problem (1.1) the operator $\Delta_{p(x)}$ denotes the $p(x)$-*Laplacian operator*, that is $\Delta_{p(x)}u = \operatorname{div}(|Du|^{p(x)-2}Du)$. The study of p-Kirchhoff equations involves the quasilinear homogeneous p-Laplace operator

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and is based on the theory of standard Sobolev spaces $W_0^{1,p}(\Omega)$ for weak solutions, cf. [6]; see also [7] for wave equations and [4] for the elliptic case. For the *nonhomogeneous* $p(x)$-Kirchhoff operators the natural setting is the one of variable exponent Sobolev spaces, which have been used in the last decades to model various phenomena, see [5,10–18], as well as [23,24] and references therein. Indeed, in recent years, there has been an increasing interest in studying systems involving somehow nonhomogeneous $p(x)$-Laplace operators, motivated by the image restoration problem, the modeling of electrorheological fluids (sometimes referred to as *smart fluids*), as well as the thermo-convective flows of non-Newtonian fluids: details and further references can be found in [2] and [17]. For the regularity of weak solutions we refer to [1].

Throughout the paper we assume

$$Q \in C\big(\mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N\big), \qquad f \in C\big(\Omega \times \mathbb{R}^N \to \mathbb{R}^N\big).$$

The function Q, representing a *nonlinear damping*, verifies the condition

$$\big(Q(t,x,u,v), v\big) \geqslant 0 \quad \text{for all arguments } t, x, u, v, \tag{1.3}$$

where $(\cdot,\cdot)$ is the inner product of $\mathbb{R}^N$. The *external force* f is assumed to be derivable from a potential F, that is

$$f(x,u) = \partial_u F(x,u), \tag{1.4}$$

where $F \in C^1(\Omega \times \mathbb{R}^N \to \mathbb{R}_0^+)$ and $F(x,0) = 0$.

Moreover, when the first eigenvalue μ_0 of $\Delta_{p(\cdot)}$ in Ω, with zero Dirichlet boundary conditions, is positive, that is

$$\mu_0 \int_\Omega \frac{|\varphi(x)|^{p(x)}}{p(x)}\,dx \leqslant \int_\Omega \frac{|D\varphi(x)|^{p(x)}}{p(x)}\,dx \tag{1.5}$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega)$, we allow $(f(x,u),u)$ to take negative values. In other words we assume that

$$\big(f(x,u), u\big) \geqslant -a\mu|u|^{p(x)} \quad \text{in } \Omega \times \mathbb{R}^N, \tag{1.6}$$

for some $\mu \in [0, \mu_0 p_-/p_+)$, with $1 < p_- \leqslant p(x) \leqslant p_+$ in Ω, where $p_+ = \sup_{x\in\Omega} p(x)$ and $p_- = \inf_{x\in\Omega} p(x)$, see Section 2 for details. The most interesting case occurs when $p_- < p_+$, that is in the so-called *nonstandard growth condition of* (p_-, p_+) *type*, cf. [2]. In particular, in the applications, the function p is supposed to satisfy the usual request $1 < p_- \leqslant p_+ < n$. In this paper we need and assume the stronger condition $2n/(n+2) \leqslant p_- \leqslant p_+ < n$, in order to get the necessary embeddings.

In studying asymptotic stability, (1.6) leads to different situations when $\mu_0 = 0$ or $\mu_0 > 0$. As a matter of fact in general μ_0 may be zero; in [10, Theorems 3.3 and 3.4] are given sufficient conditions under which $\mu_0 > 0$, as in the standard p-Laplacian model. When $\mu_0 = 0$ we take in (1.6) also $\mu = 0$. Therefore, when either $\mu_0 = 0$ or $a = 0$, that is in the latter case when (1.1) is degenerate, then (1.6) reduces to the more familiar condition $(f(x,u),u) \geqslant 0$, namely f is *of restoring type*.

Throughout the paper we consider a growth condition on f involving a continuous function q such that $p(x) \leqslant q(x)$ for all $x \in \Omega$. Moreover, when $q(y) > p^*(y)$ for some $y \in \Omega$, a further growth hypothesis on the external force f is assumed; here $p^* = p^*(x)$ denotes the variable Sobolev critical exponent for the space $W_0^{1,p(\cdot)}(\Omega)$, see Section 2. In [22] global existence of solutions is proved without imposing any bound on the exponent $q(x) \equiv q$ of the source term f, when f does not depend on t as in our setting. This justifies the importance to consider for asymptotic stability also the case in which the condition $q \leqslant p^*$ in Ω fails. We remind to [22] for a complete recent bibliography for wave equations with also nonlinear dampings, in the classical framework of Lebesgue and Sobolev spaces.

In the context of problem (1.1) the question of asymptotic stability is best considered by means of the natural energy associated with the solutions of (1.1), namely

$$Eu(t) = \frac{1}{2}\big\|u_t(t,\cdot)\big\|_2^2 + a\mathscr{I}u(t) + b\big[\mathscr{I}u(t)\big]^\gamma + \mathscr{F}u(t),$$

where $\mathscr{F}u(t) = \int_\Omega F(x,u(t,x))\,dx$. In Section 3 we provide our main result about global asymptotic stability, based on the a priori existence of a suitable auxiliary function $k = k(t)$, which was first introduced by Pucci and Serrin in [19]. To do this we follow the principal ideas of [20,21], already employed in [3], overcoming the new difficulties arisen from the more delicate setting.

Similar problems in the literature are concerned with potential energies which derive from restoring forces, while here we allow also the effect of amplifying forces, expressed by (1.6) when $\mu > 0$, as in [21] for wave systems with $p \equiv 2$. Global asymptotic stability can then no longer be expected, and should be replaced by local stability, discussed in Section 4. Such a framework requires however some stronger assumptions on the variable exponents p and q. Indeed, we suppose $p_+ < q_-$ and $q \leqslant p^*$ in Ω, see Theorem 4.1. Furthermore, because of the delicacy of the problem, only the non-degenerate case $a > 0$ is treated, assuming also $\mu_0 > 0$.

Further applications to more general models are given in Section 5. In particular, we study the problem

$$\begin{cases} u_{tt} - M\big(\mathscr{I}u(t)\big)\Delta_{p(x)}u - g(t)\Delta_{p(x)}u_t + Q(t,x,u,u_t) + f(x,u) = 0 & \text{in } \mathbb{R}_0^+ \times \Omega, \\ u(t,x) = 0 & \text{on } \mathbb{R}_0^+ \times \partial\Omega, \end{cases} \tag{1.7}$$

where $g \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+ \to \mathbb{R}_0^+)$, which involves higher dissipation terms, interesting from an applicative point of view, and includes the previous model (1.1) when $g \equiv 0$. In Theorem 5.3 we deal with global asymptotic stability while in Theorem 5.4 the question of local stability is treated, essentially following the guide lines of Sections 3 and 4, respectively.

2. Preliminaries

We consider the following setting. Let

$$C_+(\overline{\Omega}) = \Big\{h \in C(\overline{\Omega}):\ \min_{x\in\overline{\Omega}} h(x) > 1\Big\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h_+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h_- = \inf_{x\in\Omega} h(x).$$

Fix $p \in C_+(\overline{\Omega})$. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega) = [L^{p(\cdot)}(\Omega)]^N$ is the real vector space of all the measurable vector-valued functions $u : \Omega \to \mathbb{R}^N$ such that $\int_\Omega |u(x)|^{p(x)}\,dx$ is finite. This space, endowed with the so-called *Luxemburg norm*

$$\|u\|_{p(\cdot)} = \inf\Bigg\{\lambda > 0:\ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\Bigg\},$$

is a separable and reflexive Banach space. For basic properties of the variable exponent Lebesgue spaces we refer to [16]. Since $0 < |\Omega| < \infty$, if p, q are variable exponents in $C_+(\overline{\Omega})$ such that $p \leqslant q$ in Ω, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous [16, Theorem 2.8].

Let $L^{p'(\cdot)}(\Omega)$ be the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise that is, $1/p(x) + 1/p'(x) = 1$ [16, Corollary 2.7]. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder type inequality, see [16, Theorem 2.1],

$$\left|\int_\Omega (u,v)\,dx\right| \leqslant r_p \|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}, \qquad r_p := \frac{1}{p_-} + \frac{1}{p'_-}, \tag{2.1}$$

is valid.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If $(u_n)_n, u \in L^{p(\cdot)}(\Omega)$, then the following relations hold

$$\|u\|_{p(\cdot)} < 1\ (=1;\ >1) \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1\ (=1;\ >1), \tag{2.2}$$

$$\|u\|_{p(\cdot)} > 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p_-} \leqslant \rho_{p(\cdot)}(u) \leqslant \|u\|_{p(\cdot)}^{p_+}, \tag{2.3}$$

$$\|u\|_{p(\cdot)} < 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p_+} \leqslant \rho_{p(\cdot)}(u) \leqslant \|u\|_{p(\cdot)}^{p_-}, \tag{2.4}$$

$$\|u_n - u\|_{p(\cdot)} \to 0 \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u_n - u) \to 0, \tag{2.5}$$

when $p_+ < \infty$. For a proof of these facts see [16].

