Tridiagonal pairs of height one

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Received 13 December 2004; accepted 21 January 2005
Available online 19 March 2005
Submitted by R.A. Brualdi

Abstract

Let \((A, A^*)\) denote a tridiagonal pair on a vector space \(V\) over a field \(\mathbb{K}\). Let \(V_0, \ldots, V_d\) denote a standard ordering of the eigenspaces of \(A\) on \(V\), and let \(\theta_0, \ldots, \theta_d\) denote the corresponding eigenvalues of \(A\). We assume \(d \geq 3\). Let \(q\) denote a scalar taken from the algebraic closure of \(\mathbb{K}\) such that \(q^2 + q^{-2} + 1 = (\theta_3 - \theta_0)/(\theta_2 - \theta_1)\). We assume \(q\) is not a root of unity. Let \(\rho_i\) denote the dimension of \(V_i\). The sequence \(\rho_0, \rho_1, \ldots, \rho_d\) is called the \textit{shape} of the tridiagonal pair. It is known there exists a unique integer \(h(0 \leq h \leq d/2)\) such that \(\rho_{i-1} < \rho_i\) for \(1 \leq i \leq h\), \(\rho_{h-1} = \rho_{h}\) for \(h < i \leq d - h\), and \(\rho_{d-1} > \rho_{d}\) for \(d - h < i \leq d\). The integer \(h\) is known as the \textit{height} of the tridiagonal pair. In this paper we show that the shape of a tridiagonal pair of height one with \(\rho_0 = 1\) is either \(1, 2, 2, \ldots, 2, 1\) or \(1, 3, 3, 1\). In each case, we display a basis for \(V\) and give the action of \(A, A^*\) on this basis.

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AMS classification: 05E30; 05E35; 33C45; 33D45

Keywords: Tridiagonal pair; Tridiagonal relation; Leonard pair

1. Introduction

The notion of a tridiagonal pair was introduced by Ito et al. [3], generalizing the notion of a Leonard pair which had been introduced by Terwilliger [6]. See
Terwilliger’s lecture note [8] about Leonard pairs and tridiagonal pairs. A tridiagonal pair is defined as follows.

**Definition 1.1** [3]. Let \( V \) denote a nonzero finite dimensional vector space over a field \( K \). By a tridiagonal pair on \( V \), we mean a pair \((A, A^*)\), where \( A : V \longrightarrow V \) and \( A^* : V \longrightarrow V \) are linear transformations that satisfy the following conditions.

(i) \( A \) and \( A^* \) are both diagonalizable on \( V \).

(ii) There exists an ordering \( V_0, V_1, \ldots, V_d \) of the eigenspaces of \( A \) such that

\[
A^*V_i \subseteq V_{i-1} \oplus V_i \oplus V_{i+1} \quad (0 \leq i \leq d),
\]

where \( V_{-1} = 0, V_{d+1} = 0 \).

(iii) There exists an ordering \( V_0^*, V_1^*, \ldots, V_\delta^* \) of the eigenspaces of \( A^* \) such that

\[
AV_i^* \subseteq V_{i-1}^* \oplus V_i^* \oplus V_{i+1}^* \quad (0 \leq i \leq \delta),
\]

where \( V_{-1}^* = 0, V_{\delta+1}^* = 0 \).

(iv) There is no subspace \( W \) of \( V \) such that both \( AW \subseteq W, A^*W \subseteq W \), other than \( W = 0 \) and \( W = V \).

**Remark 1.2.** With reference to Definition 1.1, it is known that \( d = \delta \) [3, Corollary 5.7]. The common number \( d \) of the eigenspaces is called the **diameter** of the tridiagonal pair.

Throughout this paper, we fix the following notation. Let \( K \) denote a field and let \( V \) denote a nonzero finite dimensional vector space over \( K \). Let \((A, A^*)\) denote a tridiagonal pair on \( V \) with diameter \( d \geq 3 \). Let \( V_0, V_1, \ldots, V_d \) (respectively \( V_0^*, V_1^*, \ldots, V_\delta^* \)) denote an ordering of the eigenspaces of \( A \) (respectively \( A^* \)) that satisfies the condition (ii) (respectively (iii)) in Definition 1.1. Let \( \rho_i \) denote the dimension of \( V_i \). Let \( \theta_i \) (respectively \( \theta_i^* \)) denote the eigenvalue of \( A \) (respectively \( A^* \)) for the eigenspace \( V_i \) (respectively \( V_i^* \)). Set \( \beta = (\theta_3 - \theta_0)/(\theta_2 - \theta_1) - 1 \), and let \( q \) denote a scalar taken from the algebraic closure \( \overline{K} \) such that \( \beta = q^2 + q^{-2} \). We assume \( q \) is not a root of unity.

It is known [5, Theorem 3.3] there exists a unique integer \( h \) (\( 0 \leq h \leq d/2 \)) such that \( \rho_{i-1} < \rho_i \) for \( 1 \leq i \leq h \), \( \rho_{i-1} = \rho_i \) for \( h < i \leq d - h \), and \( \rho_{i-1} > \rho_i \) for \( d - h < i \leq d \). The integer \( h \) is known as the **height** of the tridiagonal pair.

Our first main result is the following.

**Theorem 1.3.** Suppose \( h = 1 \) and \( \rho_0 = 1 \). Then one of the following holds.

(i) \( \rho_0 = 1, \rho_1 = \rho_2 = \cdots = \rho_{d-1} = 2, \rho_d = 1 \),

(ii) \( d = 3, \rho_0 = 1, \rho_1 = \rho_2 = 2, \rho_3 = 1 \).
In each case of (i), (ii), we display a basis for $V$ and give the action of $A, A^*$ on this basis. In order to do this we review some more definitions. Set
\[ U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d) \]
for $0 \leq i \leq d$. By [3] we have
\[ V = U_0 + U_1 + \cdots + U_d \] (direct sum),
and
\[ (A - \theta_i I)U_i \subseteq U_{i+1}, \quad (A^* - \theta_i^* I)U_i \subseteq U_{i-1} \quad (0 \leq i \leq d), \]
where we set $U_{-1} = U_{d+1} = 0$. The sequence $U_0, U_1, \ldots, U_d$ is called the split decomposition of $(A, A^*)$. The raising map $R$ and the lowering map $L$ are defined by
\[ R = A - \sum_{i=0}^{d} \theta_i F_i, \quad L = A^* - \sum_{i=0}^{d} \theta_i^* F_i, \]
where $F_i : V \rightarrow U_i$ denotes the projection. The maps $R, L$ satisfy
\[ RU_i \subseteq U_{i+1}, \quad LU_i \subseteq U_{i-1} \quad (0 \leq i \leq d). \]
It is known [3] that the eigenvalues are represented as
\begin{align*}
\theta_i &= a q^{2i} + b q^{-2i} + c \quad (0 \leq i \leq d), \\
\theta_i^* &= a^* q^{2i} + b^* q^{-2i} + c^* \quad (0 \leq i \leq d),
\end{align*}
for some scalars $a, b, a^*, b^*, c, c^*$ in $K$. We set
\[ \eta_i = (q - q^{-1})^3 (aa^* q^i - bb^* q^{-i}). \]
We use the following notation;
\[ [n] = q^n - q^{-n}, \quad [n]_2 = \frac{[n][n-1]}{2}. \]
We now define our basis for the case (i) in Theorem 1.3.

**Definition 1.4.** With reference to Theorem 1.3 (i), let $u$ denote a nonzero vector in $U_0$, and define $u_i = R^i u$ for $0 \leq i \leq d$. As we will show in Section 4, there exists a unique vector $v \in U_1$ such that (i) $R^{d-1} v = 0$; and (ii) $v - L u_2$ is a scalar multiple of $u_1$. We define $v_i = R^{i-1} v$ for $1 \leq i \leq d - 1$.

**Theorem 1.5.** Suppose Theorem 1.3(i) holds. Then
\begin{enumerate}
\item $u_0$ is a basis for $U_0$,
\item $u_i, v_i$ is a basis for $U_i$ $(1 \leq i \leq d - 1)$,
\end{enumerate}
(iii) \( u_d \) is a basis for \( U_d \).
(iv) the vectors
\[
 u_0, u_1, v_1, \ldots, u_{d-1}, v_{d-1}, u_d
\]
form a basis for \( V \).

We now give the action of \( R, L \) on the basis in Theorem 1.5.

