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Decomposability for stable processes

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Abstract

We characterize all possible independent symmetric α-stable ($S\alpha S$) components of an $S\alpha S$ process, $0 < \alpha < 2$. In particular, we focus on stationary $S\alpha S$ processes and their independent stationary $S\alpha S$ components. We also develop a parallel characterization theory for max-stable processes.

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1. Introduction

Recall that a random variable Z has a *symmetric α-stable* ($S\alpha S$) distribution with $0 < \alpha \le 2$, if $\mathbb{E}\exp(itZ) = \exp(-\sigma^\alpha |t|^\alpha)$ for all $t \in \mathbb{R}$ with some constant $\sigma > 0$. A process $X = \{X_t\}_{t\in T}$ is said to be $S\alpha S$ if all its finite linear combinations follow $S\alpha S$ distributions.

In this paper, we investigate the general decomposability problem for $S\alpha S$ processes with $0 < \alpha < 2$. Namely, let $X = \{X_t\}_{t\in T}$ be an $S\alpha S$ process indexed by an arbitrary set T. Suppose that

$$\{X_t\}_{t\in T} \overset{\mathrm{d}}{=} \left\{X_t^{(1)} + \cdots + X_t^{(n)}\right\}_{t\in T}, \tag{1.1}$$

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where '$\stackrel{d}{=}$' means equality in finite-dimensional distributions, and $X^{(k)} = \{X_t^{(k)}\}_{t\in T}, k = 1,\ldots,n$ are *independent* $S\alpha S$ processes. We will write $X \stackrel{d}{=} X^{(1)} + \cdots + X^{(n)}$ in short, and each $X^{(k)}$ will be referred to as a *component* of X. The stability property readily implies that (1.1) holds with $X^{(k)} \stackrel{d}{=} n^{-1/\alpha} X \equiv \{n^{-1/\alpha} X_t\}_{t\in T}$. The components equal in finite-dimensional distributions to a constant multiple of X will be referred to as *trivial*. We are interested in the general structure of all possible *non-trivial* $S\alpha S$ components of X.

Many important decompositions (1.1) of $S\alpha S$ processes are already available in the literature: see for example [3,16,19,29,13,14,25], to name a few. These results were motivated by studies of various probabilistic and structural aspects of the underlying $S\alpha S$ processes such as ergodicity, mixing, stationarity, self-similarity, etc. Notably, Rosiński [16] established a fundamental connection between stationary $S\alpha S$ processes and non-singular flows. He developed important tools based on minimal representations of $S\alpha S$ processes and inspired multiple decomposition results motivated by connections to ergodic theory.

In this paper, we adopt a different perspective. Our main goal is to characterize *all* possible $S\alpha S$ decompositions (1.1). Our results show how the dependence structure of an $S\alpha S$ process determines the structure of its components.

Consider $S\alpha S$ processes $\{X_t\}_{t\in T}$ indexed by a complete separable metric space T with an integral representation

$$\{X_t\}_{t\in T} \stackrel{d}{=} \left\{\int_S f_t(s) M_\alpha(\mathrm{d}s)\right\}_{t\in T}, \tag{1.2}$$

where real-valued functions $\{f_t\}_{t\in T} \subset L^\alpha(S, \mathcal{B}_S, \mu)$ are referred to as the *spectral functions* of $\{X_t\}_{t\in T}$. By default, M_α is a real-valued $S\alpha S$ random measure on the standard Lebesgue space $(S, \mathcal{B}_S, \mu)$, with a σ-finite control measure μ. The spectral functions determine the finite-dimensional distributions of the process: for all $n \in \mathbb{N}, t_j \in T, a_j \in \mathbb{R}$,

$$\mathbb{E}\exp\left(-i\sum_{j=1}^n a_j X_{t_j}\right) = \exp\left(-\int_S \left|\sum_{j=1}^n a_j f_{t_j}\right|^\alpha \mathrm{d}\mu\right). \tag{1.3}$$

Every separable in probability $S\alpha S$ process X can be shown to have such a representation; see, for example, the excellent book by Samorodnitsky and Taqqu [26] for detailed discussions on $S\alpha S$ distributions and processes. Without loss of generality, we always assume that the spectral functions $\{f_t\}_{t\in T} \subset L^\alpha(S, \mathcal{B}_S, \mu)$ have *full support*, i.e., $S = \mathrm{supp}\{f_t,\ t \in T\}$.

We first state the main result of this paper. To this end, we recall that the *ratio σ-algebra* of a spectral representation $F = \{f_t\}_{t\in T}$ (of $\{X_t\}_{t\in T}$) is defined as

$$\rho(F) \equiv \rho\{f_t,\ t \in T\} := \sigma\{f_{t_1}/f_{t_2},\ t_1, t_2 \in T\}. \tag{1.4}$$

The following result characterizes the structure of all $S\alpha S$ decompositions.

Theorem 1.1. *Suppose $\{X_t\}_{t\in T}$ is an $S\alpha S$ process ($0 < \alpha < 2$) with spectral representation*

$$\{X_t\}_{t\in T} \stackrel{d}{=} \left\{\int_S f_t(s) M_\alpha(\mathrm{d}s)\right\}_{t\in T},$$

with $\{f_t\}_{t\in T} \subset L^\alpha(S, \mathcal{B}_S, \mu)$. Let $\{X_t^{(k)}\}_{t\in T},\ k = 1,\ldots,n$ be independent $S\alpha S$ processes.

(i) *The decomposition*

$$\{X_t\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{X_t^{(1)} + \cdots + X_t^{(n)}\right\}_{t\in T} \tag{1.5}$$

holds, if and only if there exist measurable functions $r_k : S \to [-1, 1],\ k = 1, \ldots, n$, *such that*

$$\{X_t^{(k)}\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{\int_S r_k(s) f_t(s) M_\alpha(\mathrm{d}s)\right\}_{t\in T}, \quad k = 1, \ldots, n. \tag{1.6}$$

In this case, necessarily $\sum_{k=1}^n |r_k(s)|^\alpha = 1$, μ*-almost everywhere on* S.

(ii) *If* (1.5) *holds, then the* r_k*'s in* (1.6) *can be chosen to be non-negative and* $\rho(F)$*-measurable. Such* r_k*'s are unique modulo* μ.

As an application, we study the structure of the *stationary* $S\alpha S$ components of a stationary $S\alpha S$ process. We obtain a characterization for all possible stationary components of stationary $S\alpha S$ processes in Theorem 3.1. As a simple example, consider the moving average process $\{X_t\}_{t\in\mathbb{R}^d}$ with spectral representation

$$\{X_t\}_{t\in\mathbb{R}^d} \stackrel{\mathrm{d}}{=} \left\{\int_{\mathbb{R}^d} f(t+s) M_\alpha(\mathrm{d}s)\right\}_{t\in\mathbb{R}^d},$$

where $d \in \mathbb{N}$, M_α is an $S\alpha S$ random measure on $\mathbb{R}^d$ with the Lebesgue control measure λ, and $f \in L^\alpha(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d}, \lambda)$ (see, e.g., [26]). We show that such a process has only trivial stationary $S\alpha S$ components, i.e. all its stationary components are rescaled versions of the original process (Corollary 3.2). Such stationary $S\alpha S$ processes will be called *indecomposable*. More examples are provided in Sections 2 and 3.

We also develop parallel decomposability theory for max-stable processes. Recently, Kabluchko [9] and Wang and Stoev [31,32] have established intrinsic connections between sum- and max-stable processes. In particular, the tools in [31] readily imply that the developed decomposition theory for $S\alpha S$ processes applies *mutatis mutandis* to max-stable processes.

The rest of the paper is structured as follows. In Section 2, we provide some consequences of Theorem 1.1 for general $S\alpha S$ processes. The stationary case is discussed in Section 3. Parallel results on max-stable processes are presented in Section 4. The proof of Theorem 1.1 is given in Section 5.

2. $S\alpha S$ Components

In this section, we provide a few examples to illustrate the consequences of our main result Theorem 1.1. The first one is about $S\alpha S$ processes with independent increments. Recall that we always assume $0 < \alpha < 2$.

