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Families of Finite Sets in Which No Set Is Covered by the Union of Two Others

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Let $f^k(n)$ denote the maximum of k-subsets of an n-set satisfying the condition in the title. It is proven that $f^{2t-1}(n) \leqslant f^{2t}(n+1) \leqslant \binom{n}{t}/\binom{2t-1}{t}$ with equalities holding iff there exists a Steiner-system $\mathscr{S}(t, 2t-1, n)$. The bounds are approximately best possible for $k \leqslant 6$ and of correct order of magnitude for $k \geqslant 7$, as well, even if the corresponding Steiner-systems do not exist.

Exponential lower and upper bounds are obtained for the case if we do not put size restrictions on the members of the family (i.e., the nonuniform case).

1. Introduction and the Statement of the Results

1.1. *Notations*

Let X be an n-element set. For an integer t, $0 \leqslant t \leqslant n$ we denote by $\binom{X}{t}$ the collection of all the t-subsets of X, while 2^X denotes the set having all the different subsets of X as its elements. A family of subsets of X is just a subset of 2^X. We shall call it *t-uniform* if it is a subset of $\binom{X}{t}$. By a Steiner-system $\mathscr{S} = \mathscr{S}(t, k, n)$ we shall mean an $\mathscr{S} \subset \binom{X}{k}$ such that for every $A \in \binom{X}{t}$ there is exactly one $B \in \mathscr{S}$ with $A \subset B$. Obviously, we have $|\mathscr{S}(t, k, n)| = \binom{n}{t}/\binom{k}{t}$. By $\lceil a \rceil$ ($\lfloor b \rfloor$) we shall denote the smallest (greatest) integer (not) exceeding a (b), respectively. We will use the Stirling formula, i.e.,

$$n! \sim (n/e)^n \sqrt{2\pi n}.$$

1.2. *The Results*

THEOREM 1. *Suppose $\mathscr{F}^k \subset \binom{X}{k}$ and there are no three distinct sets $A, B, C \in \mathscr{F}^k$ such that $A \subset B \cup C$. Let $f^k(n)$ denote the maximum of $|\mathscr{F}^k|$, subject to these constraints. Then we have*

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$$f^{2t-1}(n) \leqslant \binom{n}{t} \Big/ \binom{2t-1}{t}, \tag{1}$$

$$f^{2t}(n) \leqslant \binom{n-1}{t} \Big/ \binom{2t-1}{t}. \tag{2}$$

Moreover equality holds in (1) *iff* $\mathscr{F}^{2t-1} = \mathscr{S}(t, 2t-1, n)$, *and in* (2) *iff for some* $x \in X$ *and a Steiner-system* $\mathscr{S} = \mathscr{S}(t, 2t-1, n-1)$ *on* $X - \{x\}$ *we have* $\mathscr{F}^{2t} = \{\{x\} \cup S\colon S \in \mathscr{S}\}$.

The bounds given by this theorem are best possible only if the corresponding Steiner-systems exist. As it is well known (cf. [5]) $\mathscr{S}(2, 3, n)$ exists iff $n \geqslant 7$ and $n \equiv 1$ or $3 \pmod 6$. Thus in these cases $f^3(n) = f^4(n+1) = n(n-1)/6$. The Steiner-systems $\mathscr{S}(3, q+1, q^\alpha + 1)$, called Moebius geometries (see Hanani [3]) yield for the special case $q = 4$ some $\mathscr{S}(3, 5, 4^\alpha + 1)$. Thus for $n = 4^\alpha + 1$ we have $f^5(n) = f^6(n+1) = \binom{n}{3}/\binom{5}{3}$. Erdös and Hanani [1] proved the existence of $\mathscr{F} \subset \binom{X}{q+1}$ with $|F \cap F'| \leqslant 2$ for $F, F' \in \mathscr{F}$ and $|\mathscr{F}| = \binom{n}{3}/\binom{q+1}{3} - o(n^3)$. Let us put these observations together.

