Optimality conditions for a nonconvex set-valued optimization problem

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Abstract

In this paper we study necessary and sufficient optimality conditions for a set-valued optimization problem. Convexity of the multifunction and the domain is not required. A definition of \(K\)-approximating multifunction is introduced. This multifunction is the differentiability notion applied to the problem. A characterization of weak minimizers is obtained for invex and generalized \(K\)-convexlike multifunctions using the Lagrange multiplier rule.

Keywords: Set-valued functions; Optimality conditions; Convex and set-valued analysis; Vector optimization

1. Introduction and notation

Studies in optimization have led to the development of certain concepts of approximation of nonsmooth functions, in recent years. Some authors have investigated the properties of these approximations, such as their qualitative behaviour (see e.g., [1–3]). In set-valued optimization problems, the concept of invexity constitutes another instrument of approximation (see [4–8]).

The aim of this paper is to introduce a new concept of approximation to be applied in set-valued optimization problems using invexity properties.

We will consider the following standard assumptions:

Let \(X\) be a real normed space. Let \(Y\), \(Z\) be real normed spaces partially ordered by convex pointed cones \(K_Y \subset Y\) and \(K_Z \subset Z\) respectively. Let \(F : M \rightarrow 2^Y\), \(G : M \rightarrow 2^Z\) be set-valued maps with \(M\) a nonempty subset of \(X\).

Under these assumptions we will study the constrained set-valued optimization problem

\[
\begin{align*}
\min & \ F(x) \\
\text{subject to the constraints:} & \\
G(x) & \cap (-K_Z) \neq \phi \\
x & \in M.
\end{align*}
\]
This class of problems has been investigated by many authors (cf. [9–13]). They have established necessary and sufficient conditions under determined hypothesis and differentiability requirements. Concerning these differentiability conditions in the last years some authors have used the notion of contingent epiderivative. This epiderivative was developed by Aubin and Frankowska in [14]. It has been later applied to these problems in different research works (e.g., [15–19]). In the standard optimization theory, another assumption is that the domain of the objective function is convex.

In the present work conditions of optimality are obtained with invexity properties and a certain concept of approximating multifunction. Contingent epiderivatives will be particular cases of approximating multifunctions. Convexity of the objective set-valued map and of its domain are not required for these results.

In set-valued optimization there are different optimality concepts in use. We can find standard notions to recent research works (e.g., [20,21]). We recall two standard optimality notions (see [9,15]).

For simplicity let $\tilde{M} = \{x \in M \mid G(x) \cap (-K_Z) \neq \emptyset\}$ and let us assume that $\tilde{M}$ is nonempty. The graph, domain and image of a multifunction $F$ are denoted by $\text{graph}(F)$, $\text{Dom}(F)$ and $\text{Im}(F)$ respectively.

**Definition 1.** Let $D \subset Y$.

(a) $y_0 \in D$ is a minimal element of the set $D$ if

$$\{y_0\} \cap D = \{y_0\}.$$  

(b) Let $K_Y$ have a nonempty interior. $y_0 \in D$ is a weakly minimal element of the set $D$ if

$$\{y_0\} - \text{int}(K_Y) \cap D = \emptyset.$$  

**Definition 2.** Let $F(\tilde{M}) = \cup_{x \in \tilde{M}} F(x)$ denote the image set of $\tilde{M}$ by $F$.

(a) A point $(x_0, y_0) \in \text{graph}(F)$ is called a minimizer of the problem (1), if $y_0$ is a minimal element of the set $F(\tilde{M})$.

(b) Let $K_Y$ have a nonempty interior. A point $(x_0, y_0) \in \text{graph}(F)$, is called a weak minimizer of the problem (1) if $y_0$ is a weakly minimal element of the set $F(\tilde{M})$.