Corollary 2.1. *Let* $X = \{X_t\}_{t\in\mathbb{R}_+}$ *be an arbitrary* $S\alpha S$ *process with independent increments and* $X_0 = 0$. *Then all* $S\alpha S$ *components of* X *also have independent increments.*

Proof. Write $m(t) = \|X_t\|_\alpha^\alpha$, where $\|X_t\|_\alpha$ denotes the scale coefficient of the $S\alpha S$ random variable X_t. By the independence of the increments of X, it follows that m is a non-decreasing function with $m(0) = 0$. First, we consider the simple case when $m(t)$ is right-continuous.

Consider the Borel measure μ on $[0, \infty)$ determined by $\mu([0, t]) := m(t)$. The independence of the increments of X readily implies that X has the representation:

$$\{X_t\}_{t\in\mathbb{R}_+} \stackrel{\mathrm{d}}{=} \left\{\int_0^\infty \mathbf{1}_{[0,t]}(s) M_\alpha(\mathrm{d}s)\right\}_{t\in\mathbb{R}_+}, \tag{2.1}$$

where M_α is an $S\alpha S$ random measure with control measure μ.

Now, for any $S\alpha S$ component $Y(\equiv X^{(k)})$ of X, we have that (1.6) holds with $f_t(s) = \mathbf{1}_{[0,t]}(s)$ and some function $r(s)(\equiv r_k(s))$. This implies that the increments of Y are also independent since, for example, for any $0 \le t_1 < t_2$, the spectral functions $r(s) f_{t_1}(s) = r(s)\mathbf{1}_{[0,t_1]}(s)$ and $r(s) f_{t_2}(s) - r(s) f_{t_1}(s) = r(s)\mathbf{1}_{(t_1,t_2]}(s)$ have disjoint supports.

It remains to prove the general case. The difficulty is that $m(t)$ may have (at most countably many) discontinuities, and a representation as (2.1) is not always possible. Nevertheless, introduce the right-continuous functions $t \mapsto m_i(t)$, $i = 0, 1$,

$$m_0(t) := m(t+) - \sum_{\tau\le t}(m(\tau) - m(\tau-)) \quad \text{and} \quad m_1(t) := \sum_{\tau\le t}(m(\tau) - m(\tau-))$$

and let $\widetilde{M}_\alpha$ be an $S\alpha S$ random measure on $\mathbb{R}_+ \times \{0, 1\}$ with control measure $\mu([0, t] \times \{i\}) := m_i(t),\ i = 0, 1,\ t \in \mathbb{R}_+$. In this way, as in (2.1), one can show that

$$\{X_t\}_{t\in\mathbb{R}_+} \stackrel{\mathrm{d}}{=} \left\{\int_{\mathbb{R}_+\times\{0,1\}} \mathbf{1}_{[0,t)\times\{0\}}(s, v) + \mathbf{1}_{[0,t]\times\{1\}}(s, v)\widetilde{M}_\alpha(\mathrm{d}s, \mathrm{d}v)\right\}_{t\in\mathbb{R}_+}.$$

The rest of the proof remains similar and is omitted. □

Remark 2.1. Theorem 1.1 and Corollary 2.1 do not apply to the Gaussian case ($\alpha = 2$). For the sake of simplicity, take $T = \{1, 2\}$ and $n = 2$ (2 $S\alpha S$ components) in (1.1). In this case, all the (in)dependence information of the mean-zero Gaussian process $\{X_t\}_{t\in T}$ is characterized by the covariance matrix Σ of the Gaussian vector $({X_1}^{(1)}, {X_1}^{(2)}, {X_2}^{(1)}, {X_2}^{(2)})$. A counterexample can be easily constructed by choosing appropriately Σ. This reflects the drastic difference of the geometries of L^α spaces for $\alpha < 2$ and $\alpha = 2$.

The next natural question to ask is whether two $S\alpha S$ processes have *common components*. Namely, the $S\alpha S$ process Z is a common component of the $S\alpha S$ processes X and Y, if $X \stackrel{\mathrm{d}}{=} Z + X^{(1)}$ and $Y \stackrel{\mathrm{d}}{=} Z + Y^{(1)}$, where $X^{(1)}$ and $Y^{(1)}$ are both $S\alpha S$ processes independent of Z.

To study the common components, the *co-spectral* point of view introduced in Wang and Stoev [32] is helpful. Consider a *measurable* $S\alpha S$ process $\{X_t\}_{t\in T}$ with spectral representation (1.2), where the index set T is equipped with a measure λ defined on the σ-algebra $\mathcal{B}_T$. Without loss of generality, we take $f(\cdot, \cdot) : (S \times T, \mathcal{B}_S \times \mathcal{B}_T) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ to be jointly measurable (see Theorems 9.4.2 and 11.1.1 in [26]). The *co-spectral functions*, $f_\cdot(s) \equiv f(s, \cdot)$, are elements of $L^0(T) \equiv L^0(T, \mathcal{B}_T, \lambda)$, the space of $\mathcal{B}_T$-measurable functions modulo λ-null sets. The co-spectral functions are indexed by $s \in S$, in contrast to the spectral functions $f_t(\cdot)$ indexed by $t \in T$. Recall also that a set $\mathcal{P} \subset L^0(T)$ is a *cone*, if $c\mathcal{P} = \mathcal{P}$ for all $c \in \mathbb{R} \setminus \{0\}$ and $\{\mathbf{0}\} \in \mathcal{P}$. We write $\{f_\cdot(s)\}_{s\in S} \subset \mathcal{P}$ modulo μ, if for μ-almost all $s \in S$, $f_\cdot(s) \in \mathcal{P}$.

Proposition 2.1. *Let* $X^{(i)} = \{X_t^{(i)}\}_{t\in T}$ *be* $S\alpha S$ *processes with measurable representations* $\{f_t^{(i)}\}_{t\in T} \subset L^\alpha(S_i, \mathcal{B}_{S_i}, \mu_i)$, $i = 1, 2$. *If there exist two cones* $\mathcal{P}_i \subset L^0(T), i = 1, 2$, *such* *that* $\{f_{\cdot}^{(i)}(s)\}_{s\in S_i} \subset \mathcal{P}_i$ *modulo* μ_i*, for* $i = 1, 2$*, and* $\mathcal{P}_1 \cap \mathcal{P}_2 = \{\mathbf{0}\}$*, then the two processes have no common component.*

Proof. Suppose Z is a component of $X^{(1)}$. Then, by Theorem 1.1, Z has a spectral representation $\{r^{(1)} f_t^{(1)}\}_{t\in T}$, for some $\mathcal{B}_{S_1}$-measurable function $r^{(1)}$. By the definition of cones, the co-spectral functions of Z are included in $\mathcal{P}_1$, i.e., $\{r^{(1)}(s) f_{\cdot}^{(1)}(s)\}_{s\in S_1} \subset \mathcal{P}_1$ modulo μ_1. If Z is also a component of $X^{(2)}$, then by the same argument, $\{r^{(2)}(s) f_{\cdot}^{(2)}(s)\}_{s\in S_2} \subset \mathcal{P}_2$ modulo μ_2, for some $\mathcal{B}_{S_2}$-measurable function $r^{(2)}(s)$. Since $\mathcal{P}_1 \cap \mathcal{P}_2 = \{\mathbf{0}\}$, it then follows that $\mu_i(\mathrm{supp}(r^{(i)})) = 0, i = 1, 2$, or equivalently $Z = 0$, the degenerate case. □

We conclude this section with an application to $S\alpha S$ moving averages.