COROLLARY 1.

$$f^1(n) = n, \qquad f^2(n) = n-1,$$
$$f^3(n) = n^2/6 + O(n) = f^4(n)$$
$$f^5(n) = n^3/60 + o(n^3) = f^6(n).$$

As for $t \geqslant 4$ almost nothing is known about the existence of $\mathscr{S}(t, 2t-1, n)$, we do not have any asymptotically correct estimations of $f^k(n)$. We could only obtain:

PROPOSITION 1. $f^k(n) > \binom{n}{\lceil k/2 \rceil} / \binom{k}{\lceil k/2 \rceil}^2$.

Remark. The problems considered in this theorem belong to the so-called Turán-type problems i.e., what is the maximum number of k-subsets of an n-set if it contains no subsystem isomorphic to one member of a set of k-graphs $\{\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_q\}$. This maximum is usually denoted by $\operatorname{ext}(n, \{\mathscr{H}_1, \ldots, \mathscr{H}_q\})$. Let us define $\mathscr{U} = \{\{A, B, C\}\colon |A| = |B| = |C| = k, A \subset B \cup C\}$ and $\mathscr{H}_1 = \{\{x_1, \ldots, x_k\}, \{x_{k+1}, \ldots, x_{2k}\}, \{x_1, \ldots, x_{k-1}, x_{k+1}\}\}$. In this terminology we proved (Theorem 1 and Proposition 1) that $\operatorname{ext}(n, \mathscr{U}) = O(n^{\lceil k/2 \rceil})$. The exclusion of only one member of $\mathscr{U}$, however, leads to different results, e.g., (see [2]) $\operatorname{ext}(n, \mathscr{H}_1) = \binom{n-1}{k-1}$ $(n > n_0(k))$.

Until now we considered the uniform case, i.e., $\mathscr{F} \subset \binom{X}{k}$, but what happens if we assume only $\mathscr{F} \subset 2^X$. Let $f(n)$ denote $\max |\mathscr{F}|$ for $\mathscr{F} \subset 2^X$ and $A, B, C \in \mathscr{F}$ implies $A \not\subset B \cup C$.

THEOREM 2. $1.134^n < f(n) < 1.25^n$.

Here the upper bound follows from Theorem 1, using the Stirling formula and the obvious estimation $f(n) \leqslant \sum_{k \leqslant n} f^k(n)$. Proposition 1 would give an exponential lower bound but a weaker one. We obtain the actual bound by random construction.

If we fix $r = n - k$ only then by taking complements the problem can be stated in the following way. What is the maximum cardinality $g(r)$ of an r-uniform family if it does not contain 3 sets A, B, C with $A \cap B \subset C$. Applying Theorem 2 one can easily prove $g(r) > 1.18^r$, and trivially $g(r) \leqslant 2^r$. Kleitman *et al.* [6] have shown that $g(r) \leqslant (1.87 + o(1))^r$, i.e., $g(r)$ is exponentially smaller than 2^r.

2. THE PROOFS

2.1. *Some Preparations*

Observe that if we adjoin a fixed element outside of the ground set to all the members of $\mathscr{F}^{2t-1}$ we get an $\mathscr{F}^{2t}$ on $n+1$ points. Thus $f^{2t-1}(n) \leqslant f^{2t}(n+1)$. Hence it is sufficient to prove (2), along with the uniqueness. Let us denote $\mathscr{F}^{2t}$ by just $\mathscr{F}$ and for $T \subset X$ $\mathscr{F}(T) = \{F \in \mathscr{F} : T \subset F\}$

(i) *If $T \cup T' = F \in \mathscr{F}$, then either $|\mathscr{F}(T)| = 1$ or $|\mathscr{F}(T')| = 1$.*

Indeed, otherwise we can take F_0, F_0' different from F with $T \subset F_0, T' \subset F_0'$, and consequently, $F \subset F_0 \cup F_0'$.