**Theorem 1.6.** Suppose Theorem 1.3(i) holds. Then there exist scalars \( \lambda, \mu \) in \( \mathbb{K} \) such that the maps \( R, L \) act on the basis (3) as follows.
\[
 Ru_i = u_{i+1} \quad (0 \leq i \leq d - 1), \quad Ru_d = 0, \]
\[
 Rv_i = v_{i+1} \quad (1 \leq i \leq d - 2), \quad Rv_{d-1} = 0, \]
\[
 Lu_0 = 0, \quad Lu_1 = a_0 u_0, \quad Lu_{i+1} = a_i u_i + b_i v_i \quad (1 \leq i \leq d - 1), \]
\[
 Lv_1 = e_0 u_0, \quad Lv_{i+1} = e_i u_i + c_i v_i \quad (1 \leq i \leq d - 2),
\]
where
\[
 a_i = [i+1][d-i][\lambda - i] \eta_{d+i+1} \quad (0 \leq i \leq d - 1),
\]
\[
 b_i = \left( i + \frac{1}{2} \right) \quad (1 \leq i \leq d - 1),
\]
\[
 c_i = [i][d-i-1][\mu - i] \eta_{d+i+2} \quad (1 \leq i \leq d - 2),
\]
\[
 e_i = \left[ \frac{d-i}{2} \right] \left( -\lambda^2 + \mu^2 + \frac{4}{2} \lambda \mu + [2] \eta_{d+3} \lambda - [2] \eta_{d+1} \mu \right) \quad (0 \leq i \leq d - 2).
\]

**Remark 1.7.** The parameters \( \lambda, \mu \) are not necessarily in \( \mathbb{K} \). However since \( [d] \lambda = a_0 \) and \( [d-2]\mu = c_1 \), we find \( [d] \lambda \) and \( [d-2]\mu \) are in \( \mathbb{K} \).

**Theorem 1.8.** Suppose Theorem 1.3(i) holds. Then the maps \( A, A^* \) act on the basis (3) as follows.
\[
 Au_i = \theta_i u_i + u_{i+1} \quad (0 \leq i \leq d - 1), \quad Au_d = \theta_d u_d, \]
\[
 Av_i = \theta_i v_i + v_{i+1} \quad (1 \leq i \leq d - 2), \quad Av_{d-1} = \theta_{d-1} v_{d-1}, \]
\[
 A^* u_0 = \theta_0^* u_0, \quad A^* u_1 = a_0 u_0 + \theta_1^* u_1, \]
\[
 A^* u_{i+1} = a_i u_i + b_i v_i + \theta^{*}_{d-i+1} u_{i+1} \quad (1 \leq i \leq d - 1), \]
\[
 A^* v_1 = e_0 u_0 + \theta_1^* v_1, \quad A^* v_{i+1} = e_i u_i + c_i v_i + \theta^{*}_{d-i+1} v_{i+1} \quad (1 \leq i \leq d - 2).
\]

**Theorem 1.9.** Let \( a_0, c_1, a, a^*, b, b^*, c, c^*, q \) denote scalars in \( \mathbb{K} \). Let \( V \) denote a vector space over \( \mathbb{K} \) with dimension \( 2d \) \( (d \geq 3) \), and let \( A, A^* : V \rightarrow V \) denote linear transformations which act on some basis \( u_0, v_1, \ldots, u_{d-1}, v_{d-1}, u_d \) as in Theorems 1.6 and 1.8. Further suppose that \( V \) is irreducible as an \( (A, A^*) \)-module. Then \( (A, A^*) \) is a tridiagonal pair on \( V \).

Next we consider the case (ii) in Theorem 1.3.
Definition 1.10. With reference to Theorem 1.3(ii), we define $u_0, u_1, u_2, u_3$ and $v_1, v_2$ as in Definition 1.4. As we will show in Section 4, there exists a unique vector $w \in U_1$ such that (i) $R^2 w = 0$; and (ii) $w - L v_2$ is a scalar multiple of $u_1$. We define $w_1 = w, w_2 = Rw$.

Theorem 1.11. Suppose Theorem 1.3(i) holds. Then

(i) $u_0$ is a basis for $U_0$,
(ii) $u_i, v_i, w_i$ is a basis for $U_i$ ($1 \leq i \leq 2$),
(iii) $u_3$ is a basis for $U_3$,
(iv) the vectors

$$u_0, u_1, v_1, w_1, u_2, v_2, w_2, u_3 \quad (4)$$

form a basis for $V$.

We now give the action of $R, L$ on the basis in Theorem 1.11.

Theorem 1.12. Suppose Theorem 1.3(ii) holds. Then there exist scalars $a_0, e_1, f_1$ in $K$ such that the maps $R, L$ act on the basis $(4)$ as follows.

$$Ru_0 = u_1, \quad Ru_1 = u_2, \quad Ru_2 = u_3, \quad Ru_3 = 0,$$
$$Rv_0 = v_2, \quad Rv_1 = 0,$$
$$Rw_0 = w_2, \quad Rw_1 = 0,$$
$$Lu_0 = 0, \quad Lu_1 = a_0 u_0, \quad Lu_{i+1} = a_i u_i + b_i v_i \quad (1 \leq i \leq 2),$$
$$Lv_0 = e_0 u_0, \quad Lv_1 = e_1 u_1 + w_1,$$
$$Lw_1 = f_0 u_0, \quad Lw_2 = f_1 u_1 + s_1 v_1 + t_1 w_1,$$

where

$$a_1 = [2][2](\lambda - \eta_5), \quad a_2 = [3](\lambda - [2] \eta_6),$$
$$b_1 = 1, \quad b_2 = [3],$$
$$e_0 = [3] e_1, \quad f_0 = [3] f_1,$$
$$s_1 = -\lambda^2 - e_1 + [2] \eta_6 \lambda,$$
$$t_1 = \frac{[4] \lambda}{[2]} - [2] \eta_4,$$

and where

$$\lambda = \frac{a_0}{[3]}$$

Theorem 1.13. Suppose Theorem 1.3(ii) holds. Then the maps $A, A^*$ act on the basis $(4)$ as follows.
Remark 1.17. With reference to Theorem 1.6, let \( t \) be such that \((5)\) holds for all tridiagonal pairs when \( q \) is of the form \( 1/2 \). Then for some basis, which is different from the basis \((3)\), they assume the tridiagonal pair is irreducible with respect to the action of the affine quantum group \( U_q(\hat{sl}(2)) \) on \( V \).

Theorem 1.14. Let \( a_0, e_1, f_1, a, a^*, b, b^*, c, c^*, q \) denote scalars in \( K \). Let \( V \) denote a vector space over \( K \) with dimension 8, and let \( A, A^* : V \to V \) denote linear transformations which act on some subspace \( u_0, v_1, v_2, w_1, w_2, u_3 \) as in Theorems 1.12 and 1.13. Further suppose that \( V \) is irreducible as an \((A, A^*)\)-module. Then \((A, A^*)\) is a tridiagonal pair on \( V \).

Remark 1.15. There are some works by Alnajjar and Curtin for some family of tridiagonal pairs which satisfy Theorem 1.3(i). In [1], they give the action of \( A, A^* \) on some basis, which is different from the basis \((3)\). They assume the tridiagonal pair is irreducible as a \((A, A^*)\)-module. Then \((A, A^*)\) is a tridiagonal pair on \( V \).

Remark 1.16. It is known [4] that

\[
\rho_i \leq \left( \begin{array}{c} i \\ 2 \end{array} \right) \quad (0 \leq i \leq d)
\]

holds for tridiagonal pairs of \( q \)-Serre type with \( K \) algebraically closed. It is conjectured [4] that \((5)\) holds for all tridiagonal pairs when \( K \) is algebraically closed.

Remark 1.17. With reference to Theorem 1.6, let \( \psi_i \) denote the eigenvalue of \( L_i R_i \) on \( U_0 \) \((0 \leq i \leq d)\). Then the parameters \( \lambda, \mu \) can be written in terms of \( \psi_1, \psi_2, \psi_3 \) as follows:

\[
\lambda = \frac{\psi_1}{[d]}
\]

\[
\mu = \frac{\psi_3 - [3][d - 2] \left( -\frac{[2][d - 1]}{[d]} \psi_1^2 + \frac{[2][d - 1]}{[d]} \psi_2^2 + \frac{2}{[d]} \psi_1 \psi_2 - [2] \eta_d \psi_3 \right)}{[3][d - 2] \left( -\frac{[2][d - 1]}{[d]} \psi_1^2 + [2][d - 1] \eta_d \psi_1 + \psi_3 \right)}
\]

We remark that the denominator is equal to \([3][d - 2] \psi_0 \) and it is nonzero.
determine the action of \( L \). In Sections 5 and 6, we prove Theorem 1.3. The proofs of Theorems 1.5–1.9 are given in Section 7. Theorems 1.11–1.14 can be shown in a similar way, so we omit the proofs.

