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# Hyperbolic spatial graphs arising from strongly invertible knots

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**Abstract**

Spatial graphs in the three-dimensional sphere are constructed from strongly invertible knots. Such a graph is proved to be hyperbolic, which means that its exterior admits a hyperbolic structure with totally geodesic boundary, if the exterior has no equivalent essential torus, or a pair of tori, with respect to the involution.


*MSC:* primary 57M50; secondary 05C10

*Keywords:* Hyperbolic manifold; Spatial graph; Strongly invertible knot

## 1. Introduction

In this paper, a *graph*, say $g$, means a one-dimensional CW-complex. Additionally we always assume throughout the paper that the graph is finite, and that each connected component does not collapse to a point. The image $G := f(g)$ of a proper embedding $f : g \to M$ of $g$ into a three-dimensional manifold $M$ is called a *spatial graph*. Here *proper* means $\partial G = G \cap \partial M$, where $\partial$ indicates its boundary. Especially a spatial graph is called a *knot* (respectively *link*) if it is homeomorphic to a circle (respectively a disjoint union of (at least two) circles). A spatial graph is said to be *trivial* if it can be drawn in a plane without edges crossing. We denote by $\mathrm{N}(\cdot)$ a regular neighborhood, and by $\overline{\cdot}$ the closure.

Suppose all boundary components of $M$ are closed orientable surfaces except spheres. Then the *exterior* $\mathrm{E}(G)$ of $G$ in $M$ is defined as $\mathrm{E}(G) := \overline{M - \mathrm{N}(G)}$, and we denote by $\mathrm{C}(G)$ the set obtained from $\mathrm{E}(G)$ by deleting all toric boundary components. A spatial

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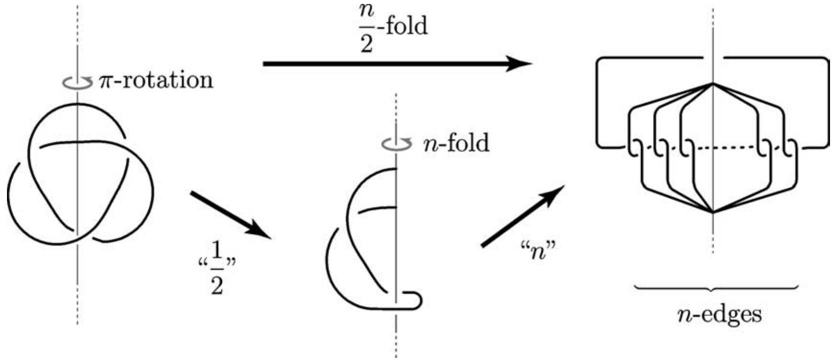


Fig. 1. $n/2$-fold cyclic branched covering of $T_{2,3}$.

graph $G$ in $M$ is said to be *hyperbolic* if $\mathrm{C}(G)$ admits a complete hyperbolic structure with each toric end being cusp neighborhood and each non-toric one being totally geodesic boundary. We note that any hyperbolic spatial graph in the three-dimensional sphere $S^3$ without circular components is not trivial, since handlebodies do not admit complete hyperbolic structure with totally geodesic boundary.

As far as the author knows, the first example of hyperbolic spatial graph in $S^3$, which has two vertices with three edges, was constructed by Thurston (see [17, Example 3.3.12]). This graph had already been known as Kinoshita's theta-curve (see [6]), and later it was generalized to Suzuki's Brunnian graph $\theta_n$ (see [14,16]; Kinoshita's theta-curve is $\theta_3$). Paoluzzi and Zimmermann proved in [12] that the graph $\theta_n$ is hyperbolic for any $n \geqslant 3$. Actually $\theta_2$ is the so-called *trefoil knot*, which is a typical non-hyperbolic knot, and $\theta_n$ (or $\mathrm{C}(\theta_n)$) is obtained by "$n/2$-fold" cyclic branched covering of $\theta_2$ (or $\mathrm{C}(\theta_2)$) along the axis of a symmetry. We here explain how to obtain $\theta_n$ from $\theta_2$.