If $p \in C_+(\overline{\Omega})$, the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega) = [W^{1,p(\cdot)}(\Omega)]^N$, consisting of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional Jacobian matrix Du exists almost everywhere and belongs to $[L^{p(\cdot)}(\Omega)]^{nN}$, endowed with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|Du\|_{p(\cdot)},$$

is a separable and reflexive Banach space. As shown by Zhikov [23,24], the smooth functions are in general not dense in $W^{1,p(\cdot)}(\Omega)$, but if the variable exponent p in $C_+(\overline{\Omega})$ is logarithmic Hölder continuous, that is

$$\left|p(x) - p(y)\right| \leqslant -\frac{M}{\log|x-y|} \quad \text{for all } x, y \in \Omega \text{ such that } |x-y| \leqslant 1/2, \tag{2.6}$$

then the smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$, and so the Sobolev space with zero boundary values, denoted by $W_0^{1,p(\cdot)}(\Omega) = [W_0^{1,p(\cdot)}(\Omega)]^N$, as the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{1,p(\cdot)}$, is meaningful, see [12,15]. Furthermore, if $p \in C_+(\overline{\Omega})$ satisfies (2.6), then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$, that is $H_0^{1,p(\cdot)}(\Omega) = W_0^{1,p(\cdot)}(\Omega)$ [13, Theorem 3.3]. Since Ω is an open bounded set and $p \in C_+(\overline{\Omega})$ satisfies (2.6), the $p(\cdot)$-*Poincaré inequality*

$$\|u\|_{p(\cdot)} \leqslant C\|Du\|_{p(\cdot)}$$

holds for all $u \in W_0^{1,p(\cdot)}(\Omega)$, where C depends on p, $|\Omega|$, $\operatorname{diam}(\Omega)$, n and N [13, Theorem 4.1], and so

$$\|u\| = \|Du\|_{p(\cdot)}$$

is an equivalent norm in $W_0^{1,p(\cdot)}(\Omega)$. Of course also the norm

$$\|u\|_{p(\cdot)} = \sum_{i=1}^n \|\partial_{x_i} u\|_{p(\cdot)}$$

is an equivalent norm in $W_0^{1,p(\cdot)}(\Omega)$. Hence $W_0^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space.

Note that if $p_+ < n$ and $h \in C(\overline{\Omega})$, with $1 \leqslant h(x) < p^*(x)$ for all $x \in \overline{\Omega}$, where

$$p^*(x) = \frac{np(x)}{n - p(x)},$$

then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega)$ is compact and continuous, see [11, Theorem 2.3, case $m=1$]; while the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ is continuous, see [14, Proposition 4.2] by virtue of (2.6) and [5, Corollary 5.3] when also $p_- > 1$. Furthermore, since $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ continuously when $p, q \in C_+(\overline{\Omega})$ are such that $p \leqslant q$ in Ω, we have $W^{1,q(\cdot)}(\Omega) \hookrightarrow W^{1,p(\cdot)}(\Omega)$, and in particular $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow W_0^{1,p_-}(\Omega) \hookrightarrow L^{p_-^*}(\Omega)$.

Clearly the canonical main case is when $1 < p_- \leqslant p_+ < n$, but, in order to have the useful embeddings for our aims, we actually assume, throughout the paper, that $p \in C_+(\overline{\Omega})$ and (2.6), together with

$$2n/(n+2) \leqslant p_- \leqslant p_+ < n, \tag{2.7}$$

hold. Details, extensions and further references about $p(x)$-spaces and embeddings can be found in [5] and [8–16].

Let $h \in C(\overline{\Omega})$ be such that $1 \leqslant h \leqslant p^*$ in Ω, and denote with $\lambda_{h(\cdot)}$ the Sobolev constant of the continuous embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega)$, that is

$$\|u\|_{h(\cdot)} \leqslant \lambda_{h(\cdot)}\|Du\|_{p(\cdot)} \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega), \tag{2.8}$$

where $\lambda_{h(\cdot)}$ depends on n, h, p, $|\Omega|$ and N, see [14, Proposition 4.2] and [16, Theorem 2.8]. Of course in (2.8) we can have $h \equiv 1$, since $|\Omega| < \infty$, and $h \equiv 2$ by (2.7).

Throughout the paper we endow the usual Lebesgue space $L^2(\Omega)$ with the canonical norm

$$\|\varphi\|_2 = \left(\int_\Omega \left|\varphi(x)\right|^2 dx\right)^{1/2},$$

with elementary bracket pairing

$$\langle\varphi,\psi\rangle \equiv \int_\Omega (\varphi,\psi)\,dx,$$

for all φ, ψ such that $(\varphi,\psi)\in L^1(\Omega)$. As stated above, for further simplicity, we set

$$L^{p(\cdot)}(\Omega)=\left[L^{p(\cdot)}(\Omega)\right]^N,\qquad X=W_0^{1,p(\cdot)}(\Omega)=\left[W_0^{1,p(\cdot)}(\Omega)\right]^N,$$

endowed with the norms $\|\cdot\|_{p(\cdot)}$ and $\|u\|=\|Du\|_{p(\cdot)}$, respectively. Now introduce

$$K'=C\left(\mathbb{R}_0^+\to X\right)\cap C^1\left(\mathbb{R}_0^+\to L^2(\Omega)\right)\quad\text{and}\quad K=\left\{\phi\in K'\colon\ E\phi \text{ is locally bounded on } \mathbb{R}_0^+\right\},$$

where $E\phi$ is the *total energy of the field* ϕ given by

$$E\phi(t)=\frac{1}{2}\left\|\phi_t(t,\cdot)\right\|_2^2+a\mathscr{I}\phi(t)+b\left[\mathscr{I}\phi(t)\right]^\gamma+\mathscr{F}\phi(t), \tag{2.9}$$

and $\mathscr{I}\phi$ is the $p(\cdot)$*-Dirichlet energy integral*

$$\mathscr{I}\phi=\mathscr{I}\phi(t)=\int_\Omega \frac{|D\phi(t,x)|^{p(x)}}{p(x)}\,dx$$

while $\mathscr{F}\phi$ is the *potential energy of the field*, defined by

$$\mathscr{F}\phi=\mathscr{F}\phi(t)=\int_\Omega F\left(x,\phi(t,x)\right)dx.$$

In writing $E\phi$ and $\mathscr{F}\phi$ we make the tacit agreement that $\mathscr{F}\phi$ is *well-defined*, namely that $F(\cdot,\phi(t,\cdot))\in L^1(\Omega)$ for all $t\in\mathbb{R}_0^+$.

We can now give our principal definition: a *strong solution of* (1.1) is a function $u\in K$ satisfying the following two conditions:

(A) *Distribution identity*

$$\langle u_t,\phi\rangle]_0^t=\int_0^t\left\{\langle u_t,\phi_t\rangle-M\left(\mathscr{I}u(t)\right)\int_\Omega |Du|^{p(x)-2}(Du,D\phi)\,dx-\left\langle Q(\tau,\cdot,u,u_t)+f(\cdot,u),\phi\right\rangle\right\}d\tau$$

for all $t\in\mathbb{R}_0^+$ and $\phi\in K$.

(B) *Conservation law*

(i) $\mathscr{D}u:=\left\langle Q(t,\cdot,u,u_t),u_t\right\rangle\in L^1_{\mathrm{loc}}\left(\mathbb{R}_0^+\right)$,

(ii) $t\mapsto Eu(t)+\displaystyle\int_0^t \mathscr{D}u(\tau)\,d\tau$ is non-increasing in $\mathbb{R}_0^+$.

Conditions (B)(ii) and (1.3) imply that *Eu is non-increasing in* $\mathbb{R}_0^+$.

We make the following natural hypothesis on f and Q, and remind that the variable exponent p is assumed to verify $2n/(n+2)\leqslant p_-\leqslant p(x)\leqslant p_+<n$, for all $x\in\Omega$.

(H) *Conditions* (1.4) *and* (1.6) *hold and there exist a variable exponent $q\in C_+(\overline{\Omega})$, with $q\geqslant p$ in Ω, and a positive constant κ such that*

(a) $\left|f(x,u)\right|\leqslant\kappa\left(1+|u|^{q(x)-1}\right)$ *for all* $(x,u)\in\Omega\times\mathbb{R}^N$.

Moreover, if there exists $y\in\Omega$ such that $q(y)>p^*(y)$, then f verifies (a) *and*

(b) $\big(f(x,u),u\big) \geqslant \kappa_1|u|^{q(x)} - \kappa_2|u|^{1/q(x)} - \kappa_3|u|^{p^*(x)}$ *for all* $(x,u) \in \Omega \times \mathbb{R}^N$

for appropriate constants $\kappa_1 > 0$, $\kappa_2, \kappa_3 \geqslant 0$.

When $f \equiv 0$, then (H)(a) holds for any fixed $q \in C_+(\overline{\Omega})$, with $p \leqslant q \leqslant p^*$ in Ω, so that (H)(b) is unnecessary.

(AS) *Condition* (1.3) *holds and there are constant exponents* m, r *satisfying*

$$p_- \leqslant m < r \leqslant s, \quad s = \max\{q_-, p_-^*\},$$

where m' *and* r' *are the Hölder conjugates of* m *and* r, *and non-negative continuous functions* $d_1 = d_1(t,x)$, $d_2 = d_2(t,x)$, *such that for all arguments* t, x, u, v,

(a) $$\big|Q(t,x,u,v)\big| \leqslant d_1(t,x)^{1/m}\big(Q(t,x,u,v),v\big)^{1/m'} + d_2(t,x)^{1/r}\big(Q(t,x,u,v),v\big)^{1/r'},$$

and the following functions δ_1 *and* δ_2 *are well-defined*

$$\delta_1(t) = \big\|d_1(t,\cdot)\big\|_{s/(s-m)}, \qquad \delta_2(t) = \begin{cases} \|d_2(t,\cdot)\|_{s/(s-r)}, & \text{if } r < s, \\ \|d_2(t,\cdot)\|_\infty, & \text{if } r = s. \end{cases}$$

Moreover, there are functions $\sigma = \sigma(t)$, $\omega = \omega(\tau)$ *such that*

(b) $\big(Q(t,x,u,v),v\big) \geqslant \sigma(t)\omega\big(|v|\big)$ for all arguments t, x, u, v,

where $\omega \in C(\mathbb{R}_0^+ \to \mathbb{R}_0^+)$ *is such that*

$$\omega(0) = 0, \qquad \omega(\tau) > 0 \quad \text{for } 0 < \tau < 1, \qquad \omega(\tau) = \tau^2 \quad \text{for } \tau \geqslant 1,$$

while $\sigma \geqslant 0$ *and* $\sigma^{1-\wp} \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+)$ *for some exponent* $\wp > 1$.