If $|\mathscr{F}(T)| = 1$, $T \subset F \in \mathscr{F}$, we say T is a *private subset* of F. For $F, F' \in \mathscr{F}$ obviously $F - F'$ is always a private subset of F. Now a $2t$-element set can be partitioned into 2 t-sets in $\frac{1}{2}\binom{2t}{t}$ different ways. Thus we have:

(ii) *If $F \in \mathscr{F}$, then it has at least $\frac{1}{2}\binom{2t}{t}$ private t-subsets.*

(Let us remark that (ii) already gives $|\mathscr{F}| \leqslant \binom{n}{t} / \binom{2t-1}{t}$ which is only slightly weaker than (2).)

Suppose $T \subset F \in \mathscr{F}$, $|T| = t$. If for some $x \in F - T$ we have $|\mathscr{F}(T \cup \{x\})| > 1$, then in view of (i) $F - T - \{x\}$ is a private subset of F and consequently for $y \notin F$ the t-set $(F - T - \{x\}) \cup \{y\}$ is not contained in any member of $\mathscr{F}$. We say that these sets are *free sets* associated with the pair (F, T). The collection of all such free sets will be denoted by $\mathscr{A}(F, T)$, i.e., $\mathscr{A}(F, T) = \{A \subset X: \ |A| = t, \ A \cap T = \varnothing, \ |A \cap F| = t - 1, |\mathscr{F}(F - A)| > 1\}$. Of course, we have:

(iii) $|\mathscr{A}(F, T)| = |X - F| \, |\{x : x \in F - T, |\mathscr{F}(T \cup \{x\})| > 1\}|$.

Next we shall prove:

(iv) *If* $|\mathscr{F}(T)| > (n-t)/t$, *then* $\sum_{F\in\mathscr{F}(T)} |\mathscr{A}(F,T)| \geqslant (n-2t) \times (t\,|\mathscr{F}(T)| - (n-t))$.

In fact in view of (iii)

$$\sum_{F\in\mathscr{F}(T)} |\mathscr{A}(F,T)| = (n-2t) \sum_{F\in\mathscr{F}(T)} |\{x\colon x\in F-T, |\mathscr{F}(T\cup\{x\})| > 1\}|$$

$$= (n-2t) \sum_{\substack{x\in X-T \\ |\mathscr{F}(T\cup\{x\})|>1}} |\{F\colon T\cup\{x\}\subset F\}|$$

$$\geqslant (n-2t) \left(\sum_{x\in X-T} |\mathscr{F}(T\cup\{x\})| - (n-t) \right)$$

$$= (n-2t)(t\,|\mathscr{F}(T)| - (n-t)).$$

(v) *For* $A\subset X$, $|A| = t$ *we have* $|\{(F,T)\colon A\in\mathscr{A}(F,T)\}| \leqslant (t+1)t$.

Indeed for $A\in\mathscr{A}(F,T)$ there exists a $y\in A$ such that $A-\{y\}$ is a private subset of F, thus the number of possibilities for F is at most t. On the other hand $T\subset F-A$, $|T| = t$, thus for T there are only $\binom{t+1}{t} = t+1$ possibilities. From now on assume:

(vi) *There are no* $F_1, F_2\in\mathscr{F}$ *with* $F_1\cup F_2 = X$.

In fact, otherwise $\mathscr{F} = \{F_1, F_2\}$, and (2) follows, also in the case $n\leqslant 2t+1$. For $n = 2t+r\leqslant 3t$ we can improve (v).

Let A belong to $\mathscr{A}(F,T)$. Then $|\mathscr{F}(F-A)| > 1$, thus we can find $F\neq F'\in F$, $F-A\subset F'$. Now (vi) yields $|F-F'|\leqslant r-1$. Suppose $y\in A-(F-F')$. Then $A-\{y\}$ cannot be contained in a member $F''\neq F$ of $\mathscr{F}$, since $F\subset F'\cup F''$ would follow. Thus the number of sets $F_0\in F$ such that $A\in\mathscr{A}(F_0, T_0)$ is at most $1+|F-F'|\leqslant 1+r-1 = r$. We infer, taking into account (v):

(vii) $|\{(F,T)\colon A\in\mathscr{A}(F,T)\}| \leqslant (t+1)\min(t, n-2t)$.

