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Asymptotic stability at infinity for differentiable vector fields of the plane

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Abstract

Let $X:\mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable (but not necessarily C^1) vector field, where $\sigma > 0$ and $\overline{D}_\sigma = \{z \in \mathbb{R}^2\colon \|z\| \leqslant \sigma\}$. Denote by $\mathcal{R}(z)$ the real part of $z \in \mathbb{C}$. If for some $\epsilon > 0$ and for all $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, no eigenvalue of D_pX belongs to $(-\epsilon, 0] \cup \{z \in \mathbb{C}\colon \mathcal{R}(z) \geqslant 0\}$, then: (a) for all $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, there is a unique positive semi-trajectory of X starting at p; (b) it is associated to X, a well-defined number $\mathcal{I}(X)$ of the extended real line $[-\infty, \infty)$ (called the index of X at infinity) such that for some constant vector $v \in \mathbb{R}^2$ the following is satisfied: if $\mathcal{I}(X)$ is less than zero (respectively greater or equal to zero), then the point at infinity ∞ of the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$ is a repellor (respectively an attractor) of the vector field $X + v$.

Keywords: Planar vector fields; Asymptotic stability; Markus–Yamabe conjecture; Injectivity

1. Introduction

The pioneer work of C. Olech [20,21] showed the existence of a strong connection between the global asymptotic stability of a vector field $X:\mathbb{R}^2 \to \mathbb{R}^2$ and the injectivity of X (considered as a map). This connection was strengthened and broadened in subsequent works (see, for instance,

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[5,8–11,13–18]). This paper proceeds with this study. We extend to the differentiable case the work, already dealt with in [15], for the C^1 case.

There has been a great interest in the local study of vector fields around their singularities. A sample of this study is the work done by C. Chicone, F. Dumortier, J. Sotomayor, R. Roussarie, F. Takens. See, for instance, [3,6,7,24,26]. To understand the global behavior of a planar vector field it is absolutely necessary to understand its behavior around infinity. At a first glance it looks as if C^r-vector fields (with $r = 0, 1, 2, \dots, \infty$) defined in a neighborhood of ∞ presented richer phase portraits than C^r-vector fields at isolated singularities. To prove that this is not the case we have included Proposition 29 in Section 7. In this context and according to the definitions below, we will be considering infinity as if it were a singularity of a vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$.

Before stating the main result, we will give some definitions. Throughout this work, we assume that $\mathbb{R}^2$ is embedded in the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$ and that "infinity" refers to the point at infinity ∞ of $\mathbb{R}^2 \cup \{\infty\}$. This applies also to subspaces of $\mathbb{R}^2 \cup \{\infty\}$ of the form $(\mathbb{R}^2 \setminus \overline{D}_\sigma) \cup \{\infty\}$, where $\sigma > 0$ and $\overline{D}_\sigma = \{z \in \mathbb{R}^2\colon\ \|z\| \leqslant \sigma\}$. Given a continuous vector field $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ of the plane, we may extend it to the vector field $\widehat{X} : ((\mathbb{R}^2 \setminus \overline{D}_\sigma) \cup \{\infty\}, \infty) \to (\mathbb{R}^2, 0)$ of the Riemann sphere which takes ∞ to 0. Notice that we allow $\widehat{X}$ to be discontinuous at the point ∞. Henceforth, we will identify X with its extension $\widehat{X}$.

Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a continuous vector field. We say that a positive (respectively a negative) semi-trajectory γ_p^+ (respectively γ_p^-) of X *goes to infinity* (respectively *comes from infinity*) if $\omega(\gamma_p^+) = \infty$ (respectively $\alpha(\gamma_p^-) = \infty$). Let $\{\Gamma_n\}_1^\infty$ be a sequence of topological circles; we say that the sequence $\{\Gamma_n\}_1^\infty$ tends to infinity if for every neighborhood V of ∞, there exists $N \in \mathbb{N}$ such that $n \geqslant N$ implies that $\Gamma_n \subset V$.

Definition 1. We say that ∞ is an attractor (respectively a repellor) of a continuous vector field $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ if

(i) there exists a sequence of C^1 circles transversal to X tending to infinity;
(ii) for some $R \geqslant \sigma$, all positive (respectively negative) semi-trajectories of X starting at $p \in \mathbb{R}^2 \setminus \overline{D}_R$ go to infinity (respectively come from infinity).

A few comments are due in order to capture the essential features of Definition 1. Firstly we shall remark that in the C^1 case, Definition 1 is equivalent to saying that the vector field $\widehat{X}$ induced by X on the Riemann sphere is locally topologically equivalent in a neighborhood of the infinity either to $p \mapsto -p$ or to $p \mapsto p$ at the origin, see [1,16]. In the differentiable or continuous case this definition is unsatisfactory because is not possible to speak here of topological equivalence. Note that saying that ∞ is an attractor or repellor of X is stronger than saying that outside a disk $\overline{D}_R$ all trajectories go to infinity. This prevents infinity from being an attractor or repellor of the constant vector field which presents elliptic sectors at infinity, see Fig. 1(a). Furthermore, condition (i) of Definition 1 cannot be weakened. Indeed, there exist vector fields which, in spite of admitting a transversal circle Γ and satisfying (ii) of Definition 1, does not admit any family of transversal circles tending to infinity, see Fig. 1(b).

Let A be a Lebesgue measurable subset of $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a measurable function. We define as usual

$$f^+(x) = \max\{f(x), 0\}, \qquad f^-(x) = \max\{-f(x), 0\}.$$

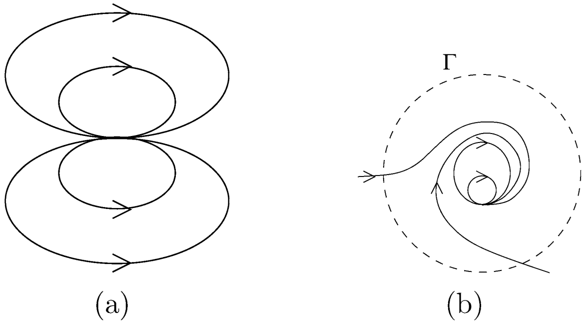

Fig. 1. Two vector fields which do not have the point at infinity as an attractor.

Accordingly, we say that $f : A \to \mathbb{R}$ is *Lebesgue almost-integrable* if

$$\min\left\{\int_A f^+\,d\lambda, \int_A f^-\,d\lambda\right\} < \infty,$$

in which case we define

$$\int_A f\,d\lambda = \int_A f^+\,d\lambda - \int_A f^-\,d\lambda,$$

which is a well-defined value of the extended real line $[-\infty, \infty]$.

Given a differentiable vector field $X : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, we let $\operatorname{Spec}(X)$ denote the set of eigenvalues of the derivative D_pX of X at p when p ranges over the whole set U. As usual, $\mathcal{R}(z)$ stands for the real part of the complex number z and $\operatorname{Trace}(DX) : U \to \mathbb{R}$ stands for the function which at each $p \in U$ takes the value $\operatorname{Trace}(D_pX)$.

Now let

$$\mathfrak{D}(U) = \big\{X : U \to \mathbb{R}^2\colon\ X \text{ is differentiable and } \operatorname{Trace}(DX) \text{ is Lebesgue almost-integrable on } U\big\}.$$

We define the index of $X \in \mathfrak{D}(\mathbb{R}^2 \setminus \overline{D}_\sigma)$ at infinity to be the number of the extended real line $[-\infty, \infty]$ defined by

$$\mathcal{I}(X) = \int_{\mathbb{R}^2} \operatorname{Trace}(D\widehat{X})\,dx \wedge dy,$$

where $\widehat{X} \in \mathfrak{D}(\mathbb{R}^2)$ is any globally differentiable extension of $X|_{\mathbb{R}^2\setminus D_s}$, for some $s > \sigma$, whose divergent is Lebesgue almost-integrable on $\mathbb{R}^2$. We will show (see Corollary 13) that $\mathcal{I}(X)$ is well-defined. We are now ready to state our main theorem.