3. Global asymptotic stability

Theorem 3.1. *Let* (H) *and* (AS) *hold. Suppose there exists a function* k *satisfying either*

$$k \in CBV\big(\mathbb{R}_0^+ \to \mathbb{R}_0^+\big) \quad \textit{and} \quad k \notin L^1\big(\mathbb{R}_0^+\big) \quad \textit{or} \tag{3.1}$$

$$k \in W^{1,1}_{\mathrm{loc}}\big(\mathbb{R}_0^+ \to \mathbb{R}_0^+\big), \quad k \not\equiv 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{\int_0^t |k'(\tau)|\,d\tau}{\int_0^t k(\tau)\,d\tau} = 0. \tag{3.2}$$

Assume finally

$$\liminf_{t\to\infty} \mathscr{A}\big(k(t)\big)\left(\int_0^t k(\tau)\,d\tau\right)^{-1} < \infty, \tag{3.3}$$

where

$$\mathscr{A}\big(k(t)\big) = \mathscr{B}\big(k(t)\big) + \left(\int_0^t \sigma^{1-\wp}k^{\wp}\,d\tau\right)^{1/\wp},$$

$$\mathscr{B}\big(k(t)\big) = \left(\int_0^t \delta_1 k^m\,d\tau\right)^{1/m} + \left(\int_0^t \delta_2 k^r\,d\tau\right)^{1/r}. \tag{3.4}$$

Then along any strong solution u *of* (1.1) *we have*

$$\lim_{t\to\infty} Eu(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty}\big\{\big\|u_t(t,\cdot)\big\|_2 + \big\|Du(t,\cdot)\big\|_{p(\cdot)}\big\} = 0. \tag{3.5}$$

The integral condition (3.3) prevents the damping term Q being either too small (*underdamping*) or too large (*overdamping*) as $t \to \infty$ and was introduced by Pucci and Serrin in [19], see also [20] and [21].

Before proving Theorem 3.1 we give two preliminary lemmas under conditions (H) and (AS)(a) which make the definition of strong solution meaningful.

Lemma 3.2. *Let u be a strong solution of* (1.1)*. Then the non-increasing energy function Eu verifies in $\mathbb{R}_0^+$*

$$Eu \geqslant \begin{cases} \frac{1}{2}\|u_t\|_2^2 + a(1-\frac{\mu}{\mu_0})\mathscr{I}u(t) + b[\mathscr{I}u(t)]^\gamma, & \text{if } \mu_0 > 0, \\ \frac{1}{2}\|u_t\|_2^2 + a\mathscr{I}u(t) + b[\mathscr{I}u(t)]^\gamma, & \text{if } \mu_0 = 0. \end{cases} \tag{3.6}$$

Moreover

$$\begin{aligned} &\|u\|_2, \|u_t\|_2, \|Du\|_{p(\cdot)}, \|u\|_{q(\cdot)}, \|u\|_{p^*(\cdot)}, M(\mathscr{I}u) \in L^\infty(\mathbb{R}_0^+), \\ &\mathscr{D}u = \langle Q(t,x,u,u_t), u_t\rangle \in L^1(\mathbb{R}_0^+). \end{aligned} \tag{3.7}$$

Proof. *Case* 1: $\mu_0 > 0$. By (1.6) we have $F(x,u) \geqslant -a\mu|u|^{p(x)}/p(x)$ in $\Omega \times \mathbb{R}^N$, so that along the strong solution $u = u(t,x)$ of (1.1)

$$\mathscr{F}u(t) \geqslant -a\mu \int_\Omega \frac{|u(t,x)|^{p(x)}}{p(x)}\,dx \geqslant -a\frac{\mu}{\mu_0}\mathscr{I}u(t).$$

Hence (3.6) follows at once.

In order to prove conditions (3.7) first note that Eu is bounded above by $Eu(0)$, as well as $\mathscr{I}u$, by the definition of Eu and the fact that $a+b>0$. Hence $M(\mathscr{I}u) \in L^\infty(\mathbb{R}_0^+)$. Moreover $\mathscr{I}u(t) \geqslant \rho_{p(\cdot)}(Du(t,\cdot))/p_+$ and

$$\|Du\|_{p(\cdot)} \leqslant \max\{[\rho_{p(\cdot)}(Du)]^{1/p_-}, [\rho_{p(\cdot)}(Du)]^{1/p_+}\}, \tag{3.8}$$

so that $\|u_t\|_2, \|Du\|_{p(\cdot)} \in L^\infty(\mathbb{R}_0^+)$ by (3.6). Hence $\|u\|_2 \in L^\infty(\mathbb{R}_0^+)$, since $X = W_0^{1,p(\cdot)}(\Omega)$ is continuously embedded in $L^2(\Omega)$ by (2.7), that is (2.8) holds for $h \equiv 2$. Furthermore, when $p \leqslant q \leqslant p^*$ in Ω, since the Sobolev embeddings $X \hookrightarrow L^{q(\cdot)}(\Omega)$ and $X \hookrightarrow L^{p^*(\cdot)}(\Omega)$ are continuous by (2.6), as noted in (2.8), also $\|u\|_{q(\cdot)}, \|u\|_{p^*(\cdot)} \in L^\infty(\mathbb{R}_0^+)$.

Let us now consider the case in which there exists $y \in \Omega$ such that $q(y) > p^*(y)$. Hence, by (H)(b) we get for all $\phi \in K$

$$\begin{aligned} F(x,\phi) &= \int_0^1 (f(x,\tau\phi),\phi)\,d\tau \\ &\geqslant \int_0^1 (\kappa_1|\phi|^{q(x)}\tau^{q(x)-1} - \kappa_2|\phi|^{1/q(x)}\tau^{-1/q'(x)} - \kappa_3|\phi|^{p^*(x)}\tau^{p^*(x)-1})\,d\tau \\ &= \frac{\kappa_1}{q(x)}|\phi|^{q(x)} - q(x)\kappa_2|\phi|^{1/q(x)} - \frac{\kappa_3}{p^*(x)}|\phi|^{p^*(x)}, \end{aligned}$$

and, since $\kappa_1 > 0$, we then have along the solution $u \in K$

$$\rho_{q(\cdot)}(u(t,\cdot)) \leqslant \frac{q_+}{\kappa_1}\left(\mathscr{F}u(t) + K\max\{\|u(t,\cdot)\|_1^{1/q_-}, \|u(t,\cdot)\|_1^{1/q_+}\} + \frac{\kappa_3}{p_-^*}\rho_{p^*(\cdot)}(u(t,\cdot))\right), \tag{3.9}$$

where $K = (1/q_- + 1/q'_-)q_+\kappa_2\max\{|\Omega|^{1/q'_-}, |\Omega|^{1/q'_+}\}$. Now observe that $\mathscr{F}u$ is bounded above being $Eu(t) \leqslant Eu(0)$ and also below by (1.6). Moreover, since $\|u_t\|_2, \|Du\|_{p(\cdot)} \in L^\infty(\mathbb{R}_0^+)$, we also get $\|u\|_1 \in L^\infty(\mathbb{R}_0^+)$ by (2.8) when $h \equiv 1$, as well as

$$\rho_{p^*(\cdot)}(u(t,\cdot)) \leqslant \max\{\|u(t,\cdot)\|_{p^*(\cdot)}^{p_-^*}\|u(t,\cdot)\|_{p^*(\cdot)}^{p_+^*}\} \leqslant \text{constant},$$

since again $\|u\|_{p^*(\cdot)} \in L^\infty(\mathbb{R}_0^+)$ by (2.8) with $h = p^*$. Therefore $\rho_{q(\cdot)}(u) \in L^\infty(\mathbb{R}_0^+)$. Hence $\|u\|_{q(\cdot)} \in L^\infty(\mathbb{R}_0^+)$ by (2.3)–(2.4) and (3.9). This completes the proof of $(3.7)_1$.

Case 2: $\mu_0 = 0$. The situation is much simpler since the external force f is of restoring type. It follows that $\mathscr{F}u(t) \geqslant 0$ for all $t \in \mathbb{R}_0^+$ and (3.6) follows at once. Hence $Eu \geqslant 0$ since all the three terms in the definition of Eu, with $u \in K$, are non-negative, and clearly bounded by $Eu(0)$. Hence $\|u_t\|_2 \in L^\infty(\mathbb{R}_0^+)$ and from the fact that

$$a\frac{\rho_{p(\cdot)}(Du(t,\cdot))}{p_+} + b\left[\frac{\rho_{p(\cdot)}(Du(t,\cdot))}{p_+}\right]^\gamma \leqslant a\mathscr{I}u(t) + b\big[\mathscr{I}u(t)\big]^\gamma,$$

and the right-hand side is bounded by $Eu(t) \leqslant Eu(0)$, in turn $M(\mathscr{I}u)$ and $\rho_{p(\cdot)}(Du(t,\cdot))$ are also bounded in $\mathbb{R}_0^+$ since $a+b>0$ and $\gamma>1$. Hence $\|Du\|_{p(\cdot)} \in L^\infty(\mathbb{R}_0^+)$ by (3.8). From now on the proof can proceed as in the previous case word by word. Therefore $(3.7)_1$ is valid also in the case $\mu_0 = 0$.