The trefoil knot $\theta_2$ is a non-trivial knot in $S^3$, shown in the left-hand side figure of Fig. 1. This knot has a symmetry of order two (in other words *involution*), namely there is an automorphism, say $\varphi$, of $S^3$ preserving the knot as a set and $\varphi^2$ being the identity of $S^3$. Now we take the quotient of $S^3$ by $\varphi$, and obtain a graph in it, consisting of a trivial circle arising from the axis of $\varphi$ and an arc from the knot. We might say the quotient space is a "$1/2$" of $S^3$. We then take the $n$-fold cyclic branched covering of the quotient space along the (axial) circle. Thus we obtain the graph $\theta_n$, the right-hand side figure of Fig. 1, after moving the lower vertex to the bottom. So, by this construction, we could say $\theta_n$ (or $\mathrm{C}(\theta_n)$) is obtained by "$n/2$-fold" cyclic branched covering of $\theta_2$ (or $\mathrm{C}(\theta_2)$) along the axis of the involution.

A knot or a link $L$ in $S^3$ is called *strongly invertible* if there is an orientation preserving involution on $S^3$, which induces an involution in each component of $L$ with exactly two fixed points. The trefoil knot is a typical example of strongly invertible knot. Using $n/2$-fold cyclic branded covering, we can construct a family of spatial graphs from any strongly invertible knot or link in a similar way. So the following question naturally arises:

**Question 1.1.** *For any strongly invertible knot or link and any integer* $n \geqslant 3$, *is the* $n/2$*-fold cyclic branched covering a hyperbolic spatial graph*?

The main result of this paper is to answer this question for knots. To denote the main theorem, we review some definitions on three-dimensional manifolds, say $M$. Most of them are seen in [4]. Let $F$ be a surface (i.e., two-dimensional compact connected submanifold) in $M$. Suppose it is either properly embedded in $M$ or contained in $\partial M$. Then $F$ is called *compressible* if one of the following three conditions is satisfied:

(1) $F$ is a sphere bounding a ball in $M$, or
(2) $F$ is a disc and either $F \subset \partial M$ or there is a ball $B^3 \subset M$ with $\partial B^3 \subset F \cup \partial M$, or
(3) there is a disc $D^2 \subset M$ with $D^2 \cap F = \partial D^2$ and with $\partial D^2$ not shrinking to a point in $F$.

The disc $D^2$ is called the *compressing disc*. We say that $F$ is *incompressible* if $F$ is not compressible.

A manifold $M$ is said to be *irreducible* if each sphere is compressible. When $\partial M \neq \emptyset$, $M$ is called $\partial$*-irreducible* if $\partial M$ is incompressible in $M$. Suppose $F$ is a two-sided surface properly embedded in $M$. Then $F$ is called *essential* if it is incompressible in $M$ and not parallel to a surface in $\partial M$.

Now the main theorem is denoted as follows:

**Theorem 1.2.** *Let $K$ be a non-trivial strongly invertible knot in $S^3$, and let $a$ be the axis of the involution. Then the $n/2$-fold cyclic branched covering is a hyperbolic spatial graph for any $n \geqslant 3$ if and only if there is no essential torus, or a pair of tori, in $\mathrm{E}(K \cup a)$, being equivalent to the action of the involution.*

A $(p,q)$*-torus knot* $T_{p,q}$ is obtained by looping a string through the hole of a standard torus $p$ times with $q$ revolutions before joining its ends, where $p$ and $q$ are relatively prime. Torus knots are strongly invertible, and $T_{p,q}$ is non-trivial if and only if $|p| \geqslant 2$ and $|q| \geqslant 2$. The trefoil knot is $T_{3,2}$. An $l$ component *torus link* $T_{lp,lq}$ is a union of $l$ string torus knots $T_{p,q}$ running parallel to them. A reference of these results is [5, §2.2]. We note that torus links with less than three components are strongly invertible (see Fig. 2).

Seeing the construction of $n/2$-fold cyclic branched covering for torus knots case precisely, we obtain the following result:

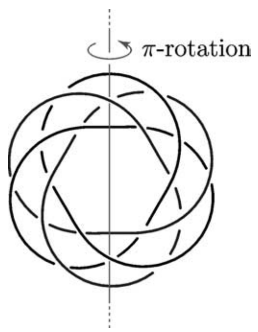


Fig. 2. Torus link $T_{4,6}$ with the axis of an involution.