In order to obtain necessary and sufficient conditions we will mainly use the concept of weak minimizer. In Section 2 we introduce the concept of $K$-approximating multifunction. Some properties about the images of this multifunction are proved (Propositions 11 and 13). Section 3 deals with a necessary condition. Via an alternative theorem (Theorem 14) we establish a multiplier rule for the problem (1) in the case of an invex set-valued map $F \times G$ with $K$-approximating multifunction (Theorem 15). Finally in Section 4 we prove a case of invexity as a sufficient condition so that the point is a weak minimizer of the problem (1).

The following notions of set-valued maps will be used throughout this work.

The epigraph of $F$ is the set

$$\text{epi}(F) = \{(x, y) \in X \times Y \mid x \in M, \ y \in F(x) + K_Y\},$$

the epiran of $F$ is the set

$$\text{epiran}(F) = \{y \in Y \mid \text{there exists} \ x \in M, \ y \in F(x) + K_Y\}.$$  

We observe that $\text{epiran}(F) = \text{Pr}_Y(\text{epi}(F))$.

Let $D$ a subset of a real normed space $X$. The contingent cone of the subset $D$ at $x_0 \in D$ is denoted by $T(D; x_0)$ and consists of all tangent vectors $h = \lim_{n \to \infty} \mu_n (x_n - x_0)$, with $\lim_{n \to \infty} x_n = x_0$. $(x_n)_{n \in \mathbb{N}} \subset D$ and $(\mu_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, $\mu_n > 0$ for all $n \in \mathbb{N}$. Or equivalently, there exists a sequence of real numbers $(t_n)_{n \in \mathbb{N}} \to 0$, $t_n > 0$, and a sequence of vectors $(h_n)_{n \in \mathbb{N}} \to h$ such that $x_0 + t_n h_n \in D$ for all $n \in \mathbb{N}$.

It is useful to observe that $T(D; x_0) \subset \text{cl}(\text{cone}(D - x_0))$.

The dual cone of $K_Y$ is the set

$$K_Y^* = \{y^* \in Y^* \mid y^*(y) \geq 0 \ \text{for all} \ y \in K_Y\}.$$  

The cone generated by a nonempty subset $B$ of $Y$ is the set

$$\text{cone}(B) = \{\lambda y \in Y \mid \lambda \geq 0, \ y \in B\}.$$
Definition 3. (a) The set-valued map $F : M \to 2^Y$ is called $K_Y$-convexlike if the set $F(M) + K_Y$ is convex.
(b) $F$ is called generalized $K_Y$-convexlike at $(x_0, y_0) \in \text{graph}(F)$ if the set cone($F(M) - y_0$) + $K_Y$ is convex.

2. $K$-approximating multifunctions

From the basic idea of J. Aubin and H. Frankowska of contingent epiderivative (see [14]) and the definitions of generalized contingent epiderivative of Jahn and Khan [22], we define the concept of $K_Y$-approximating multifunction as follows:

Definition 4. Let $A$ be a set-valued map from $X$ to $Y$. $A$ is a $K_Y$-approximating multifunction of $F$ at $(x_0, y_0) \in \text{graph}(F)$ if it verifies:
(a) epi$(A)$ is a closed cone.
(b) epi$(A) \subset T$(epi$(F)$; $(x_0, y_0)$).

Example 5. An example of $K_Y$-approximating multifunction is, under certain conditions the generalized contingent epiderivative. Jahn and Khan [22] defined this epiderivative as a set-valued map,

$$D_g F(x_0, y_0) : X \rightarrow 2^Y,$$

given by

$$D_g F(x_0, y_0)(x) = \text{Min}(\gamma(x), K_Y),$$

where $\gamma(x) = \{y \in Y | (x, y) \in T(\text{epi}(F); (x_0, y_0))\}$ and $\text{Min}(\gamma(x), K_Y)$ is the set of minimal points of $\gamma(x)$ (see Definition 1).