Theorem A. *Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable (but not necessarily C^1) vector field. If for some $\epsilon > 0$,* $\operatorname{Spec}(X)$ *is disjoint from* $(-\epsilon, 0] \cup \{z \in \mathbb{C}\colon\ \mathcal{R}(z) \geqslant 0\}$*, then:*

(a) *for all* $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, *there is a unique positive semi-trajectory of* X *starting at* p;
(b) $\mathcal{I}(X)$, *the index of* X *at infinity, is a well-defined number of the extended real line* $[-\infty, \infty)$;
(c) *there exists a constant vector* $v \in \mathbb{R}^2$ *such that if* $\mathcal{I}(X)$ *is less than zero* (*respectively greater or equal to zero*), *then the point at infinity* ∞ *of the Riemann sphere* $\mathbb{R}^2 \cup \{\infty\}$ *is a repellor* (*respectively an attractor*) *of the vector field* $X + v$.

2. Differentiable vector fields

Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field defined on an open set $U \subset \mathbb{R}^2$. We say that a C^1 curve $\gamma_p : I \to U$ is a *solution of the vector field X passing through p* if $\gamma_p(0) = p$ and $\gamma_p'(t) = X(\gamma_p(t))$, for all $t \in I$, where $I \subset \mathbb{R}$ is an open interval containing zero. By Peano's existence theorem, through each $p \in U$, there exists a solution $\gamma_p : J(\gamma_p) \to U$ defined on some open maximal interval $J(\gamma_p)$ which depends on both the solution γ_p and on the starting point p. For the sake of simplicity, we identify the solution γ_p with its range which we refer to as a *trajectory of X passing through p* defined on $J(\gamma_p)$. Likewise, γ_p^+ (respectively γ_p^-) will denote the positive (respectively negative) semi-trajectory of X contained in γ_p and starting at p. Accordingly, $\gamma_p = \gamma_p^- \cup \gamma_p^+$. Given a positive (respectively negative) semi-trajectory γ_p^+ (respectively γ_p^-), we denote by $\omega(\gamma_p^+)$ (respectively $\alpha(\gamma_p^-)$) its ω-limit set (respectively α-limit set).

We say that $p \in U$ is a *singularity* (respectively *a regular point*) of X if $X(p) = 0$ (respectively $X(p) \neq 0$). A trajectory γ is said to be *periodic* if it is defined on $\mathbb{R}$ and there exits $\tau > 0$ such that $\gamma(t + \tau) = \gamma(t)$ for all $t \in \mathbb{R}$. We recall that trajectories of continuous vector fields may cross themselves or each other. If a trajectory cross itself then it naturally contains a periodic trajectory defined on $\mathbb{R}$. If U is simply connected then it follows by index theory that every periodic trajectory of X has to surround a singularity.

Given a vector field $X = (f, g)$, let $X^* = (-g, f)$ be the orthogonal vector field to X. The same notation as that for intervals of $\mathbb{R}$ will be used for oriented arcs of trajectory $[p, q], [p, q), \ldots$ (respectively $[p, q]^*, [p, q)^*, \ldots$) of X (respectively X^*), connecting the points p and q. The orientation of theses arcs is that induced by X (respectively X^*).

Definition 2. A compact rectangle $R = R(p_1, p_2; q_1, q_2) \subset U$ of a continuous vector field $X : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is the compact region the boundary of which is made up of two arcs of trajectory $[p_1, p_2], [q_1, q_2]$ of X and two arcs of trajectory $[p_1, q_1]^*, [p_2, q_2]^*$ of X^*. Notice that we assume that the flow induced by X goes into R by $[p_1, q_1]^*$ and leaves R by $[p_2, q_2]^*$.

For any arc of trajectory $[p, q]^*$ of X^*, let

$$L\big([p, q]^*\big) = \left| \int_{[p,q]^*} \|X^*\| \, ds \right|,$$

where ds denotes the arc length element. Given an arc of trajectory $[p, q]$ (respectively $[p, q]^*$), we denote by $\ell([p, q])$ (respectively $\ell([p, q]^*)$) the arc length of it. Next formula is a corollary of Green's formula as presented in [23].

Lemma 3. *Let* $R = R(p_1, p_2; q_1, q_2) \subset U$ *be a compact rectangle of* $X \in \mathfrak{D}(U)$. *Then*

$$L\big([p_2, q_2]^*\big) - L\big([p_1, q_1]^*\big) = \int_R \operatorname{Trace}(DX) \, dx \wedge dy.$$

Next result says that a vector field $X \in \mathfrak{D}(U)$ whose divergent is strictly negative on U generates a positive semiflow.

Theorem 4. *Let $X \in \mathfrak{D}(U)$ be a vector field without singularities such that* $\mathrm{Trace}(DX) < 0$ *on U. Then for each $p \in V$, there is a unique positive semi-trajectory of X passing through p.*

Proof. Assume, by contradiction, that there are two positive semi-trajectories $\gamma_p^+, \sigma_p^+ \subset U$ starting at p. So we may take a triangle (i.e., a degenerate rectangle) $R = R(p, q_1; p, q_2) \subset U$ with $[p, q_1] \subset \gamma_p^+$ and $[p, q_2] \subset \sigma_p^+$. By Lemma 3,

$$0 < L\big([q_1, q_2]^*\big) = \int_R \mathrm{Trace}(DX)\, dx \wedge dy < 0,$$

which is a contradiction. □

Lemma 5. *Let $X \in \mathfrak{D}(U)$ be a vector field such that* $\mathrm{Trace}(DX) < 0$ *on U. Assume that U is free of singularities and periodic trajectories and that $K \subset U$ is a compact set. Then there is no positive* (*respectively negative*) *semi-trajectory of X contained in K.*

Proof. In the case of a positive semi-trajectory the proof follows easily from Theorem 4 and the Poincaré–Bendixson theorem for semiflows (see [4]). In the case of a negative semi-trajectory, we will give an explicit proof based on the negativeness of the divergent of X. So we assume that γ^- is a negative semi-trajectory of X contained in a compact set $K \subset U$. Let $p \in \alpha(\gamma^-)$ and let Σ be a compact orthogonal section to X passing through p. We know that no negative semi-trajectory can intersect itself, otherwise it would contain a periodic trajectory. So γ^- intersects Σ monotonically and infinitely many times. Let $\{p_n\}_1^\infty$ denote the corresponding sequence of intersection points, where $p_n \to p$ as $n \to \infty$. Then, from Lemma 3:

$$L\big([p_{j-1}, p_j]^*\big) - L\big([p_j, p_{j+1}]^*\big) < 0, \quad \forall j \in \mathbb{N}^*,$$

where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Hence, for all $n \in \mathbb{N}^*$,

$$L\big([p_0, p_1]^*\big) - L\big([p_n, p_{n+1}]^*\big) = \sum_{j=1}^{n} L\big([p_{j-1}, p_j]^*\big) - L\big([p_j, p_{j+1}]^*\big) < 0.$$

That is,

$$0 < L\big([p_0, p_1]^*\big) < L\big([p_n, p_{n+1}]^*\big), \quad \forall n \in \mathbb{N}^*.$$

But this is an absurd since $L([p_n, p_{n+1}]^*) \to 0$ as $n \to \infty$. So $\alpha(\gamma^-) = \emptyset$. As K is a compact and $\gamma^- \subset K$, $\alpha(\gamma^-)$ cannot be empty. This contradiction finishes the proof. □

Definition 6. We denote by $\mathfrak{D}_\sigma$ the set of the differentiable vector fields $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ such that $\mathrm{Spec}(X)$ is disjoint from $(-\epsilon, 0] \cup \{z \in \mathbb{C}\colon\ \mathcal{R}(z) \geqslant 0\}$ for some $\epsilon > 0$.

First, we derive some useful properties of the vector fields in the class $\mathfrak{D}_\sigma$. Next result shows that if $X \in \mathfrak{D}_\sigma$ then $X|_{\mathbb{R}^2 \setminus \overline{D}_s} \in \mathfrak{D}(\mathbb{R}^2 \setminus \overline{D}_s)$ for all $s \geqslant \sigma$.