Property $(3.7)_2$ follows at once by (B)(ii) since in $\mathbb{R}_0^+$

$$0 \leqslant \int_0^t \mathscr{D}u(\tau)\,d\tau \leqslant Eu(0)$$

by (1.3). □

By (B)(ii) and Lemma 3.2 it is clear that there exists $l \geqslant 0$ such that

$$\lim_{t\to\infty} Eu(t) = l. \tag{3.10}$$

Lemma 3.3. *Let u be a strong solution of* (1.1) *and suppose $l > 0$ in* (3.10). *Then there exists a constant $\alpha = \alpha(l) > 0$ such that in $\mathbb{R}_0^+$*

$$\big\|u_t(t,\cdot)\big\|_2^2 + a\rho_{p(\cdot)}\big(Du(t,\cdot)\big) + \frac{b}{p_+^{\gamma-1}}\big[\rho_{p(\cdot)}\big(Du(t,\cdot)\big)\big]^\gamma + \big\langle f(\cdot,u),u\big\rangle \geqslant \alpha. \tag{3.11}$$

Proof. Since $Eu(t) \geqslant l$ for all $t \in \mathbb{R}_0^+$ it follows that

$$\big\|u_t(t,\cdot)\big\|_2^2 + a\rho_{p(\cdot)}\big(Du(t,\cdot)\big) + b\big[\rho_{p(\cdot)}\big(Du(t,\cdot)\big)\big]^\gamma \geqslant \eta(l - \mathscr{F}u) \quad \text{in } \mathbb{R}_0^+,$$

where $\eta = \min\{2, p_-\} > 1$. Let

$$J_1 = \big\{t \in \mathbb{R}_0^+\colon\ \mathscr{F}u(t) \leqslant l/\eta'\big\}, \qquad J_2 = \big\{t \in \mathbb{R}_0^+\colon\ \mathscr{F}u(t) > l/\eta'\big\},$$

where $\eta' = \eta/(\eta-1)$ is the Hölder conjugate of η. For $t \in J_1$

$$\|u_t\|^2 + a\rho_{p(\cdot)}\big(Du(t,\cdot)\big) + b\big[\rho_{p(\cdot)}\big(Du(t,\cdot)\big)\big]^\gamma \geqslant l,$$

so that

$$\|u_t\|^2 + a\rho_{p(\cdot)}\big(Du(t,\cdot)\big) + \frac{b}{p_+^{\gamma-1}}\big[\rho_{p(\cdot)}\big(Du(t,\cdot)\big)\big]^\gamma \geqslant \frac{l}{p_+^{\gamma-1}}. \tag{3.12}$$

Before dividing the proof into two parts, we observe that in $\mathbb{R}_0^+$

$$|\mathscr{F}u| \leqslant \kappa\big[\|u\|_1 + \rho_{q(\cdot)}(u)\big] \leqslant \kappa\big(\|u\|_1 + \max\big\{\|u\|_{q(\cdot)}^{q_-}, \|u\|_{q(\cdot)}^{q_+}\big\}\big) \tag{3.13}$$

by (H)(a) and (2.3)–(2.4).

Case 1: $q(x) \leqslant p^*(x)$ *for all* $x \in \Omega$. Let us first consider the case in which $\mu_0 > 0$. By Lemma 3.1 of [10] we also have $\overline{\mu}_0 > 0$, where

$$\overline{\mu}_0 \int_\Omega \big|\varphi(x)\big|^{p(x)}\,dx \leqslant \int_\Omega \big|D\varphi(x)\big|^{p(x)}\,dx$$

for all $\varphi \in X$. Hence by (1.6) and the fact that $\overline{\mu}_0 \geqslant \mu_0 p_-/p_+$, since $u \in K$,

$$\langle f(\cdot,u),u\rangle \geqslant -a\mu\int_\Omega \big|u(t,x)\big|^{p(x)}\,dx \geqslant -a\frac{\mu}{\overline{\mu}_0}\int_\Omega \big|Du(t,x)\big|^{p(x)}\,dx \geqslant -a\frac{\mu}{\mu_0}\frac{p_+}{p_-}\rho_{p(\cdot)}\big(Du(t,\cdot)\big). \tag{3.14}$$

Denoting by $\mathscr{L}u$ the left-hand side of (3.11) and using (3.12) and (3.14), we have for all $t\in J_1$

$$\begin{aligned}\mathscr{L}u(t) &\geqslant a\left(1-\frac{\mu}{\mu_0}\cdot\frac{p_+}{p_-}\right)\rho_{p(\cdot)}\big(Du(t,\cdot)\big)+\big\|u_t(t,\cdot)\big\|_2^2+\frac{b}{p_+^{\gamma-1}}\big[\rho_{p(\cdot)}\big(Du(t,\cdot)\big)\big]^\gamma\\ &\geqslant \left(1-\frac{\mu}{\mu_0}\cdot\frac{p_+}{p_-}\right)\frac{l}{p_+^{\gamma-1}}+\frac{\mu}{\mu_0}\cdot\frac{p_+}{p_-}\left(\big\|u_t(t,\cdot)\big\|_2^2+\frac{b}{p_+^{\gamma-1}}\big[\rho_{p(\cdot)}\big(Du(t,\cdot)\big)\big]^\gamma\right)\\ &\geqslant \left(1-\frac{\mu}{\mu_0}\cdot\frac{p_+}{p_-}\right)\frac{l}{p_+^{\gamma-1}}.\end{aligned}$$

Next consider $t\in J_2$. By (3.13), (2.8) and (2.2)–(2.4) we have in $\mathbb{R}_0^+$

$$|\mathscr{F}u|\leqslant C\big(\|Du\|_{p(\cdot)}+\max\big\{\|Du\|_{p(\cdot)}^{q_-},\|Du\|_{p(\cdot)}^{q_+}\big\}\big), \tag{3.15}$$

for an appropriate constant $C>0$, depending on κ, λ_1, $\lambda_{q(\cdot)}$ introduced in (2.8) and p. Hence in J_2

$$\frac{l}{\eta'}<\mathscr{F}u(t)\leqslant 2C\begin{cases}\|Du(t,\cdot)\|_{p(\cdot)}, & \text{if } \|Du(t,\cdot)\|_{p(\cdot)}\leqslant 1,\\ \|Du(t,\cdot)\|_{p(\cdot)}^{q_+}, & \text{if } \|Du(t,\cdot)\|_{p(\cdot)}>1,\end{cases} \tag{3.16}$$

that is

$$\big\|Du(t,\cdot)\big\|_{p(\cdot)}\geqslant \min\left\{\frac{l}{2C\eta'},\left(\frac{l}{2C\eta'}\right)^{1/q_+}\right\}=C_2(l)>0.$$

By (3.14) for all $t\in J_2$

$$\begin{aligned}\mathscr{L}u(t) &\geqslant a\left(1-\frac{\mu}{\mu_0}\cdot\frac{p_+}{p_-}\right)\rho_{p(\cdot)}\big(Du(t,\cdot)\big)+\frac{b}{p_+^{\gamma-1}}\big[\rho_{p(\cdot)}\big(Du(t,\cdot)\big)\big]^\gamma\\ &\geqslant a\left(1-\frac{\mu}{\mu_0}\cdot\frac{p_+}{p_-}\right)\min\big\{\big\|Du(t,\cdot)\big\|_{p(\cdot)}^{p_-},\big\|Du(t,\cdot)\big\|_{p(\cdot)}^{p_+}\big\}\\ &\quad+\frac{b}{p_+^{\gamma-1}}\big[\min\big\{\big\|Du(t,\cdot)\big\|_{p(\cdot)}^{p_-},\big\|Du(t,\cdot)\big\|_{p(\cdot)}^{p_+}\big\}\big]^\gamma.\end{aligned}$$

Denoting by $C_3=C_3(l)$ the positive number $\min\{C_2^{p_-}(l),C_2^{p_+}(l)\}$, then in J_2

$$\mathscr{L}u\geqslant a\left(1-\frac{\mu}{\mu_0}\cdot\frac{p_+}{p_-}\right)C_3+\frac{b}{p_+^{\gamma-1}}C_3^\gamma.$$

Therefore (3.11) holds with

$$\alpha=\alpha(l)=\left(1-\frac{\mu}{\mu_0}\cdot\frac{p_+}{p_-}\right)\min\left\{\frac{l}{p_+^{\gamma-1}},aC_3\right\}+\frac{b}{p_+^{\gamma-1}}C_3^\gamma, \tag{3.17}$$

provided that either $a\neq 0$ or $J_2\neq\emptyset$, being $a+b>0$.

Now, if $a=0$ and $J_2=\emptyset$, then (1.6) reduces to $(f(x,u),u)\geqslant 0$, and so (3.11) holds with $\alpha=l/p_+^{\gamma-1}>0$.

When $\mu_0=0$, that is when also $\mu=0$ in (1.6), then the proof simplifies and (3.11) holds with

$$\alpha=\alpha(l)=\min\left\{\frac{l}{p_+^{\gamma-1}},aC_3\right\}+\frac{b}{p_+^{\gamma-1}}C_3^\gamma>0,$$

since $l>0$ and $a+b>0$.