Corollary 2.2. *Let* $X^{(1)}$ *and* $X^{(2)}$ *be two* $S\alpha S$ *moving averages*

$$\{X_t^{(i)}\}_{t\in\mathbb{R}^d} \overset{\mathrm{d}}{=} \left\{\int_{\mathbb{R}^d} f^{(i)}(t+s) M_\alpha{}^{(i)}(\mathrm{d}s)\right\}_{t\in\mathbb{R}^d}$$

with kernel functions $f^{(i)} \in L^\alpha(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d}, \lambda),\ i = 1, 2$. *Then, either*

$$X^{(1)} \overset{\mathrm{d}}{=} cX^{(2)} \quad \textit{for some } c > 0, \tag{2.2}$$

or $X^{(1)}$ *and* $X^{(2)}$ *have no common component. Moreover, (2.2) holds, if and only if for some* $\tau \in \mathbb{R}^d$ *and* $\epsilon \in \{\pm 1\}$*,*

$$f^{(1)}(s) = \epsilon c f^{(2)}(s+\tau), \quad \mu\textit{-almost all } s \in S. \tag{2.3}$$

Proof. Clearly (2.3) implies (2.2). Conversely, if (2.2) holds, then (2.3) follows as in the proof of Corollary 4.2 in [32], with slight modification (the proof therein was for *positive* cones). When (2.2) (or equivalently (2.3)) does not hold, consider the smallest cones containing $\{f^{(i)}(s + \cdot)\}_{s\in\mathbb{R}}, i = 1, 2$ respectively. Since these two cones have trivial intersection $\{\mathbf{0}\}$, Proposition 2.1 implies that $X^{(1)}$ and $X^{(2)}$ have no common component. □

3. Stationary $S\alpha S$ Components and Flows

Let $X = \{X_t\}_{t\in T}$ be a stationary $S\alpha S$ process with representation (1.2), where now $T = \mathbb{R}^d$ or $T = \mathbb{Z}^d, d \in \mathbb{N}$. The seminal work of Rosński [16] established an important connection between stationary $S\alpha S$ processes and *flows*. A family of functions $\{\phi_t\}_{t\in T}$ is said to be a flow on $(S, \mathcal{B}_S, \mu)$, if for all $t_1, t_2 \in T$, $\phi_{t_1+t_2}(s) = \phi_{t_1}(\phi_{t_2}(s))$ for all $s \in S$, and $\phi_0(s) = s$ for all $s \in S$. We say that a flow is *non-singular*, if $\mu(\phi_t(A)) = 0$ is equivalent to $\mu(A) = 0$, for all $A \in \mathcal{B}_S, t \in T$. Given a flow $\{\phi_t\}_{t\in T}$, $\{c_t\}_{t\in T}$ is said to be a *cocycle* if $c_{t+\tau}(s) = c_t(s) c_\tau \circ \phi_t(s)$ μ-almost surely for all $t, \tau \in T$ and $c_t \in \{\pm 1\}$ for all $t \in T$.

To understand the relation between the structure of stationary $S\alpha S$ processes and flows, it is necessary to work with *minimal* representations of $S\alpha S$ processes, introduced by Hardin [7,8]. The minimality assumption is crucial in many results on the structure of $S\alpha S$ processes, although it is in general difficult to check (see e.g. Rosiński [18] and Pipiras [12]).

Definition 3.1. The spectral functions $F \equiv \{f_t\}_{t\in T}$ (and the corresponding spectral representation (1.2)) are said to be minimal, if the ratio σ-algebra $\rho(F)$ in (1.4) is equivalent to $\mathcal{B}_S$, i.e., for all $A \in \mathcal{B}_S$, then there exists $B \in \rho(F)$ such that $\mu(A\Delta B) = 0$, where $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

Rosiński [16, Theorem 3.1] proved that if $\{f_t\}_{t\in T}$ is minimal, then there exists a modulo μ unique non-singular flow $\{\phi_t\}_{t\in T}$, and a corresponding cocycle $\{c_t\}_{t\in T}$, such that for all $t \in T$,

$$f_t(s) = c_t(s)\left(\frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}(s)\right)^{1/\alpha} f_0\circ\phi_t(s), \quad \mu\text{-almost everywhere.} \tag{3.1}$$

Conversely, suppose that (3.1) holds for some non-singular flow $\{\phi_t\}_{t\in T}$, a corresponding cocycle $\{c_t\}_{t\in T}$, and a function $f_0 \in L^\alpha(S,\mu)$ ($\{f_t\}_{t\in T}$ not necessarily minimal). Then, clearly the $S\alpha S$ process X in (1.2) is stationary. In this case, we shall say that X is generated by the flow $\{\phi_t\}_{t\in T}$.

Consider now an $S\alpha S$ decomposition (1.1) of X, where the independent components $\{X_t^{(k)}\}_{t\in T}$'s are *stationary*. This will be referred to as a *stationary $S\alpha S$ decomposition*, and the $\{X_t^{(k)}\}_{t\in T}$'s as *stationary components* of X. Our goal in this section is to characterize the structure of all possible stationary components. This characterization involves the invariant σ-algebra with respect to the flow $\{\phi_t\}_{t\in T}$:

$$\mathcal{F}_\phi = \{A \in \mathcal{B}_S : \mu(\phi_\tau(A)\Delta A) = 0, \text{ for all } \tau \in T\}. \tag{3.2}$$

Given a function g and a σ-algebra $\mathcal{G}$, we write $g \in \mathcal{G}$, if g is measurable with respect to $\mathcal{G}$.

Theorem 3.1. *Let $\{X_t\}_{t\in T}$ be a stationary and measurable $S\alpha S$ process with spectral functions $\{f_t\}_{t\in T}$ given by*

$$f_t(s) = \int_S c_t(s)\left(\frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}(s)\right)^{1/\alpha} f_0\circ\phi_t(s) M_\alpha(\mathrm{d}s), \quad t \in T.$$

(i) *Suppose that $\{X_t\}_{t\in T}$ has a stationary $S\alpha S$ decomposition*

$$\{X_t\}_{t\in T} \overset{\mathrm{d}}{=} \left\{X_t^{(1)} + \cdots + X_t^{(n)}\right\}_{t\in T}. \tag{3.3}$$

Then, each component $\{X_t^{(k)}\}_{t\in T}$ has a representation

$$\{X_t^{(k)}\}_{t\in T} \overset{\mathrm{d}}{=} \left\{\int_S r_k(s) f_t(s) M_\alpha(\mathrm{d}s)\right\}_{t\in T}, \quad k = 1,\ldots,n, \tag{3.4}$$

where the r_k's can be chosen to be non-negative and $\rho(F)$-measurable. This choice is unique modulo μ and these r_k's are ϕ-invariant, i.e. $r_k \in \mathcal{F}_\phi$.
(ii) *Conversely, for any ϕ-invariant r_k's such that $\sum_{k=1}^n |r_k(s)|^\alpha = 1$, μ-almost everywhere on S, decomposition (3.3) holds with $X^{(k)}$'s as in (3.4).*

Proof. By using (3.1), a change of variables, and the ϕ-invariance of the functions r_k's, one can show that the $X^{(k)}$'s in (3.4) are stationary. This fact and Theorem 1.1 yield part (ii).

We now show (i). Suppose that $X^{(k)}$ is a stationary ($S\alpha S$) component of X. Theorem 1.1 implies that there exists unique modulo μ non-negative and $\rho(F)$-measurable function r_k for which (3.4) holds. By the stationarity of $X^{(k)}$, it also follows that for all $\tau \in T$, $\{r_k(s)f_{t+\tau}(s)\}_{t\in T}$ is also a spectral representation of $X^{(k)}$. By the flow representation (3.1), it follows that for all $t,\tau \in T$,

$$f_{t+\tau}(s) = c_\tau(s) f_t\circ\phi_\tau(s)\left(\frac{\mathrm{d}(\mu\circ\phi_\tau)}{\mathrm{d}\mu}\right)^{1/\alpha}(s), \quad \mu\text{-almost everywhere,} \tag{3.5}$$

and we obtain that for all $\tau, t_j \in T, a_j \in \mathbb{R},\ j = 1, \ldots, n$:

$$\int_S \left| \sum_{j=1}^n a_j r_k(s) f_{t_j+\tau}(s) \right|^\alpha \mu(\mathrm{d}s) = \int_S \left| \sum_{j=1}^n a_j r_k \circ \phi_{-\tau}(s) f_{t_j}(s) \right|^\alpha \mu(\mathrm{d}s),$$

which shows that $\{r_k \circ \phi_{-\tau}(s) f_t(s)\}_{t\in T}$ is also a representation for $X^{(k)}$, for all $\tau \in T$.