Lemma 7. *Let $X \in \mathcal{D}_\sigma$ be a differentiable vector field. Then for all $s \geqslant \sigma$ we have that* $\mathrm{Trace}(DX) < 0$ *on* $\mathbb{R}^2 \setminus \overline{D}_s$ *and so* $\mathrm{Trace}(DX)|_{\mathbb{R}^2\setminus\overline{D}_s} : \mathbb{R}^2 \setminus \overline{D}_s \to \mathbb{R}$ *is Lebesgue almost-integrable.*

Proof. By the constraints on $\mathrm{Spec}(X)$, for each $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, all the eigenvalues of D_pX have negative real parts so that $\mathrm{Trace}(DX) < 0$ on $\mathbb{R}^2 \setminus \overline{D}_s \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$. The Lebesgue integrability of $\mathrm{Trace}(DX)|_{\mathbb{R}^2\setminus\overline{D}_s} : \mathbb{R}^2 \setminus \overline{D}_s \to \mathbb{R}$ follows from the definition. □

In the proof of next theorem we make use of the following result due to Gutierrez and Rabanal [14].

Theorem 8. *Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable vector field. If for some $\epsilon > 0$,* $\mathrm{Spec}(X) \cap (-\epsilon, +\infty) = \emptyset$*, then there exists $s_0 \geqslant \sigma$ such that $X|_{\mathbb{R}^2\setminus\overline{D}_{s_0}}$ can be extended to a globally injective local homeomorphism $\widetilde{X} : \mathbb{R}^2 \to \mathbb{R}^2$.*

Remark 9. An immediate consequence of Theorem 8 is that if $X \in \mathcal{D}_\sigma$ then outside a big disk $\overline{D}_R \supset \overline{D}_\sigma$, the vector field X has no singularity. In addition, by Lemma 7, the divergent of X is negative on $\mathbb{R}^2 \setminus \overline{D}_\sigma$ so that by Lemma 3, X admits at most one periodic trajectory contained in $\mathbb{R}^2 \setminus \overline{D}_\sigma$. So we may take R big enough so that $\mathbb{R}^2 \setminus \overline{D}_R$ is a region free of singularities and periodic trajectories. Put differently, X has neither singularities nor periodic trajectories at infinity. As $\mathcal{D}_\sigma$ is invariant by translation (i.e., $X + v \in \mathcal{D}_\sigma$ whenever $X \in \mathcal{D}_\sigma$ and $v \in \mathbb{R}^2$), we have that if $X \in \mathcal{D}_\sigma$ and $v \in \mathbb{R}^2$, then $X + v \in \mathcal{D}_\sigma$ and so has neither singularities nor periodic trajectories at infinity.

Theorem 10. *Let $X \in \mathcal{D}_\sigma$ be a differentiable vector field. Then for some $s_0 \geqslant \sigma$, there exist $v \in \mathbb{R}^2$, $c > 0$ and a globally injective local homeomorphism $Y : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

(1) $Y(0) = 0$;
(2) $Y|_{\mathbb{R}^2\setminus\overline{D}_{s_0}} = X|_{\mathbb{R}^2\setminus\overline{D}_{s_0}} + v$;
(3) $\|Y(p)\| > c$ *for any* $p \in \mathbb{R}^2 \setminus \overline{D}_{s_0}$;
(4) $\mathrm{Trace}(DY)|_{\mathbb{R}^2\setminus\overline{D}_{s_0}} : \mathbb{R}^2 \setminus \overline{D}_{s_0} \to \mathbb{R}_-$ *is Lebesgue almost-integrable*;
(5) $Y|_{\mathbb{R}^2\setminus\overline{D}_{s_0}}$ *preserves orientation*;
(6) *Y has neither singularities nor periodic trajectories in* $\mathbb{R}^2 \setminus \overline{D}_{s_0}$;
(7) $Y|_{\mathbb{R}^2\setminus\overline{D}_{s_0}}$ *generates a positive semiflow.*

Proof. By the assumptions on $\mathrm{Spec}(X)$, we have that $\mathrm{Spec}(X) \cap (-\epsilon, +\infty) = \emptyset$. So by Theorem 8 there exist $s_0 \geqslant \sigma$ and a global injective local homeomorphism $\widetilde{X} : \mathbb{R}^2 \to \mathbb{R}^2$ which extends $X|_{\mathbb{R}^2\setminus\overline{D}_{s_0}}$. Set $v = -\widetilde{X}(0)$ and $Y = \widetilde{X} + v$ to get the desired map Y. (1) and (2) follow at once. (3) follows from (1) together with the global injectivity and openness of the map Y. (4) follows from (2), from the invariance of $\mathcal{D}_\sigma$ by translation, and from Lemma 7. To prove (5), observe that $\mathrm{Det}(D_pY) = \mathrm{Det}(D_pX) > 0$ for all $p \in \mathbb{R}^2 \setminus \overline{D}_{s_0}$. (6) follows from Remark 9 under the assumption that s_0 is large enough. Finally, (7) follows from (4), (6) and Theorem 4. □

In the forthcoming sections, we will exploit Theorem 10 as fully as possible. We now turn ourselves to an integration theory problem. In order that $\mathcal{I}(X)$ be well-defined, we have to show that there exists some differentiable global extension of $X|_{\mathbb{R}^2\setminus\overline{D}_r}$, for some $r>\sigma$, whose divergent is Lebesgue almost-integrable on $\mathbb{R}^2$. This is the purpose of next theorem. Notice that the continuous extension $\widetilde{X}:\mathbb{R}^2\to\mathbb{R}^2$ provided by Theorem 8 may be not differentiable on $\overline{D}_{s_0}$.

Theorem 11. *Let $X\in\mathfrak{D}(\mathbb{R}^2\setminus\overline{D}_\sigma)$. Then, for some $r>\sigma$, $X|_{\mathbb{R}^2\setminus\overline{D}_r}$ admits a differentiable global extension $\widetilde{X}\in\mathfrak{D}(\mathbb{R}^2)$ whose divergent is Lebesgue almost-integrable on $\mathbb{R}^2$.*

Proof. Let $r_1>\sigma$ and $\lambda:\mathbb{R}^2\to[0,1]$ be a smooth bump function such that $\lambda(z)=0$ for $\|z\|\leqslant r_1$ and $\lambda(z)=1$ for $\|z\|\geqslant r_1+1$. Given $\epsilon>0$, let $X_1:\mathbb{R}^2\to\mathbb{R}^2$ be a C^1 map such that $\|X_1(z)-X(z)\|<\epsilon$ for all $r_1\leqslant\|z\|\leqslant r_1+1$. Define $\widetilde{X}:\mathbb{R}^2\to\mathbb{R}^2$ to be the differentiable map satisfying

$$\widetilde{X}(z)=\lambda(z)X(z)+\big(1-\lambda(z)\big)X_1(z),$$

where as usual we define $\lambda(z)X(z)=0$ for $z\in\overline{D}_\sigma$.

Let $A=\overline{D}_{r_1}$, $B=\overline{D}_{r_1+1}\setminus D_{r_1}$ and $C=\mathbb{R}^2\setminus D_{r_1+1}$. We have that $\mathbb{R}^2=A\cup B\cup C$. Furthermore,

$$\widetilde{X}|_A=X_1|_A, \tag{1}$$

$$\widetilde{X}|_B=\lambda|_B X|_B+(1-\lambda|_B)X_1|_B, \tag{2}$$

$$\widetilde{X}|_C=X|_C. \tag{3}$$

Since $X\in\mathfrak{D}(\mathbb{R}^2\setminus\overline{D}_\sigma)$, we have that

$$\min\Bigg\{\int\limits_{\mathbb{R}^2\setminus\overline{D}_\sigma}\operatorname{Trace}^+(DX)\,dx\wedge dy,\int\limits_{\mathbb{R}^2\setminus\overline{D}_\sigma}\operatorname{Trace}^-(DX)\,dx\wedge dy\Bigg\}<\infty.$$

Without loss of generality, we may assume that

$$\int\limits_{\mathbb{R}^2\setminus\overline{D}_\sigma}\operatorname{Trace}^+(DX)\,dx\wedge dy<\infty. \tag{4}$$

From the smoothness of X_1 and (1), we get that $\int_A\operatorname{Trace}^+(D\widetilde{X})\,dx\wedge dy<\infty$. On the other side, from (3) and (4),

$$\int\limits_C\operatorname{Trace}^+(D\widetilde{X})\,dx\wedge dy\leqslant\int\limits_{\mathbb{R}^2\setminus\overline{D}_\sigma}\operatorname{Trace}^+(DX)\,dx\wedge dy<\infty.$$