2. Background

In this section, we recall some known facts about the tridiagonal pairs. For \( 0 \leq i \leq d \), we set

\[
U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d).
\]  

(6)

**Lemma 2.1** [3, Theorem 4.6]. The space \( V \) is decomposed as

\[
V = U_0 + U_1 + \cdots + U_d \quad \text{(direct sum)}.
\]  

(7)

The decomposition given in (7) is called the **split decomposition** of the tridiagonal pair.

**Lemma 2.2** [3, Corollary 5.7]. For \( 0 \leq i \leq d \),

(i) \( \dim V_i = \dim V_i^* = \dim U_i = \rho_i \),

(ii) \( \rho_i = \rho_{d-i} \).

Let \( F_i : V \to U_i \) denote the projection with respect to the direct sum (7). Then for \( 0 \leq i, j \leq d \),

\[
F_0 + F_1 + \cdots + F_d = I, \quad F_iF_j = F_j \quad \text{if } i \neq j.
\]  

(8)

The **raising map** \( R \) and the **lowering map** \( L \) are defined as follows.

\[
R = A - \sum_{i=0}^{d} \theta_i F_i, \quad L = A^* - \sum_{i=0}^{d} \theta_i^* F_i.
\]  

(9)

**Lemma 2.3** [3, Corollary 6.3]

(i) \( RU_i \subseteq U_{i+1} \quad (0 \leq i \leq d-1), \quad RU_d = 0 \).

(ii) \( LU_i \subseteq U_{i-1} \quad (1 \leq i \leq d), \quad LU_0 = 0 \).

**Lemma 2.4.** Let \( W \) denote a subspace of \( V \). Suppose that \( RW \subseteq W, LW \subseteq W \) and \( F_iW \subseteq W \) for \( 0 \leq i \leq d \). Then \( W = 0 \) or \( W = V \).

**Proof.** Observe that \( A \) and \( A^* \) are represented as linear combinations of \( R, L, F_i \) (\( 0 \leq i \leq d \)) by (9), so that \( AW \subseteq W \) and \( A^* W \subseteq W \). Now the result follows from Definition 1.1(iv). \( \square \)
Lemma 2.5 [3, Theorem 10.1]. There is a sequence of scalars $\beta, \gamma, \varrho, \varepsilon$ taken from $\mathbb{K}$ such that

$$[A, A^2] = \beta AA^* A + A^* A^2 - \gamma(AA^* + A^* A) - \varrho A^* A = 0,$$  \hspace{1cm} (10)

$$[A^*, A^2] = -\beta A^* AA^* + AA^* A^2 - \gamma^*(A^* A + AA^*) - \varrho^* A = 0,$$  \hspace{1cm} (11)

where $[B, C] = BC - CB$. The sequence is unique if the diameter is at least three.

The above relations are known as the tridiagonal relations. These relations imply the following relations between $R$ and $L$. Let $\varepsilon_i$ ($0 \leq i \leq d - 2$) denote the scalar defined by

$$\varepsilon_i = (\theta_i - \theta_{i+2})(\theta_{i+1}^* - \theta_{i+2}^*) - (\theta_{i+2}^* - \theta_i^*)(\theta_{i+1} - \theta_i).$$  \hspace{1cm} (12)

Lemma 2.6 [3, Theorem 12.1]. For $0 \leq i \leq d - 2$,

$$(R^3 - (\beta + 1)R^2L + (\beta + 1)RLR^2 - LR^3 + (\beta + 1)\varepsilon_i R^2)F_i = 0,$$  \hspace{1cm} (13)

$$(L^3R - (\beta + 1)L^2RL + (\beta + 1)LRL^2 - RL^3 - (\beta + 1)\varepsilon_i L^2)F_{i+2} = 0.$$  \hspace{1cm} (14)

Lemma 2.7. Let $V$ denote a vector space over $\mathbb{K}$. Suppose $V$ is decomposed into direct sum of subspaces $U_0, U_1, \ldots, U_d$ ($d \geq 3$), and let $F_i$ denote the projection onto $F_i$. Let $\beta$ denote a scalar in $\mathbb{K}$, and let $\theta_0, \theta_1, \ldots, \theta_d$ (respectively $\theta_0^*, \ldots, \theta_d^*$) denote distinct scalars in $\mathbb{K}$ such that the expressions

$$\frac{\theta_{i+3} - \theta_i}{\theta_{i+2} - \theta_{i+1}}, \quad \frac{\theta_{i+3}^* - \theta_i^*}{\theta_{i+2}^* - \theta_{i+1}^*}$$

both equal to $\beta + 1$ for $0 \leq i \leq d - 3$. Define scalars $\gamma, \varrho, \varepsilon_i$ ($0 \leq i \leq d - 2$) by

$$\gamma = \theta_0 - \beta \theta_1 + \theta_2,$$

$$\varrho = \theta_0^* - \beta \theta_1^* + \theta_2^* - \gamma(\theta_0 + \theta_1),$$

$$\varepsilon_i = (\theta_i - \theta_{i+2})(\theta_{i+1}^* - \theta_{i+2}^*) - (\theta_{i+2}^* - \theta_i^*)(\theta_{i+1} - \theta_i).$$

Let $R, L : V \rightarrow V$ denote linear transformations such that $RU_i \subseteq U_{i+1}$ and $LU_i \subseteq U_{i-1}$ holds for $0 \leq i \leq d$, where we set $U_{-1} = U_{d+1} = 0$. Suppose $R, L$ satisfy (13) for $0 \leq i \leq d - 2$. Define maps $A, A^*$ by

$$A = R + \sum_{i=0}^d \theta_i F_i, \quad A^* = L + \sum_{i=0}^d \theta_i^* F_i.$$  

Then $A, A^*$ satisfy (10).
Proof. Let $C$ denote the left side of (10). Replace $A$ (respectively $A^*$) in each term of $C$ by $R + \sum_{i=0}^{d} \theta_i F_i$ (respectively $L + \sum_{i=0}^{d} \theta_i^* F_i$). After expanding each term of $CF_j$ ($0 \leq j \leq d - 2$), collect the resulting expression in $R$, $L$, and verify that each term vanishes. □

Lemma 2.8 [7, Theorem 3.10]. Let $\beta$, $\gamma$, $\gamma^*$, $\varrho$, $\varrho^*$ denote scalars in $K$, and assume $q$ is not a root of unity, where $\beta = q^2 + q^{-2}$. Let $T$ denote the algebra generated by two symbols $A$, $A^*$ subject to the relations (10), (11). Let $V$ denote an irreducible finite dimensional $T$-module and assume each of $A$, $A^*$ is diagonalizable on $V$. Then $A$, $A^*$ act on $V$ as a tridiagonal pair.

3. The refined split decomposition

In this section, we pick up some results concerning the refined split decomposition from [5]. For the rest of this paper, let $h$ denote the height of the tridiagonal pair.

For $0 \leq r \leq h$ and $r \leq i \leq d - r$, we set

$$U_i^{(r)} = R^{i-r} (U_r \cap \text{Ker } R^{d-2r+1}).$$

Lemma 3.1 [5, Lemma 4.1]. The following hold for $0 \leq r \leq h$.

(i) $U_0^{(0)} = U_0$ and $U_d^{(0)} = U_d$.
(ii) $U_i^{(r)} \subseteq U_i$ ($r \leq i \leq d - r$).
(iii) $U_r^{(r)} = U_r \cap \text{Ker } R^{d-2r+1}$.
(iv) $U_i^{(r)} = R^{i-r} U_r^{(r)}$ ($r \leq i \leq d - r$).
(v) $RU_i^{(r)} = U_{i+1}^{(r)}$ ($r \leq i \leq d - r - 1$), $RU_{d-r}^{(r)} = 0$.
(vi) The restriction $R|_{U_i^{(r)}} : U_i^{(r)} \rightarrow U_{i+1}^{(r)}$ is a bijection ($r \leq i \leq d - r - 1$).

Lemma 3.2 [5, Lemma 4.3]. For $0 \leq r \leq h$,

$$\dim U_i^{(r)} = \rho_r - \rho_{r-1} \quad (r \leq i \leq d - r),$$

where we set $\rho_{-1} = 0$.