In [22] it is proved that if the cone $K_Y$ is regular (cf. [23]) and the previous set $\gamma(x)$ has a $K_Y$-lower bound for all $x \in \text{Dom}(\gamma)$, then

$$\text{epi}(D_g F(x_0, y_0)) = T(\text{epi}(F); (x_0, y_0)).$$

In this case $D_g F(x_0, y_0)$ is a clear example of a $K_Y$-approximating multifunction.

Remark 6. When $Y = \mathbb{R}$, $K_Y = \mathbb{R}_+$ and $F$ is a function, some particular examples of $K_Y$-approximating multifunctions are the contingent, adjacent and circatangent epiderivatives (see [14]) which are functions.

In a more general way Jahn and Rauh in [15] developed the concept of contingent epiderivative for a multifunction defined between two arbitrary real normed spaces. They also prove that when this epiderivative exists, it is a unique and positively homogeneous function. This contingent epiderivative constitutes another example of $K_Y$-approximating multifunction.

However $K_Y$-approximating multifunctions are of special interest when the contingent epiderivative and the generalized contingent epiderivative do not exist, as is shown in the next example.

Example 7. Let $F : [-1, 1] \rightarrow 2^{\mathbb{R}^2}$ be a set-valued map defined by

$$F(x) = \begin{cases} \{(y, z) | y = x, \ y \leq z\} & \text{if } x \geq 0 \\ \{(y, z) | y = -x, \ -\sqrt{y} \leq z\} & \text{if } x < 0. \end{cases}$$

Let us consider the cone $K_Y = \mathbb{R}^2_+ \{ (x, y) \in \mathbb{R}^2 | x \geq 0, \ y \geq 0 \}$. We define the multifunction $A : \mathbb{R} \rightarrow 2^{\mathbb{R}^2}$ by

$$A(x) = \begin{cases} \{(y, z) | y = x, \ y \leq z\} & \text{if } x \geq 0 \\ \{(y, z) | y = -x, \ z \leq -y\} & \text{if } x < 0. \end{cases}$$

It is easy to prove that $A$ is a $K_Y$-approximating multifunction of $F$ at $(0, (0, 0))$. Observe that the contingent epiderivative and the generalized contingent epiderivative of $F$ at $(0, (0, 0))$ do not exist.

In order to establish in the next section a Lagrange multiplier rule for the problem (1), we will first prove some properties about the images of $K_Y$-approximating multifunctions. With this purpose we will consider the set $L = \text{cone}(M - \{x_0\})$ and we will use the concepts of asymptotically compact set and asymptotic cone. Let us recall these concepts (see [24]). $B(0, 1)$ will denote the unit open ball in $X$. 


Definition 8 ([24]). A subset \( L \) of \( X \) is said to be asymptotically compact if there exists \( \epsilon \) such that \([0, \epsilon]L \cap B(0, 1)\) is a relatively compact set.

Definition 9 ([24]). Let \( D \subset X \). The asymptotic cone of \( D \) is the set
\[
D_\infty = \bigcap_{\epsilon > 0} \text{cl}([0, \epsilon]D).
\]

We will use the concept of recession multifunction of a set-valued map \( F : X \to 2^Y \), defined as the multifunction \( F_\infty : X \to 2^Y \) whose epigraph is \((\text{epi} F)_\infty\), (see [24]).

\( N(F) \) will denote the set \( \{x \in X \mid 0_Y \in F(x)\} \).

To simplify the notation, let us consider in the problem (1): \( E = Y \times Z \), \( H = F \times G \), \( K = K_Y \times K_Z \). Let us suppose that the next conditions are satisfied

\[
\begin{aligned}
&P(24) \Rightarrow \text{we conclude that } \text{Im} P(10) = \text{cl} D_\infty = \text{Im} P(24). \\
\end{aligned}
\]

Proposition 10 ([24], Corollary 3.9). Let \( \Psi : X \to 2^E \) be a closed multifunction. Suppose that \( \text{Dom}(\Psi) \) is asymptotically compact and \( N(\Psi_\infty) = \{0_X\} \). Then \( \text{Im}(\Psi) \) is closed.

