Nonlinear $(2 + 1)$-dimensional field equations from incomplete Lie algebra structures

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Abstract

We show that the nonlinear $(2 + 1)$-dimensional Three-Wave Resonant Interaction equations, describing several important physical phenomena, can be generated starting from incomplete Lie algebras in the framework of multidimensional prolongation structures. We make use of an *ansatz* involving the structure equations of a principal prolongation connection induced by an admissible Bäcklund map.

1. Introduction

As it is well known, completely integrable nonlinear field equations admit Lax pairs, (multi-)soliton solutions and an infinite set of conservation laws. It was pointed out that all these properties can be related with the existence of specific kinds of Bäcklund transformations (see, e.g., [1,2]).

In this context the prolongation structures method, which can be interpreted as the construction of an Ehresmann connection [3,4], plays a relevant role (see, e.g., [5–10]). One of the most interesting features of the arising algebraic structures is that they are homomorphic to infinite-dimensional *loop* algebras (see, e.g., [11–14] and references quoted therein). The inverse prolongation structures procedure can be thought as the search of the Bäcklund map which generates a given (incomplete) Lie algebra structure [4–8,10]. This inverse procedure, based on the Cartan method of moving frames, was first outlined by Estabrook [5–7] who showed how exterior differential systems can be obtained from incomplete Lie algebra structures. Further results concerning the $(1 + 1)$-dimensional case were exploited, e.g., in [5–7,11–16]. Starting from a given incomplete Lie algebra

structure, it is then possible to generate the field equations whose prolongation structure is the given algebra.[3] In fact, the extension of both the direct and inverse prolongation structure procedure to the multi-dimensional case is not trivial and presents several technical difficulties (see, e.g., [14,17]).

This Letter is aimed to give a contribution in the geometrical characterization of *integrability properties* of nonlinear field equations in the framework of the inverse procedure in $2 + 1$ dimensions. It is in fact well known that the study of higher-dimensional systems is a central theme in the theory of integrable systems, nevertheless the investigation was always made in the direction of extension of $(1 + 1)$-dimensional systems to $(2 + 1)$-dimensional ones just *via* the extension either of the Lax pair (see, e.g., [18]) or of the so-called prolongation forms (see, e.g., [19,20]), as well as of the moving frames setting [21]. In this Letter the more general approach of *generating new $(2+1)$-dimensional integrable systems from a given abstract algebraic structure via the extension of a Bäcklund map* is tackled.

We are, in general, interested in the case when incomplete Lie algebras arise as necessary conditions for integrability of a (formal) connection induced by a Bäcklund map [4], and the choice of a realization of such algebras (particularly in loop-algebras form) gives us the solution of the so-called Bäcklund problem. Conversely, if the algebraic structure is known and the 'new' variable (also called pseudopotentials [10]) dependence of the Bäcklund map is fixed, the integrability condition for the formal connection induced by a Bäcklund map provides the whole family of exterior differential systems which admits the given Bäcklund map.

We shall make use of an *ansatz* which is based on the fact that, from a geometric point of view, the connection forms (induced by an admissible Bäcklund map) play the relevant role. This approach is then slightly different from the one formulated in the $(2 + 1)$-dimensional direct prolongation procedure by Morris and Tondo [19,20] and it should be seen

within the geometric approach in the framework of jet bundles and connections provided by Pirani et al. (see, e.g., [4]). The *ansatz* is in fact a slightly modified version of the structure equations of the connection, which will be the starting point for the application of the inverse method. According to this, in Section 2 we recall how a connection can be induced by a Bäcklund map in the jet bundles framework and provide a characterization of completely integrable systems in terms of Bäcklund structures.