The proof will be finished if we show that $\int_B\operatorname{Trace}^+(D\widetilde{X})\,dx\wedge dy<\infty$. By differentiating Eq. (2) we reach for $z\in B$,

$$\begin{aligned}\operatorname{Trace}(D_z\widetilde{X})&=\lambda(z)\operatorname{Trace}(D_zX)+\big(1-\lambda(z)\big)\operatorname{Trace}(D_zX_1)\\&\quad+\lambda_x(z)\big(f(z)-f_1(z)\big)+\lambda_y(z)\big(g(z)-g_1(z)\big),\end{aligned} \tag{5}$$

where $X = (f, g)$ and $X_1 = (f_1, g_1)$. Since $\|X_1 - X\| < \epsilon$ on B, we have that $f_1(z) - f(z)$ and $g(z) - g_1(z)$ are bounded in B. The function λ and its partial derivatives are also bounded. Moreover, $\mathrm{Trace}(D_z X_1)$ is a smooth function on the compact B. Finally, from (4) it follows that $\int_B \mathrm{Trace}^+(DX)\,dx \wedge dy < \infty$. By Eq. (5) we get that $\int_B \mathrm{Trace}^+(D\widetilde{X})\,dx \wedge dy < \infty$. Hence, by the above and by using that $\mathbb{R}^2 = A \cup B \cup C$, it follows that $\int_{\mathbb{R}^2} \mathrm{Trace}^+(D\widetilde{X})\,dx \wedge dy < \infty$ so that $\mathrm{Trace}(D\widetilde{X})$ is Lebesgue almost-integrable. To finish the proof take $r = r_1 + 1$ and use (3). □

We will need the following lemma.

Lemma 12. *Let $X \in \mathfrak{D}(\mathbb{R}^2 \setminus \overline{D}_\sigma)$ and $\widehat{X}_1, \widehat{X}_2 \in \mathfrak{D}(\mathbb{R}^2)$ be differentiable global extensions of $X|_{\mathbb{R}^2 \setminus \overline{D}_r}$ for some $r > \sigma$, that is, $\widehat{X}_i(z) = X(z)$, for all z with $\|z\| > r$ and for $i = 1, 2$. Then*

$$\int_{\mathbb{R}^2} \mathrm{Trace}(D\widehat{X}_1)\,dx \wedge dy = \int_{\mathbb{R}^2} \mathrm{Trace}(D\widehat{X}_2)\,dx \wedge dy.$$

Proof. Thanks to Green's formula as presented in [23], the proof of Proposition 2.1 of [1] (which is the C^1 version of Lemma 12) also works in this case. □

Corollary 13. *Let $X \in \mathfrak{D}_\sigma$ be a differentiable vector field. Then the index $\mathcal{I}(X)$ of X at infinity is a well-defined number of the extended real line $[-\infty, \infty)$.*

Proof. It follows from Lemma 7 and Theorem 11 that, for some $r > \sigma$, $X|_{\mathbb{R}^2 \setminus \overline{D}_r}$ admits a differentiable global extension $\widehat{X} \in \mathfrak{D}(\mathbb{R}^2)$ whose divergent is Lebesgue almost integrable on $\mathbb{R}^2$. From Lemma 12, $\mathcal{I}(X)$ does not depend on the extension so that it is well-defined. Since at infinity $\mathrm{Trace}(DX)$ is negative, we have that $\mathcal{I}(X) < \infty$. □

3. Transversal sections to continuous vector fields

When constructing transversal sections to smooth vector fields we can take advantage of many tools such as the continuous dependence of the flow with respect to initial conditions and the flow box theorem. In the continuous case, the picture turns out to be different because the local uniqueness of solutions fails. Meanwhile, as the following result shows, we still have some kind of continuous dependence with respect to initial conditions.

We first introduce some notation. Let $F : V \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. If γ_p is a trajectory of F passing through $p \in V$, then $J(\gamma_p)$ denote its maximal interval of existence. We denote by $J(p)$ the subset of the real line

$$J(p) = \bigcap_{\gamma_p} \bigl\{ J(\gamma_p) \colon \gamma_p \text{ is a trajectory of } F \text{ passing through } p \bigr\},$$

which, by Peano's existence theorem, is an interval containing p (see [25, Corollary 4]).

Lemma 14. *Let $F : V \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field defined on an open set V. Let $p_0 \in V$ and assume that $J(p_0) \supset [0, \tau]$. Then for each $\epsilon > 0$, there exist $\delta > 0$, such that if $\|p - p_0\| < \delta$ then*:

(i) $J(p) \supset [0, \tau]$;
(ii) *for each trajectory γ_p of F passing through p, there exists some trajectory γ_{p_0} of F passing through p_0 such that $\|\gamma_p(t) - \gamma_{p_0}(t)\| < \epsilon$ for all $t \in [0, \tau]$.*

Proof. We refer the reader to [25, Theorem 4] (see also [2]). □

In the next theorem we assume that the positive semi-trajectories of X are unique and so that X generates a positive semiflow.

Theorem 15. *Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field with unique positive semi-trajectories, defined on an open set U free of singularities*; *γ be a positive semi-trajectory of X with maximal interval of existence $J(\gamma) \supset [0, \tau]$, $z_1 = \gamma(0)$ and $z_2 = \gamma(\tau)$; and let Σ_2 be a local transversal section to X passing through z_2. Then, in each connected component of $\Sigma_2 \setminus \{z_2\}$, there exist a point $\tilde{z}_2$ arbitrarily close to z_2, and a C^1 segment Δ transversal to X, starting at z_1, ending at $\tilde{z}_2$, and close to the subarc of trajectory $[z_1, z_2] \subset \gamma$ of X.*

Proof. Since $J(\gamma)$ is open, we may choose $\bar{\tau} > \tau$ in $J(\gamma)$. Let $\widetilde{X} : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field transversal to X. We wish to find a transversal segment to X that, for some $\lambda > 0$, is a trajectory of the perturbed vector field $X_\lambda : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined by $X_\lambda = X + \lambda\widetilde{X}$. For so we expand the phase space to include the parameter λ by considering the extended vector field $F : U \times [0, 1] \to \mathbb{R}^3$ defined by $F(z, \lambda) = (X_\lambda(z), 0)$. Let $\pi_1 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ and $\pi_2 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be the canonical projections. It is plain that if $\gamma_{(z,\lambda)}$ is a trajectory of F passing through $(z, \lambda) \in U \times [0, 1]$, then $\pi_1 \circ \gamma_{(z,\lambda)}$ is a trajectory of X_λ passing through z and $(\pi_2 \circ \gamma_{(z,\lambda)})(t) \equiv \lambda$. In particular, as $X_0 = X$ generates a positive semiflow, all positive semi-trajectories of F passing through $(z, 0) \in U \times [0, 1]$ are unique. So the only positive semi-trajectory of F passing through $(z_1, 0)$ is $\gamma_{(z_1,0)}(t) = (\gamma(t), 0)$. Hence $J(z_1, 0) = J(\gamma) \supset [0, \bar{\tau}]$. It follows from Lemma 14 that given $\epsilon > 0$, there exists $\delta > 0$ such that if $\|(z_1, \lambda) - (z_1, 0)\| < \delta$ then all trajectory $\gamma_{(z_1,\lambda)}$ of F passing through (z_1, λ) satisfies $J(\gamma_{(z_1,\lambda)}) \supset [0, \bar{\tau}]$ and $\|\gamma_{(z_1,\lambda)}(t) - (\gamma(t), 0)\| < \epsilon$, $\forall t \in [0, \bar{\tau}]$. For each $(z_1, \lambda) \in U \times [0, 1]$, choose some trajectory $\gamma_{(z_1,\lambda)}$ of F starting at (z_1, λ) and set $\gamma_\lambda = \pi_1 \circ \gamma_{(z_1,\lambda)}$. So γ_λ is a trajectory of X_λ starting at z_1. By the above, if λ is small enough, then $J(\gamma_\lambda) \supset [0, \bar{\tau}]$ and $\sup_{t\in[0,\bar{\tau}]} \|\gamma_\lambda(t) - \gamma(t)\| < \epsilon$. Hence, since γ cross Σ_2 transversally at $z_2 = \gamma(\tau)$, we have that there exists $\tau_2 \in [0, \bar{\tau}]$ such that $\gamma_\lambda(\tau_2) \in \Sigma_2$. Set $\tilde{z}_2(\lambda) = \gamma_\lambda(\tau_2)$ and let $\Delta(\lambda) = [z_1, \tilde{z}_2(\lambda)] \subset \gamma_\lambda$ be the subarc of trajectory of γ_λ connecting z_1 to $\tilde{z}_2(\lambda)$. It is easy to see that if $\lambda > 0$ is small enough then $\tilde{z}_2 = \tilde{z}_2(\lambda)$ and the segment $\Delta = \Delta(\lambda)$ has all the properties required. To get a point $\tilde{z}_2$ in the other connected component of $\Sigma_2 \setminus \{z_2\}$, replace $\widetilde{X}$ by $-\widetilde{X}$ and proceed in the same way. □