Proposition 11. Let us assume conditions (2) and let \( A : X \to 2^E \) be a \( K \)-approximating multifunction of \( H \) at \((x_0, u_0)\). Then the set
\[
C = \text{cone}(A (M - x_0)) + K
\]
is closed.

Proof. Because \( \text{epi}(A) \) is a cone we get that \( C = (\text{epi} A)(L) \), where \( \text{epi} A \) is identified with the multifunction whose graph is \( \text{epi} A \). Let \( \Psi : X \to 2^E \) be the multifunction which verifies:
\[
\text{graph}(\Psi) = (L \times E) \cap \text{epi}(A).
\]

Then \( \text{Dom}(\Psi) \subset L \). Since \( L \) is asymptotically compact, from (2) and [24] (Proposition 2.2) we deduce that \( \text{Dom}(\Psi) \) is asymptotically compact. Moreover by (2) and because \( A \) is a \( K \)-approximating multifunction of \( H \) at \((x_0, u_0)\) we get
\[
N(\Psi_\infty) \times 0_E \subset ((L \times E) \cap \text{epi}(A)) \cap (X \times 0_E) \subset 0_X \times 0_E.
\]

Applying Proposition 10 we conclude that \( \text{Im}(\Psi) = (\text{epi} A)(L) = C \) is closed. \( \Box \)

The concept of invex multifunction will be necessary to prove that the previous set \( C \) is convex. Luc and Malivert in [8] defined this concept by means of another multifunction \( A \) as follows:

Definition 12. Let \((x_0, y_0) \in \text{graph}(F)\) and \( \eta_{(x_0, y_0)} : X \to X \) be a function. Let \( A \) be a set-valued map from \( X \) to \( Y \). We say that \( F \) is \( A \)-invex at \((x_0, y_0)\) with respect to \( \eta_{(x_0, y_0)} \) if for all \( x \in M \)
\[
F(x) \subset y_0 + A(\eta_{(x_0, y_0)}(x)) + K_Y.
\]

Proposition 13. Let us suppose assumptions (2) and that \( A \) is a \( K \)-approximating multifunction of \( H \) at \((x_0, u_0)\). Let the multifunction \( H \) be \( A \)-invex at \((x_0, u_0)\) with respect to \( \eta_{(x_0, y_0)}(x) = x - x_0 \) and generalized \( K \)-convexlike. Then the set
\[
C = \text{cone}(A (M - x_0)) + K
\]
is convex.
Proposition 11. If $S$ is convex, furthermore, if $u$ (a) If $S$ is convex, we get successively
\[
H(M) - u_0 \subset A(M - x_0) + K,
\]
\[
\text{cone}(H(M) - u_0) \subset \text{cone}(A(M - x_0)) + K = C,
\]
\[
C' = \text{cone}(H(M) - u_0) + K \subset C.
\]
But
\[
\text{epi}(H) - (x_0, u_0) \subset L \times (\text{cone}(H(M) - u_0) + K) = L \times C',
\]
whence
\[
\text{epi}(A) \subset T(\text{epi}(H); (x_0, u_0)) \subset \text{cl}(L \times C') = L \times \text{cl}C'.
\]
Hence
\[
C' \subset C = (\text{epi} A)(L) \subset \text{cl}C',
\]
where $\text{epi} A$ is identified with the multifunction whose graph is $\text{epi}(A)$. Since $C$ is closed by Proposition 11 and $C'$ is convex we get that $C = \text{cl}C'$ is convex. \(\square\)

3. A necessary condition for a weak minimizer

Theorem 14. Let the cones $K_Y$, $K_Z$ have nonempty interiors. Let $S \subset Y \times Z$ be a set with $0_Y, 0_Z \in S$. Let us consider the next assertions:

(i) $S \cap \{(-\text{int}(K_Y)) \times (-\text{int}(K_Z))\} = \emptyset$

(ii) there exists $(u, v) \in K_Y^* \times K_Z^*$, $(u, v) \neq (0_Y^*, 0_Z^*)$ such that for all $(y, z) \in S$
\[
u(y) + v(z) \geq 0.
\]