Case 2: *There exists* $y\in\Omega$ *such that* $q(y)>p^*(y)$. As before we first suppose $\mu_0>0$. Using $(3.13)_1$, (H)(b) and Hölder's inequality, we have for $t\in J_2$, since $\kappa_1>0$,

$$\frac{l}{\eta'} < \mathscr{F}u(t) \leqslant \kappa_0\big[\big\langle f\big(\cdot, u(t,\cdot)\big), u(t,\cdot)\big\rangle + \kappa_1\big\|u(t,\cdot)\big\|_1 + \tilde{\kappa}_2 \max\big\{\big\|u(t,\cdot)\big\|_1^{1/q_-}, \big\|u(t,\cdot)\big\|_1^{1/q_+}\big\} + \kappa_3\rho_{p^*(\cdot)}\big(u(t,\cdot)\big)\big],$$

where $\kappa_0 = \kappa/\kappa_1 > 0$ and $\tilde{\kappa}_2 = (1/q_- + 1/q'_-)\kappa_2 \max\{|\Omega|^{1/q'_-}, |\Omega|^{1/q'_+}\}$. Therefore, by (2.8), when $h \equiv 1$,

$$\big\langle f(\cdot,u),u\big\rangle + c_1\|Du\|_{p(\cdot)} + c_2 \max\big\{\|Du\|_{p(\cdot)}^{1/q_-}, \|Du\|_{p(\cdot)}^{1/q_+}\big\} + c_3 \max\big\{\|Du\|_{p(\cdot)}^{p_-^*}, \|Du\|_{p(\cdot)}^{p_+^*}\big\} > l/\kappa_0\eta',$$

where $c_1 = \kappa_1\lambda_1 > 0$, $c_2 = \tilde{\kappa}_2 \max\{\lambda_1^{1/q_-}, \lambda_1^{1/q_+}\} \geqslant 0$ and $c_3 = \kappa_3 \max\{\lambda_{p^*(\cdot)}^{p_-^*}, \lambda_{p^*(\cdot)}^{p_+^*}\} \geqslant 0$. Hence for $t \in J_2$ and $\langle f(\cdot,u(t,\cdot)), u(t,\cdot)\rangle \geqslant 0$, then

$$\text{either} \quad \big\langle f\big(\cdot,u(t,\cdot)\big), u(t,\cdot)\big\rangle \geqslant l/2\kappa_0\eta' \quad \text{or} \quad \big\|Du(t,\cdot)\big\|_{p(\cdot)} \geqslant c_4, \tag{3.18}$$

where $c_4 = c_4(l,\kappa_0,\eta) > 0$ is an appropriate constant, arising when

$$c_1\|Du\|_{p(\cdot)} + c_2 \max\big\{\|Du\|_{p(\cdot)}^{1/q_-}, \|Du\|_{p(\cdot)}^{1/q_+}\big\} + c_3 \max\big\{\|Du\|_{p(\cdot)}^{p_-^*}, \|Du\|_{p(\cdot)}^{p_+^*}\big\} \geqslant l/2\kappa_0\eta'$$

at the time t. On the other hand, if $t \in J_2$ and $\langle f(\cdot,u(t,\cdot)), u(t,\cdot)\rangle < 0$, then $\|Du(t,\cdot)\|_{p(\cdot)} \geqslant c_5$, where $c_5 \geqslant c_4$ is an appropriate number arising from

$$c_1\|Du\|_{p(\cdot)} + c_2 \max\big\{\|Du\|_{p(\cdot)}^{1/q_-}, \|Du\|_{p(\cdot)}^{1/q_+}\big\} + c_3 \max\big\{\|Du\|_{p(\cdot)}^{p_-^*}, \|Du\|_{p(\cdot)}^{p_+^*}\big\} \geqslant l/\kappa_0\eta'.$$

Now, when $\mu_0 > 0$ in (1.6), putting $c_6 = \min\{c_5^{p_-}, c_5^{p_+}\}$, the conclusion (3.11) holds, with

$$\alpha = \min\left\{\left(1 - \frac{\mu}{\mu_0}\frac{p_+}{p_-}\right)\frac{l}{p_+^{\gamma-1}}, a\left(1 - \frac{\mu}{\mu_0}\frac{p_+}{p_-}\right)c_6 + \frac{b}{p_+^{\gamma-1}}c_6^{\gamma}, ac_4 + \frac{b}{p_+^{\gamma-1}}c_4^{\gamma}, \frac{l}{2\kappa_0\eta'}\right\} > 0,$$

since $l > 0$, $\mu \in [0, \mu_0 p_-/p_+)$, $c_4 > 0$, $c_6 > 0$ and $a + b > 0$.

While if $\mu_0 = 0$, hence $\mu = 0$ in (1.6), then (3.11) holds, with

$$\alpha = \min\left\{\frac{l}{p_+^{\gamma-1}}, ac_6 + \frac{b}{p_+^{\gamma-1}}c_6^{\gamma}, ac_4 + \frac{b}{p_+^{\gamma-1}}c_4^{\gamma}, \frac{l}{2\kappa_0\eta'}\right\} > 0.$$

This completes the proof. □

Proof of Theorem 3.1. The approach is analogous to the one of [3, Theorem 3.1], which is related to the main ideas of the proof of [20, Theorem 3.1] and [21, Theorem 1]. Initially we treat case (3.1) in the simpler situation in which k is not only $CBV(\mathbb{R}_0^+)$, but also of class $C^1(\mathbb{R}_0^+)$. Suppose, for contradiction that $l > 0$ in (3.10). Define a Lyapunov function by

$$V(t) = k(t)\langle u, u_t\rangle = \langle u_t, \phi\rangle, \qquad \phi = k(t)u.$$

Since $k \in C^1(\mathbb{R}_0^+)$ and $\phi_t = k'u + ku_t$, it is clear that $\phi \in K$. Thus, by the distribution identity (A) in Section 2, we get for any $t \geqslant T \geqslant 0$

$$V(\tau)]_T^t = \int_T^t \big\{k'\langle u,u_t\rangle + 2k\|u_t\|_2^2 - k\big[\|u_t\|_2^2 + M\big(\mathscr{I}u(t)\big)\rho_{p(\cdot)}\big(Du(t,\cdot)\big) + \big\langle f(\cdot,u),u\big\rangle\big]\big\}\,d\tau - \int_T^t k\big\langle Q(\tau,\cdot,u,u_t),u\big\rangle\,d\tau. \tag{3.19}$$

We now estimate the right-hand side of (3.19). First note that

$$\sup_{\mathbb{R}_0^+}\big|\big\langle u(t,\cdot), u_t(t,\cdot)\big\rangle\big| \leqslant \sup_{\mathbb{R}_0^+}\big\|u(t,\cdot)\big\|_2 \cdot \big\|u_t(t,\cdot)\big\|_2 = U < \infty \tag{3.20}$$

by (2.7) and (3.7) of Lemma 3.2, that is $\|u\|_2, \|u_t\|_2 \in L^\infty(\mathbb{R}_0^+)$. Now, using Lemma 3.3

$$-\int_T^t k\{\|u_t\|_2^2 + M(\mathscr{I}u(t))\rho_{p(\cdot)}(Du(t,\cdot)) + \langle f(\cdot,u),u\rangle\}\,d\tau \leqslant -\alpha\int_T^t k\,d\tau, \tag{3.21}$$

and by Lemmas 3.2 and 3.3 of [21]

$$-\int_T^t k\langle Q(\tau,\cdot,u,u_t),u\rangle\,d\tau \leqslant \varepsilon_1(T)\mathscr{B}(k(t)), \tag{3.22}$$

$$\int_T^t k\|u_t\|_2^2\,d\tau \leqslant \theta\int_T^t k\,d\tau + \varepsilon_2(T)C(\theta)\left(\int_0^t \sigma^{1-\wp}k^{\wp}\,d\tau\right)^{1/\wp}, \tag{3.23}$$

where $C(\theta)=\omega_\theta^{1/\wp'}$, $\omega_\theta=\sup\{\tau^2/\omega(\tau)\colon\ \tau\geqslant\sqrt{\theta/|\Omega|}\}$,

$$\varepsilon_1(T)=\sup_{\mathbb{R}_0^+}\|u(t,\cdot)\|_s\cdot\left[\left(\int_T^\infty \mathscr{D}u(t)\,dt\right)^{1/m'}+\left(\int_T^\infty \mathscr{D}u(t)\,dt\right)^{1/r'}\right], \tag{3.24}$$

and

$$\varepsilon_2(T)=\sup_{\mathbb{R}_0^+}\|u_t(t,\cdot)\|_2^{2/\wp}\cdot\left(\int_T^\infty \mathscr{D}u(t)\,dt\right)^{1/\wp'}, \tag{3.25}$$

with $\varepsilon_1(T)=o(1)$ and $\varepsilon_2(T)=o(1)$ as $T\to\infty$ by (3.7) of Lemma 3.2. Thus, by (3.19) it follows

$$V(\tau)]_T^t \leqslant U\int_T^t |k'|\,d\tau + 2\theta\int_T^t k\,d\tau + 2\varepsilon(T)C(\theta)\left(\int_0^t \sigma^{1-\wp}k^{\wp}\,d\tau\right)^{1/\wp} - \alpha\int_T^t k\,d\tau + \varepsilon(T)\mathscr{B}(k(t)),$$

where $\varepsilon(T)=\max\{\varepsilon_1(T),\varepsilon_2(T)\}$. From now on the proof can proceed exactly as in Theorem 3.1 of [3]. Hence $Eu(t)$ approaches zero as $t\to\infty$. Thus, by (3.6) and the facts that $a+b>0$ and

$$p_+\mathscr{I}u(t)\geqslant\min\{\|Du(t,\cdot)\|_{p(\cdot)}^{p_-},\ \|Du(t,\cdot)\|_{p(\cdot)}^{p_+}\},$$

then (3.5) holds. □

4. Local asymptotic stability

In this section we require (1.6) only for u sufficiently small, but under the stronger requirement that $a>0$, that is (1.1) is non-degenerate, and also $\mu_0>0$, see [10]. For simplicity and clarity we denote by (H)$'$ the corresponding modified condition (H), where (1.6) is now replaced by

$$\liminf_{u\to 0}\frac{(f(x,u),u)}{|u|^{p(x)}}\geqslant -a\overline{\mu},\quad \text{with } a\mu_0>0 \text{ and } \overline{\mu}\in[0,\mu_0p_-/p_+). \tag{4.1}$$

Theorem 4.1. *Suppose that* (H)$'$, *with* $p_+<q_-$ *and* $q\leqslant p^*$ *in* Ω, *and* (AS) *hold. Assume that* k *is the auxiliary function as in Theorem* 3.1, *which verifies* (3.1)–(3.4). *If* u *is a strong solution of* (1.1) *with sufficiently small initial data* $\|Du(0,\cdot)\|_{p(\cdot)}$, $\|u_t(0,\cdot)\|_2$, *then* (3.5) *continues to hold.*

We start with a series of lemmas, in which we assume that f verifies (H)$'$.