In fact, we have proved that for fixed A

$$|\{F'\colon A\in\mathscr{A}(F',T)\}| \leqslant \min(t, n-2t)$$

holds. Moreover, for fixed A and T;

(viii) $|\{F'\colon A\in\mathscr{A}(F',T)\}| \leqslant \min\{t, n-2t, |\mathscr{F}(T)|\}$ trivially follows.

2.2. *The Proof of* (2)

For every pair (F,T), $T\subset F\in\mathscr{F}$, $|T| = t$ we define a nonnegative weight function on the t-subsets of X (i.e., $w_{(F,T)}\colon \binom{X}{t}\to R$). For convenience we set $t_0 = \min(t, n-2t)$.

(a) If $|\mathscr{F}(T)| = 1$, then $w_{(F,T)}(A) = 1$ for $A = T$ and 0, otherwise.

(b) If $1 < |\mathscr{F}(T)| \leqslant (n-t)/t$, then $w_{(F,T)}(A) = t/(n-t)$ for $A = T$ and 0, otherwise.

(c) If $|\mathscr{F}(T)| > (n-t)/t$, then

$$\begin{aligned} w_{(F,T)}(A) &= 1/|\mathscr{F}(T)|, && \text{for } A = T,\\ &= 1/(t+1)t_0, && \text{if } A \in \mathscr{A}(F', T) \text{ for some } F',\\ &= 0, && \text{otherwise.} \end{aligned}$$

Let us estimate the sum of the weights in the case (c), for brevity we set $|\mathscr{F}(T)| = d$, $\min(t_0, d) = d_0$, and use (viii) and (iv).

$$\begin{aligned} \sum_{A\in\binom{X}{t}} W_{(F,T)}(A) &= \frac{1}{d} + \frac{1}{(t+1)t_0}\left|\left\{A : A \in \bigcup_{F'\in\mathscr{F}(T)} \mathscr{A}(F', T)\right\}\right|\\ &\geqslant \frac{1}{d} + \frac{1}{(t+1)t_0}\frac{1}{d_0}\sum_{F'\in\mathscr{F}(T)} |\mathscr{A}(F', T)|\\ &\geqslant \frac{1}{d} + \frac{(n-2t)(dt-(n-t))}{t_0(t+1)d_0} \geqslant \frac{1}{d} + \frac{dt-(n-t)}{(t+1)d_0}\\ &= \frac{t}{n-t} + \frac{(dt-(n-t))(d(n-t)-d_0(t+1))}{(t+1)\,d_0 d(n-t)} > \frac{t}{n-t}. \end{aligned}$$

Using (ii) we infer for any fixed $F \in \mathscr{F}$

$$\begin{aligned} \sum_{T\subset F}\sum_{A\in\binom{X}{t}} w_{(F,T)}(A) &\geqslant \frac{1}{2}\binom{2t}{t} 1 + \frac{1}{2}\binom{2t}{t}\frac{t}{n-t}\\ &= \binom{2t-1}{t}\frac{n}{n-t}. \end{aligned} \tag{3}$$

On the other hand for any fixed $A \in \binom{X}{t}$ we prove

$$\sum_{(F,T)} w_{(F,T)}(A) \leqslant 1. \tag{4}$$

Indeed, if $|\mathscr{F}(A)| \geqslant 1$, then $w_{(F,T)}(A)$ is positive only for $T = A$ and even then it is at most $1/|\mathscr{F}(A)|$; if $|\mathscr{F}(A)| = 0$, then it follows from (vii).