Then:

(a) if $S$ is convex, (i) implies (ii)
(b) if the regularity assumption
\[
Z = \{z| \text{there exists } y \in Y \text{ with } (y, z) \in S\} + K_Z
\]
is satisfied, (ii) implies $u \neq 0_Y^*$
(c) furthermore, if $u \neq 0_Y^*$, (ii) implies
\[
S \cap \{(-\text{int}(K_Y)) \times (-K_Z + \text{Ker}(v))\} = \emptyset.
\]

Proof. (a) If $S$ is convex with $S \cap \{(-\text{int}(K_Y)) \times (-\text{int}(K_Z))\} = \emptyset$, then by the Hahn–Banach theorem, there exist $u \in Y^*$, $v \in Z^*$, $(u, v) \neq (0_Y^*, 0_Z^*)$ such that
\[
u(y') + v(z') \leq u(y) + v(z),
\]
for all $(y', z') \in (-\text{int}(K_Y)) \times (-\text{int}(K_Z))$ and for all $(y, z) \in S$.

Taking into account the continuity of $u$ and $v$ and that $0_Y \in \text{cl}(-\text{int}(K_Y))$ and $0_Z \in \text{cl}(-\text{int}(K_Z))$ we have that $u(y') + v(z) \geq 0$ for all $(y, z) \in S$. Furthermore since $(0_Y, 0_Z) \in S$, for all $y' \in (-\text{int}(K_Y))$ and for all $z' \in (-\text{int}(K_Z))$ we get
\[
u(y') + v(z') \leq u(0_Y) + v(0_Z) = 0.
\]
As $0_Y \in \text{cl}(-\text{int}(K_Y))$ and $u$ is continuous it follows that $v(z') \leq 0$ for all $z' \in (-\text{int}(K_Z))$. And from $K_Z \subset \text{cl}(\text{int}(K_Z))$ we obtain that $v \in K_Z^*$. Similarly for $0_Z$ we deduce that $u(y') \leq 0$ for all $y' \in (-\text{int}(K_Y))$ and hence $u \in K_Y^*$.

(b) The regularity condition asserts that $Z = \text{Pr}_Z(S) + K_Z$. Assuming that $u = 0_Y^*$ we get $v(z) \geq 0$ for every $z \in \text{Pr}_Z(S)$. Since $v(z') \geq 0$ for every $z' \in K_Z$, we get $v(z''') \geq 0$ for every $z'' \in \text{Pr}_Z(S) + K_Z = Z$, which implies that $v = 0_{Z^*}$, a contradiction with $(u, v) \neq (0_Y^*, 0_Z^*)$. 

\[
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\]
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(c) Since $K_Y$ is a convex cone with nonempty interior and $u \in K_{Y^*}$, $u \neq 0_{Y^*}$, then $u(y) < 0$ for every $y \in (-\text{int}(K_Y))$. If there exist $(y, z) \in S$, $y \in (-\text{int}(K_Y))$, $z \in (-K_Z + \text{Ker}(v))$, we get that $u(y) + v(z) < 0$ which contradicts (ii). □

From now on we will suppose that $x_0 \in M$, $u_0 = (y_0, z_0)$ where $y_0 \in F(x_0)$ and $z_0 \in G(x_0) \cap (-K_Z)$.