**Theorem 1.3.** *For any torus link and for any integer $n \geqslant 3$, the $n/2$-fold cyclic branched covering is not a hyperbolic spatial graph.*

All proofs are given in the next section. The last section is devoted to some comments related to these results.

## 2. Proofs

We start this section with reviewing several basic definitions on tangles following [5, Chapter 3]. A *tangle* $(B^3, t)$ is the pair consisting of a ball $B^3$ and a proper one-dimensional submanifold $t$ with $\partial t \neq \emptyset$. In particular, it is called an *$n$-string tangle* if $t$ consists of $n$ arcs, and it is called a *trivial $n$-*string tangle if it is homeomorphic to the pair $(D^2, \{a_1, a_2, \ldots, a_n\}) \times [0, 1]$ for some interior points $a_1, a_2, \ldots, a_n$ of $D^2$. Trivial 2-string tangles are also called *rational* tangles, and they are encoded by rational numbers (called *slope*; see Fig. 3).

A tangle $(B^3, t)$ is called *non-split* if any proper disc in $B^3$ does not split $t$ in it. A tangle $(B^3, t)$ is called *locally trivial* if any sphere in $B^3$ intersecting $t$ transversely at two points bounds a trivial 1-string tangle. A tangle $(B^3, t)$ is called *indivisible* if any proper disc in $B^3$ intersecting $t$ transversely in one point divides the tangle into two tangles, at least one of which is a trivial 1-string tangle. A tangle is called *prime* if it is non-split, locally trivial, indivisible, and if it is not a trivial 1-string tangle. A tangle $(B^3, t)$ is called *atoroidal* if there is no essential torus in it, namely, any torus in $\mathrm{E}(t)$ is compressible or parallel to a component of $\overline{\partial \mathrm{N}(t) - (\partial \mathrm{N}(t) \cap \partial B^3)}$.

**Proof of Theorem 1.2.** Let $(B^3, t)$ be the 2-string tangle obtained from $(S^3, K)$ by the so-called *Montesinos trick* (see [9,1]). Namely $B^3$ is the quotient of $\mathrm{E}(K)$ by the strong inversion, and $t$ is the projection of $a$ in $B^3$. Since $K$ is a non-trivial knot in $S^3$, $\mathrm{E}(K)$ is irreducible and $\partial$-irreducible. So $(B^3, t)$ is a prime tangle by [5, Theorem 3.5.17]. Let $(DB^3, Dt)$ be the double of $(B^3, t)$, namely $(DB^3, Dt)$ is the tangle sum of $(B^3, t)$ and its mirror image in a natural way. Here we note that $DB^3$ is a three-dimensional sphere, and $Dt$ is a link of two components.

Let $M_{t,n}$ be the $n$-fold cyclic branched covering of $B^3$ along $t$, and let $M_{Dt,n}$ be the $n$-fold cyclic branched covering of $DB^3$ along $Dt$. Then the double $DM_{t,n}$ of $M_{t,n}$ is homeomorphic to $M_{Dt,n}$. Now suppose $M_{t,n}$ admits a hyperbolic structure with totally geodesic boundary. Then this structure naturally induces that of $DM_{t,n}$ (cf. [8]). Thus $M_{Dt,n}$ is also admits a complete hyperbolic structure by Mostow–Prasad's rigidity theorem. Conversely suppose $M_{Dt,n}$ admits a complete hyperbolic structure. Then,

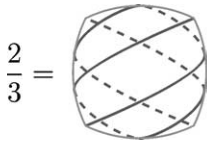


Fig. 3. The rational tangle of slope $2/3$.

applying the rigidity theorem again, the involution on $M_{Dt,n}$ exchanging $M_{t,n}$ for its copy can be regarded as an isometry. Since the fixed point set of the isometry is totally geodesic (cf. [7, p. 61]), this isometry induces the hyperbolic structure on $M_{t,n}$ with totally geodesic boundary. Thus the $n/2$-fold cyclic branched covering is a hyperbolic spatial graph if and only if the $n$-fold cyclic branched covering of $(DB^3, Dt)$ is a hyperbolic manifold.