4. Pseudo-hyperbolic sector at infinity

Definition 16 *(Pseudo-hyperbolic sector).* Given a vector field $X \in \mathcal{D}_\sigma$, let $S = S(p_1, p_2; q_1, q_2, \{\sigma_i\}) \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ be the unbounded region whose boundary ∂S is made up of two unbounded semi-trajectories $[q_1, \infty)$ and $(\infty, q_2]$ of X, a compact arc of trajectory $[p_1, p_2]$ of X, two arcs of trajectory $[p_1, q_1]^*$, $[p_2, q_2]^*$ of X^*, and a set at most countable (which may be empty) of pairwise disjoint trajectories $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ that start and end at ∞ (see Fig. 2). We call such a region a pseudo-hyperbolic sector of X if the following conditions are satisfied:

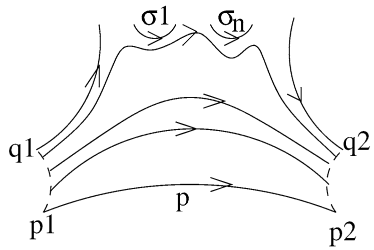

Fig. 2. Pseudo-hyperbolic sector.

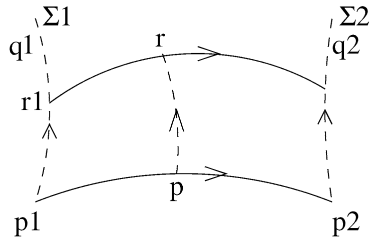

Fig. 3.

(1) for each $z \in [p_1, q_1)^*$, there exists an arc of trajectory $[z, \pi(z)] \subset S$ of X starting at $z \in [p_1, q_1)^*$ and ending at $\pi(z) \in [p_2, q_2]^*$;
(2) $\overline{\bigcup_{z\in[p_1,q_1)}[z, \pi(z)]} = S$.

In this way, the map $\pi : [p_1, q_1)^* \to [p_2, q_2]^*$ is nothing but the forward Poincaré map induced by the positive semiflow associated to $X|_{\mathbb{R}^2\setminus\overline{D}_\sigma}$ (see Lemma 7 and Theorem 4). Let us call the unbounded part of ∂S the set

$$\partial_+ S = [q_1, \infty) \cup (\infty, q_2] \cup \bigcup_{i=1}^{\infty} \sigma_i \subset \partial S.$$

Hereafter all efforts we make are towards proving the main theorem of this section, stated below. In what follows, the vector field Y and the positive constant s_0 are as in Theorem 10.

Theorem 17. *There is no pseudo-hyperbolic sector of Y contained in $\mathbb{R}^2 \setminus \overline{D}_s$, for any $s \geqslant s_0$.*

Before proving Theorem 17, we give some preparatory lemmas.

Lemma 18. *Let $s \geqslant s_0$ and let $[p_1, q_1]^* \in \mathbb{R}^2 \setminus \overline{D}_s$ be a fixed arc of trajectory of Y^*. Then, there exists $K > 0$ such that for any compact rectangle $R = R(p_1, p, r_1, r) \subset \mathbb{R}^2 \setminus \overline{D}_s$ of Y satisfying $[p_1, r_1]^* \subset [p_1, q_1]^*$ we have that $\ell([p, r]^*) < K$. See Fig. 3.*

Proof. By Lemma 3, for any rectangle $R(p_1, p, r_1, r) \subset \mathbb{R}^2 \setminus \overline{D}_s$,

$$L([p, r]^*) - L([p_1, r_1]^*) = \int_R \mathrm{Trace}(DY)\, dx \wedge dy < 0.$$

Setting $d = \sup\{\|Y(z)\|\colon z \in [p_1, q_1]^*\}$ and using (3) of Theorem 10 yields

$$c\ell\big([p,r]^*\big) \leqslant \left| \int\limits_{[p,r]^*} \|Y\|\, ds \right| = L\big([p,r]^*\big) < L\big([p_1,r_1]^*\big) = \left| \int\limits_{[p_1,r_1]^*} \|Y\|\, ds \right| \leqslant d\ell\big([p_1,r_1]^*\big).$$

Therefore, setting $K = \frac{d}{c}\ell([p_1,q_1]^*)$, we obtain

$$\ell\big([p,r]^*\big) \leqslant \frac{d}{c}\ell\big([p_1,r_1]^*\big) \leqslant \frac{d}{c}\ell\big([p_1,q_1]^*\big) = K. \qquad \square$$

Lemma 19. *Let $S = S(p_1, p_2, q_1, q_2, \{\sigma_i\})$ be a pseudo-hyperbolic sector of Y contained in $\mathbb{R}^2 \setminus \overline{D}_s$ for some $s \geqslant s_0$. Then for each $q \in \partial_+ S$, there exist $p \in [p_1, p_2]$ and arc of trajectory $[p,q]^* \subset S$ of Y^* departing from p and ending at q.*

Proof. Let $q \in \partial_+ S$ and $\pi : [p_1, q_1)^* \to [p_2, q_2]^*$ be the forward Poincaré map induced by the positive semiflow generated by $Y|_{\mathbb{R}^2\setminus\overline{D}_{s_0}}$. Let $\{z_n\}_1^\infty \to q_1$ be a sequence in $[p_1, q_1]^*$. Set $w_n = \pi(z_n) \in [p_2, q_2]^*$. Then $w_n \to q_2$ as $n \to \infty$ and the arc of trajectory $[z_n, w_n]$ of Y accumulates in $\partial_+ S$. Let γ_q^- be any negative semi-trajectory of Y^* starting at q. Hence, for some $n \in \mathbb{N}$, γ_q^- goes into the compact rectangle $R(p_1, p_2, z_n, w_n)$. Now, by Lemma 5, γ_q^- meets $[p_1, p_2] \cup [p_1, z_n]^* \cup [p_2, w_n]^*$ and so γ_q^- meets $A = [p_1, p_2] \cup [p_1, q_1]^* \cup [p_2, q_2]^*$. By a patching-arcs procedure, as described right below, we can find an arc of trajectory $[p,q]^*$ of Y^* as requested in this lemma. In fact, if γ_q^- meets A, for the first time, at $p \in [p_1, p_2]$, then the subarc $[p,q]$ of γ_q^- satisfies the conditions requested in this lemma; if γ_q^- meets A, for the first time, at $r \in [p_1, q_1]^*$ (respectively at $r \in [p_2, q_2]^*$), the arc $[p,q]$ made up by the union of the subarc $[p_1, r]^*$ of $[p_1, q_1]^*$ (respectively $[p_2, r]^*$ of $[p_2, q_2]^*$) with the subarc $[r,q]^*$ of γ_q^- satisfies the conditions requested in this lemma. $\square$

Lemma 20. *Let $s \geqslant s_0$ and let $S = S(p_1, p_2; q_1, q_2, \{\sigma_i\}) \subset \mathbb{R}^2 \setminus \overline{D}_s$ be a pseudo-hyperbolic sector of Y. Then there exists constant $K > 0$ such that any arc of trajectory $\gamma^* = [p,q]^* \subset S$ of Y^* connecting a point $p \in [p_1, p_2]$ with a point $q \in \partial S$ satisfies $\ell(\gamma^*) \leqslant K$.*