In Section 3, we give an example of physical application of such a generalized *ansatz* providing some nonlinear $(2 + 1)$-dimensional field equations describing various physical phenomena [19,20,23–26]. We show that the $(2 + 1)$-dimensional Three Wave Resonant Interaction (3WRI) equations, together with a condition on the group velocities which is related with the resonance condition, can be generated from an incomplete Lie algebra, as the integrability condition for a special class (which we shall call *admissible*) of Bäcklund connections. Our *ansatz* enables us on the other hand to generate a whole family of nonlinear field equations related with the 3WRI equations by transformations of coordinates and fields of the Miura type. All the family is in fact 'contained' in the exterior differential system admitted by a postulated (admissible) Bäcklund map as the integrability condition of the corresponding induced Bäcklund connection.

2. Bäcklund transformations and induced connections

In the following we shall shortly recall few basics concepts and set the notation. We shall assume the reader is familiar with the basic notions from the theory of bundles, jet prolongations, principal bundles and connections (for references and details see, e.g., [27–29]).

Let $\pi : U \to X$, $\tau : Z \to X$, be two (vector) bundles with local fibered coordinates (x^α, u^A) and (x^α, z^i), respectively, where $\alpha = 1, \dots, m = \dim X$, $A = 1, \dots, n = \dim U - \dim X$, $i = 1, \dots, N = \dim Z - \dim X$. We shall assume that sections of the bundles $U = (U, X, \pi)$ and $Z = (Z, X, \tau)$ represent physical fields sastifying a given system of (nonlinear) field equations.

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[3] As well as the associated linear spectral problem useful for the integration *via* the Inverse Spectral Transform (IST) method. For a review of the IST method see, e.g., [22] and references quoted therein.

A system of nonlinear field equations of order k on U is geometrically described as an exterior differential system ν on $J^k U$ [5–10,29,30]. The solutions of the field equations are (local) sections of $U \to X$—i.e., (local) mappings $\sigma : X \to U$ such that $\sigma \circ \pi = \mathrm{id}_X$— such that $(j^k \sigma)^* \nu = 0$. They define a submanifold in $J^k U$. We shall also denote by $J^\infty \nu$ (respectively, $j^\infty \sigma$) the infinite order jet prolongation of ν (respectively, σ).

2.1. Admissible Bäcklund transformations

We recall that the group of *contact transformations* of a bundle is the group of its infinitesimal fibered automorphisms. A contact transformation preserves then, by definition, the fibering (see, e.g., [29,30]). Let then $\boldsymbol{B}$ be the infinite-order contact transformations group on $J^\infty U$ [2].

In the following we recall some basic definitions (see, e.g., [2]) and stress some important properties which will be used later.

Definition 2.1. The group of *Bäcklund transformations for the system ν* is the closed subgroup $\boldsymbol{K}$ of $\boldsymbol{B}$ which leaves invariant $J^\infty \nu$.

Definition 2.2. The group of *admissible Bäcklund transformations for the system ν* is the closed subgroup $\widetilde{\boldsymbol{K}}$ of $\boldsymbol{B}$ which leaves invariant solution submanifolds of $J^\infty \nu$.

Remark 2.3. The contact transformations group $\boldsymbol{B}$ acts freely on the infinite-dimensional submanifold of $J^\infty U$ defined by $J^\infty \nu$. The group $\widetilde{\boldsymbol{K}}$ of admissible Bäcklund transformations is then the compact subgroup of $\boldsymbol{B}$ which preserves contact elements along the solutions of the given (system of) nonlinear field equations. Thus $\widetilde{\boldsymbol{K}}$ can be seen as the isotropy subgroup of $\boldsymbol{B}$.