Lemma 3.3 [5, Lemma 4.7]. For $0 \leq i \leq d$,

$$U_i = \sum_{r=0}^{m} U_i^{(r)} \quad (\text{direct sum}),$$

where $m = \min\{i, h, d - i\}$. 
For $0 \leq r \leq h$, we set
\[ U(r) = \sum_{i=r}^{d-r} U_i^{(r)}. \]  
(18)

**Lemma 3.4** [5, Lemma 5.1]. \( V \) is decomposed as
\[ V = \sum_{r=0}^{h} U^{(r)} \]  
(direct sum).  
(19)

**Lemma 3.5** [5, Lemma 5.2]. For $0 \leq r \leq h$ and $0 \leq i \leq d$,
\[ U^{(r)} \cap U_i = \begin{cases} U_i^{(r)} & \text{if } r \leq i \leq d - r, \\ 0 & \text{otherwise}. \end{cases} \]  
(20)

**Lemma 3.6** [5, Lemma 5.3]. For $0 \leq r \leq h$,
\[ RU^{(r)} \subseteq U^{(r)}. \]  
(21)

**Lemma 3.7** [5, Theorem 5.6]. For $0 \leq r \leq h$,
\[ LU^{(r)} \subseteq U^{(r-1)} + U^{(r)} + U^{(r+1)}, \]  
(22)
where we set $U^{-1} = U^{(h+1)} = 0$.

Let
\[ F^{(r)} : V \longrightarrow U^{(r)} \quad (0 \leq r \leq h) \]
denote the projection with respect to the direct sum $V = \sum_{r=0}^{h} U^{(r)}$. Observe that for $0 \leq r \leq h$ and $0 \leq s \leq h$,
\[ F^{(0)} + F^{(1)} + \cdots + F^{(h)} = I, \quad F^{(r)} F^{(s)} = \begin{cases} F^{(r)} & \text{if } r = s, \\ 0 & \text{if } r \neq s. \end{cases} \]  
(23)

We set
\[ F_i^{(r)} = F_i F^{(r)} \quad (0 \leq r \leq h, \ 0 \leq i \leq d). \]

**Lemma 3.8** [5, Lemma 6.1]. For $0 \leq r \leq h$ and $0 \leq i \leq d$,
(i) $F_i^{(r)} = F^{(r)} F_i = F_i F^{(r)}$,
(ii) $F_0^{(0)} = F_0$ and $F_d^{(0)} = F_d$,
(iii) $F_i^{(r)} \neq 0$ if and only if $r \leq i \leq d - r$. 

Lemma 3.9 [5, Lemma 6.2]. For $0 \leq r \leq h$ and $r \leq i \leq d - r$, $F_i^{(r)} V = U_i^{(r)}$, and

$$F_i^{(r)} : V \rightarrow U_i^{(r)}$$

is the projection with respect to the direct sum $V = \sum_{r=0}^{h} \sum_{i=r}^{d-r} U_i^{(r)}$.

Lemma 3.10 [5, Lemma 6.3]. For $0 \leq r \leq h$.

(i) $F^{(r)} R = RF^{(r)}$,
(ii) $F^{(r)} R = RF^{(i-1)}_{i-1}$ $(1 \leq i \leq d)$,
(iii) $RF^{(r)} = 0$.

We set

$$L^{(-)} = \sum_{r=1}^{h} F^{(r-1)} L F^{(r)}, \quad L^{(0)} = \sum_{r=0}^{h} F^{(r)} L F^{(r)},$$

$$L^{(+)} = \sum_{r=0}^{h-1} F^{(r+1)} L F^{(r)}.$$  \hspace{1cm} (24)

Lemma 3.11 [5, Lemma 6.5]

$$L = L^{(-)} + L^{(0)} + L^{(+)}.$$  \hspace{1cm} (25)

Lemma 3.12 [5, Lemma 6.6]. The following hold.

(i) $F^{(r-1)} L F^{(r)} = L^{(-)} F^{(r)}$ $(1 \leq r \leq h)$,
(ii) $F^{(r)} L F^{(r)} = L^{(0)} F^{(r)}$ $(0 \leq r \leq h)$,
(iii) $F^{(r+1)} L F^{(r)} = L^{(+)} F^{(r)}$ $(0 \leq r \leq h - 1)$.

Lemma 3.13 [5, Lemma 6.7]. The following hold for $0 \leq r \leq h$.

(i) $L^{(0)} F^{(r)} = 0$.
(ii) $L^{(+)} F^{(r)} = L^{(+)} F^{(r+1)} = 0$.

Lemma 3.14 [5, Theorem 8.4]. For $0 \leq r \leq h - 1$ and $r + 2 \leq i \leq d - r - 1$,

$$RL^{(+) = \frac{i - r - 1}{i - r + 1} L^{(+)} R}$$

vanishes on $U_i^{(r)}$. 
Lemma 3.15 [5, Theorem 9.4]. For $1 \leq r \leq h$ and $r \leq i \leq d - r - 1$,
\[
RL(-) - \frac{[d - r - i + 2]}{[d - r - i]}L(-)R
\]
vanishes on $U_i^{(r)}$.

Lemma 3.16 [3, Lemma 8.6, Theorem 11.1]. There are scalars $a, a^*, b, b^*, c, c^*$ in the algebraic closure $\mathbb{K}$ such that
\[
\theta_i = aq^{2i} + bq^{-2i} + c \quad (0 \leq i \leq d), \tag{26}
\]
\[
\theta_i^* = a^*q^{2i} + b^*q^{-2i} + c^* \quad (0 \leq i \leq d). \tag{27}
\]
We fix scalars $a, a^*, b, b^*, c, c^*$ which satisfy (26) and (27).

Lemma 3.17 [5, Theorem 10.7]. For $0 \leq i \leq d - 2$,
\[
\varepsilon_i = [2](q - q^{-1})^3(aa^*q^{4i+4} - bb^*q^{-(4i+4)}). \tag{28}
\]
We set
\[
\mu_i = (q - q^{-1})^3(aa^*q^{d+2i} - bb^*q^{-d-2i}).
\]

Lemma 3.18 [5, Theorem 10.9]. For $0 \leq r \leq h$ and $r + 1 \leq i \leq d - r - 1$, the following map vanishes on $U_i^{(r)}$:
\[
RL^{(0)} - \frac{[i - r][d - r - i + 1]}{[i - r + 1][d - r - i]}L^{(0)}R - [i - r][d - r - i + 1]\mu_i I. \tag{29}
\]

Lemma 3.19. Let $r$ denote an integer with $0 \leq r \leq h$. Let $Y$ denote a subspace of $U_i^{(r)}$ such that $L^{(0)}RY \subseteq Y$. We set $W = \sum_{i=0}^{d-2r} R^iY$. Then $L^{(0)}W \subseteq W$.

Proof. We show
\[
L^{(0)}R^iY \subseteq R^{i-1}Y \quad (1 \leq i \leq d - 2r), \tag{30}
\]
by induction. Clearly (30) holds for $i = 1$ by our assumption. Assume $2 \leq i \leq d - 2r$. Pick any vector $u$ in $Y$ and observe that $R^{i-1}u$ belongs to $U_{r+i-1}^{(r)}$ by Lemma 3.1. Applying Lemma 3.18 to $R^{i-1}u$,
\[
L^{(0)}R^iu = \frac{[i][d - 2r - i + 1]}{[i - 1][d - 2r - i + 2]}RL^{(0)}R^{i-1}u - [i][d - 2r - i + 1]\mu_{r+i-1}R^{i-1}u.
\]
This implies that $L^{(0)} R^i u$ lies in the span of $\{ R L^{(0)} R^{i-1} u, R^{i-1} u \}$, where we have $R^{i-1} u \in R^{i-1} Y$, and $L^{(0)} R^{i-1} u \in R^{i-2} Y$ by induction. Hence $L^{(0)} R^i u$ belongs to $R^{i-1} Y$. We have also $L^{(0)} Y \subseteq L^{(0)} U_r(r) = 0$ by Lemma 3.13. Thus $L^{(0)} W \subseteq W$. □

4. Determining the action of $L$

For the rest of this paper, we assume $\rho_0 = \rho_d = 1$ and $\rho_1 = \rho_2 = \cdots = \rho_{d-1} \geq 2$, so that $h = 1$.

Lemma 4.1. $V$ is decomposed as $V = \sum_{i=0}^{d} U_i^{(0)} + \sum_{i=1}^{d-1} U_i^{(1)}$ (direct sum).

Proof. Follows from (18) and (19) with $h = 1$. □

We fix a nonzero vector $u_0$ in $U_0$, and we set $u_i = R^i u_0$ ($1 \leq i \leq d$).

Lemma 4.2. For $0 \leq i \leq d$, $\{ u_i \}$ is a basis of $U_i^{(0)}$.