Observe that from (3.5), for all $t_1, t_2, \tau \in T$ and $\lambda \in \mathbb{R}$,

$$\left\{ \frac{f_{t_1+\tau}}{f_{t_2+\tau}} \le \lambda \right\} = \phi_\tau{}^{-1} \left\{ \frac{f_{t_1}}{f_{t_2}} \le \lambda \right\} \text{ modulo } \mu.$$

It then follows that for all $\tau \in T$, the σ-algebra $\phi_{-\tau}(\rho(F)) \equiv (\phi_\tau)^{-1}(\rho(F))$ is equivalent to $\rho(F)$. This, by the uniqueness of $r_k \in \rho(F)$ (Theorem 1.1), implies that $r_k \circ \phi_\tau = r_k$ modulo μ, for all τ. Then, $r_k \in \mathcal{F}_\phi$ follows from standard measure-theoretic argument. The proof is complete. □

Remark 3.1. The structure of the *stationary* $S\alpha S$ components of stationary $S\alpha S$ processes (including random fields) has attracted much interest since the seminal work of Rosiński [16,17]. See, for example, [14,25,20,21,24,22,23,30]. In view of Theorem 3.1, the components considered in these works correspond to indicator functions $r_k(s) = \mathbf{1}_{A_k}(s)$ of certain disjoint flow-invariant sets A_k's arising from ergodic theory (see e.g. [11,1]).

Theorem 3.1 can be applied to check *indecomposability* of stationary $S\alpha S$ processes. Recall that a stationary $S\alpha S$ process is said to be *indecomposable*, if all its stationary $S\alpha S$ components are trivial (i.e. constant multiples of the original process).

Corollary 3.1. *Consider $\{X_t\}_{t\in T}$ as in Theorem 3.1. If $\mathcal{F}_\phi$ is trivial, then $\{X_t\}_{t\in T}$ is indecomposable. The converse is true when, in addition, $\{f_t\}_{t\in T}$ is minimal.*

Proof. If $\mathcal{F}_\phi$ is trivial, the result follows from Theorem 3.1. Conversely, let $\{f_t\}_{t\in T}$ be minimal and X indecomposable. Then, one can choose $A \in \mathcal{F}_\phi$, such that $\mu(A) > 0$ and $\mu(S \setminus A) > 0$. Then, consider

$$\{X_t^A\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{ \int_S \mathbf{1}_A(s) f_t(s) M_\alpha(\mathrm{d}s) \right\}_{t\in T}.$$

By Theorem 3.1, X^A is a stationary component of X. It suffices to show that X^A is a non-trivial of X, which would contradict the indecomposability.

Suppose that X^A is trivial, then $cX^A \stackrel{\mathrm{d}}{=} X$, for some $c > 0$. Thus, by Theorem 3.1, cX^A has a representation as in (3.4), with $r_k := c\mathbf{1}_A$. On the other hand, since $cX^A \stackrel{\mathrm{d}}{=} X$, we also have the trivial representation with $r_k := 1$. Since $A \in \rho(F)$, the uniqueness of r_k implies that $1 = c\mathbf{1}_A$ modulo μ, which contradicts $\mu(A^c) > 0$. Therefore, X^A is non-trivial. □

The indecomposable stationary $S\alpha S$ processes can be seen as the elementary building blocks for the construction of general stationary $S\alpha S$ processes. We conclude this section with two examples.

Example 3.1 (*Mixed Moving Averages*)**.** Consider a *mixed moving average* in the sense of [28]:

$$\{X_t\}_{t\in\mathbb{R}^d} \stackrel{\mathrm{d}}{=} \left\{ \int_{\mathbb{R}^d \times V} f(t+s, v) M_\alpha(\mathrm{d}s, \mathrm{d}v) \right\}_{t\in\mathbb{R}^d}. \tag{3.6}$$

Here, M_α is an $S\alpha S$ random measure on $\mathbb{R}^d \times V$ with the control measure $\lambda \times \nu$, where λ is the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ and ν is a probability measure on $(V, \mathcal{B}_V)$, and $f(s, v) \in L^\alpha(\mathbb{R}^d \times V, \mathcal{B}_{\mathbb{R}^d \times V}, \lambda \times \nu)$. Given a disjoint union $V = \bigcup_{j=1}^n A_j$, where A_j's are measurable subsets of V, the mixed moving averages can clearly be decomposed as in (3.3) with

$$\{X_t^{(k)}\}_{t\in\mathbb{R}^d} \stackrel{\mathrm{d}}{=} \left\{\int_{\mathbb{R}^d \times A_k} f(t+s, v) M_\alpha(\mathrm{d}s, \mathrm{d}v)\right\}_{t\in\mathbb{R}^d}, \quad \text{for all } k = 1, \ldots, n.$$

Any moving average process

$$\{X_t\}_{t\in\mathbb{R}^d} \stackrel{\mathrm{d}}{=} \left\{\int_{\mathbb{R}^d} f(t+s) M_\alpha(\mathrm{d}s)\right\}_{t\in\mathbb{R}^d} \tag{3.7}$$

trivially has a mixed moving average representation. The next result shows when the converse is true.

Corollary 3.2. *The mixed moving average X in (3.6) is indecomposable, if and only if it has a moving average representation as in (3.7).*

Proof. By Corollary 3.1, the moving average process (3.7) is indecomposable, since in this case $\phi_t(s) = t + s$, $t, s \in \mathbb{R}^d$ and therefore $\mathcal{F}_\phi$ is trivial. This proves the 'if' part.

Suppose now that X in (3.6) is indecomposable. In Section 5 of Pipiras [12], it was shown that $S\alpha S$ processes with mixed moving average representations and *stationary increments* also have minimal representations of the mixed moving average type. By using similar arguments, one can show that this is also true for the class of *stationary* mixed moving average processes.

Thus, without loss of generality, we assume that the representation in (3.6) is minimal. Suppose now that there exists a set $A \in \mathcal{B}_V$ with $\nu(A) > 0$ and $\nu(A^c) > 0$. Since $\mathbb{R}^d \times A$ and $\mathbb{R}^d \times A^c$ are flow-invariant, we have the stationary decomposition $\{X_t\}_{t\in\mathbb{R}^d} \stackrel{\mathrm{d}}{=} \{X_t^A + X_t^{A^c}\}_{t\in\mathbb{R}^d}$, where

$$X_t^B := \int_{\mathbb{R}\times V} \mathbf{1}_B(v) f(t+s, v) M_\alpha(\mathrm{d}s, \mathrm{d}v), \quad B \in \{A, A^c\}.$$

Note that both components $X^A = \{X_t^A\}_{t\in\mathbb{R}^d}$ and $X^{A^c} = \{X_t^{A^c}\}_{t\in\mathbb{R}^d}$ are non-zero because the representation of X has full support.

Now, since X is indecomposable, there exist positive constants c_1 and c_2, such that $X \stackrel{\mathrm{d}}{=} c_1 X^A \stackrel{\mathrm{d}}{=} c_2 X^{A^c}$. The minimality of the representation and Theorem 3.1 imply that $c_1 \mathbf{1}_A = c_2 \mathbf{1}_{A^c}$ modulo ν, which is impossible. This contradiction shows that the set V cannot be partitioned into two disjoint sets of positive measure. That is, V is a singleton and the mixed moving average is in fact a moving average. □

Example 3.2 (*Doubly Stationary Processes*)**.** Consider a stationary process $\xi = \{\xi_t\}_{t\in T}$ ($T = \mathbb{Z}^d$) supported on the probability space $(E, \mathcal{E}, \mu)$ with $\xi_t \in L^\alpha(E, \mathcal{E}, \mu)$. Without loss of generality, we may suppose that $\xi_t(u) = \xi_0 \circ \phi_t(u)$, where $\{\phi_t\}_{t\in T}$ is a μ-measure-preserving flow.

Let M_α be an $S\alpha S$ random measure on $(E, \mathcal{E}, \mu)$ with control measure μ. The stationary $S\alpha S$ process $X = \{X_t\}_{t\in T}$

$$X_t := \int_E \xi_t(u) M_\alpha(\mathrm{d}u), \quad t \in T \tag{3.8}$$

is said to be *doubly stationary* (see [2]). By Corollary 3.1, if ξ is ergodic, then X is indecomposable.

A natural and interesting question raised by a referee is: what happens when X is decomposable and hence ξ is non-ergodic? Can we have a direct integral decomposition of the process X into indecomposable components? The following remark partly addresses this question.