Let $\pi : U \to X$, $\tau : Z \to X$, be vector bundles as the above and $\pi^1 : J^1 U \to X$, $\tau^1 : J^1 Z \to X$, the first order jet prolongations bundles,[4] with local fibered coordinates $(x^\alpha, u^A, u^A_\alpha)$, $(x^\alpha, z^i, z^i_\alpha)$, respectively. Fur-

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[4] A Bäcklund transformation can be analogously defined at any jet order [4], but here we will consider only the jet order which is involved with the physical application we are concerned with.

thermore, let $(\partial_\beta, \partial_A, \partial^\beta_A)$, $(\partial_\beta, \partial_i, \partial^\beta_i)$ and $(dx^\beta, du^A, du^A_\beta)$, $(dx^\beta, dz^i, dz^i_\beta)$ be local bases of tangent vector fields and 1-forms on $J^1 U$ and $J^1 Z$, respectively. In the sequel we will be concerned with the pull-backs $\pi^*(Z) \simeq Z \times_X U$, $\tau^*(U) \simeq U \times_X Z$, $\eta^*(J^1 U) \simeq J^1 U \times_X Z$, where $\eta := \tau^*(\pi)$.

Definition 2.4. Following [4], we define a Bäcklund map to be the fibered morphism over Z:

$$\phi : J^1 U \times_X Z \to J^1 Z : (x^\alpha, u^A, u^A_\alpha; z^i)$$
$$\mapsto (x^\alpha, z^i, z^i_\alpha), \qquad (1)$$

with $z^i_\alpha = \phi^i_\alpha(x^\beta, u^A, u^A_\beta; z^j)$.

Definition 2.5. The fibered morphism ϕ is said to be an *admissible* Bäcklund transformation for the differential system ν if $z^i_\alpha = \phi^i_\alpha(x^\beta, u^A, u^A_\beta; z^j)$, $\phi^i_\alpha = \mathcal{D}_\alpha \phi^i$ and the integrability conditions

$$\mathcal{D}_\alpha \phi^j_\omega = \mathcal{D}_\omega \phi^j_\alpha, \qquad (2)$$

(with $\mathcal{D}_\alpha = \partial_\alpha + u^A_\alpha \partial_A + u^A_{\alpha\beta} \partial^\beta_A + \phi^i_\alpha \partial_i$) coincide with the exterior differential system ν [2,4].

Remark 2.6. By pull-back of the contact structure on $J^1 Z$, the Bäcklund morphism induces in a natural way an horizontal (with respect to π^{1*}_0) distribution on the bundle $(J^1 U \times_X Z, J^1 U, \pi^{1*}_0(\eta))$. The local expression of generators of such a horizontal distribution is given by the following connection forms

$$\Theta^i = dz^i - \phi^i_\beta(x^\alpha, u^A, u^A_\alpha; z^j) dx^\beta. \qquad (3)$$

Definition 2.7. The horizontal distribution locally defined by Eq. (3) is called the *induced Bäcklund connection*.

Definition 2.8. The system ν is said to be *completely integrable* if there exists a normal subgroup $\boldsymbol{K}_0 \subset (\widetilde{\boldsymbol{K}} \cap \boldsymbol{K}) \subset \boldsymbol{B}$ leaving invariant (the infinite order prolongation of) ν and its solutions.

Let $\chi_k = \chi^i_k \partial_i$ be generators of the Lie algebra $\mathfrak{k}_\circ$ of the Lie subgroup $\boldsymbol{K}_0$, which satisfy the commutation relations

$$[\chi_l, \chi_m] = -\mathcal{C}^k_{lm} \chi_k, \qquad (4)$$

where $\mathcal{C}^n_{lm}$ are the structure constants of the Lie algebra $\mathfrak{k}_\circ$.

Theorem 2.9. *The following statements are equivalent.*

(1) ϕ *is an admissible Bäcklund transformation for the differential system ν.*
(2) *The horizontal distribution* (3) *is $\mathbf{K}_0$-invariant.*

Proof. If the horizontal distribution (3) is $\mathbf{K}_0$-invariant, we have (see, e.g., [29]):

$$\phi^i_\beta(x^\alpha, u^A, u^A_\alpha; z^j) = \omega^k_\beta(x^\alpha, u^A, u^A_\alpha)\chi^i_k(z^j). \qquad (5)$$