Now using the inequalities (3) and (4) we infer

$$\begin{aligned} |\mathscr{F}|\binom{2t-1}{t}\frac{n}{n-t} &\leqslant \sum_{F\in\mathscr{F}}\sum_{\substack{T\subset F\\|T|=t}}\left(\sum_{A\in\binom{X}{t}} w_{(F,T)}(A)\right)\\ &= \sum_{A\in\binom{X}{t}}\left(\sum_{F\in\mathscr{F}}\sum_{\substack{T\subset F\\|T|=t}} w_{(F,T)}(A)\right) \leqslant \binom{n}{t}, \end{aligned}$$

$$|\mathscr{F}| \leqslant \binom{n-1}{t} \Big/ \binom{2t-1}{t} \qquad \text{Q.E.D.}$$

2.3. *The Case of Equality*

If $|\mathscr{F}| = \binom{n-1}{t}/\binom{2t-1}{t}$, then we must have equality for every $F \in \mathscr{F}$ in (3), thus F has excactly $\frac{1}{2}\binom{2t}{t}$ private t-subsets and another $\frac{1}{2}\binom{2t}{t}$ for which $1 \leqslant |\mathscr{F}(T)| \leqslant (n-t)/t$.

We must have equality in (4) as well yielding that there are no free t-sets A, i.e., with $|\mathscr{F}(A)| = 0$; more exactly $|\mathscr{F}(A)|$ is either 1 or $(n-t)/t$ (in particular $(n-t)/t$ is an integer, i.e., $t \mid n$). Let us set

$$\varepsilon = \left\{ A \in \binom{X}{t} : |\mathscr{F}(A)| = 1 \right\}, \qquad \tau = \left\{ A \in \binom{X}{t} : |\mathscr{F}(A)| = \frac{n-t}{t} \right\}$$

Then $|\varepsilon| = \frac{1}{2}\binom{2t}{t}|\mathscr{F}| = \binom{n-1}{t}$ and $|\tau| = \binom{n}{t} - |\varepsilon| = \binom{n-1}{t-1}$.

Now we show:

(ix) *For $F, F' \in \mathscr{F}$ we have $|F \cap F'| \leqslant t$.*

Indeed, otherwise $|F - F'| \leqslant t-1$ and $F - F'$ is a private subset of F if we choose A such that $|A| = t$, $F - F' \subset A \not\subset F$, then A is a free t-subset, a contradiction. Thus for $T \in \tau$ the sets $F - T$ for $F \in \mathscr{F}(T)$ partition $X - T$. Let us take some fixed $B \in \binom{X}{t-1}$ and set

$$D = \{x \in X - B\colon (B \cup \{x\}) \in \tau\}, \qquad |D| = d,$$
$$c_i = |\{F \in \mathscr{F} : F \supset B, |F \cap D| = i\}|, \qquad 0 \leqslant i \leqslant t+1.$$

We have

$$\sum_{0 \leqslant i \leqslant t+1} i c_i = \sum_{B \subset F \in \mathscr{F}} |F \cap D| = \sum_{x \in D} |\mathscr{F}(B \cup \{x\})| = d\,\frac{n-t}{t}. \tag{5}$$

Also we have

$$\sum_{0 \leqslant i \leqslant t+1} c_i(t+1-i) = \sum_{B \subset F \in \mathscr{F}} |F \cap (X - B - D)|$$
$$= \sum_{x \in X - B - D} |\mathscr{F}(B \cup \{x\})| = n - t + 1 - d. \tag{6}$$

As for every $x, y \in D$, we have $|\mathscr{F}(B \cup \{x, y\})| = 1$, we deduce

$$\sum_{i=0}^{t+1} c_i \binom{i}{2} = \sum_{B \subset F \in \mathscr{F}} \binom{|F \cap D|}{2} = \sum_{x, y \in D} |\mathscr{F}(B \cup \{x, y\})| = \binom{d}{2}. \tag{7}$$

If we subtract the double of (7) from (5) multiplied by t we obtain

$$\sum_{0\leqslant i\leqslant t+1} c_i i(t+1-i) = d(n-t+1-d). \tag{8}$$

If $d\neq 0$ and $d\neq n-t+1$, then, using $c_i=0$ for $i>d$, a comparison of (6) and (8) yields $dc_i(t+1-i)=ic_i(t+1-i)$ for $0\leqslant i\leqslant t+1$, consequently $c_d(t+1-d)=n-t+1-d$ and $c_d\geqslant 1$, $c_0=c_1=\cdots=c_{d-1}=0$. Now if $d\geqslant 2$ from (7) we deduce $c_d=1$ which leads to the contradiction $n-t+1-d=t+1-d$, i.e., $n=2t$. Thus we have proved:

(x) $|\tau(B)|=0$ *or* 1 *or* $n-t+1$.