Remark 3.2. The doubly stationary $S\alpha S$ processes are a special case of stationary $S\alpha S$ processes generated by *positively recurrent flows (actions)*. As shown in Samorodnitsky [25, Remark 2.6], each such stationary $S\alpha S$ process $X = \{X_t\}_{t\in T}$ can be expressed through a measure-preserving flow (action) on a *finite* measure space. Namely,

$$\{X_t\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{\int_E f_t(u) M_\alpha^{(\mu)}(\mathrm{d}u)\right\}_{t\in T}, \quad \text{with } f_t(u) := c_t(u) f_0 \circ \phi_t(u), \tag{3.9}$$

where $M_\alpha^{(\mu)}$ is an $S\alpha S$ random measure with a *finite* control measure μ on $(E, \mathcal{E})$, $\phi = \{\phi_t\}_{t\in T}$ is a μ-preserving flow (action), and $\{c_t\}_{t\in T}$ is a co-cycle with respect to ϕ. In the case when the co-cycle is trivial ($c_t \equiv 1$) and $\mu(E) = 1$, the process X is *doubly stationary*.

For simplicity, suppose that $T = \mathbb{Z}^d$ and without loss of generality, let $(E, \mathcal{E}, \mu)$ be a standard Lebesgue space with $\mu(E) = 1$. The ergodic decomposition theorem (see e.g. [10, Theorem 2.3.3]) implies that there exists conditional probability distributions $\{\mu_u\}_{u\in E}$ with respect to $\mathcal{I}$ such that ϕ is measure-preserving and ergodic with respect to the measures μ_u for μ-almost all $u \in E$. Let ν be another ϕ-invariant measure on $(E, \mathcal{E})$ dominating the conditional probabilities μ_u so that the Radon–Nikodym derivatives $p(x, u) = (\mathrm{d}\mu_u/\mathrm{d}\nu)(x)$ are jointly measurable on $(E \times E, \mathcal{E} \otimes \mathcal{E}, \nu \times \mu)$. Consider

$$g_t(x, u) = f_t(x) p(\phi_t(x), u)^{1/\alpha}.$$

Recall that ν and μ_u are ϕ-invariant, whence

$$p(\phi_t(x), u) = \frac{\mathrm{d}\mu_u}{\mathrm{d}\nu}(\phi_t(x)) = \frac{\mathrm{d}\mu_u}{\mathrm{d}\nu}(x) = p(x, u), \quad \text{modulo } \nu \times \mu.$$

Thus, $g_t(x, u) = f_t(x)(\mathrm{d}\mu_u/\mathrm{d}\nu)^{1/\alpha}(x)$, and for all $a_j \in \mathbb{R},\ t_j \in T,\ j = 1, \ldots, n$, we have

$$\begin{aligned}
\int_{E^2} \left|\sum_{j=1}^n a_j g_{t_j}(x, u)\right|^\alpha \nu(\mathrm{d}x)\mu(\mathrm{d}u) &= \int_{E^2} \left|\sum_{j=1}^n a_j f_{t_j}(x)\right|^\alpha \frac{\mathrm{d}\mu_u}{\mathrm{d}\nu}(x)\nu(\mathrm{d}x)\mu(\mathrm{d}u) \\
&= \int_{E^2} \left|\sum_{j=1}^n a_j f_{t_j}(x)\right|^\alpha \mathrm{d}\mu_u(\mathrm{d}x)\mu(\mathrm{d}u) \\
&= \int_E \left|\sum_{j=1}^n a_j f_{t_j}(x)\right|^\alpha \mu(\mathrm{d}x),
\end{aligned}$$

where the last equality follows from the identity that $\int_E h(x)\mu(\mathrm{d}x) = \int_{E^2} h(x)\mu_u(\mathrm{d}x)\mu(\mathrm{d}u)$, for all $h \in L^1(E, \mathcal{E}, \mu)$. We have thus shown that $\{X_t\}_{t\in T}$ defined by (3.9) has another spectral representation

$$\{X_t\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{\int_{E\times E} g_t(x,u) M_\alpha^{(\nu\times\mu)}(\mathrm{d}x, \mathrm{d}u)\right\}_{t\in T}, \tag{3.10}$$

where $M_\alpha^{(\nu\times\mu)}$ is an $S\alpha S$ random measure on $E \times E$ with control measure $\nu \times \mu$. It also follows that for μ-almost all $u \in E$, the process defined by

$$X_t^{(u)} := \int_E g_t(x,u) M_\alpha^{(\nu)}(\mathrm{d}x), \quad t \in T,$$

is indecomposable, where $M_\alpha^{(\nu)}$ has control measure ν. Indeed, as above, one can show that

$$\{X_t^{(u)}\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{\int_E f_t(u,x) M_\alpha^{(\mu_u)}(\mathrm{d}x)\right\}_{t\in T},$$

where $M_\alpha^{(\mu_u)}$ has control measure μ_u. The ergodic decomposition theorem implies that the flow (action) ϕ is ergodic with respect to μ_u, which by Corollary 3.1 implies the indecomposability of $X^{(u)} = \{X_t^{(u)}\}_{t\in T}$. In this way, (3.10) parallels the mixed moving average representation for stationary $S\alpha S$ processes generated by *dissipative flows* (see e.g. [16]).

Remark 3.3. The above construction of the decomposition (3.10) assumes the existence of a ϕ-invariant measure ν dominating all conditional probabilities μ_u, $u \in E$. If the measure μ, restricted on the invariant σ-algebra $\mathcal{F}_\phi$ is discrete, i.e. $\mathcal{F}_\phi$ consists of countably many atoms under μ, then one can take $\nu \equiv \mu$. In this case, the process X is decomposed into a sum (possibly infinite) of its indecomposable components:

$$X_t = \sum_k \int_{E_k} f_t(x) M_\alpha^{(\mu)}(\mathrm{d}x),$$

where the E_k's are disjoint ϕ-invariant measurable sets, such that $E = \cup_k E_k$ and $\phi|_{E_k}$ is ergodic, for each k. In this case, the E_k's are the atoms of $\mathcal{F}_\phi$.

In general, when $\mu|_{\mathcal{F}_\phi}$ is not discrete, the dominating measure ν if it exists, may not be σ-finite. Indeed, since the ϕ_t's are ergodic for μ_u, it follows that either $\mu_{u'} = \mu_{u''}$ or $\mu_{u'}$ and $\mu_{u''}$ are singular, for μ-almost all $u', u'' \in E$. Thus, if $\mathcal{F}_\phi$ is "too rich", this singularity feature implies that the measure ν may not be chosen to be σ-finite.

4. Decomposability of max-stable processes

Max-stable processes are central objects in the extreme value theory. They arise in the limit of independent maxima and thus provide canonical models for the dependence of the extremes (see e.g. [6] and the references therein). Without loss of generality, we focus here on α-Fréchet processes.

Recall that a random variable Z has an α-Fréchet distribution, if $\mathbb{P}(Z \le x) = \exp(-\sigma^\alpha x^{-\alpha})$ for all $x > 0$ with some constant $\sigma > 0$. A process $Y = \{Y_t\}_{t\in T}$ is said to be α-Fréchet if for all $n \in \mathbb{N}, a_i \ge 0,\ t_i \in T, i = 1, \ldots, n$, the max-linear combinations $\max\{a_i Y_{t_i},\ i = 1, \ldots, n\} \equiv \bigvee_{i=1}^n a_i Y_{t_i}$ are α-Fréchet. It is well known that a max-stable process is α-Fréchet, if and only if it has α-Fréchet marginals [4]. In the seminal paper [5], de Haan developed convenient spectral representations of these processes. An extremal integral representation, which parallels the integral representations of $S\alpha S$ processes, was developed by Stoev and Taqqu [27].