Then the integrability conditions (2) for the Bäcklund map ϕ are equivalent to the integrable exterior differential system:

$$\nu := \Omega^k_{\alpha\beta} = 0, \qquad (6)$$

where

$$\Omega^k_{\alpha\beta} = \mathcal{D}_\alpha \omega^k_\beta - \mathcal{D}_\alpha \omega^k_\alpha + \mathcal{C}^k_{lm}\omega^l_\alpha \omega^m_\beta.$$

In fact, taking Eqs. (3) into account, from the equivariance condition, the structure equations for the connection induced by the admissible Bäcklund map are (up to pull-backs):

$$d\Theta^i = \left(d\omega^k + \frac{1}{2}\mathcal{C}^k_{lm}\omega^l \wedge \omega^m\right)\chi^i_k \quad (\mathrm{mod}\ \Theta^j) \qquad (7)$$

$$= \frac{1}{2}\Omega^k_{\alpha\beta}\chi^i_k\,dx^\alpha \wedge dx^\beta, \qquad (8)$$

where $\omega^k = \omega^k_\alpha dx^\alpha$ are 1-forms on $J^1\mathbf{U}$, horizontal with respect to π^1. The integrability condition $d\Theta^i = 0$ is then equivalent to $\Omega^k_{\alpha\beta}\chi^i_k = 0$, i.e., to Eq. (6).

Conversely, if the Bäcklund map is admissible for the system ν, then the group $\mathbf{K}_0$ leaves invariant the solution (integral maximal) submanifolds of ν, then the horizontal distribution induced by the Bäcklund map is $\mathbf{K}_0$-invariant. □

Corollary 2.10. *A nonlinear exterior system ν is completely integrable if and only if there exists an admissible Bäcklund transformation ϕ for the differential system ν.*

Proof. It follows from Definition 2.8 and the above theorem. □

3. The $(2+1)$-3WRI from incomplete Lie algebras

From a geometric point of view, the connection forms induced by an admissible Bäcklund map play a relevant role. Once that the dependence of the connection forms on the z^i variables is known,[5] the integrability condition (i.e., *the zero curvature* condition) for such a connection provides, *via the inverse procedure and thanks to Corollary 2.10, whole families of *integrable* differential systems [5–7,11–16].

This suggests the introduction of an *ansatz* which is aimed to generalize the geometric approach in the framework of jet bundles provided by Pirani et al. [4] and Hoenselaers [15]. The *ansatz* is in fact a slightly modified version of the structure equations, which will be the starting point for the application of the inverse method.

Let us then consider the following *ansatz*:

$$d\Omega^k = 0, \quad \text{with } \Omega^k = \theta \wedge \Theta^k, \qquad (9)$$

where θ is a closed 1-form on $J^1\mathbf{U}$ horizontal over $\mathbf{X}$, and $\Theta^k = dz^k + \chi^k_j(z^m)\omega^j$ are *admissible Bäcklund connection forms* on $J^1\mathbf{U} \times \mathbf{X}\mathbf{Z}$, with $k = 1, \ldots, N = \dim \mathfrak{b}, j = 1, \ldots, M = \dim \mathfrak{k}_\circ$.[6]

Assume now that the dependence on the z^i variables is provided by the following incomplete[7] Lie algebra structure

$$[\chi_1, \chi_5] = 0, \qquad [\chi_2, \chi_4] = 0,$$
$$[\chi_1, \chi_6] = 0, \qquad [\chi_3, \chi_4] = 0,$$
$$[\chi_2, \chi_6] = 0, \qquad [\chi_3, \chi_5] = 0,$$
$$[\chi_1, \chi_7] = 0, \qquad [\chi_4, \chi_7] = 0,$$
$$[\chi_2, \chi_8] = 0, \qquad [\chi_5, \chi_8] = 0,$$
$$[\chi_7, \chi_8] = 0, \qquad [\chi_1, \chi_2] = a\chi_6,$$
$$[\chi_1, \chi_3] = b\chi_5, \qquad [\chi_2, \chi_3] = c\chi_4,$$
$$[\chi_4, \chi_5] = -a\chi_3, \qquad [\chi_4, \chi_6] = -b\chi_2,$$

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[5] This is the case when any abstract incomplete Lie algebra structure is given.