Proof. As $\gamma^* = [p,q]^*$ ends at $q \in \partial S$, so either γ^* ends at $[p_1, q_1]^* \cup [p_2, q_2]^*$, or it ends at $\partial_+ S$. By a patching-arcs procedure, as described in the proof of Lemma 19, we may assume that $q \in \partial_+ S$. Let $\{r_1^{(n)}\}_1^\infty \to q_1$ be a sequence in $[p_1, q_1]^*$. Denote by γ_n the positive semi-trajectory of $Y|_{R^2\setminus\overline{D}_s}$ starting at $r_1^{(n)}$, whose uniqueness follows from item (7) of Theorem 10. Set $r^{(n)} = \gamma_n \cap \gamma^*$. See Fig. 4. As γ_n accumulates in $\partial_+ S$ as n tends to infinity, we have that $\gamma^* = \limsup [p, r^{(n)}]^*$. Then, from Lemma 18, there exists constant $K > 0$, not depending on γ^*, such that $\ell(\gamma^*) = \lim_{n\to\infty} \ell([p, r^{(n)}]^*) \leqslant K$. $\square$

Lemma 21. *Let $s \geqslant s_0$ and let $S = S(p_1, p_2; q_1, q_2, \{\sigma_i\}) \subset \mathbb{R}^2 \setminus \overline{D}_s$ be a pseudo-hyperbolic sector of Y. Then there exists constant $K > 0$ such that $d(q, [p_1, p_2]) \leqslant K$, for all $q \in \partial_+ S$.*

Proof. Let $q \in \partial_+ S$. From Lemmas 19 and 20, it follows that there exist constant $K > 0$ not depending on q, and arc of trajectory $[p,q]^* \subset S$ of Y^* with $p \in [p_1, p_2]$ and $\ell([p,q]^*) \leqslant K$. So $d(q, [p_1, p_2]) \leqslant K$, for all $q \in \partial_+ S$. $\square$

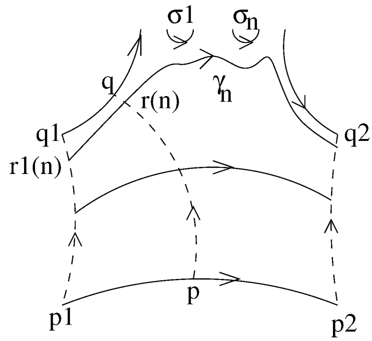

Fig. 4.

Proof of Theorem 17. Assume, for contradiction, that Y admits a pseudo-hyperbolic sector $S = S(p_1, p_2; q_1, q_2, \{\sigma_i\})$ contained in $\mathbb{R}^2 \setminus \overline{D}_s$ for some $s \geqslant s_0$. By Lemma 21, there exists constant $K > 0$ such that $d(q, [p_1, p_2]) \leqslant K$, for all $q \in \partial_+ S$. In particular, as $[p_1, p_2]$ is compact, we have that $\partial_+ S$ is a bounded set. This is an absurd. □

5. Transversal circles around infinity

This section is devoted to the construction of a C^1 circle, contained in $\mathbb{R}^2 \setminus \overline{D}_s$, transversal to the differentiable vector field Y, for s arbitrarily large. Let $\mathcal{C} = \mathcal{C}_s$ denote the class of the piecewise C^1 circles contained in $\mathbb{R}^2 \setminus \overline{D}_s$. A circle $C \in \mathcal{C}$ is said to be internally (respectively externally) tangent to a differentiable vector field $X : \mathbb{R}^2 \setminus \overline{D}_s \to \mathbb{R}^2$ at $p \in C$ if for each trajectory γ passing through p, there exists $\epsilon > 0$ such that $\gamma(t) \in D(C)$ (respectively $\gamma(t) \in \mathbb{R}^2 \setminus \overline{D}(C)$) for all $0 < |t| < \epsilon$, where $D(C)$ (respectively $\overline{D}(C)$) denotes the open (respectively compact) disk bounded by C. If this is the case, we say that C has an internal (respectively external) tangency with X at p. A circle $C \in \mathcal{C}$ is said to be in general-position with the differentiable vector field $X : \mathbb{R}^2 \setminus \overline{D}_s \to \mathbb{R}^2$ if there exists a subset F of C at most finite such that: (i) X is transversal to C in $C \setminus F$; (ii) C is internally or externally tangent to X at each point of F; (iii) any trajectory of X meets C tangentially at most at one point. We denote the class of circles in $\mathbb{R}^2 \setminus \overline{D}_s$ in general position with X by $\mathcal{GP}(X, s)$. In what follows, Y is the vector field as in Theorem 10.

Lemma 22. *For each $s \geqslant s_0$, $\mathcal{GP}(Y, s) \neq \emptyset$.*

Proof. Let $C = \{p \in \mathbb{R}^2\colon \|p\| = s + 1\}$ and let $0 < \varepsilon < 0.1$. By (4) of Theorem 10, $p \mapsto \frac{Y(p)}{\|Y(p)\|}$ is a continuous map defined on $\mathbb{R}^2 \setminus \overline{D}_s \subset \mathbb{R}^2 \setminus \overline{D}_{s_0}$. So there exists a cover $\{B_i\}_{i=1}^N$ of C by open balls contained in $\mathbb{R}^2 \setminus \overline{D}_s$ so small that

(a) if p, q belong to the same ball B_i then $\|\frac{Y(p)}{\|Y(p)\|} - \frac{Y(q)}{\|Y(q)\|}\| < \varepsilon$.

Let $m > 0$ be a natural number so large that $\frac{8(s+1)}{m}$ is a Lebesgue number for the cover above. For all $j \in \{0, 1, 2, \ldots, m\}$, let $p_j = (s + 1)(\cos \frac{2\pi j}{m}, \sin \frac{2\pi j}{m}) \in C$. In this way, for all $j \in \{0, 1, 2, \ldots, m - 1\}$, $\|p_{j+1} - p_j\| < \frac{2\pi(s+1)}{m} < \frac{8(s+1)}{m}$. For every $j \in \{0, 1, 2, \ldots, m - 1\}$, select $q_j \in \mathbb{R}^2$ so that $\Delta_j = \{p_j, p_{j+1}, q_j\}$ consists of the vertices of an equilateral triangle; certainly, the diameter of Δ_j is less than the Lebesgue number $\frac{8(s+1)}{m}$ and so $\Delta_j \subset \mathbb{R}^2 \setminus \overline{D}_s$, for all j. If the arc $[p_j, p_{j+1}]_C \subset C$ is transversal to Y, define $\Gamma_j = [p_j, p_{j+1}]_C$; otherwise, define Γ_j as the union of the linear segments $[p_j, q_j]$ and $[q_j, p_{j+1}]$. Take m large enough, say $m > 16$, so that the angular variation of the unit tangent vector to C within $[p_j, p_{j+1}]_C$ is less than $\frac{\pi}{8}$ for all $j \in \{0, 1, 2, \ldots, m-1\}$. From this and from (a) it follows that $\Gamma_j \setminus \Delta_j$ is transversal to Y. The circle $\Gamma = \bigcup_{j=0}^{m-1} \Gamma_j$ is transversal to Y except possibly at a finite subset of $\bigcup_{j=0}^{m-1} \Delta_j$. As $Y|_{\mathbb{R}^2 \setminus \overline{D}_s}$ has neither singularities nor closed orbits, by the Poincaré–Bendixson theorem for semi-flows (see [4]) no positive semi-trajectory of $Y|_{\mathbb{R}^2 \setminus \overline{D}_s}$ is recurrent. It is not difficult to conclude from this that Γ may be approximated by a piecewise C^1 circle of $\mathcal{GP}(Y, s)$. $\square$

Remark 23. Let $s \geqslant s_0$ and let $C \in \mathcal{GP}(Y, s)$ be a piecewise C^1 circle in general position with Y. Assume that C has an internal tangency with Y at the point q. Then looking at the trajectories of Y around q we see that there must exist closed subintervals $[p, q]_C \subset C$ and $[q, r]_C \subset C$, with $[p, q]_C \cap [q, r]_C = \{q\}$ and an orientation reversing, continuous, surjective map $T : [p, q]_C \to [q, r]_C$ induced by the positive semiflow associated to $Y|_{\mathbb{R}^2 \setminus \overline{D}_{s_0}}$ with the following properties:

P.1. For each $z \in (p, q)$, there exists an arc of trajectory $[z, T(z)] \subset \mathbb{R}^2 \setminus D(C)$ of Y that meets C transversally and precisely at $\{z, T(z)\}$.