Let $Y = \{Y_t\}_{t\in T}$ be an α-Fréchet ($\alpha > 0$) process. As in the $S\alpha S$ case, if Y is separable in probability, it has the extremal representation

$$\{Y_t\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{ {}^{\mathrm{e}}\!\!\int_S f_t(s) M_\alpha^\vee(\mathrm{d}s) \right\}_{t\in T}, \tag{4.1}$$

where '${}^{\mathrm{e}}\!\!\int$' stands for an extremal integral, $\{f_t\}_{t\in T} \subset L_+^\alpha(S, \mathcal{B}_S, \mu) = \{f \in L^\alpha(S, \mathcal{B}_S, \mu) : f \geq 0\}$ are non-negative deterministic functions, and where $M_\alpha^\vee$ is an *α-Fréchet random sup-measure with control measure μ* (see [27] for more details). The finite-dimensional distributions of Y are characterized in terms of the spectral functions f_t's as follows:

$$\mathbb{P}(Y_{t_i} \leq y_i,\ i = 1, \ldots, n) = \exp\left\{ -\int_S \Big(\max_{1\leq i\leq n} \frac{f_{t_i}(s)}{y_i} \Big)^\alpha \mu(\mathrm{d}s) \right\}, \tag{4.2}$$

for all $y_i > 0, t_i \in T,\ i = 1, \ldots, n$.

The above representations of max-stable processes mimic those of $S\alpha S$ processes (1.2) and (1.3). The cumulative distribution functions and max-linear combinations of spectral functions, in the max-stable setting, play the role of characteristic functions and linear combinations in the sum-stable setting, respectively. In fact, the deep connection between the two classes of processes has been clarified via the notion of *association* by Kabluchko [9] and Wang and Stoev [31], independently through different perspectives.

In the sequel, assume $0 < \alpha < 2$. An $S\alpha S$ process X and an α-Fréchet process Y are said to be *associated* if they have a common spectral representation. That is, if for *some* non-negative $\{f_t\}_{t\in T} \subset L_+^\alpha(S, \mathcal{B}_S, \mu)$, Relations (1.2) and (4.1) hold. The association is well defined in the following sense: any other set of functions $\{g_t\}_{t\in T} \subset L_+^\alpha(S, \mathcal{B}_S, \mu)$ is a spectral representation of X, if and only if, it is a spectral representation of Y (see [31, Theorem 4.1]).

Remark 4.1. It is well known that $\widetilde{Y} = \{Y_t^\alpha\}_{t\in T}$ is a 1-Fréchet process (see e.g. [27, Proposition 2.9]). Moreover, if (4.1) holds, then $\widetilde{Y}$ has spectral functions $\{f_t^\alpha\}_{t\in T} \subset L_+^1(S, \mathcal{B}_S, \mu)$. Thus, the exponent $\alpha > 0$ plays no essential role in the dependence structure of α-Fréchet processes. Consequently, the notion of association (defined for $\alpha \in (0, 2)$) can be used to study α-Fréchet processes with arbitrary positive α's.

The association method can be readily applied to transfer decomposability results for $S\alpha S$ processes to the max-stable setting, where now sums are replaced by maxima. Namely, let $Y = \{Y_t\}_{t\in T}$ be an α-Fréchet process. If

$$\{Y_t\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{ Y_t^{(1)} \vee \cdots \vee Y_t^{(n)} \right\}_{t\in T}, \tag{4.3}$$

for some independent α-Fréchet processes $Y^{(k)} = \{Y_t^{(k)}\}_{t\in T},\ i = 1, \ldots, n$, then we say that the $Y^{(k)}$'s are *components* of Y. By the max-stability of Y, (4.3) trivially holds if the $Y^{(k)}$'s are independent copies of $\{n^{-1/\alpha} Y_t\}_{t\in T}$. The constant multiples of Y are referred to as trivial components of Y and as in the $S\alpha S$ case, we are interested in the structure of the non-trivial ones.

To illustrate the association method, we prove the max-stable counterpart of our main result Theorem 1.1. From the proof, we can see that the other results in the sum-stable setting have their natural max-stable counterparts by association. We briefly state some of these results at the end of this section.

Theorem 4.1. *Suppose $\{Y_t\}_{t\in T}$ is an α-Fréchet process with spectral representation (4.1), where $F \equiv \{f_t\}_{t\in T} \subset L^\alpha_+(S, \mathcal{B}_S, \mu)$. Let $\{Y_t^{(k)}\}_{t\in T}$, $k = 1, \ldots, n$, be independent α-Fréchet processes. Then the decomposition (4.3) holds, if and only if there exist measurable functions $r_k : S \to [0, 1]$, $k = 1, \ldots, n$, such that*

$$\{Y_t^{(k)}\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{ {\int^{\mathrm{e}}_S} r_k(s) f_t(s) M^\vee_\alpha(\mathrm{d}s) \right\}_{t\in T}, \quad k = 1, \ldots, n. \tag{4.4}$$

In this case, $\sum_{k=1}^n r_k(s)^\alpha = 1$, μ-almost everywhere on S and the r_k's in (4.4) can be chosen to be $\rho(F)$-measurable, uniquely modulo μ.

Proof. The 'if' part follows from the straight-forward calculation of the cumulative distribution functions (4.2). To show the 'only if' part, suppose (4.3) holds and $Y^{(k)}$ has spectral functions $\{g_t^{(k)}\}_{t\in T} \subset L^\alpha_+(V_k, \mathcal{B}_{B_k}, \nu_k)$, $k = 1, \ldots, n$. Without loss of generality, assume $\{V_k\}_{k=1,\ldots,n}$ to be mutually disjoint and define $g_t(v) = \sum_{k=1}^n {g_t}^{(k)}(v)\mathbf{1}_{V_k} \in L^\alpha_+(V, \mathcal{B}_V, \nu)$ for appropriately defined $(V, \mathcal{B}_V, \nu)$ (see the proof of Theorem 1.1).

Now, consider the $S\alpha S$ process X associated to Y. It has spectral functions $\{f_t\}_{t\in T}$ and $\{g_t\}_{t\in T}$. Consider the $S\alpha S$ processes $X^{(k)}$ associated to $Y^{(k)}$ via spectral functions $\{g_t^{(k)}\}_{t\in T}$ for $k = 1, \ldots, n$. By checking the characteristic functions, one can show that $\{X^{(k)}\}_{k=1,\ldots,n}$ form a decomposition of X as in (1.1). Then, by Theorem 1.1, each $S\alpha S$ component $X^{(k)}$ has a spectral representation (1.6) with spectral functions $\{r_k f_t\}_{t\in T}$. But we introduced $X^{(k)}$ as the $S\alpha S$ process associated to $Y^{(k)}$ via spectral representation $\{g_t^{(k)}\}_{t\in T}$. Hence, $X^{(k)}$ has spectral functions $\{g_t^{(k)}\}_{t\in T}$ and $\{r_k f_t\}_{t\in T}$, and so does $Y^{(k)}$ by the association [31, Theorem 4.1]. Therefore, (4.4) holds and the rest of the desired results follow. □

Further parallel results can be established by the association method. Consider a stationary α-Fréchet process Y. If $Y^{(k)}$, $k = 1, \ldots, n$ are independent stationary α-Fréchet processes such that (4.3) holds, then we say that each $Y^{(k)}$ is a *stationary α-Fréchet component* of Y. The process Y is said to be *indecomposable*, if it has no non-trivial stationary component. The following results on *(mixed) moving maxima* (see e.g. [27,9] for more details) follow from Theorem 4.1 and the association method, in parallel to Corollary 3.2 on (mixed) moving averages in the sum-stable setting.

Corollary 4.1. *The mixed moving maxima process*

$$\{Y_t\}_{t\in\mathbb{R}^d} \stackrel{\mathrm{d}}{=} \left\{ {\int^{\mathrm{e}}_{\mathbb{R}^d\times V}} f(t+s, v) M^\vee_\alpha(\mathrm{d}s, \mathrm{d}v) \right\}_{t\in\mathbb{R}^d}$$

is indecomposable, if and only if it has a moving maxima representation

$$\{Y_t\}_{t\in\mathbb{R}^d} \stackrel{\mathrm{d}}{=} \left\{ {\int^{\mathrm{e}}_{\mathbb{R}^d}} f(t+s) M^\vee_\alpha(\mathrm{d}s) \right\}_{t\in\mathbb{R}^d}.$$

5. Proof of Theorem 1.1

We will first show that Theorem 1.1 is true when $\{f_t\}_{t\in T}$ is minimal (Proposition 5.1), and then we complete the proof by relating a general spectral representations to a minimal one. This technique is standard in the literature of representations of $S\alpha S$ processes (see e.g. [16, Remark 2.3]). We start with a useful lemma.