Theorem 15. Let the cones $K_Y$, $K_Z$ have nonempty interiors. Let $(x_0, y_0)$ be a weak minimizer of the problem (1). Let $A : X \to 2^{1 \times 2}$ be a $(K_Y \times K_Z)$-approximating multifunction of $F \times G$ at $(x_0, (y_0, z_0))$. Assume conditions (2). If the set-valued map $F \times G$ is generalized $(K_Y \times K_Z)$-convexlike and $A$-invex at $(x_0, (y_0, z_0))$ with respect to $\eta(x_0, y_0)(x) = x - x_0$, then:

(a) there exist $u \in K_{Y^*}$ and $v \in K_{Z^*}$, $(u, v) \neq (0_{Y^*}, 0_{Z^*})$ such that $v(z_0) = 0$ and $u(y) + v(z) \geq 0$ for all $(y, z) \in A(M - x_0)$;

(b) if in addition to the above hypothesis, the regularity assumption

$$Z = \{z | \exists \ y \in Y \text{ with } (y, z) \in \text{cone}\{A(M - x_0)\} + (K_Y \times (K_Z + \{z_0\}))\}$$

is satisfied, then $u \neq 0_{Y^*}$.

Proof. (a) By the Proposition 13, the set $C + (0_Y, z_0)$ is convex. $(0_Y, 0_Z) \in C + (0_Y, z_0)$ because $(0_Y, 0_Z) \in \text{cone}\{A(M - x_0)\}$ and $z_0 \in G(x_0) \cap (-K_Z)$. If we prove that

$$(C + (0_Y, z_0)) \cap [(-\text{int}(K_Y)) \times (-\text{int}(K_Z))] = \emptyset,$$

then by Theorem 14 there exist $(u, v) \in K_{Y^*} \times K_{Z^*}$, $(u, v) \neq (0_{Y^*}, 0_{Z^*})$ such that $u(y) + v(z) \geq 0$ for all $(y, z) \in C + (0_Y, z_0)$.

Furthermore as $z_0 \in G(x_0) \cap (-K_Z)$, then $(0_Y, 0_Z) \in K + (0_Y, z_0)$ and we obtain that $u(y) + v(z) \geq 0$ for all $(y, z) \in A(M - x_0)$.

Moreover as $(0_Y, z_0) \in C + (0_Y, z_0)$ we get that $v(z_0) \geq 0$ and from $z_0 \in (-K_Z)$ and $v \in K_{Z^*}$, we have that $v(z_0) \leq 0$. Thus $v(z_0) = 0$.

Let us prove (3.1). We will suppose that it is false and we shall see that $(x_0, y_0)$ is not a weak minimizer.

In fact, let us assume that there exists $(y, z) \in Y \times Z$ such that

$$(y, z + z_0) \in (C + (0_Y, z_0)) \cap [(-\text{int}(K_Y)) \times (-\text{int}(K_Z))],$$

therefore there exist

$$x \in M, \quad \lambda > 0, \quad (y^1, z^1) \in A(x - x_0), \quad y^2 \in K_Y, \quad z^2 \in K_Z$$

such that $y = \lambda y^1 + y^2$, $z = \lambda z^1 + z^2$ and it verifies

$$(x - x_0, (y^1 + y^2/\lambda, z^1 + z^2/\lambda)) \in \text{epi}(A),$$

hence by Definition 4(b), there exist a sequence $(x_n, (y_n, z_n))_{n \in \mathbb{N}} \subset \text{epi}(F \times G)$ and a sequence $(\mu_n)_{n \in \mathbb{N}}$ of real positive numbers such that $(x_0, (y_0, z_0)) = \lim_{n \to \infty}(x_n, (y_n, z_n))$ and

$$(x - x_0, (y^1 + y^2/\lambda, z^1 + z^2/\lambda)) = \lim_{n \to \infty} \mu_n(x_n - x_0, (y_n - y_0, z_n - z_0)).$$