Lemma 4.2. *There exist two numbers* $\mu\in(\overline{\mu},\mu_0p_-/p_+)$ *and* $c>0$ *such that*

$$(f(x,u),u)\geqslant -a\mu|u|^{p(x)}-c|u|^{q(x)}\quad \text{in } \Omega\times\mathbb{R}^N,$$

$$\mathscr{F}\phi(t)\geqslant -a\mu\int_\Omega\frac{|\phi(t,x)|^{p(x)}}{p(x)}\,dx - c\int_\Omega\frac{|\phi(t,x)|^{q(x)}}{q(x)}\,dx\quad \text{in } \mathbb{R}_0^+ \text{ for all } \phi\in K. \tag{4.2}$$

Furthermore, if u is a strong solution of (1.1)*, then for all* $t \in \mathbb{R}_0^+$

$$Eu(t) \geqslant \frac{1}{2}\|u_t(t,\cdot)\|_2^2 + \frac{a}{2}\left(1-\frac{\mu}{\mu_0}\right)\mathscr{I}u(t) + \tilde{a}\min\left\{\|u(t,\cdot)\|_{q(\cdot)}^{p_-}, \|u(t,\cdot)\|_{q(\cdot)}^{p_+}\right\}$$
$$- \tilde{c}\max\left\{\|u(t,\cdot)\|_{q(\cdot)}^{q_-}, \|u(t,\cdot)\|_{q(\cdot)}^{q_+}\right\}, \tag{4.3}$$

where $\tilde{c} = c/q_-$ *and by* (2.8) *and* $(4.2)_1$

$$\tilde{a} = \frac{a}{2p_+\tilde{\lambda}_{q(\cdot)}}\left(1-\frac{\mu}{\mu_0}\cdot\frac{p_+}{p_-}\right) > 0, \qquad \tilde{\lambda}_{q(\cdot)} = \max\left\{\lambda_{q(\cdot)}^{p_+}, \lambda_{q(\cdot)}^{p_-}\right\}.$$

Proof. Inequality $(4.2)_1$ is an immediate consequence of (H)$'$, and so $(4.2)_2$ follows at once by integration. Indeed, $(f(x,u),u) \geqslant -a\mu|u|^{p(x)}$ for all $(x,u) \in \Omega \times \mathbb{R}^N$, with $|u| < \delta$, by (4.1), provided $\delta \in (0,1]$ is sufficiently small; while $(f(x,u),u) \geqslant -c|u|^{q(x)}$ for all $(x,u) \in \Omega \times \mathbb{R}^N$, with $|u| \geqslant \delta$, by (H)$'$, provided that $c > 0$ is sufficiently large. Hence $(4.2)_1$ holds with c as large as we wish.

By (4.2), the definition of E, (1.5) and (2.3)–(2.4) for p and q, we have in $\mathbb{R}_0^+$

$$Eu(t) \geqslant \frac{1}{2}\|u_t(t,\cdot)\|_2^2 + \frac{a}{2}\left(1-\frac{\mu}{\mu_0}\right)\mathscr{I}u(t) + \frac{a}{2p_+}\left(1-\frac{\mu}{\mu_0}\cdot\frac{p_+}{p_-}\right)\min\left\{\|Du(t,\cdot)\|_{p(\cdot)}^{p_-}, \|Du(t,\cdot)\|_{p(\cdot)}^{p_+}\right\}$$
$$- \frac{c}{q_-}\max\left\{\|u(t,\cdot)\|_{q(\cdot)}^{q_-}, \|u(t,\cdot)\|_{q(\cdot)}^{q_+}\right\},$$

and so (4.3) follows at once by application of (2.8) with $h = q$. □

Furthermore, if $p_+ < q_-$, we introduce

$$\Sigma = \left\{(\upsilon, E) \in \mathbb{R}^2\colon\ 0 \leqslant \upsilon < \upsilon_1,\ 0 \leqslant E < E_1\right\}, \tag{4.4}$$

where

$$\upsilon_1 = \left(\frac{\tilde{a}}{c}\right)^{1/(q_- - p_+)}, \qquad E_1 = \tilde{a}\left(2 - \frac{1}{q_-}\right)\upsilon_1^{p_+},$$

and the numbers $\tilde{a}$ and c are given in Lemma 4.2. Without loss of generality, we also assume that $\tilde{a}/c \leqslant 1$, by taking c sufficiently large, if necessary.

Lemma 4.3. *Assume* $p_+ < q_-$*. Let u be a strong solution of* (1.1) *and denote by* $\upsilon(t)$ *the number* $\|u(t,\cdot)\|_{q(\cdot)}$*. If* $(\upsilon(0), Eu(0)) \in \Sigma$*, then*

$$\big(\upsilon(t), Eu(t)\big) \in \Sigma \quad \textit{for all } t \in \mathbb{R}_0^+. \tag{4.5}$$

Moreover in $\mathbb{R}_0^+$

$$2Eu(t) \geqslant \|u_t(t,\cdot)\|_2^2 + a\left(1-\frac{\mu}{\mu_0}\right)\mathscr{I}u(t). \tag{4.6}$$

Proof. Since $\mathscr{I}u(t) \geqslant \min\{\|Du(t,\cdot)\|_{p(\cdot)}^{p_-}, \|Du(t,\cdot)\|_{p(\cdot)}^{p_+}\}/p_+$ by (2.3) and (2.4), then (4.3) and another use of (2.8), with $h = q$, yield

$$Eu(t) \geqslant 2\tilde{a}\min\left\{\|u(t,\cdot)\|_{q(\cdot)}^{p_-}, \|u(t,\cdot)\|_{q(\cdot)}^{p_+}\right\} - \tilde{c}\max\left\{\|u(t,\cdot)\|_{q(\cdot)}^{q_-}, \|u(t,\cdot)\|_{q(\cdot)}^{q_+}\right\}$$
$$= \begin{cases} 2\tilde{a}\upsilon(t)^{p_+} - \tilde{c}\upsilon(t)^{q_-}, & \text{if } 0 \leqslant \upsilon(t) \leqslant 1,\\ 2\tilde{a}\upsilon(t)^{p_-} - \tilde{c}\upsilon(t)^{q_+}, & \text{if } \upsilon(t) > 1.\end{cases}$$

Now, if there would exist t such that $\upsilon(t) = \upsilon_1 \leqslant 1$, then

$$2\tilde{a}\upsilon_1^{p_+} - \tilde{c}\upsilon_1^{q_-} = E_1 > Eu(0) \geqslant Eu(t) \geqslant 2\tilde{a}\upsilon_1^{p_+} - \tilde{c}\upsilon_1^{q_-},$$

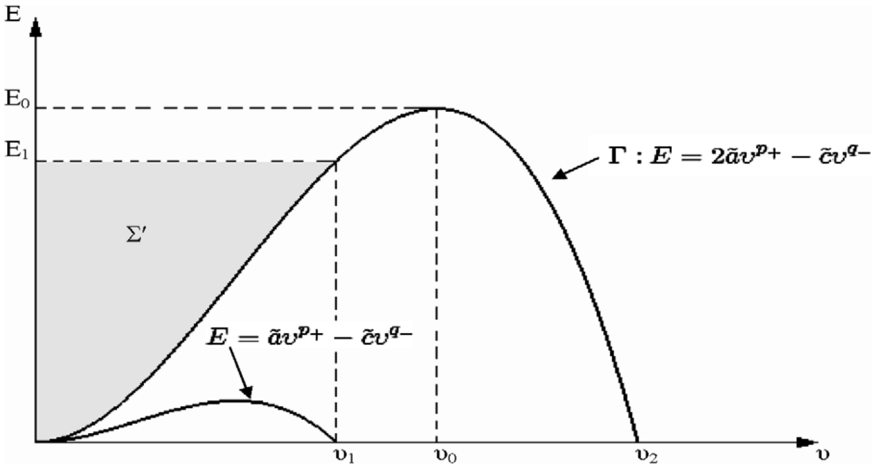

Fig. 1. The phase plane (υ, E).