[6] The geometrical interpretation of this *ansatz*, will be investigated in a separate paper [31].

[7] It is incomplete in the sense that not all of the commutators are known, then the algebra is not closed as a Lie algebra structure.

$$[\chi_5, \chi_6] = -c\chi_1, \qquad [\chi_3, \chi_7] = [\chi_3, \chi_8]$$
$$[\chi_6, \chi_7] = [\chi_6, \chi_8],$$

where $a = \frac{i}{\lambda_{12}}$, $b = \frac{i}{\lambda_{13}}$, $c = \frac{i}{\lambda_{23}}$, with λ_{12}, λ_{23}, λ_{13} different from zero.

Thus, by requiring the structure equations (8) hold true, from the *ansatz* (9), we obtain the following *integrable* exterior differential system:

$$\theta \wedge d\omega^7 = 0,$$
$$\theta \wedge d\omega^8 = 0, \tag{10}$$

$$\theta \wedge (d\omega^1 + c\omega^5 \wedge \omega^6) = 0,$$
$$\theta \wedge (d\omega^2 + b\omega^4 \wedge \omega^6) = 0, \tag{11}$$

$$\theta \wedge (d\omega^3 + a\omega^4 \wedge \omega^5) = 0,$$
$$\theta \wedge (d\omega^4 - c\omega^2 \wedge \omega^3) = 0, \tag{12}$$

$$\theta \wedge (d\omega^5 - b\omega^1 \wedge \omega^3) = 0,$$
$$\theta \wedge (d\omega^6 - a\omega^1 \wedge \omega^2) = 0, \tag{13}$$

$$\theta \wedge \omega^3 \wedge (\omega^7 - \omega^8) = 0,$$
$$\theta \wedge \omega^6 \wedge (\omega^7 - \omega^8) = 0, \tag{14}$$

$$\theta \wedge \omega^1 \wedge \omega^4 = \theta \wedge \omega^1 \wedge \omega^8 = \theta \wedge \omega^2 \wedge \omega^5$$
$$= \theta \wedge \omega^2 \wedge \omega^7 = \theta \wedge \omega^3 \wedge \omega^6 = \theta \wedge \omega^4 \wedge \omega^8$$
$$= \theta \wedge \omega^5 \wedge \omega^7 = 0. \tag{15}$$

The choice of fibrations of U over a basis manifold provides a whole family of nonlinear fields equations related by Miura's transformations (see, e.g., [11–14, 16]). In the sequel we will find out the 3WRI equations making a specific choice.

Along a section of $Z \to X$ the closed form θ can be chosen as

$$\theta = m_1\, dx + m_2\, dy + m_3\, dt, \tag{16}$$

where m_i, $(i = 1, 2, 3)$ are some constants.[8] Then (10) imply

$$\omega^7 = n_1\, dx + n_2\, dy + n_3\, dt, \tag{17}$$
$$\omega^8 = p_1\, dx + p_2\, dy + p_3\, dt, \tag{18}$$

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[8] In this way we obtain a case which turns out to be degenerate in the standard AKNS formulation (see, e.g., [18] for further details).

with n_i, p_i, $(i = 1, 2, 3)$ constants. Moreover, from (15), we have that

$$\omega^1 = u\omega^8, \qquad \omega^2 = v\omega^7, \tag{19}$$
$$\omega^3 = r(\omega^7 - \omega^8), \qquad \omega^4 = z\omega^8, \tag{20}$$
$$\omega^5 = w\omega^7, \qquad \omega^6 = s(\omega^7 - \omega^8). \tag{21}$$