From (3.2) and (3.3) we deduce that for a sufficiently large $n$

$$\lambda \mu_n(y_n - y_0) \in -\text{int}(K_Y),$$
$$\lambda \mu_n(z_n - z_0) + z_0 \in -\text{int}(K_Z).$$

therefore

$$y_n \in y_0 - \text{int}(K_Y),$$
$$z_n \in z_0 \left(1 - \frac{1}{\lambda \mu_n}\right) - \text{int}(K_Z).$$
Furthermore there exist sequences \((y_n^*, z_n^*)_{n \in \mathbb{N}}, (z_n^*, y_n^*)_{n \in \mathbb{N}}\), with \(y_n^* \in F(x_n), z_n^* \in G(x_n)\), such that

\[ y_n \in y_n^* + K_Y, \quad z_n \in z_n^* + K_Z \quad \text{for all } n \in \mathbb{N}, \]

and from this, taking into account (3.6) and (3.7) we get that for a sufficiently large \(n \in \mathbb{N}\)

\[ y_n \in y_n - K_Y \subset y_0 - \text{int}(K_Y) - y_0 - \text{int}(K_Y), \]

we deduce that (3.10)

\[ z_n \in z_n - K_Z \subset z_0 \left(1 - \frac{1}{\lambda \mu_n}\right) - \text{int}(K_Z) - K_Z \]

and for a sufficiently large \(n \in \mathbb{N}\)

\[ z_n \in z_0 \left(1 - \frac{1}{\lambda \mu_n}\right) - \text{int}(K_Z). \] (3.9)

Then, on the one hand from (3.8) we have that for sufficiently large \(n\)

\[ F(x_n) \cap (y_0 - \text{int}(K_Y)) \neq \emptyset. \] (3.10)

On the other hand from (3.2) we deduce that \(y = \lambda y^1 + y^2 \neq 0_Y\). In consequence \(y/\lambda = y^1 + y^2/\lambda \neq 0_Y\). So \((\mu_n) \to \infty\) and for a sufficiently large \(n\) we obtain that \(\lambda \mu_n > 1\). As by hypothesis \(z_0 \in (-K_Z)\), then

\[ z_0 \left(1 - \frac{1}{\lambda \mu_n}\right) \in (-K_Z). \] (3.9)

From (3.9) we deduce that \(z_n^* \in (-\text{int}(K_Z))\), and in consequence

\[ z_n^* \in G(x_n) \cap (-\text{int}(K_Z)). \] (3.11)

Hence for sufficiently large \(n \in \mathbb{N}\) we have \(x_n \in M\) which verifies (3.10) and (3.11). Then \(x_n \in \tilde{M}\) verifies (3.10) and we conclude that \((x_0, y_0)\) is not a weak minimizer of problem (1). Thus (3.1) is proved.

(b) The imposed condition is equivalent to the condition (b) of Theorem 14 replacing \(z_0\) by \(0_Z\). Then it implies \(u \neq 0_Y^*\). \(\square\)

Example 16. Let \(F, K_Y\) be defined as in Example 7. Let \(G : \mathbb{R} \to 2^{\mathbb{R}}\) be the multifunction given by \(G(x) = [-x^2, \infty)\). Let \(K_Z = \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}\).

The problem

\[
\begin{align*}
& \text{min } F(x) \\
& \text{subject to} \\
& x \in [-1, 1]
\end{align*}
\]

is a particular case of problem (1). It is easy to see that \((0, (0, 0))\) is a weak minimizer of this problem.

We consider the multifunction \(A : \mathbb{R} \to 2^{\mathbb{R}^3}\) defined by

\[ A(x) = \begin{cases} 
{(y, z, w) \mid y = x, y \leq z, -x^2 \leq w} & \text{if } x \geq 0 \\
{(y, z, w) \mid y = -x, z \leq -y, -x^2 \leq w} & \text{if } x < 0.
\end{cases} \]

It is not difficult to check that \(A\) is a \(K\)-approximating multifunction of \(F \times G\) at \((0, ((0, 0), 0))\) and that conditions (2) are verified. Moreover \(F \times G\) is \(A\)-invex at \((0, ((0, 0), 0))\) with respect to \(\eta_{(0, 0)}(x) = x\). Furthermore

\[(F \times G)([-1, 1]) = \{(y, z, w) \mid y \in [0, 1], -\sqrt{y} \leq z, -y^2 \leq w\}. \]

Therefore the set

\[ \text{cone}(F \times G)([-1, 1]) + (\mathbb{R}_+^2 \times \mathbb{R}_+) \]

is convex and \(F \times G\) is generalized \(K_Y \times K_Z\)-convexlike. Observe that it is not \(K_Y \times K_Z\)-convex, hence the results for convex multifunctions (see [16]) cannot be applied.