which is impossible. Therefore $\upsilon(t) \neq \upsilon_1$ for all $t \in \mathbb{R}_0^+$. Hence by the continuity of υ we have $\upsilon(\mathbb{R}_0^+) \subset [0, \upsilon_1)$, being $\upsilon(0) < \upsilon_1$. In particular, the case $\upsilon(t) > 1$ can never occur. Consequently, we have proved that along any solution $u \in K$

$$E_1 > Eu(0) \geqslant Eu(t) \geqslant 2\tilde{a}\upsilon(t)^{p_+} - \tilde{c}\upsilon(t)^{q_-} \geqslant 0 \quad \text{for all } t \in \mathbb{R}_0^+, \tag{4.7}$$

since $0 \leqslant \upsilon(t) < \upsilon_1 \leqslant 1$ for all $t \in \mathbb{R}_0^+$. Hence (4.5) is proved and (4.3) reduces to

$$Eu(t) \geqslant \frac{1}{2}\|u_t(t,\cdot)\|_2^2 + \frac{a}{2}\left(1 - \frac{\mu}{\mu_0}\right)\mathscr{I}u(t) + \tilde{a}\upsilon(t)^{p_+} - \tilde{c}\upsilon(t)^{q_-}. \tag{4.8}$$

Since $\tilde{a}\upsilon^{p_+} - \tilde{c}\upsilon^{q_-} \geqslant 0$ in $[0, \upsilon_1]$ we get (4.6) at once. □

Remark. Lemma 4.3 is easily visualized using the two-dimensional phase plane (υ, E) shown in Fig. 1. In particular, by (4.7) any point $(\upsilon(t), Eu(t))$ on the trajectory of a solution $u \in K$ must lie above the curve

$$\Gamma : E = 2\tilde{a}\upsilon^{p_+} - \tilde{c}\upsilon^{q_-}.$$

The region Σ' is shaded in Fig. 1, with υ_1 defined by $\tilde{a}\upsilon^{p_+} - \tilde{c}\upsilon^{q_-} = 0$. As we shall see if $(\upsilon(0), Eu(0)) \in \Sigma'$, then $\lim_{t\to\infty} Eu(t) = 0$.

Lemma 4.4. *Let the assumptions of Lemma* 4.3 *hold and assume also that* $p \leqslant q \leqslant p^*$ *in* Ω. *Then*

$$\begin{aligned} &\|u\|_2, \|u_t\|_2, \|Du\|_{p(\cdot)}, \|u\|_{q(\cdot)}, \|u\|_{p^*(\cdot)}, M(\mathscr{I}u) \in L^\infty(\mathbb{R}_0^+), \\ &\mathscr{D}u = \langle Q(t,x,u,u_t), u_t\rangle \in L^1(\mathbb{R}_0^+). \end{aligned} \tag{4.9}$$

Proof. The fact that $\|u_t\|_2$, $M(\mathscr{I}u)$ and $\|Du\|_{p(\cdot)}$ are in $L^\infty(\mathbb{R}_0^+)$ follow at once by (4.6) and (3.8); moreover $\|u\|_2 \in L^\infty(\mathbb{R}_0^+)$ by (2.7). The latter part of $(4.9)_1$ follows by the continuity of the Sobolev embeddings $X \hookrightarrow L^{q(\cdot)}(\Omega)$ and $X \hookrightarrow L^{p^*(\cdot)}(\Omega)$, being $p \leqslant q \leqslant p^*$ in Ω, as assumed in Theorem 4.1. Property $(4.9)_2$ can be proved exactly as in Lemma 3.2. □

Of course Lemma 3.3 continues to hold since in the setting of this section we consider the special case in which a, $\mu_0 > 0$ and $p \leqslant q \leqslant p^*$ in Ω. In particular, (3.11) holds, with α simply given in (3.17).

Proof of Theorem 4.1. Let $(\upsilon(0), Eu(0)) \in \Sigma$. Using Lemma 3.3 and Lemmas 4.2–4.4 and the estimates (3.22) and (3.23), we derive as in the proof of Theorem 3.1 that $Eu(t) \to 0$ as $t \to \infty$. This shows that also $(3.5)_2$ holds by virtue of (4.6). It remains to show that if the data $\|u_t(0,\cdot)\|_2$ and $\|Du(0,\cdot)\|_{p(\cdot)}$ are sufficiently small, then

$(\upsilon(0), Eu(0)) \in \Sigma$. But $\upsilon(0) = \|u(0,\cdot)\|_{q(\cdot)} < \upsilon_1 \leqslant 1$ if $\|Du(0,\cdot)\|_{p(\cdot)}$ is sufficiently small by the continuity of the embedding $X \hookrightarrow L^{q(\cdot)}(\Omega)$, while the definition of Eu, (3.15) and (2.4) give

$$Eu(0) \leqslant \frac{1}{2}\|u_t(0,\cdot)\|_2^2 + \left(\frac{a+b}{p_-} + 2C\right)\|Du(0,\cdot)\|_{p(\cdot)}.$$

This shows that $Eu(0) < E_1$ for sufficiently small data. Finally, since $0 \leqslant \upsilon(0) < \upsilon_1$ it follows that $\tilde{a}\upsilon(0)^{p_+} - \tilde{c}\upsilon(0)^{q_-} \geqslant 0$ and so $Eu(0) \geqslant 0$ by (4.3). □

5. Higher order damping terms

In this section we consider the more delicate system (1.7), in which $g \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+ \to \mathbb{R}_0^+)$, and take

$$X = W_0^{1,p(\cdot)}(\Omega) = \left[W_0^{1,p(\cdot)}(\Omega)\right]^N, \qquad K' = C^1\left(\mathbb{R}_0^+ \to X\right)$$

and K in the usual way as in Section 2. Moreover through this section we assume $p_+ - p_- < 1$ and again the function E, defined in (2.9), can be taken as the *natural energy function* along a solution of (1.7), while, as we shall see below in (B)(ii), the terms involving the new internal damping arising from $g = g(t)$ are treated in the same way as we did for the external damping Q. When $p \geqslant 2$ in Ω, then K' reduces to $C^1(\mathbb{R}_0^+ \to X)$, as in [3,20,21].

By a *strong solution* of (1.7) we mean a function $u \in K$ satisfying the following two conditions

(A) *Distribution identity* for all $t \in \mathbb{R}_0^+$ and $\phi \in K$

$$\langle u_t, \phi\rangle]_0^t = \int_0^t \left\{\langle u_t, \phi_t\rangle - M\left(\mathscr{I}u(t)\right)\left\langle |Du|^{p(x)-2}Du, D\phi\right\rangle - g(t)\left\langle |Du_t|^{p(x)-2}Du_t, D\phi\right\rangle - \left\langle Q(t,\cdot,u,u_t) + f(\cdot,u), \phi\right\rangle\right\} d\tau.$$

(B) *Conservation law*

(i) $$\mathscr{D}u := \left\langle Q(t,\cdot,u,u_t), u_t\right\rangle \in L^1_{\mathrm{loc}}\left(\mathbb{R}_0^+\right),$$

(ii) $$t \mapsto Eu(t) + \int_0^t \left\{\mathscr{D}u(\tau) + g(\tau)\rho_{p(\cdot)}\left(Du_t(\tau,\cdot)\right)\right\} d\tau$$

is non-increasing in $\mathbb{R}_0^+$.

The function $t \mapsto g(t)\rho_{p(\cdot)}(Du_t(t,\cdot))$ in (B)(ii) is the *internal material damping of Kelvin–Voigt type*. It is easy to see that the definition of strong solution is meaningful when hypotheses (H) and (AS)(a) hold. Also in this new context Eu is *non-increasing in* $\mathbb{R}_0^+$ by (1.3) and the fact that the integrand in (B)(ii) is non-negative by the non-negativity of g.

Before proving the main result of the section, we observe that the discussion already given in Section 3 must take into account the more delicate terms in (A) and (B) involving g. Lemmas 3.2 and 3.3 hold also in this new context; so (3.10) is true for some $l \geqslant 0$.

As in Section 3 we give some preliminary lemmas under the structural hypothesis (H) and (AS)(a).

Lemma 5.1. *Let u be a strong solution of* (1.7). *Then the function $t \mapsto g(t)\rho_{p(\cdot)}(Du_t(t,\cdot))$ is in $L^1(\mathbb{R}_0^+)$.*

Proof. By (B), (1.3) and the fact that g is non-negative, as explained above,

$$0 \leqslant \int_0^t \left\{\mathscr{D}u(\tau) + g(\tau)\rho_{p(\cdot)}\left(Du_t(\tau,\cdot)\right)\right\} d\tau \leqslant Eu(0) - Eu(t) \leqslant Eu(0),$$

since $Eu \geqslant 0$ in $\mathbb{R}_0^+$ by Lemma 3.2, and so $g(t)\rho_{p(\cdot)}(Du_t(t,\cdot)) \in L^1(\mathbb{R}_0^+)$. □

Lemma 5.2. *Let u be a strong solution of* (1.7) *and assume $p_+ - p_- < 1$. Moreover suppose that $k = k(t)$ is an auxiliary function as in Theorem* 3.1. *Then there exists T sufficiently large such that for all $t \geqslant T$ we have*

$$\int_T^t k(\tau)g(\tau)\int_\Omega \big|Du_t(\tau,x)\big|^{p(x)-1}\big|Du(\tau,x)\big|\,dx\,d\tau \leqslant \varepsilon_3(T)\mathscr{C}\big(k(t)\big),$$
$$\mathscr{C}\big(k(t)\big) = \left(\int_T^t gk^{p_1}\,d\tau\right)^{1/p_1} + \left(\int_T^t gk^{p_2}\,d\tau\right)^{1/p_2},$$
$$p_1 = \frac{p_+}{1+p_+-p_-}, \qquad p_2 = \frac{p_-}{1+p_--p_+}, \tag{5.1}$$

where

$$\varepsilon_3(T) = \mathcal{K}\left(\int_T^\infty g(t)\rho_{p(\cdot)}\big(Du_t(t,\cdot)\big)\,dt\right)^{(p_--1)/p_+} \to 0 \quad \textit{as } T\to\infty,$$

and

$$\mathcal{K} = r_p \cdot \sup_{t\in\mathbb{R}_0^+}\big\|Du(t,\cdot)\big\|_{p(\cdot)}, \quad r_p = \frac{1}{p_-}+\frac{1}{p'_-}.$$