Let us now consider r, s, u, v, z and w as complex functions of the independent variables (this is equivalent to a choice of a fibration of U over the manifold X) as follows:

$$u := u_1(x, y, t), \quad v := u_2(x, y, t),$$
$$r := u_3(x, y, t), \tag{22}$$

$$z := u_1^*(x, y, t), \quad w := u_2^*(x, y, t),$$
$$s := u_3^*(x, y, t). \tag{23}$$

Furthermore, for notational convenience, we put:

$$a_1 = \frac{m_3 p_2 - m_2 p_3}{m_2 p_1 - m_1 p_2}, \quad b_1 = \frac{m_1 p_3 - m_3 p_1}{m_2 p_1 - m_1 p_2}, \tag{24}$$
$$a_2 = \frac{m_3 n_2 - m_2 n_3}{m_2 n_1 - m_1 n_2}, \quad b_2 = \frac{m_1 n_3 - m_3 n_1}{m_2 n_1 - m_1 n_2}, \tag{25}$$
$$a_3 = \frac{m_3 q_2 - m_2 q_3}{m_2 q_1 - m_1 q_2}, \quad b_3 = \frac{m_1 q_3 - m_3 q_1}{m_2 q_1 - m_1 q_2}, \tag{26}$$

and

$$\lambda_{23} = n_3 + \frac{m_3 p_2 - m_2 p_3}{m_2 p_1 - m_1 p_2} n_1 + \frac{m_1 p_3 - m_3 p_1}{m_2 p_1 - m_1 p_2} n_2, \tag{27}$$
$$\lambda_{13} = -p_3 - \frac{m_3 n_2 - m_2 n_3}{m_2 n_1 - m_1 n_2} p_1 - \frac{m_1 n_3 - m_3 n_1}{m_2 n_1 - m_1 n_2} p_2, \tag{28}$$
$$\lambda_{12} = -n_3 - \frac{m_3 q_2 - m_2 q_3}{m_2 q_1 - m_1 q_2} n_1 - \frac{m_1 q_3 - m_3 q_1}{m_2 q_1 - m_1 q_2} n_2, \tag{29}$$

where $\lambda_{12} = a_1 b_2 - a_2 b_1$, $\lambda_{23} = a_2 b_3 - a_3 b_2$, $\lambda_{13} = a_1 b_3 - a_3 b_1$, and $q_i = n_i - p_i$, $i = 1, 2, 3$.

Then Eqs. (11)–(13) with (16)–(21) provide the following system of integrable NFEs

$$u_{1t} + a_1 u_{1x} + b_1 u_{1y} = i u_2^* u_3^*, \tag{30}$$
$$u_{2t} + a_2 u_{2x} + b_2 u_{2y} = i u_1^* u_3^*, \tag{31}$$
$$u_{3t} + a_3 u_{3x} + b_3 u_{3y} = i u_1^* u_2^*, \tag{32}$$

together with their complex conjugates.

Remark 3.1. The compatibility condition for the system of Eqs. (24)–(29) is

$$\lambda_{12} + \lambda_{23} = \lambda_{13}. \tag{33}$$

Remark 3.2. Eqs. (30)–(32), together with their complex conjugate, are just the equations describing the resonant interaction of the three waves envelope in $2 + 1$ dimensions [19,20,23]. Eq. (33) is related to the resonance condition for the wave envelope (see, e.g., [23]).

4. Conclusions

We provided a characterization of completely integrable nonlinear field equations in terms of algebraic properties of associated Bäcklund structures by investigating the relation between Bäcklund transformations and connections theory. As an example of application, by resorting to the structure equations induced by an integrable admissible Bäcklund map, we have shown how nonlinear $(2 + 1)$-dimensional field equations can be generated starting from incomplete Lie algebras.

References

[1] M.J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM Stud. in Appl. Math., 1981.

[2] R.L. Anderson, N.H. Ibragimov, Lie–Bäcklund Transformations in Applications, SIAM, Philadelphia, PA, 1979.