Lemma 5.1. *Let $\{f_t\}_{t\in T} \subset L^\alpha(S, \mathcal{B}_S, \mu)$ be a minimal representation of an $S\alpha S$ process. For any two bounded $\mathcal{B}_S$-measurable functions $r^{(1)}$ and $r^{(2)}$, we have*

$$\left\{\int_S r^{(1)} f_t \mathrm{d}M_\alpha\right\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{\int_S r^{(2)} f_t \mathrm{d}M_\alpha\right\}_{t\in T},$$

if and only if $|r^{(1)}| = |r^{(2)}|$ modulo μ.

Proof. The 'if' part is trivial. We shall prove now the 'only if' part. Let $S^{(k)} := \mathrm{supp}(r^{(k)})$, $k = 1, 2$ and note that since $\{f_t\}_{t\in T}$ is minimal, then $\{r^{(k)} f_t\}_{t\in T}$, are minimal representations, restricted to $S^{(k)}$, $k = 1, 2$, respectively. Since the latter two representations correspond to the same process, by Theorem 2.2 in [16], there exist a bi-measurable, one-to-one and onto point mapping $\Psi : S^{(1)} \to S^{(2)}$ and a function $h : S^{(1)} \to \mathbb{R} \setminus \{0\}$, such that, for all $t \in T$,

$$r^{(1)}(s) f_t(s) = r^{(2)} \circ \Psi(s) f_t \circ \Psi(s) h(s), \quad \text{almost all } s \in S^{(1)}, \tag{5.1}$$

and

$$\frac{\mathrm{d}(\mu \circ \Psi)}{\mathrm{d}\mu} = |h|^\alpha, \quad \mu\text{-almost everywhere.} \tag{5.2}$$

It then follows that, for almost all $s \in S^{(1)}$,

$$\frac{f_{t_1}(s)}{f_{t_2}(s)} = \frac{r^{(1)}(s) f_{t_1}(s)}{r^{(1)}(s) f_{t_2}(s)} = \frac{f_{t_1} \circ \Psi(s)}{f_{t_2} \circ \Psi(s)}. \tag{5.3}$$

Define $R_\lambda(t_1, t_2) = \{s : f_{t_1}(s)/f_{t_2}(s) \le \lambda\}$ and note that by (5.3), for all $A \equiv R_\lambda(t_1, t_2)$,

$$\mu(\Psi(A \cap S^{(1)}) \Delta (A \cap S^{(2)})) = 0. \tag{5.4}$$

In fact, one can show that Relation (5.4) is also valid for all $A \in \rho(F) \equiv \sigma(R_\lambda(t_1, t_2) : \lambda \in \mathbb{R}, t_1, t_2 \in T)$. Then, by minimality, (5.4) holds for all $A \in \mathcal{B}_S$. In particular, taking A equal to $S^{(1)}$ and $S^{(2)}$, respectively, it follows that $\mu(S^{(1)} \Delta S^{(2)}) = 0$. Therefore, writing $\widetilde{S} := S^{(1)} \cap S^{(2)}$, we have

$$\mu(\Psi(A \cap \widetilde{S}) \Delta (A \cap \widetilde{S})) = 0, \quad \text{for all } A \in \mathcal{B}_S. \tag{5.5}$$

This implies that $\Psi(s) = s$, for μ-almost all $s \in \widetilde{S}$. To see this, let $\mathcal{B}_{\widetilde{S}} = \mathcal{B}_S \cap \widetilde{S}$ denote the σ-algebra $\mathcal{B}_S$ restricted to $\widetilde{S}$. Observe that for all $A \in \mathcal{B}_{\widetilde{S}}$, we have $\mathbf{1}_A = \mathbf{1}_A \circ \Psi$, for μ-almost all $s \in \widetilde{S}$, and trivially $\sigma(\mathbf{1}_A : A \in \mathcal{B}_{\widetilde{S}}) = \mathcal{B}_{\widetilde{S}}$. Thus, by the second part of Proposition 5.1 in [18], it follows that $\Psi(s) = s$ modulo μ on $\widetilde{S}$. This and (5.2) imply that $h(s) \in \{\pm 1\}$, almost everywhere. Plugging Ψ and h into (5.1) yields the desired result. □

Proposition 5.1. *Theorem 1.1 is true when $\{f_t\}_{t\in T}$ is minimal.*

Proof. *We first prove the 'if' part*. The result follows readily by using characteristic functions. Indeed, suppose that the $X^{(k)} = \{X_t^{(k)}\}_{t\in T}, k = 1,\ldots,n$ are independent and have representations as in (1.6). Then, for all $a_j \in \mathbb{R},\ t_j \in T,\ j = 1,\ldots,m$, we have

$$\begin{aligned}\mathbb{E}\exp\left(i\sum_{j=1}^m a_j X_{t_j}\right) &= \exp\left(-\int_S\left|\sum_{j=1}^m a_j f_{t_j}\right|^\alpha \mathrm{d}\mu\right)\\ &= \prod_{k=1}^n \exp\left(-\int_S\left|\sum_{j=1}^m a_j r_k f_{t_j}\right|^\alpha \mathrm{d}\mu\right)\\ &= \prod_{k=1}^n \mathbb{E}\exp\left(i\sum_{j=1}^m a_j X_{t_j}{}^{(k)}\right),\end{aligned} \tag{5.6}$$

where the second equality follows from the fact that $\sum_{k=1}^n |r_k(s)|^\alpha = 1$, for μ-almost all $s \in S$. Relation (5.6) implies the decomposition (1.1).

We now prove the 'only if' part. Suppose that (1.1) holds and let $\{f_t^{(k)}\}_{t\in T} \subset L^\alpha(V_k, \mathcal{B}_{V_k}, \nu_k),\ k = 1,\ldots,n$ be representations for the independent components $\{X_t^{(k)}\}_{t\in T}, k = 1,\ldots,n$, respectively, and without loss of generality, assume that $\{V_k\}_{k=1,\ldots,n}$ are mutually disjoint. Introduce the measure space $(V, \mathcal{B}_V, \nu)$, where $V := \bigcup_{k=1}^n V_k, \mathcal{B}_V := \{\bigcup_{k=1}^n A_k,\ A_k \in \mathcal{B}_{V_k},\ k = 1,\ldots,n\}$ and $\nu(A) := \sum_{k=1}^n \nu_k(A \cap V_k)$ for all $A \in \mathcal{B}_V$.

By (1.1), it follows that $\{X_t\}_{t\in T} \overset{\mathrm{d}}{=} \{\int_V g_t \mathrm{d}\overline{M}_\alpha\}_{t\in T}$, with $g_t(u) := \sum_{k=1}^n f_t{}^{(k)}(u)\mathbf{1}_{V_k}(u)$ and $\overline{M}_\alpha$ an $S\alpha S$ random measure on $(V, \mathcal{B}_V)$ with control measure ν.

Thus, $\{f_t\}_{t\in T} \subset L^\alpha(S, \mathcal{B}_S, \mu)$ and $\{g_t\}_{t\in T} \subset L^\alpha(V, \mathcal{B}_V, \nu)$ are two representations of the same process X, and by assumption the former is *minimal*. Therefore, by Remark 2.5 in [16], there exist modulo ν unique functions $\Phi : V \to S$ and $h : V \to \mathbb{R}\setminus\{0\}$, such that, for all $t \in T$,

$$g_t(u) = h(u) f_t \circ \Phi(u), \quad \text{almost all } u \in V, \tag{5.7}$$

where moreover $\mu = \nu_h \circ \Phi^{-1}$ with $\mathrm{d}\nu_h = |h|^\alpha \mathrm{d}\nu$.