Furthermore, by the following theorem, the $n/2$-fold cyclic branched covering is a hyperbolic spatial graph if and only if $Dt$ is a hyperbolic link in $S^3$:

**Theorem 2.1** (cf. [13, Theorem 1]). *Let $L$ be a link in $S^3$. Then $L$ is a hyperbolic link if and only if the $n$-fold cyclic branched covering of $S^3$ along $L$ is a hyperbolic manifold for any $n \geqslant 3$.*

Suppose there is no essential torus, or a pair of tori, in $\mathrm{E}(K \cup a)$, being equivalent to the action of the involution. Let $T_B$ be a torus in $(B^3, t)$. Since $T_B$ does not intersect with $t$, The lift of $T_B$ in $(S^3, K)$ is also a torus, or a pair of tori, in $\mathrm{E}(K \cup a)$, being equivalent to the action of the involution. By the assumption, we can have a compression disc of the lifted torus, and its projection is a compression disc of $T_B$. So $(B^3, t)$ is atoroidal. Since $(B^3, t)$ is a 2-string prime atoroidal tangle, $Dt$ is a hyperbolic link (see [5, Exercise 3.6.4 and Theorem 3.6.6]).

Conversely suppose the $n/2$-fold cyclic branched covering is a hyperbolic spatial graph for any $n \geqslant 3$. As we saw previously, this is equivalent to the condition that $Dt$ is a hyperbolic link in $DB^3$. Let $T_S$ be a torus in $\mathrm{E}(K \cup a)$, being equivalent to the involution. Since $T_S$ does not intersect the axis of the involution, the projection, say $T_B$, of $T_S$ in $(B^3, t)$ is also a torus. Since $Dt$ is a hyperbolic link, $(B^3, t)$ has no essential torus. So there is a compression disc of $T_B$, and its lift is a compression disc of $T_S$. Similarly we can show that there is no pair of essential tori, being equivalent to the involution.

We have thus finished the proof of Theorem 1.2. □

*Torus knots and $n/2$-fold cyclic branched coverings*

Let $T_{p,q}$ be a torus knot in $S^3$. Then it is known that $\mathrm{E}(T_{p,q})$ is a Seifert fibred manifold with two exceptional fibres (cf. [15, p. 402]). Its base orbifold is a disc with two cone points of orders $p$ and $q$, respectively. The Seifert invariant of the exceptional fibre corresponding to the cone point of order $p$ (respectively $q$) is $(p, \beta_1)$ (respectively $(q, \beta_2)$), where $\beta_1$ (respectively $\beta_2$) is an integer uniquely determined from the following two conditions: $0 < \beta_1 < p$ and $q\beta_1 \equiv 1 \bmod p$ (respectively $0 < \beta_2 < q$ and $p\beta_2 \equiv 1 \bmod q$).

Let $\varphi$ be an involution of $T_{p,q}$. Then $\varphi$ preserves the fibre structure and reverses the orientation of the fibres. The axis of $\varphi$ intersects each exceptional fibre at two points.

Let $M$ be a three-dimensional manifold of trivial $S^1$ bundle over a disc (i.e., $M$ is a trivial solid torus), and $L$ a link in $M$ of two (trivial) fibres. Then $\mathrm{E}(T_{p,q})$ is a result of Dehn surgery on $M$ along $L$ with surgery coefficients $\beta_1/p$ and $\beta_2/q$. We here note that, since $\varphi$ acts on $\mathrm{E}(T_{p,q})$ as an involution, the link $L$ must be preserved by the action of $\varphi$, and that the axis of $\varphi$ must intersect each component of $L$ at two points. So the quotient space $(B^3, t)$ is a tangle sum of two rational tangles with slopes $\beta_1/p$ and $\beta_2/q$.

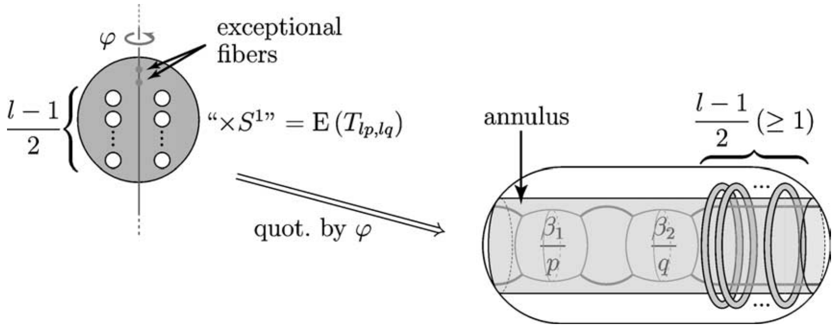


Fig. 4. The quotient space of $\mathrm{E}(T_{lp,lq})$ when $l(\geqslant 3)$ is odd.