P.2. The family $\{[z, T(z)]\colon z \in (p, q)\}$ depends continuously on z and tends to the one point set $\{q\}$ as $z \to q$.

Lemma 24. *Let $s \geqslant s_0$ and $C \in \mathcal{GP}(Y, s)$ be a piecewise C^1 circle in general position with Y. Assume that C has an internal tangency with Y at the point q. If $[p, q]_C \subset C$ is maximal with respect to property* P.1 *of Remark* 23 *then*:

(i) *The positive semi-trajectory γ_p^+ starting at p contains an arc of trajectory $[p, r]$ of Y that meets C precisely at $\{p, r\}$.*

(ii) *C is transversal to $[p, r]$ at one of its endpoints and has an external tangency at the other endpoint.*

(iii) *Let $\Gamma = [p, r]_C \cup [p, r]$. If r (respectively p) is the external tangency then $\overline{D}(\Gamma)$ is contained in $\mathbb{R}^2 \setminus D(C)$ and the points of $\gamma_p^+ \setminus [p, r]$ nearby r (respectively p) do not belong to $\overline{D}(\Gamma)$.*

Proof. (i) First, we show that $\gamma_p^+ \cap C \supsetneq \{p\}$. Assume the contrary, that is, that $\gamma_p^+ \cap C = \{p\}$. So either $\gamma_p^+ \subset \overline{D}(C)$ or $\gamma_p^+ \subset \mathbb{R}^2 \setminus D(C)$. By property P.1 it is not difficult to see that $\gamma_p^+ \subset \mathbb{R}^2 \setminus D(C)$. By (6) of Theorem 10 there are neither periodic orbits nor singularities in $\mathbb{R}^2 \setminus D(C) \subset \mathbb{R}^2 \setminus \overline{D}_{s_0}$. So, by Lemma 5, $\omega(\gamma_p^+) = \infty$. Now let $r \in C$ be the unique point satisfying $[q, r) = T((p, q])$ and let γ_r^- be any negative semi-trajectory of Y starting at r. Let us show that $\alpha(\gamma_r^-) = \infty$. Assume, by contradiction, that there exists some circle C_1 with $C \subset D(C_1)$ and $\gamma_r^- \subset D(C_1)$. Once more, by Lemma 5, as $\overline{D}(C_1) \setminus D(C)$ is a compact region free of singularities and periodic orbits, and as all tangencies of C with Y are either external or internal (C is in generic position), we have that γ_r^- has to cross C transversally at some point $r_1 \neq r$. Take now $z_n \to p$, $z_n \in (p, q]$. From the assumption of maximality of $[p, q]_C$, the sequence of arcs of trajectory $\{[z_n, T(z_n)]\}$ of Y accumulates in the positive arc of trajectory $[r_1, r]$ of Y. So for n big enough $[z_n, T(z_n)] \cap C \supsetneq \{z_n, T(z_n)\}$, which contradicts P.1. Therefore, $\alpha(\gamma_r^-) = \infty$. It is not difficult to see that γ_p^+ and γ_r^- form the boundary of a pseudo-hyperbolic sector, even in

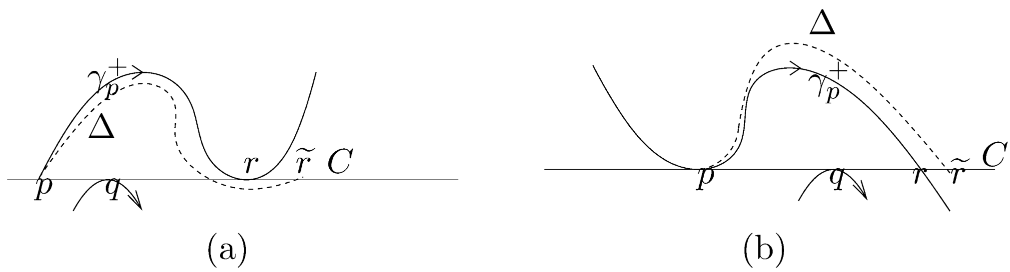

Fig. 5. Transversal section to a positive semiflow.

the case when $p = r$. This contradiction with Theorem 17 proves (i). Item (ii) follows from the maximality of $[p, q]_C$. The proof of item (iii) is the same as that of Lemma 2 in [15]. □

Lemma 25. *Let $s \geqslant s_0$ and $C \in \mathcal{GP}(Y, s)$ be a piecewise C^1-circle in general position with Y. Assume that C has an internal tangency with Y at the point q. Take all the notation of Lemma* 24. *Then there exists $\tilde{r} \in C$ arbitrarily close to r such that the subinterval $[p, \tilde{r}]_C$ of C contains the subinterval $[p, r]_C \subset C$, and the following holds*:

(i) *We can deform the circle C into a new circle $C_1 \in \mathcal{GP}(Y, s)$ in such a way that the deformation fixes $C \setminus (p, \tilde{r})_C$ and takes $[p, \tilde{r}]_C \subset C$ to an interval $[p, \tilde{r}]_{C_1} \subset C_1$ transversal to Y, and so free of tangencies with Y, which is close to the arc of trajectory $[p, r]$ of Y, see Fig.* 5.
(ii) *The number of internal tangencies of C_1 with Y is strictly smaller than that of C.*

Proof. (i) Let γ_p^+ be the positive semi-trajectory of $Y|_{\mathbb{R}^2 \setminus \overline{D}_s}$ starting at p and let Σ_2 be a local transversal section to Y passing through $z_2 \in \gamma_p^+ \setminus [p, r]$, where $[p, r]$ is the (unique) arc of trajectory of $Y|_{\mathbb{R}^2 \setminus \overline{D}_s}$ which starts at p and ends at r. By Theorem 15, we may choose some vector field $Y_\lambda : \mathbb{R}^2 \setminus \overline{D}_s \to \mathbb{R}^2$, transversal to $Y|_{\mathbb{R}^2 \setminus \overline{D}_s}$, and some arc of trajectory Δ of Y_λ which departs from p, ends at $\tilde{z}_2 \in \Sigma_2 \setminus \{z_2\}$ and is close to the arc of trajectory $[p, z_2] \subset \gamma_p^+$ of $Y|_{\mathbb{R}^2 \setminus \overline{D}_s}$. Furthermore, by adjusting Y_λ, we may take $\tilde{z}_2$ arbitrarily close to z_2 and in any of the two connected components of $\Sigma_2 \setminus \{z_2\}$. So by taking $\tilde{z}_2$ in the appropriate connected component of Σ_2, we have that the corresponding arc of trajectory $\Delta = [p, \tilde{z}_2]$ of Y_λ intersects C at some point $\tilde{r}$ close to r and in such a way that $[p, r]_C \subset [p, \tilde{r})_C$. The subarc of trajectory $[p, \tilde{r}] \subset \Delta$ of Y_λ has all the properties required. By replacing $[p, \tilde{r}]_C$ in C by $[p, \tilde{r}] \subset \Delta$ we get the circle C_1.

(ii) We just observe that in the gluing points p and $\tilde{r}$ of $C \setminus (p, \tilde{r})_C$ with $[p, \tilde{r}] \subset \Delta$ the vector field Y is still transversal to C_1. So the deformation replace the interval $[p, r]_C$ by the segment $[p, \tilde{r}] \subset \Delta$, which eliminates at least two tangencies of C with Y leaving the other ones unchanged. □

Theorem 26. *For each $s \geqslant s_0$, there exist a C^1 circle transversal to Y contained in $\mathbb{R}^2 \setminus \overline{D}_s$.*

Proof. Take a circle $C \in \mathcal{GP}(Y, s)$. As C has finitely many internal tangencies with Y, by applying Lemma 25 finitely many times, we can get a circle $\widetilde{C} \in \mathcal{GP}(Y, s)$ with finitely many tangencies, all external. Let $\deg(Y|_{\widetilde{C}})$ denote the Brower degree of the map $Y|_{\widetilde{C}}$. By Theorem 10, the map $Y|_{\widetilde{C}}$ is injective and preserves orientation; this implies that $\deg(Y|_{\widetilde{C}}) = 1$. On the other hand, as $\widetilde{C} \in \mathcal{GP}(Y, s)$, we have that

$$\deg(Y|_{\widetilde{C}}) = \frac{2 - n_e(Y, \widetilde{C}) + n_i(Y, \widetilde{C})}{2}, \tag{6}$$

where $n_e(Y, \widetilde{C})$ (respectively $n_i(Y, \widetilde{C})$) is the number of external (respectively internal) tangencies of $\widetilde{C}$ with Y (see [19, Theorems 9.1 and 9.2, pp. 166–174]).