Proof. Follows from Lemma 3.1. □

Thus $L^{(0)} u_{i+1}$ is a scalar multiple of $u_i$. We set

$$L^{(0)} u_{i+1} = a_i u_i \quad (0 \leq i \leq d-1). \quad (31)$$

Lemma 4.3. For $0 \leq i \leq d-1$,

$$a_i = [i+1][d-i] \left( a_0 \frac{a_1}{d} - \sum_{k=1}^{i} \mu_k \right). \quad (32)$$

Proof. We show (32) by induction. Clearly (32) holds for $i = 0$, so we assume $1 \leq i \leq d - 1$. Applying Lemma 3.18 to $u_i$,

$$L^{(0)} R u_i = \frac{[i+1][d-i]}{[i][d-i+1]} R L^{(0)} u_i - [i+1][d-i] \mu_i u_i,$$

where we have $L^{(0)} R u_i = L^{(0)} u_{i+1} = a_i u_i$ and $R L^{(0)} u_i = R(a_i u_{i-1}) = a_{i-1} u_i$. Hence

$$a_i = \frac{[i+1][d-i]}{[i][d-i+1]} a_{i-1} - [i+1][d-i] \mu_i.$$
By induction,
\[ a_{i-1} = [i][d - i + 1] \left( \frac{a_0}{d} - \sum_{k=1}^{i-1} \mu_k \right). \]
Now (32) follows. □

**Lemma 4.4**

(i) \( L^{(+)} u_0 = 0, L^{(+)} u_1 = 0, \)

(ii) \( L^{(+)} u_2 \neq 0. \)

**Proof.** (i) Follows from Lemma 3.13.

(ii) Suppose \( L^{(+)} u_2 = 0. \) Applying Lemma 3.14 to \( u_i, \)

\[ L^{(+)} u_{i+1} = L^{(+)} Ru_i = \left[ \frac{i + 1}{i - 1} \right] RL^{(+)} u_i \quad (2 \leq i \leq d - 1). \]
Combining with (i), this implies \( L^{(+)} u_i = 0 \) for \( 0 \leq i \leq d, \) so that \( L^{(+)} U^{(0)} = 0. \)
Using (24) and (25), this implies
\[ \nabla U^{(0)} = L^{(-)} U^{(0)} + L^{(0)} U^{(0)} + L^{(+)} U^{(0)} = L^{(0)} U^{(0)} \subseteq U^{(0)}. \]
Thus \( U^{(0)} \) is invariant under \( L. \) Also we have \( R U^{(0)} \subseteq U^{(0)} \) by Lemma 3.6, and \( F_i U^{(0)} = U_i^{(0)} \subseteq U^{(0)} \) (\( 0 \leq i \leq d \)) by Lemma 3.8. These imply \( U^{(0)} = \nabla \) by Lemma 2.4, a contradiction. □

We set \( v_1 = L^{(+)} u_2, v_i = R^{i-1} v_1 \) (\( 2 \leq i \leq d - 1 \)).

**Lemma 4.5.** For \( 1 \leq i \leq d - 1, \) \( v_i \) lies in \( U^{(1)}_i, \) and \( v_i \neq 0. \)

**Proof.** Follows from Lemmas 4.4 and 3.1. □

We set
\[ b_i = \left[ \frac{i + 1}{2} \right] \quad (1 \leq i \leq d - 1). \]

**Lemma 4.6.** For \( 1 \leq i \leq d - 1, \)
\[ L^{(+)} u_{i+1} = b_i v_i. \]

**Proof.** We show (34) by induction. Clearly (34) holds for \( i = 1, \) so we assume \( 2 \leq i \leq d - 1. \) Applying Lemma 3.14 to \( u_i, \)
L^{(+)}u_{i+1} = L^{(+)}Ru_i = \left[ \frac{i + 1}{i - 1} \right] RL^{(+)}u_i.

By induction,

L^{(+)}u_i = b_{i-1}v_{i-1} = \left[ \begin{array}{c} \frac{i}{2} \\ \end{array} \right] v_{i-1}.

Hence

L^{(+)}u_{i+1} = \left[ \frac{i + 1}{i - 1} \right] \left[ \begin{array}{c} \frac{i}{2} \\ \end{array} \right] v_i. \quad \square

Observe that \( L^{(-)}v_{i+1} \) is a scalar multiple of \( u_i \) by Lemmas 4.5 and 4.2, so we may write

\[ L^{(-)}v_{i+1} = e_i u_i \quad (0 \leq i \leq d - 2). \tag{35} \]

**Lemma 4.7.** For \( 0 \leq i \leq d - 2 \),

\[ e_i = \frac{[d - i][d - i - 1]}{[d][d - 1]} e_0. \tag{36} \]

**Proof.** We show (36) by induction. Clearly (36) holds for \( i = 0 \), so we assume \( 1 \leq i \leq d - 2 \). Applying Lemma 3.15 to \( v_i \),

\[ L^{(-)}Rv_i = \left[ \frac{d - i - 1}{d - i + 1} \right] \right] RL^{(-)}v_i, \]

where we have

\[ L^{(-)}Rv_i = L^{(-)}v_{i+1} = e_i u_i, \]

and

\[ RL^{(-)}v_i = R(e_{i-1}u_{i-1}) = e_{i-1}u_i, \]

so that

\[ e_i = \frac{[d - i - 1]}{[d - i + 1]} e_{i-1}. \]

By induction

\[ e_{i-1} = \frac{[d - i + 1][d - i]}{[d][d - 1]} e_0. \]

Now (36) follows. \( \square \)
Lemma 4.8. Suppose $\rho_1 = 2$. Then $\{v_i\}$ is a basis of $U_i^{(1)}$ ($1 \leq i \leq d - 1$).

Proof. Follows from Lemmas 4.8 and 3.1. □

Hence, when $\rho_1 = 2$,

\[ L^{(0)} v_1 = 0, \quad L^{(0)} v_{i+1} = c_i v_i \quad (1 \leq i \leq d - 2) \]  \hspace{1cm} (37)

hold for some scalars $c_1, \ldots, c_{d-2}$.

Lemma 4.9. Suppose $\rho_1 = 2$. Then

\[ c_i = [i][d - i - 1] \left( \frac{c_1}{[d-2]} - \sum_{k=2}^{i} \mu_k \right) \quad (1 \leq i \leq d - 2). \]  \hspace{1cm} (38)

Proof. We show (38) by induction. Clearly (38) holds for $i = 1$, so we assume $2 \leq i \leq d - 2$. Applying (29) to $v_i$,

\[ L^{(0)} R v_i = \left[ \frac{i}{i-1} \right] [d-i] R L^{(0)} v_i - [i][d-i-1] \mu_i v_i. \]

This implies

\[ c_i = \left[ \frac{i}{i-1} \right] [d-i] c_{i-1} - [i][d-i] \mu_i. \]

By induction,

\[ c_{i-1} = [i-1][d-i] \left( \frac{c_1}{[d-2]} - \sum_{k=2}^{i-1} \mu_k \right). \]

Now (38) follows. □

Lemma 4.10. Suppose $\rho_1 \geq 3$. Then $L^{(0)} v_2$ and $v_1$ are linearly independent.

Proof. By way of contradiction, we assume $L^{(0)} v_2$ lies in the span $Y$ of $\{v_1\}$. We set $W = \sum_{i=0}^{d-2} R Y$ and $Z = U^{(0)} + W$. Clearly $Z$ is invariant under $R$ and $F_i$ ($0 \leq i \leq d$). If $Z$ is invariant under $L$, then $Z$ is invariant under $L$, $R$ and $F_i$ ($0 \leq i \leq d$), so that $W = V$ by Lemma 2.4. This contradicts our assumption $\rho_1 \geq 3$. So, it is enough to show that $Z$ is invariant under $L$.