[3] C. Ehresmann, Le Connexions Infinitésimales dans un Espace Fibré Différentiable, Colloque de Topologie, Masson, Paris, 1951.

[4] F.A.E. Pirani, D.C. Robinson, W.F. Shadwick, Local Jet Bundle Formulation of Bäcklund Transformations, Math. Phys. Stud., Reidel, Dordrecht, 1979.

[5] F.B. Estabrook, J. Math. Phys. 23 (11) (1982) 2071.

[6] F.B. Estabrook, H.D. Wahlquist, Class. Quantum Grav. 6 (3) (1989) 263.

[7] F.B. Estabrook, Differential geometry techniques for sets of nonlinear partial differential equations, in: Partially Integrable Evolution Equations in Physics, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 310, Kluwer Academic, Dordrecht, 1990, pp. 413–434.

[8] R. Hermann, in: The Geometry of Non-linear Differential Equations, Bäcklund Transformations and Solitons, Interdisciplinary Mathematics A, Vol. XII, Math. Sci. Press, Brookline, MA, 1976.

[9] R. Hermann, Phys. Rev. Lett. 36 (15) (1976) 835.

[10] H.D. Whalquist, F.B. Estabrook, J. Math. Phys. 16 (1975) 1.

[11] E. Alfinito, M. Leo, R.A. Leo, M. Palese, G. Soliani, Lett. Math. Phys. 32 (1994) 241.

[12] E. Alfinito, M. Leo, R.A. Leo, M. Palese, G. Soliani, Phys. Lett. B 352 (1995) 314.

[13] M. Palese, E. Alfinito, M. Leo, R.A. Leo, G. Soliani, Algebraic and geometrical properties of integrable nonlinear field equations, in: E. Alfinito et al. (Eds.), Proc. Nonlinear Physics. Theory and Experiments, World Scientific, Singapore, 1996, pp. 249–252.

[14] M. Palese, Prolongation structures of nonlinear field equations (Italian), Doctoral Thesis, unpublished, University of Lecce, 1993.

[15] C. Hoenselaers, Prog. Theor. Phys. 75 (1986) 1014.

[16] R.A. Leo, G. Soliani, Phys. Lett. B 222 (1989) 415.

[17] M. Palese, R.A. Leo, G. Soliani, The prolongation problem for the heavenly equation, in: B. Casciaro et al. (Eds.), Recent Developments in General Relativity, Springer, Berlin, 2000, pp. 337–344.

[18] M.J. Ablowitz, R. Haberman, Phys. Rev. Lett. 35 (18) (1975) 1185.

[19] H.C. Morris, J. Math. Phys. 17 (1976) 1870.

[20] G.S. Tondo, Lett. Nuovo Cimento 44 (1985) 297.

[21] M. Lakshmanan, R. Myrzakulov, S. Vijayalakshmi, A.K. Danlybaeva, J. Math. Phys. 39 (7) (1998) 3765.

[22] A.S. Fokas, I.M. Gelfand, Lett. Math. Phys. 32 (3) (1994) 189.

[23] D.J. Kaup, Physica 1D (1980) 45.

[24] D.J. Benney, J. Fluid Mech. 14 (1962) 577.

[25] D.J. Benney, A.C. Newell, J. Math. Phys. 46 (1967) 133.

[26] D.E. Amundsen, D.J. Benney, Stud. Appl. Math. 105 (3) (2000) 277.

[27] S. Kobayashi, Transformation Groups in Differential Geometry, Springer, Berlin, 1972.

[28] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Vol. I, Wiley, Interscience, New York, 1963.

[29] I. Kolář, P.W. Michor, J. Slovák, Natural Operations in Differential Geometry, Springer, Berlin, 1993.

[30] D.J. Saunders, The Geometry of Jet Bundles, Cambridge Univ. Press, Cambridge, 1989.

[31] M. Palese, E. Winterroth, Remarks on the geometry of Bäcklund transformations, in preparation.