We note that the regularity condition (b) of Theorem 15 is satisfied. And for instance the functions \(u(x, y) = x, v(x) = x\) belong to \(K_Y^*\) and \(K_Z^*\) respectively and comprise a pair of multipliers for this problem.

4. A sufficient condition of a weak minimizer

In this section we will consider the multifunctions \(F, G\) and the cones \(K_Y, K_Z\) defined as in the problem (1).
**Theorem 17.** Let the cone $K_Y$ have a nonempty interior. Let $x_0 \in M$, $y_0 \in F(x_0)$, $z_0 \in G(x_0) \cap (-K_Z)$ and let $A : X \to 2^{Y \times Z}$ be a $(K_Y \times K_Z)$-approximating multifunction of $F \times G$ at $(x_0, (y_0, z_0))$. Assume that there exists $(u, v) \in K_{Y^*} \times K_{Z^*}$ with $u \neq 0_{Y^*}$, such that
\[ u(y) + v(z) \geq 0 \quad \text{for all } (y, z) \in A(M - x_0) \text{ and } v(z_0) = 0. \]

If the multifunction $(F \times G)$ is $A$-invex at $(x_0, (y_0, z_0))$ with respect to $\eta_{(x_0, y_0)}(x) = x - x_0$, then $(x_0, y_0)$ is a weak minimizer of the problem $(1)$.

**Proof.** Let $C = \text{cone}\{A(M - x_0)\} + K$ as in the Proposition 11. Let $u_0 = (y_0, z_0)$. From $u(y) + v(z) \geq 0$ for all $(y, z) \in A(M - x_0)$ and $(u, v) \in K_{Y^*} \times K_{Z^*} = K$, we obtain that $u(y) + v(z) \geq 0$ for all $(y, z) \in C$. Since $H = F \times G$ is $A$-invex at $(x_0, u_0)$, we have that $u(y) + v(z) \geq 0$ for all $(y, z) \in H(M) - u_0$. Assume that $(x_0, y_0)$ is not a weak minimizer. Then there exist $x \in M$, $y \in F(x)$ with $y_0 - y \in \text{int}(K_Y)$. Take $z \in G(x) \cap (-K_Z)$. Hence $(y, z) \in H(M)$ and $u(y - y_0) + v(z - z_0) \geq 0$. Since $u \neq 0_{Y^*}$, $y_0 - y \in \text{int}(K_Y)$ and $v(z_0) = 0$ it follows that
\[ 0 > u(y - y_0) \geq -v(z - z_0) = v(-z) \geq 0, \]
which is a contradiction. □

As a consequence of Theorems 15 and 17 we obtain a characterization of weak minimizers as follows

**Corollary 18.** Let us suppose that the conditions (2) are satisfied and that $\text{int}(K_Y) \neq \emptyset$, $\text{int}(K_Z) \neq \emptyset$. Assume that $A$ is a $(K_Y \times K_Z)$-approximating multifunction of $F \times G$ at $(x_0, (y_0, z_0))$ and that $F \times G$ is generalized $(K_Y \times K_Z)$-convexlike and $A$-invex at this point with respect to $\eta_{(x_0, y_0)}(x) = x - x_0$. Moreover let the regularity condition (b) of Theorem 15 be satisfied. Then $(x_0, y_0)$ is a weak minimizer of problem $(1)$ if and only if there are $u \in K_{Y^*}$, $v \in K_{Z^*}$ with $u \neq 0_{Y^*}$ such that $v(z_0) = 0$ and
\[ u(y) + v(z) \geq 0, \]
for all $(y, z) \in A(x - x_0)$ with $x \in M$.

**References**