Observe that $W \subseteq U^{(1)}$ and $L^{(1)} U^{(1)} = 0$ by $h = 1$, so that $LW \subseteq L^{(-)} W + L^{(0)} W \subseteq U^{(0)} + L^{(0)} W$. We have $L^{(0)} R Y \subseteq Y$ from our assumption $L^{(0)} v_2 \in Y$, and this implies $L^{(0)} W \subseteq W$ by Lemma 3.19. Observe that $L^{(+)} U^{(0)} \subseteq W$ by Lemma 4.6, so that $LU^{(0)} \subseteq L^{(-)} U^{(0)} + L^{(0)} U^{(0)} + L^{(+)} U^{(0)} \subseteq U^{(0)} + W$, since $L^{(-)} U^{(0)} = 0$. Therefore $U^{(0)} + W$ is invariant under $L$. □
When $\rho_1 \geq 3$, we set
\[ w_1 = L^{(0)}v_2, \quad w_i = R^{i-1}w_1 \quad (2 \leq i \leq d - 1). \]

**Lemma 4.11.** Suppose $\rho_1 \geq 3$. Then $v_i$ and $w_i$ are linearly independent for $1 \leq i \leq d - 1$.

**Proof.** Follows from Lemmas 4.10 and 3.1. \(\square\)

When $\rho_1 \geq 3$, we set
\[ L_{(-)}w_{i+1} = f_i u_i \quad (0 \leq i \leq d - 2). \]

**Lemma 4.12.** Suppose $\rho_1 \geq 3$. Then
\[ f_i = \left[ \frac{d-i}{d} \right] \mu_k, \quad n_i = \left[ \frac{i}{d} \right] \mu_k. \] (39)

**Proof.** Similar to the proof of Lemma 4.7. \(\square\)

**Lemma 4.13.** Suppose $\rho_1 \geq 3$. Then
\[ L^{(0)}v_{i+1} = m_i v_i + n_i w_i \quad (1 \leq i \leq d - 2), \] (40)

where
\[ m_i = -\left[ i \right] [d-i-1] \sum_{k=2}^{i} \mu_k, \quad n_i = \left[ \frac{i}{d} \right] \mu_k. \] (41)

**Proof.** We show (40) by induction. Observe that (40) holds for $i = 1$ with $m_1 = 0$, $n_1 = 1$, since $L^{(0)}v_2 = w_1$. We assume $2 \leq i \leq d - 2$. Applying (29) to $v_i$,
\[ L^{(0)} R_{v_i} = \left[ i \right] [d-i-1] R L^{(0)} v_i - [i][d-i-1] \mu_i v_i. \] (42)

Observe that $L^{(0)} R_{v_i} = L^{(0)} v_{i+1}$. By induction, we have
\[ R L^{(0)} v_i = R(m_{i-1} v_{i-1} + n_{i-1} w_{i-1}) = m_{i-1} v_i + n_{i-1} w_i, \]

with
\[ m_{i-1} = -[i-1][d-i] \sum_{k=2}^{i-1} \mu_k, \quad n_{i-1} = \left[ \frac{i-1}{d} \right]. \]
Thus (42) becomes
\[ L(0)v_{i+1} = \frac{[i][d-i-1]}{[i-1][d-i]} \left( -[i-1][d-i] \sum_{k=2}^{i-1} \mu_k v_i + \frac{[i-1][d-i]}{[d-2]} w_i \right) - [i][d-i-1] \mu_i v_i \]
\[ = -[i][d-i-1] \sum_{k=2}^{i} \mu_k v_i + \frac{[i][d-i-1]}{[d-2]} w_i. \]
□

Lemma 4.14. Suppose \( \rho_1 = 3 \). Then \( \{v_i, w_i\} \) is a basis of \( U_i^{(1)} (1 \leq i \leq d - 1) \).

Proof. Follows from Lemmas 4.11 and 3.2. □

Hence, when \( \rho_1 = 3 \),
\[ L(0)w_{i+1} = s_i v_i + t_i w_i \quad (1 \leq i \leq d - 2) \]
holds for some scalars \( s_i, t_i \).

Lemma 4.15. Suppose \( \rho_1 = 3 \). Then
\[ s_i = \frac{[i][d-i-1]}{[d-2]} s_1 \quad (1 \leq i \leq d - 2), \]
\[ t_i = \frac{[i][d-i-1]}{[d-2]} \left( \frac{t_1}{[d-2]} - \sum_{k=2}^{i} \mu_k \right) \quad (1 \leq i \leq d - 2). \]

Proof. Similar to the proof of Lemma 4.13. □

Lemma 4.16. The following hold with the values of \( a_i, b_i, e_i, f_i, m_i, n_i \) given by (32), (33), (36), (39) and (41).

(i) \( Lu_0 = 0, Lu_1 = a_0 u_0, Lu_{i+1} = a_i u_i + b_i v_i \) (\( 1 \leq i \leq d - 1 \)),
(ii) \( Lv_1 = e_0 u_0, Lv_{i+1} = e_i u_i + m_i v_i + n_i w_i \) (\( 1 \leq i \leq d - 2 \)),
(iii) \( L(\cdot)w_{i+1} = f_i u_i \) (\( 0 \leq i \leq d - 2 \)).

Proof. Follows from (25) and Lemmas 4.3, 4.6, 4.7, 4.12 and 4.13. □

5. Proof of \( \rho_1 \leq 3 \)

In this section, we show \( \rho_1 \leq 3 \). By way of contradiction, we assume \( \rho_1 \geq 4 \).
Lemma 5.1. The vectors \( v_1, w_1, L^{(0)}w_2 \) are linearly independent.

Proof. Suppose \( v_1, w_1, L^{(0)}w_2 \) are linearly dependent. Since \( v_1, w_1 \) are linearly independent by Lemma 4.11, \( L^{(0)}w_2 \) lies in \( \text{span} \{v_1, w_1\} \). Observe that \( RY = \text{span} \{v_2, w_2\} \) and \( L^{(0)}v_2 = w_1 \), so that \( L^{(0)}RY \subseteq Y \). Hence the subspace \( W = \sum_{i=0}^{d-2} R^iY \) is invariant under \( L^{(0)} \) by Lemma 3.19. This implies \( LW \subseteq L^{(0)}W + L^{(0)}Y \subseteq U^{(0)} + Y \). Moreover, \( L^{(2)}U^{(0)} \subseteq W \) by Lemma 4.6. Hence \( U^{(0)} + W \) is invariant under \( L \). Clearly \( U^{(0)} + W \) is invariant under \( R \) and \( F_i \) \((0 \leq i \leq d)\). These imply \( U^{(0)} + W = V \) by Lemma 2.4, so that \( U_1 = \{u_1, v_1, w_1\} \), contradicting our assumption \( \rho_1 \geq 4 \). \( \Box \)

We set \( L^{(0)}w_2 = x_1 \), so that

\[
Lw_2 = f_1u_1 + x_1. \tag{46}
\]

Observe that \( u_1, v_1, x_1 \) are linearly independent by \( (19) \) and Lemma 5.1. Applying \( (14) \) to \( u_3 \),

\[
\]

We compute each term of \( (47) \) using Lemma 4.16 and \( (46) \). We need to divide our computation into two cases. First we consider the case of \( d = 3 \). Observe that \( Ru_3 = Rv_2 = 0 \) by Lemma 3.1.

\[
L^3Ru_3 = 0,
\]

\[
L^2RLu_3 = L^2R(a_2u_2 + b_2v_2) = L^2(a_2u_3) = L(a_2(a_2u_2 + b_2v_2)) = L(a_2a_2u_2 + a_2b_2v_2) = a_2a_2(a_1u_1 + v_1) + a_2b_2(e_1u_1 + w_1) = (a_2a_2 + e_1a_2b_2)u_1 + a_2b_2v_1 + a_2b_2w_1,
\]

\[
LRL^2u_3 = LRL(a_2u_2 + b_2v_2) = LR(a_2(a_1u_1 + v_1) + b_2(e_1u_1 + w_1)) = LR((a_1a_2 + e_1b_2)u_1 + a_2v_1 + b_2w_1) = L((a_1a_2 + e_1b_2)u_1 + a_2v_1 + b_2w_1) = (a_1a_2 + e_1b_2)(a_1u_1 + v_1) + a_2(e_1u_1 + w_1) + b_2(f_1u_1 + x_1) = (a_1(a_1a_2 + e_1b_2) + e_1a_2 + f_1b_2)u_1 + (a_1a_2 + e_1b_2)v_1 + a_2w_1 + b_2x_1,
\]

\[
RL^3u_3 \in \text{span} \{u_1\},
\]

\[
L^2u_3 = (a_1a_2 + e_1b_2)u_1 + a_2v_1 + b_2w_1.
\]