Now we shall need the following special instance of the Kruskal–Katona theorem (see [4, 7]).

(xi) *Suppose for some integers* m, g, $m\geqslant g\geqslant 1$ *we have a family* $\mathscr{G}$ *of* g*-sets with* $|\mathscr{G}|=\binom{m}{g}$. *Then* $|\{H\subset G\in\mathscr{G}\ |H|=g-1\}|\geqslant\binom{m}{g-1}$ *with equality holding iff* $|\bigcup_{G\in\mathscr{G}} G|=m$.

Using (xi) we show next:

(xii) *There exist an* $x_0\in X$ *such that* $\tau=\{T\in\binom{X}{t}: x_0\in T\}$.

Applying (xi) to ε we infer

$$\left|\left\{B\in\binom{X}{t-1}: \left|\varepsilon(B)\right|\geqslant 1\right\}\right|\geqslant\binom{n-1}{t-1}. \tag{9}$$

As $|\varepsilon(B)|+|\tau(B)|=n-t+1$, in view of (x) we deduce

$$\begin{gathered}\left|\left\{B\in\binom{X}{t-1}: \left|\tau(B)\right|\leqslant 1\right\}\right|\geqslant\binom{n-1}{t-1},\\ \left|\left\{B\in\binom{X}{t-1}: \left|\tau(B)\right|=n-t+1\right\}\right|\leqslant\binom{n}{t-1}-\binom{n-1}{t-1}=\binom{n-1}{t-2}.\end{gathered} \tag{10}$$

Using these two inequalities we infer

$$\begin{aligned} t\binom{n-1}{t-1} &= |\{(B,T): B\subset T\in\tau, |B|=t-1\}| \\ &= \sum_{B\in\binom{X}{t-1}} |\tau(B)| \\ &\leqslant \binom{n-1}{t-1}1+\binom{n-1}{t-2}(n-t+1)=t\binom{n-1}{t-1}. \end{aligned}$$

Thus we must have equality in (10) and consequently in (9), too. Now (xi) yields the existence of $x_0\in X$ such that $\bigcup_{E\in\varepsilon} E=X-\{x_0\}$ which implies (xii). Thus $x_0 \in F$ for every $F \in \mathscr{F}$. Set $\mathscr{F}_0 = \{F - \{x_0\} : F \in \mathscr{F}\}$. Now $\mathscr{F}_0 \subset \binom{X - \{x_0\}}{2t-1}$, $|\mathscr{F}_0| = \binom{n-1}{t-1} / \binom{2t-1}{t-1}$ and in view of (ix) for $F_0, F_0' \in \mathscr{F}_0$ we have $|F_0 \cap F_0'| \leqslant t-1$, thus $\mathscr{F}_0$ is an $(n-1, 2t-1, t)$ Steiner system.

Q.E.D.

2.4. *The Proof of Proposition* 1

We will actually describe an algorithm to find a family with so many sets. We start with $\mathscr{F}_0 = \varnothing$, $\mathscr{G}_0 = \binom{X}{k}$. If $\mathscr{F}_i, \mathscr{G}_i$ are defined, then let F be an arbitrary member of $\mathscr{G}_i$ and set

$$\mathscr{F}_{i+1} = \mathscr{F}_i \cup \{F\}, \mathscr{G}_{i+1} = \mathscr{G}_i - \left\{ G \in \binom{X}{k} : |G \cap F| \geqslant \frac{\lceil k \rceil}{2} \right\}.$$

Of course we have $\mathscr{F}_{i+1} \cap \mathscr{G}_{i+1} = \varnothing$.

$$|\mathscr{G}_i - \mathscr{G}_{i+1}| < \binom{k}{\lceil k/2 \rceil} \binom{n - \lceil k/2 \rceil}{\lfloor k/2 \rfloor} \tag{11}$$