Proof. By (3.7) clearly $\mathcal{K} < \infty$. By Lemma 5.1 we first take T so large that

$$\int_T^\infty g(t)\rho_{p(\cdot)}\big(Du_t(t,\cdot)\big)\,dt \leqslant 1.$$

By Hölder's inequality, see [16, Theorem 2.1],

$$\int_\Omega \big|Du_t(t,x)\big|^{p(x)-1}\big|Du(t,x)\big|\,dx \leqslant r_p\cdot \Big\|\big|Du_t(t,\cdot)\big|^{p(\cdot)-1}\Big\|_{p'(\cdot)}\cdot \Big\|\big|Du(t,\cdot)\big|\Big\|_{p(\cdot)},$$

while by Lemma 2.1 of [8] we also have

$$\big\||Du_t|^{p(\cdot)-1}\big\|_{p'(\cdot)} \leqslant \|Du_t\|_{p(\cdot)}^{p_--1} \leqslant \rho_{p(\cdot)}(Du_t)^{(p_--1)/p_+} \quad \text{if } \|Du_t\|_{p(\cdot)}\leqslant 1,$$

by (2.4), while

$$\big\||Du_t|^{p(\cdot)-1}\big\|_{p'(\cdot)} \leqslant \|Du_t\|_{p(\cdot)}^{p_+-1} \leqslant \rho_{p(\cdot)}(Du_t)^{(p_+-1)/p_-} \quad \text{if } \|Du_t\|_{p(\cdot)} > 1$$

by (2.3). Let $D_1 = \{\tau\in[T,t]\colon \|Du_t(\tau,\cdot)\|_{p(\cdot)}\leqslant 1\}$ and $D^1 = [T,t]\setminus D_1$. Hence, by Hölder's inequality,

$$\begin{aligned}
&\int_{D_1} k(\tau)g(\tau)\big[\rho_{p(\cdot)}\big(Du_t(\tau,\cdot)\big)\big]^{(p_--1)/p_+}\,d\tau \\
&\quad\leqslant \left(\int_{D_1} g(\tau)\rho_{p(\cdot)}\big(Du_t(\tau,\cdot)\big)\,d\tau\right)^{(p_--1)/p_+}\left(\int_{D_1} g(\tau)\big[k(\tau)\big]^{p_1}\,d\tau\right)^{1/p_1} \\
&\quad\leqslant \left(\int_T^\infty g(\tau)\rho_{p(\cdot)}\big(Du_t(\tau,\cdot)\big)\,d\tau\right)^{(p_--1)/p_+}\left(\int_T^t g(\tau)\big[k(\tau)\big]^{p_1}\,d\tau\right)^{1/p_1},
\end{aligned}$$

where $p_1 > 1$ is given in (5.1); similarly

$$\int_{D^1} k(\tau)g(\tau)\big[\rho_{p(\cdot)}\big(Du_t(\tau,\cdot)\big)\big]^{(p_+-1)/p_-}\,d\tau$$
$$\leqslant \Bigg(\int_{D^1} g(\tau)\rho_{p(\cdot)}\big(Du_t(\tau,\cdot)\big)\,d\tau\Bigg)^{(p_+-1)/p_-}\Bigg(\int_{D^1} g(\tau)\big[k(\tau)\big]^{p_2}\,d\tau\Bigg)^{1/p_2}$$
$$\leqslant \Bigg(\int_T^\infty g(\tau)\rho_{p(\cdot)}\big(Du_t(\tau,\cdot)\big)\,d\tau\Bigg)^{(p_+-1)/p_-}\Bigg(\int_T^t g(\tau)\big[k(\tau)\big]^{p_2}\,d\tau\Bigg)^{1/p_2},$$

being $p_2 = p_-/(1+p_- - p_+) > 1$ by the assumption $p_+ - p_- < 1$. Moreover, also $(p_+ - 1)/p_- \leqslant (p_- - 1)/p_+$, since $p_+ - p_- < 1$. Combining all these facts and the choice of T, we conclude

$$\int_T^t k(\tau)g(\tau)\int_\Omega \big|Du_t(\tau,x)\big|^{p(x)-1}\big|Du(\tau,x)\big|\,dx\,d\tau$$
$$\leqslant \varepsilon_3(T)\cdot\Bigg\{\Bigg(\int_T^t g(\tau)\big[k(\tau)\big]^{p_1}\,d\tau\Bigg)^{1/p_1} + \Bigg(\int_T^t g(\tau)\big[k(\tau)\big]^{p_2}\,d\tau\Bigg)^{1/p_2}\Bigg\}.$$

Finally, $\varepsilon_3(T) \to 0$ as $T \to \infty$ by Lemma 5.1. □

Theorem 5.3. *Let the assumptions of Theorem* 3.1 *hold, with the only exception that* (3.3) *is replaced by*

$$\liminf_{t\to\infty}\big\{\mathscr{C}\big(k(t)\big) + \mathscr{A}\big(k(t)\big)\big\}\Big/\int_0^t k\,d\tau < \infty, \tag{5.2}$$

where $t \mapsto \mathscr{A}(k(t))$ is given in (3.4) *and $t \mapsto \mathscr{C}(k(t))$ in* (5.1)*. If $p_+ - p_- < 1$, then along any strong solution u of* (1.7) *property* (3.5) *holds.*

Proof. Suppose for contradiction that $Eu(t)$ approaches a limit $l > 0$ as $t \to \infty$. As in the proof of Theorem 3.1 we first treat the case (3.1) when k is also of class $C^1(\mathbb{R}_0^+)$. Consider the Lyapunov function

$$V(t) = \langle u_t, \phi\rangle, \quad \phi = k(t)u \in K.$$

Hence by the distribution identity (A) above, for any $t \geqslant T \geqslant 0$, we have

$$V(\tau)\big]_T^t = \int_T^t \big\{k'\langle u, u_t\rangle + 2k\|u_t\|_2^2 - k\big[\|u_t\|_2^2 + M(\mathscr{I}u)\big\langle |Du|^{p(\cdot)-2}Du, Du\big\rangle + \big\langle f(\cdot,u), u\big\rangle\big]\big\}\,d\tau$$
$$- \int_T^t kg\int_\Omega |Du_t|^{p(x)-2}(Du_t, Du)\,dx\,d\tau - \int_T^t k\big\langle Q(\tau,\cdot,u,u_t), u\big\rangle\,d\tau. \tag{5.3}$$

We first estimate the right-hand side of (5.3). Clearly (3.20) holds by definition of K', being u in K. Moreover, as in the proof of Theorem 3.1 the estimates (3.11) and (3.21)–(3.23) continue to hold. Now, taking T so large that Lemma 5.2 holds, from (5.3) we obtain

$$V(\tau)\big]_T^t \leqslant U\int_T^t |k'|\,d\tau + 2\theta\int_T^t k\,d\tau + 2\varepsilon_2(T)C(\theta)\Bigg(\int_0^t \sigma^{1-\wp}k^{\wp}\,d\tau\Bigg)^{1/\wp} - \alpha\int_T^t k\,d\tau$$
$$+ \varepsilon_3(T)\mathscr{C}\big(k(t)\big) + \varepsilon_1(T)\mathscr{B}\big(k(t)\big), \tag{5.4}$$

where $\varepsilon_1(T)$ is defined in (3.24), $\varepsilon_2(T)$ in (3.25), $\varepsilon_3(T)$ in (5.1), $k \mapsto \mathscr{B}(k)$ in (3.4) and $k \mapsto \mathscr{C}(k)$ in (5.1). By (5.2) there is a sequence $t_i \nearrow \infty$ and a number $\ell > 0$ such that

$$\mathscr{C}\big(k(t_i)\big) + \mathscr{A}\big(k(t_i)\big) \leqslant \ell \int_0^{t_i} k\,d\tau. \tag{5.5}$$

From now on the proof can proceed exactly as in Theorem 4.1 of [3]. Hence $Eu(t)$ approaches zero as $t \to \infty$. Thus, by (3.6) and the facts that $a + b > 0$ and

$$p_+ \mathscr{I} u(t) \geqslant \min\big\{\big\|Du(t,\cdot)\big\|_{p(\cdot)}^{p_-}, \big\|Du(t,\cdot)\big\|_{p(\cdot)}^{p_+}\big\}$$

(3.5) holds. □

Theorem 5.4. *Let the assumptions of Theorem* 4.1 *hold, with the only exception that* (3.3) *is replaced by* (5.2). *If u is a strong solution of* (1.7) *with sufficiently small initial data $\|Du(0,\cdot)\|_{p(\cdot)}$, $\|u_t(0,\cdot)\|_2$, then* (3.5) *continues to hold, provided that $p_+ - p_- < 1$.*

Proof. Lemma 3.3, Lemmas 4.2–4.4 continue to hold. Hence the proof of Theorem 4.1 can be repeated word by word, with the only exception that the derivation of (3.5) now follows from the proof of Theorem 5.3 instead of Theorem 3.1. □

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