Recall that V is the union of mutually disjoint sets $\{V_k\}_{k=1,\ldots,n}$. For each $k = 1,\ldots,n$, let $\Phi_k : V_k \to S_k := \Phi(V_k)$ be the restriction of Φ to V_k, and define the measure $\mu_k(\cdot) := \nu_{h,k} \circ \Phi_k{}^{-1}(\cdot \cap S_k)$ on $(S, \mathcal{B}_S)$ with $\mathrm{d}\nu_{h,k} := |h|^\alpha \mathrm{d}\nu_k$. Note that μ_k has support S_k, and the Radon–Nikodym derivative $\mathrm{d}\mu_k/\mathrm{d}\mu$ exists. We claim that (1.6) holds with $r_k := (\mathrm{d}\mu_k/\mathrm{d}\mu)^{1/\alpha}$. To see this, observe that for all $m \in \mathbb{N}, a_1,\ldots,a_m \in \mathbb{R}, t_1,\ldots,t_m \in T$,

$$\int_S\left|\sum_{j=1}^m a_j r_k f_{t_j}\right|^\alpha \mathrm{d}\mu = \int_{S_k}\left|\sum_{j=1}^m a_j f_{t_j}\right|^\alpha \mathrm{d}\mu_k = \int_{V_k}\left|\sum_{j=1}^m a_j h f_{t_j} \circ \Phi_k\right|^\alpha \mathrm{d}\nu_k,$$

which, combined with (5.7), yields (1.6) because $g_t|_{V_k} = f_t^{(k)}$.

Note also that $\sum_{k=1}^n \mu_k = \mu$ and thus $\sum_{k=1}^n r_k^\alpha = 1$. This completes the proof of part *(i)* of Theorem 1.1 in the case when $\{f_t\}_{t\in T}$ is minimal.

To prove part (ii), note that the r_k's above are in fact non-negative and $\mathcal{B}_S$-measurable. Note also that by minimality, the r_k's have versions $\widetilde{r}_k$'s that are $\rho(F)$-measurable, i.e. $r_k = \widetilde{r}_k$ modulo μ. Their uniqueness follows from Lemma 5.1. □

Proof of Theorem 1.1. (i) The 'if' part follows by using characteristic functions as in the proof of Proposition 5.1 above.

Now, we prove the 'only if' part. Let $\{\widetilde{f}_t\}_{t\in T} \subset L^\alpha(\widetilde{S}, \mathcal{B}_{\widetilde{S}}, \widetilde{\mu})$ be a minimal representation of X. As in the proof of Proposition 5.1, by Remark 2.5 in [16], there exist modulo μ unique functions $\Phi : S \to \widetilde{S}$ and $h : S \to \mathbb{R} \setminus \{0\}$, such that, for all $t \in T$,

$$f_t(s) = h(s)\widetilde{f}_t \circ \Phi(s), \quad \text{almost all } s \in S, \tag{5.8}$$

and $\widetilde{\mu} = \mu_h \circ \Phi^{-1}$ with $\mathrm{d}\mu_h = |h|^\alpha \mathrm{d}\mu$.

Now, by Proposition 5.1, if the decomposition (1.1) holds, then there exist unique non-negative functions $\widetilde{r}_k$, $k = 1, \dots, n$, such that

$$\{X_t^{(k)}\}_{t\in T} \stackrel{d}{=} \left\{ \int_{\widetilde{S}} \widetilde{r}_k \widetilde{f}_t \mathrm{d}\widetilde{M}_\alpha \right\}_{t\in T}, \quad k = 1, \dots, n, \tag{5.9}$$

and $\sum_{k=1}^n \widetilde{r}_k^\alpha = 1$ modulo $\widetilde{\mu}$. Here $\widetilde{M}_\alpha$ is an $S\alpha S$ measure on $(\widetilde{S}, \mathcal{B}_{\widetilde{S}})$ with control measure $\widetilde{\mu}$. Let $r_k(s) := \widetilde{r}_k \circ \Phi(s)$ and note that by using (5.8) and a change of variables, for all $a_j \in \mathbb{R}, t_j \in T$, $j = 1, \dots, m$, we obtain

$$\int_S \left| \sum_{j=1}^m a_j r_k(s) f_{t_j}(s) \right|^\alpha \mu(\mathrm{d}s) = \int_{\widetilde{S}} \left| \sum_{j=1}^m a_j \widetilde{r}_k(s) \widetilde{f}_{t_j}(s) \right|^\alpha \widetilde{\mu}(\mathrm{d}s). \tag{5.10}$$

This, in view of Relation (5.9), implies (1.6). Further, the fact that $\sum_{k=1}^n \widetilde{r}_k^\alpha = 1$ implies $\sum_{k=1}^n r_k^\alpha = 1$, modulo μ, because the mapping Φ is non-singular, i.e. $\widetilde{\mu} \circ \Phi^{-1} \sim \mu$. This completes the proof of part (i).

We now focus on proving part (ii). Suppose that (1.6) holds for two choices of r_k, namely r'_k and r''_k. Let also r'_k and r''_k be non-negative and measurable with respect to $\rho(F)$. We claim that

$$\rho(F) \sim \Phi^{-1}(\rho(\widetilde{F})) \tag{5.11}$$

and defer the proof to the end. Then, since the minimality implies that $\mathcal{B}_{\widetilde{S}} \sim \rho(\widetilde{F})$. r'_k and r''_k are measurable with respect to $\rho(F) \sim \Phi^{-1}(\mathcal{B}_{\widetilde{S}})$. Now, Doob–Dynkin's lemma (see e.g. [15, p. 30]) implies that

$$r'_k(s) = \widetilde{r}'_k \circ \Phi(s) \quad \text{and} \quad r''_k(s) = \widetilde{r}''_k \circ \Phi(s), \quad \text{for } \mu \text{ almost all } s, \tag{5.12}$$

where $\widetilde{r}'_k$ and $\widetilde{r}''_k$ are two $\mathcal{B}_{\widetilde{S}}$-measurable functions. By using the last relation and a change of variables, we obtain that (5.10) holds with $(r_k, \widetilde{r}_k)$ replaced by $(r'_k, \widetilde{r}'_k)$ and $(r''_k, \widetilde{r}''_k)$, respectively. Thus, both $\{\widetilde{r}'_k \widetilde{f}_t\}_{t\in T}$ and $\{\widetilde{r}''_k \widetilde{f}_t\}_{t\in T}$ are representations of the k-th component of X. Since $\{\widetilde{f}_t\}_{t\in T}$ is a minimal representation of X, Lemma 5.1 implies that $\widetilde{r}'_k = \widetilde{r}''_k$ modulo $\widetilde{\mu}$. This, by (5.12) and the non-singularity of Φ yields $r'_k = r''_k$ modulo μ.

It remains to prove (5.11). Relation (5.8) and the fact that $h(s) \neq 0$ imply that for all λ and $t_1, t_2 \in T$, $\{f_{t_1}/f_{t_2} \le \lambda\} = \Phi^{-1}(\{\widetilde{f}_{t_1}/\widetilde{f}_{t_2} \le \lambda\})$ modulo μ.Thus the classes of sets $\mathcal{C} := \{\{f_{t_1}/f_{t_2} \le \lambda\},\ t_1, t_2 \in T,\ \lambda \in \mathbb{R}\}$ and $\widetilde{\mathcal{C}} := \{\Phi^{-1}(\{\widetilde{f}_{t_1}/\widetilde{f}_{t_2} \le \lambda\}),\ t_1, t_2 \in T,\ \lambda \in \mathbb{R}\}$ are equivalent. That is, for all $A \in \mathcal{C}$, there exists $\widetilde{A} \in \widetilde{\mathcal{C}}$, with $\mu(A \Delta \widetilde{A}) = 0$ and vice versa.

Define

$$\widetilde{\mathcal{G}} = \left\{ \Phi^{-1}(A) : A \in \rho(\widetilde{F}) \quad \text{such that } \mu(\Phi^{-1}(A)\Delta B) = 0 \text{ for some } B \in \sigma(\mathcal{C}) \right\}.$$

Note that $\widetilde{\mathcal{G}}$ is a σ-algebra and since $\widetilde{\mathcal{C}} \subset \widetilde{\mathcal{G}} \subset \Phi^{-1}(\rho(\widetilde{F}))$, we obtain that $\sigma(\widetilde{\mathcal{C}}) = \Phi^{-1}(\rho(\widetilde{F})) \equiv \widetilde{\mathcal{G}}$. This, in view of definition of $\widetilde{\mathcal{G}}$, shows that for all $\widetilde{A} \in \sigma(\widetilde{\mathcal{C}})$, exists $A \in \sigma(\mathcal{C})$ with

$\mu(A\Delta\widetilde{A}) = 0$. In a similar way, one can show that each element of $\sigma(\mathcal{C})$ is equivalent to an element in $\sigma(\widetilde{\mathcal{C}})$, which completes the proof of the desired equivalence of the σ-algebras. □

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