Observe that the coefficient of \( x_1 \) in \( (47) \) becomes \( [3]b_2 \), so that \( [3]b_2 = 0 \), contradicting our assumption that \( q \) is not a root of unity. Next we consider the case of \( d \geq 4 \).
\[ L^3 R u_3 = L^3 u_4 = L^2 (a_3 u_3 + b_3 v_3) \]
\[ = L((a_2 a_3 + e_2 b_3) u_2 + (b_2 a_3 + m_2 b_3) v_2 + n_2 b_3 u_2) \]
\[ = (a_2 a_3 + e_2 b_3)(a_1 u_1 + v_1) \]
\[ + (b_2 a_3 + m_2 b_3)(e_1 u_1 + v_1) + n_2 b_3(f_1 u_1 + x_1), \]
\[ L^2 R L u_3 = L^2 R (a_2 u_2 + b_2 v_2) = L^2 (a_2 u_3 + b_2 v_3) \]
\[ = L((a_2 a_2 + b_2 e_2) u_2 + (a_2 b_2 + b_2 m_2) v_2 + b_2 n_2 u_2) \]
\[ = (a_2 a_2 + b_2 e_2)(a_1 u_1 + v_1) + (a_2 b_2 + b_2 m_2)(e_1 u_1 + w_1) \]
\[ + b_2 n_2(f_1 u_1 + x_1), \]
\[ L R L^2 u_3 = L R L (a_2 u_2 + b_2 v_2) = L R (a_2 (a_1 u_1 + v_1) + b_2 (e_1 u_1 + w_1)) \]
\[ = L((a_1 a_2 + e_1 b_2) u_1 + a_2 v_1 + b_2 w_1) \]
\[ = L((a_1 a_2 + e_1 b_2) u_2 + a_2 v_2 + b_2 w_2) \]
\[ = (a_1 a_2 + e_1 b_2)(a_1 u_1 + v_1) + a_2 (e_1 u_1 + w_1) + b_2 (f_1 u_1 + x_1), \]
\[ R L^3 u_3 \in \text{span } \{u_1\}, \]
\[ L^2 u_3 = (a_1 a_2 + e_1 b_2) u_1 + a_2 v_1 + b_2 w_1. \]

Now looking at the coefficients of \( x_1 \) in (47),
\[ n_2 b_3 - [3] b_2 n_2 + [3] b_2 = 0, \]
so that
\[ \frac{[2][d - 3]}{[d - 2]} \cdot \frac{[4][3]}{[2]} - [3][3] \cdot \frac{[2][d - 3]}{[d - 2]} + [3][3] = \frac{[3][d]}{[d - 2]} = 0, \]
contradicting our assumption that \( q \) is not a root of unity. This completes the proof of \( \rho_1 = 3 \).

6. Proof of \( d = 3 \)

In this section, we assume \( \rho_1 = 3 \), and we show \( d = 3 \). By way of contradiction, we assume \( d \geq 4 \). Applying (14) to \( v_3 \),
\[ L^3 R v_3 - [3] L^2 R L v_3 + [3] L R L^2 v_3 - R L^3 v_3 = [3] \delta_1 v_3 = 0. \]

We compute each term of (49) using Lemmas 4.16 and 3.17. The term of \( L^3 R v_3 \) vanishes when \( d = 4 \). When \( d \geq 5 \), it becomes
\[ L^3 R v_3 = L^3 v_4 = L^2 (e_3 w_3 + m_3 v_3 + n_3 w_3) = \\
L(e_1 (a_2 u_2 + b_2 v_2) + m_2 (e_2 u_2 + m_2 v_2 + n_2 w_2) \\
+ n_3 (f_2 u_2 + s_2 v_2 + t_2 w_2)) = \\
L((a_2 e_3 + e_2 m_3 + f_2 n_3) u_2 + (b_2 e_3 + m_2 m_3 + s_2 n_3) v_2 \\
+ (n_2 m_3 + t_2 n_3) w_2) = \\
(a_2 e_3 + e_2 m_3 + f_2 n_3) (a_1 u_1 + v_1) \\
+ (b_2 e_3 + m_2 m_3 + s_2 n_3) (e_1 u_1 + w_1) \\
+ (n_2 m_3 + t_2 n_3) (f_1 u_1 + s_1 v_1 + t_1 w_1) = \\
(a_1 (a_2 e_2 + e_2 m_2 + f_2 n_2) + e_1 (b_2 e_2 + m_2 m_2 + s_2 n_2) \\
+ f_1 (m_2 n_2 + t_2 n_2)) u_1 \\
+ ((a_2 e_2 + e_2 m_2 + f_2 n_2) + s_1 (m_2 n_2 + t_2 n_2)) v_1 \\
+ ((b_2 e_2 + m_2 m_2 + s_2 n_2) + t_1 (m_2 n_2 + t_2 n_2)) w_1. \\
\]

The other terms become

\[ L^2 RL v_3 = L^2 R (e_2 u_2 + m_2 v_2 + n_2 w_2) = L^2 (e_2 u_3 + m_2 v_3 + n_2 w_3) = \\
L(e_2 (a_2 u_2 + b_2 v_2) + m_2 (e_2 u_2 + m_2 v_2 + n_2 w_2) \\
+ n_2 (f_2 u_2 + s_2 v_2 + t_2 w_2)) = \\
L((a_2 e_2 + e_2 m_2 + f_2 n_2) u_2 + (b_2 e_2 + m_2 m_2 + s_2 n_2) v_2 \\
+ (m_2 n_2 + t_2 n_2) w_2) = \\
(a_2 e_2 + e_2 m_2 + f_2 n_2) (a_1 u_1 + v_1) \\
+ (b_2 e_2 + m_2 m_2 + s_2 n_2) (e_1 u_1 + w_1) \\
+ (m_2 n_2 + t_2 n_2) (f_1 u_1 + s_1 v_1 + t_1 w_1) = \\
(a_1 (a_2 e_2 + e_2 m_2 + f_2 n_2) + e_1 (b_2 e_2 + m_2 m_2 + s_2 n_2) \\
+ f_1 (m_2 n_2 + t_2 n_2)) u_1 \\
+ ((a_2 e_2 + e_2 m_2 + f_2 n_2) + s_1 (m_2 n_2 + t_2 n_2)) v_1 \\
+ ((b_2 e_2 + m_2 m_2 + s_2 n_2) + t_1 (m_2 n_2 + t_2 n_2)) w_1. \\
\]

\[ LRL^2 v_3 = LRL (e_2 u_2 + m_2 v_2 + n_2 w_2) = \\
LRL(e_2 (a_2 u_2 + v_1) + m_2 (e_1 u_1 + v_1) + n_2 (f_1 u_1 + s_1 v_1 + t_1 w_1)) = \\
LRL((a_1 e_2 + e_1 m_2 + f_1 n_2) u_1 + (e_2 + s_1 n_2) v_1 + (m_2 + t_1 n_2) w_1) = \\
L((a_1 e_2 + e_1 m_2 + f_1 n_2) u_1 + (e_2 + s_1 n_2) (e_1 u_1 + w_1) \\
+ (m_2 + t_1 n_2) (f_1 u_1 + s_1 v_1 + t_1 w_1) = \\
(a_1 (a_1 e_2 + e_1 m_2 + f_1 n_2) + e_1 (e_2 + s_1 n_2) + f_1 (m_2 + t_1 n_2)) u_1 \\
+ ((a_1 e_2 + e_1 m_2 + f_1 n_2) + s_1 (m_2 + t_1 n_2)) v_1 \\
+ ((e_2 + s_1 n_2) + t_1 (m_2 + t_1 n_2)) w_1, \\
\]

\[ RL^3 v_3 \in \text{span} \{u_1\}. \\
L^2 v_3 = (a_1 e_2 + e_1 m_2 + f_1 n_2) u_1 + (e_2 + s_1 n_2) v_1 + (m_2 + t_1 n_2) w_1. \]
When $d \geq 5$, by a routine computation, the coefficient of $w_1$ in (49) becomes

$$b_2 e_3 + m_2 n_3 + s_2 n_3 + t_1 (n_2 m_3 + t_2 n_3) - [3]((b_2 e_2 + m_2 m_2 + s_2 n_2) + t_1 (m_2 n_2 + t_2 n_2)) + [3](e_2 + s_1 n_2 + t_1 (m_2 + t_1 n_2)) - [3] e_1 (m_2 + t_1 n_2) = -\frac{[3][d - 3]}{[d - 1]} e_0,$$

so that $e_0 = 0$, and this implies $e_i = 0$ ($1 \leq i \leq d - 1$). The coefficient of $v_1$ becomes

$$f_2 n_3 + s_1 (n_2 m_3 + t_2 n_3) - [3](f_2 n_2 + s_1 (m_2 + t_2 n_2)) + [3](f_1 n_2 + s_1 (m_2 + t_1 n_2)) - [3] e_1 s_1 n_2 = -\frac{[3][d - 3]}{[d - 1]} f_0,$$

so that $f_0 = 0$. When $d = 4$, the coefficient of $w_1$ becomes

$$-[3]((b_2 e_2 + m_2 m_2 + s_2 n_2) + t_1 (m_2 n_2 + t_2 n_2)) + [3](e_2 + s_1 n_2 + t_1 (m_2 + t_2 n_2)) - [3] e_1 (m_2 + t_1 n_2) = -e_0,$$

so that $e_i = 0$ ($1 \leq i \leq d - 1$). The coefficient of $v_1$ becomes

$$-[3](f_2 n_2 + s_1 (m_2 n_2 + t_2 n_2)) + [3](f_1 n_2 + s_1 (m_2 + t_1 n_2)) - [3] e_1 s_1 n_2 = f_0.$$

In either case, $e_i = f_i = 0$ ($1 \leq i \leq d - 2$), so that $L^{(-)} U^{(1)} = 0$ and hence $L U^{(1)} \subseteq U^{(1)}$. Since $U^{(1)}$ is invariant under $R$ and $F_i$ ($0 \leq i \leq d$), we get $U^{(1)} = V$ by Lemma 2.4, a contradiction. This completes the proof of Theorem 1.3.