We go on with this procedure until we reach an m such that $\mathscr{G}_m = \varnothing$. Then by (11) we have

$$|\mathscr{F}_m| = m > \binom{n}{k} \Big/ \left(\binom{k}{\lceil k/2 \rceil} \binom{n - \lceil k/2 \rceil}{\lfloor k/2 \rfloor} \right) = \binom{n}{\lceil k/2 \rceil} \Big/ \binom{k}{\lceil k/2 \rceil}^2.$$

By the definition of $\mathscr{F}_m$ for $F, F' \in \mathscr{F}_m$ we have $|F \cap F'| < \lceil k/2 \rceil$ thus $F \subset F' \cup F''$ is impossible.

Q.E.D.

2.5. *The Proof of Theorem* 2

Let us choose independently and with probability $2m/\binom{n}{k}$ each of the k-subsets of X, the value of m will be fixed later. Let $\mathscr{S}$ denote the obtained random hypergraph. Obviously, the expectation of the number of edges in $\mathscr{S}$ is $E(|\mathscr{S}|) = 2m$. We will need the expression for the number of ordered pairs of k-sets B, C such that for a given k-set A the relation $A \subset B \cup C$ holds:

$$\begin{aligned} R(n, k) &= \sum_{0 \leqslant x \leqslant k} \binom{k}{x} \binom{n-k}{k-x} \sum_{0 \leqslant y \leqslant x} \binom{x}{y} \binom{n-k}{x-y} \\ &= \sum_{0 \leqslant x \leqslant k} \binom{k}{x} \binom{n-k}{k-x} \binom{n-k+x}{x} \\ &= \sum_{0 \leqslant x \leqslant k} \binom{k}{x}^2 \binom{n-k+x}{k} \leqslant \max_{x \leqslant k} \binom{k}{x}^2 \binom{n-k+x}{k} n. \end{aligned}$$

Thus for $m < 1/2 \sqrt{2} \binom{n}{k} / \sqrt{n \max_{x \leqslant k} \binom{k}{x}^2 \binom{n-k+x}{k}}$ we have $R(n, k)\, 4m^2 / \binom{n}{k}^2 < \frac{1}{2}$, yielding that the probability for a given edge $A \in \mathscr{S}$ to be covered by $B \cup C$. $B, C \in \mathscr{S}$ is less than $\frac{1}{2}$. Hence the expected number of edges to remain in $\mathscr{S}$ after the omission of the covered edges is greater than $2m - \frac{1}{2}2m = m$, and in that hypergraph the conditions are already satisfied.

So we have shown the existence of a desired hypergraph with at least $1/2\sqrt{2}\binom{n}{k}/\sqrt{n \max_{x \leqslant k} \binom{k}{x}^2 \binom{n-k+x}{k}}$ edges. All we need is a lower bound on this expression. The ratio of the term to be maximized, for consecutive values of x, is $(k-x+1)^2(n-k+x)/(n-2k+x)x^2$. This function is monotone decreasing in x, thus the maximum is taken at the value where this ratio is about 1. We get a quadratic equation in x; the solution of which is $(0 \leqslant x \leqslant k)$

$$x_{\max} \sim \tfrac{1}{2}(3k - 2n + \sqrt{5k^2 - 8kn + 4n^2}).$$

Setting $k = 0.26n$ we obtain $x_{\max} = 0.1413 \cdots n$. Putting this value back into the expression for m and applying the Stirling formula we see that m can be as large as $(1.1348)^n$. Q.E.D.

Added in proof. The first and third authors observed that the characteristic vectors of the members of the set-system $\mathscr{F}$ in Theorem 2 yield a point set $\mathscr{P}$ of cardinality at least 1.13^n in R^n such that all angles determined by the triples of $\mathscr{P}$ are less than $\pi/2$. This disproves the old conjecture $|\mathscr{P}| \leqslant 2n - 1$. Moreover one can give 1.001^n points in R^n all the angles of which are less than 61° (and greater than 58°). These and other related results can be found in [8].

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