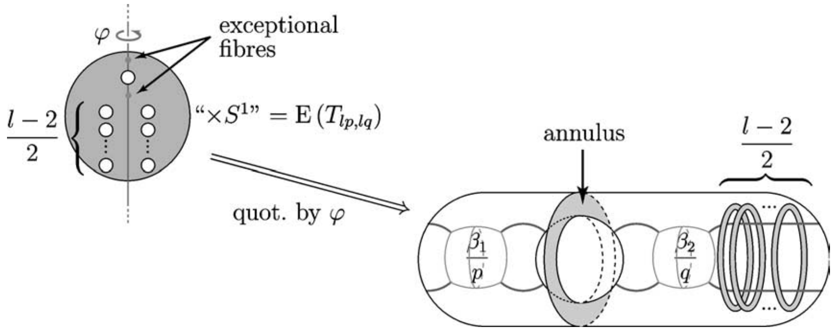


Fig. 5. The quotient space of $\mathrm{E}(T_{lp,lq})$ when $l$ is even.

**Proof of Theorem 1.3.** We first suppose that $l \geqslant 3$ is an odd integer, and consider a torus link $T_{lp,lq}$. Then $\mathrm{E}(T_{lp,lq})$ has $l$ toric boundary components, and one of them intersects the axis of $\varphi$. So the quotient space is the tangle sum of two rational tangles with slopes $\beta_1/p$ and $\beta_2/q$, minus the interior of $(l-1)/2$ solid tori.

We consider an annulus decomposing the quotient space into two manifolds; one is the tangle sum of the two rational tangles, and the other is a solid torus minus the interior of $(l-1)/2$ solid tori (see Fig. 4). Since the annulus does not intersect with the branch set, its lift in the $n$-fold cyclic branched covering of the quotient space is also an annulus. Furthermore, since $(l-1)/2 \neq 0$, the lifted annulus is essential. We have thus proved Theorem 1.3 in the odd components links case.

Next we suppose that $l$ is an even integer. Then $\mathrm{E}(T_{lp,lq})$ has $l$ toric boundary components, and two of them intersect the axis of $\varphi$. So the quotient space of $\mathrm{E}(T_{lp,lq})$ by $\varphi$ is topologically $S^2 \times [0,1]$ minus the interior of $(l-2)/2$ solid tori.

We consider an annulus naturally connecting two spherical boundary components (see Fig. 5). Since the slopes of the tangles are not $\infty$, any possible disc compressing the annulus must intersect $t$ at least two points. So the annulus is essential in the $n$-fold cyclic branched covering of the quotient space by Riemann–Hurwitz theorem. We have thus proved Theorem 1.3 in the even components links case too. $\square$

## 3. Comments

When we apply the method of the construction of $n/2$-fold cyclic branched coverings to the case when the symmetry is not of order two but of order $m$, then we might call the method *$n/m$-fold cyclic branched covering*. A known example of hyperbolic manifolds obtained by $n/3$-fold cyclic branched covering is in [11]. The three-dimensional torus $T^3$ has a symmetry of order three, with axis the diagonal of the cube of the fundamental region. One of the results in [11] is that the $n/3$-fold cyclic branched covering over $T^3$ along this axis is a hyperbolic manifold with totally geodesic boundary when $n \geqslant 4$. In this case the boundary arises from a sphere in $T^3$ intersecting the axis at two points.

It is known that the 2-fold cyclic branched covering of $S^3$ along any Montesinos link is a Seifert fibred space (see [2]). So, using Theorem 2.1, we can say that the $n/2$-fold cyclic branched covering of a manifold obtained from the 2-fold cyclic branched coverings of $S^3$ along a hyperbolic Montesinos link are hyperbolic manifolds for any $n \geqslant 3$. This gives another proof of Theorem 1.2 for torus knots case.

There are other known examples of hyperbolic spatial graphs; see [3,10]. Especially the examples in [3] are obtained from the so-called *Whitehead link* by $n/2$-fold cyclic branched coverings, and in this case the symmetry is not strongly invertible (see [3, Fig. 1]).

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