### 7. Proof of Theorems 1.5–1.9

**Proof of Theorem 1.5.** Follows from Lemmas 2.1, 3.3, 4.2 and 4.5. □

**Lemma 7.1.** Suppose Theorem 1.3(ii) holds. Then the maps $R$, $L$ act on the basis (3) as follows.

- $Ru_i = u_{i+1}$ ($0 \leq i \leq d - 1$),
- $Ru_d = 0$,
- $Ru_i = v_{i+1}$ ($1 \leq i \leq d - 2$),
- $Lu_0 = 0$,
- $Lu_1 = a_0 u_0$,
- $Lu_{i+1} = a_i u_i + b_i v_i$ ($1 \leq i \leq d - 1$),
\[ \begin{align*}
Lv_1 &= e_0u_0, \\
Lv_{i+1} &= e_iu_i + c_i v_i \quad (1 \leq i \leq d-2),
\end{align*} \]

where the coefficients satisfy (32), (33), (36) and (38).

**Proof.** Follows from Theorem 1.5 and Eqs. (24), (31)–(38). \(\square\)

**Proof of Theorem 1.6.** First observe the following formulas hold, which can be verified by routine computations.

\[ \begin{align*}
\sum_{k=1}^{i} \mu_k &= [i] \eta_{d+i+1}, \\
\sum_{k=2}^{i} \mu_k &= [i-1] \eta_{d+i+2}.
\end{align*} \] (50) (51)

Now the expressions for \(a_i, b_i, c_i\) follow from Lemma 7.1. Applying (14) to \(v_2\),

\[ L^3 R v_2 - [3] L^2 R L v_2 + [3] L R L^2 v_2 - RL^2 v_2 - [3] \epsilon_0 v_2 = 0. \] (52)

We compute each term of (52) as follows using Lemma 7.1.

\[ \begin{align*}
L^3 R v_2 & = L^3 v_3 = L^2(e_2 u_2 + c_2 v_2) \\
& = L(e_2(a_1 u_1 + v_1) + c_2(e_1 u_1 + c_1 v_1)) \\
& = L((a_1e_2 + e_1c_2)u_1 + (e_2 + c_1 c_2) v_1) \\
& = (a_0(a_1e_2 + e_1c_2) + e_0(e_2 + c_1 c_2))u_0,
\end{align*} \]

\[ \begin{align*}
L^2 R L v_2 & = L^2 R(e_1 u_1 + c_1 v_1) = L^2(e_1 u_2 + c_1 v_2) \\
& = L((a_1 u_1 + v_1) + c_1(e_1 u_1 + c_1 v_1)) \\
& = L((a_1 e_1 + c_1 e_1) u_1 + (e_1 + c_1 c_1) v_1) \\
& = (a_0(a_1 e_1 + c_1 e_1) + e_0(e_1 + c_1 c_1))u_0,
\end{align*} \]

\[ \begin{align*}
L R L^2 v_2 & = LR L(e_1 u_1 + c_1 v_1) = L R(a_0 e_1 + e_0 c_1) u_0 \\
& = a_0(a_0 e_1 + e_0 c_1) u_0,
\end{align*} \]

\[ RL^2 = 0, \]

\[ L^2 v_2 = (a_0 e_1 + e_0 c_1) u_0. \]

Hence (52) implies

\[ a_0(a_1 e_2 + e_1 c_2) + e_0(e_2 + c_1 c_2) - [3] (a_0(a_1 e_1 + c_1 e_1) + e_0(e_1 + c_1 c_1)) \\
+ [3] a_0(a_0 e_1 + e_0 c_1) - [3] \epsilon_0(a_0 e_1 + e_0 c_1) = 0. \]
Using Lemma 7.1, we get an equation in terms of $a_0$, $c_1$ and $e_0$, in which $e_0$ has degree one, so that we may solve it in $e_0$. After a routine computation, we get

\[
e_0 = -\frac{[d-1]}{2[d]} a_0^2 - \frac{[d][d-1]}{2[d-2]} c_1^2 + [4][d-1] \frac{[d-1]}{2[d-2]} b_0 c_1 \\
+ [d-1] \eta_{d+3} a_0 - \frac{[d][d-1]}{[d-2]} \eta_{d+1} c_1.
\]

This implies the expression for $e_i$ in Theorem 1.6. □

**Proof of Theorem 1.8.** Follows from Theorem 1.6 and the definition of $R$, $L$. □

**Proof of Theorem 1.9.** Let $R$, $L$ denote the maps which act on the given basis as in Theorem 1.6. Define subspaces $U_0, \ldots, U_d$ by

\[
U_0 = \text{span}\{u_0\}, \\
U_i = \text{span}\{u_i, v_i\} \quad (1 \leq i \leq d-1), \\
U_d = \text{span}\{u_d\}.
\]

Observe that $V$ is decomposed into direct sum of $U_0, \ldots, U_d$. Let $F_i : V \longrightarrow U_i$ denote the projection. By Lemmas 2.7 and 2.8, it is enough to show that $R$, $L$ satisfy the relations (13) and (14).

Let $C_1$ denote the left side of (13). It is routine to verify that $C_1 u_i = 0$ ($0 \leq i \leq d-2$) and $C_1 v_i = 0$ ($1 \leq i \leq d-2$). So (13) holds.

Let $C_2$ denote the left side of (14). We shall verify $C_2 u_i = 0$ for $2 \leq i \leq d$ and $C_2 v_i = 0$ for $2 \leq i \leq d-1$. When $4 \leq i \leq d-2$, after routine computation,

\[
C_2 u_i = ((a_1 a_i - 1 + b_i e_i - 1) a_i - 2 + (a_1 b_i - 1 + b_i c_i - 1) e_i - 2 \\
- [3]([a_1 a_i - 1 + b_i e_i - 1] a_i - 2 + (a_1 b_i - 1 + b_i - 1 c_i - 1) e_i - 2) \\
+ [3]([a_1 a_i - 1 + b_i e_i - 1] a_i - 2 + (a_1 - 1 b_i - 2 + b_i - 1 c_i - 2) e_i - 2) \\
- ((a_1 a_i - 1 + b_i c_i - 1) a_i - 2 + (a_1 - 1 b_i - 2 + b_i - 1 c_i - 2) e_i - 3) \\
- [3] e_i - 2 (a_i - 1 a_i - 2 + b_i - 1 e_i - 2) u_i - 2 \\
+ (a_1 a_i - 1 + b_i e_i - 1) b_i - 2 + (a_1 b_i - 1 + b_i - 1 c_i - 1) c_i - 2 \\
- [3] ([a_1 a_i - 1 + b_i e_i - 1] b_i - 2 + (a_1 b_i - 1 + b_i - 1 c_i - 1) c_i - 2) \\
+ [3] ([a_1 a_i - 1 + b_i e_i - 1] b_i - 2 + (a_1 - 1 b_i - 2 + b_i - 1 c_i - 2) c_i - 2) \\
- ((a_1 a_i - 1 + b_i e_i - 1) b_i - 3 + (a_1 - 1 b_i - 2 + b_i - 1 c_i - 2) c_i - 3) \\
- [3] e_i - 2 (a_i - 1 b_i - 2 + b_i - 1 c_i - 2) v_i - 2.
\]

It is routine to verify that both coefficients vanish, so that $C_2 u_i = 0$. In the same way, we can verify $C_2 v_i = 0$. Similarly, we can verify $C_2 u_i = 0$ and $C_2 v_i = 0$ for the case of $i = 2, 3, d - 1, d$. □
Acknowledgments

The author would like to thank Paul Terwilliger for his helpful suggestions.

References