As $n_i(Y, \widetilde{C}) = 0$, formula (6) implies that $n_e(Y, \widetilde{C}) = n_i(Y, \widetilde{C}) = 0$. Observing that $\widetilde{C}$ is a piecewise C^1 circle transversal to Y, we can deform it into a C^1 circle $C_1 \in \mathcal{GP}(Y, s)$ transversal to Y. □

6. Asymptotic stability at infinity

In this section we prove the main theorem. In what follows, $X \in \mathcal{D}_\sigma$ is a differentiable vector field and $Y : \mathbb{R}^2 \to \mathbb{R}^2$ is the vector field associated to X through Theorem 10. The constant vector v is as in Theorem 10.

Lemma 27. *The point ∞ is an attractor or repellor of Y.*

Proof. By Theorem 26, there exists a nested family $\{\Gamma_n \subset \mathbb{R}^2 \setminus \overline{D}_{s_0}:\ n \in \mathbb{N}\}$ of C^1 circles transversal to Y tending to infinity. Let $A_n = \overline{D}(\Gamma_n) \setminus D(\Gamma_{n-1})$ denote the corresponding sequence of annulus. By item (6) of Theorem 10, there are neither singularities nor periodic trajectories in A_n so that by Lemma 5 no trajectory of Y has accumulation points in A_n, for all $n \in \mathbb{N}$. This implies that the trajectories of Y that meet Γ_1 have to cross all circles Γ_n. It is plain that under these conditions ∞ is either an attractor or a repellor of Y. □

Theorem 28. *The point at infinity of $\mathbb{R}^2 \cup \{\infty\}$ is an attractor or repellor of $X + v$. More specifically, if $\mathcal{I}(X)$ is less than* 0 (*respectively greater or equal to* 0)*, then ∞ is a repellor* (*respectively an attractor*) *of the vector field $X + v$.*

Proof. That ∞ is an attractor or repellor of $X + v$ follows directly from the previous lemma by recalling that Y and $X + v$ agree around infinity. To finish the proof notice that $\mathcal{I}(X) = \mathcal{I}(X + v) = \mathcal{I}(Y)$. Now we proceed as in [16]. Assume that ∞ is a repellor of $X + v$. Take a C^1 circle $C \subset \mathbb{R}^2 \setminus \overline{D}_s$ transversal to Y such that $Y|_C$ points inwards the disk $D(C)$ bounded by C. By Green's formula $\int_{D(C)} \operatorname{Trace}(DY) < 0$. On the other hand, by statement (4) of Theorem 10 $\int_{\mathbb{R}^2 \setminus D(C)} \operatorname{Trace}(DY) < 0$. So

$$\begin{aligned} \mathcal{I}(X) = \mathcal{I}(Y) &= \int\limits_{\mathbb{R}^2} \operatorname{Trace}(DY)\, dx \wedge dy \\ &= \int\limits_{\overline{D}(C)} \operatorname{Trace}(DY)\, dx \wedge dy + \int\limits_{\mathbb{R}^2 \setminus \overline{D}(C)} \operatorname{Trace}(DY)\, dx \wedge dy < 0. \end{aligned}$$

Hence, if $\mathcal{I}(X) \geqslant 0$ then ∞ is a attractor of $X + v$. The proof of the other case is similar. □

Now we proof our main theorem.

Theorem A. *Let $X:\mathbb{R}^2\setminus\overline{D}_\sigma\to\mathbb{R}^2$ be a differentiable (but not necessarily C^1) vector field. If for some $\epsilon>0$, $\operatorname{Spec}(X)$ is disjoint from $(-\epsilon,0]\cup\{z\in\mathbb{C}\colon \mathcal{R}(z)\geqslant 0\}$, then:*

(a) *for all $p\in\mathbb{R}^2\setminus\overline{D}_\sigma$, there is a unique positive semi-trajectory of X starting at p;*
(b) *$\mathcal{I}(X)$, the index of X at infinity, is a well-defined number of the extended real line $[-\infty,\infty)$;*
(c) *there exists a constant vector $v\in\mathbb{R}^2$ such that if $\mathcal{I}(X)$ is less than 0 (respectively greater or equal to 0), then the point at infinity of the Riemann sphere $\mathbb{R}^2\cup\{\infty\}$ is a repellor (respectively an attractor) of the vector field $X+v$.*

Proof. We have that $X\in\mathfrak{D}_\sigma$ so that by Lemma 7, $X\in\mathfrak{D}(\mathbb{R}^2\setminus\overline{D}_\sigma)$. The proof of (a) is finished applying Theorem 4. The proof of (b) and (c) follows from Corollary 13 and Theorem 28, respectively. □

7. Final remarks

Using methods of [12], we present here a result that shows that in some sense ∞ can be considered as an isolated singularity of a vector field defined in a neighborhood of ∞. More precisely, the vector fields X and Z of the proposition below are topologically equivalent.

Proposition 29. *Let $r\in\{0,1,\dots,\infty\}$ and $X:\mathbb{R}^2\setminus D\to\mathbb{R}^2$ be a C^r-vector field without singularities, where D is the compact unit disc of $\mathbb{R}^2$. Let $Y:D\setminus\{0\}\to\mathbb{R}^2$ be the vector field given by*

$$Y(p)=DH_{H^{-1}(p)}\circ X\circ H^{-1}(p),$$

where $H:\mathbb{R}^2\setminus\{0\}\to D\setminus\{0\}$ is given by $H(x,y)=(x/(x^2+y^2),y/(x^2+y^2))$. Then there exists a smooth function $\varphi:D\to[0,1]$, with $\varphi^{-1}(0)=0$, such that $Z=\varphi Y$ extends to a C^r-vector field defined on D (having 0 as its only singularity).

Proof. We assume that X is a C^∞-vector field (the other cases are similar). So, by definition, Y is also a C^∞-vector field.

Let $D(\rho)$ (respectively $\mathring{D}(\rho)$) denote the compact disc (respectively open disc) of ratio $\rho>0$ centered at the origin, and let

$$A_n=D\left(\frac{1}{n}\right)-\mathring{D}\left(\frac{1}{n+2}\right),\quad n=1,2,\dots.$$

Notice that the sequence of annulus $\{A_n\}_{n=1}^\infty$ is a covering of the punctured unit disc $D\setminus\{0\}$. Let $\{\varphi_n\}_{n=1}^\infty$ be a partition of unity subordinate to the covering $\{A_n\}_{n=1}^\infty$. We denote by $\mathfrak{X}^r(D)$ the Banach space formed by the C^r-vector fields on the compact unit disc D endowed with the C^r-norm $\|\cdot\|_r$ (see [22, Proposition 2.1]). By the compactness of A_n, for each $n\in\mathbb{N}-0$, we can find a positive constant $c_n>0$ such that

$$\|c_n\varphi_nY\|_n<\frac{1}{2^n}. \tag{7}$$

Since $\|\cdot\|_p \leqslant \|\cdot\|_q$ whenever $p \leqslant q$, we obtain from (7) that for each $k \in \mathbb{N}$, the series $\sum_n \|c_n\varphi_n Y\|_k$ of positive real numbers has an upper bound. Then, by Cauchy's criterium, there exists a C^∞-vector field $Z : D \to \mathbb{R}^2$ such that the series $\sum_n c_n\varphi_n Y$ converges to Z in $\mathfrak{X}^k(D)$ for all $k \in \mathbb{N}$. Notice that $Z = \varphi Y$, where $\varphi = \sum_n c_n\varphi_n$. It is plain that the origin is the only singularity of Z. $\square$

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